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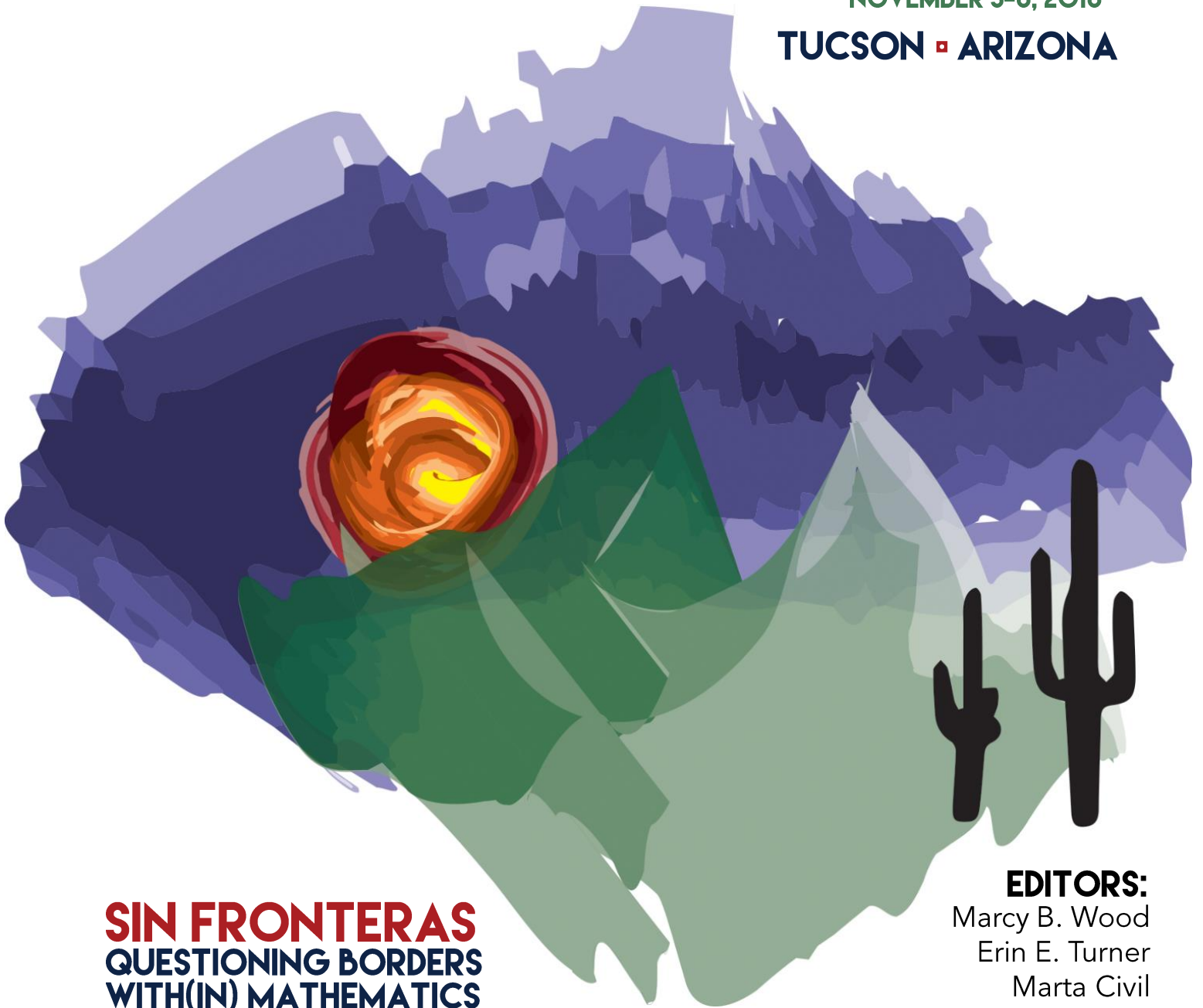
THIRTY-EIGHTH ANNUAL MEETING
OF THE NORTH AMERICAN CHAPTER OF THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION

NOVEMBER 3–6, 2016
TUCSON ▫ ARIZONA

SIN FRONTERAS
QUESTIONING BORDERS WITH(IN) MATHEMATICS EDUCATION

EDITORS:
Marcy B. Wood
Erin E. Turner
Marta Civil
Jennifer A. Eli

THE UNIVERSITY OF ARIZONA
College of Education

THE UNIVERSITY OF ARIZONA
COLLEGE OF SCIENCE
Mathematics

Proceedings of the Thirty-Eighth Annual Meeting of the North American Chapter of the International Group for the

Psychology of Mathematics Education

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PME-NA History and Goals

PME came into existence at the Third International Congress on Mathematical Education (ICME-3) in Karlsrühe, Germany in 1976. It is affiliated with the International Commission for Mathematical Instruction. PME-NA is the North American Chapter of the International Group of Psychology of Mathematics Education. The first PME-NA conference was held in Evanston, Illinois in 1979.

The major goals of the International Group and the North American Chapter are:

1. To promote international contacts and the exchange of scientific information in the psychology of mathematics education;
2. To promote and stimulate interdisciplinary research in the aforesaid area, with the cooperation of psychologists, mathematicians, and mathematics teachers;
3. To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

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Membership is open to people involved in active research consistent with PME-NA's aims or professionally interested in the results of such research. Membership is open on an annual basis and depends on payment of dues for the current year. Membership fees for PME-NA (but not PME International) are included in the conference fee each year. If you are unable to attend the conference but want to join or renew your membership, go to the PME-NA website at http://pmena.org. For information about membership in PME, go to http://www.igpme.org and click on "Membership" at the left of the screen.

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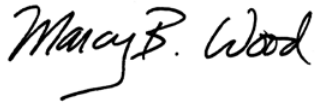
Welcome

On behalf of the 2016 PME-NA Steering Committee, the 2016 PME-NA Local Organizing Committee, and the University of Arizona, we welcome you to the 38th Annual Meeting of the International Group for the Psychology of Mathematics Education – North American Chapter held at the Marriott Starr Pass Resort in Tucson, Arizona.

The theme of this year's conference is *Sin Fronteras: Questioning Borders with(in) Mathematics Education*. This theme is intended to encourage research presentations, discussion, and reflection on the variety of borders within mathematics education, as well as those that might be probed, challenged, explained, enhanced and/or potentially transformed by mathematics education. Sessions examine topics such as geographic, political, cultural and language borders with(in) mathematics education, as well as borders between mathematical content areas and grade levels, and how borders can limit access to mathematical content or impact mathematics teaching practices.

To catalyze conversations about the conference theme, we organized three plenary talks that each address different perspectives on borders with(in) mathematics education. In the Thursday evening plenary*, Sandra Crespo* reflects on three intellectual divides that have challenged and motivated her scholarship across her many years of participation in PME-NA. *Anna Sfard* will follow with comments, and further discussion of the conference theme. In the Friday joint plenary session, *Maria Trigueros Gaisman* and *Chris Rasmussen* discuss their respective work in coordinating multiple theoretical perspectives and frameworks in analyzing mathematics learning at the undergraduate level. In the Saturday panel plenary session, scholars that represent the three PME-NA member countries (*Julia Aguirre, Lisa Lunney Borden*, and *Olimpia Figueras*) will address the idea of challenging borders in mathematics education from different perspectives (e.g., a focus on students, focus on teachers) and in different contexts (Canada, Mexico, US).

We are thrilled to report record attendance for PME-NA 2016! This year's conference will be attended by over 550 researchers, faculty and graduate students from around the world including the US, Mexico, Canada, Sweden, Australia, Chile, Puerto Rico, Israel, Trinidad and Tobago, and Germany. We received 581 submissions. The acceptance rate was 92% for working groups, 48% for research reports, 36% for brief research reports and 61% for posters. The conference program includes 104 Research Report sessions, 102 Brief Research Report sessions, 222 Posters, and 11 Working Groups. A significant aspect of this year's conference is that for the first time in PME-NA, there will be some presentations in Spanish, as well as simultaneous oral interpretation (from English to Spanish, and from Spanish to English) for selected sessions, thus embracing the theme of the conference, Sin Fronteras/Without Borders. It is the hope of the organizers that this would a model to follow in future PME-NA conferences.

We would also like to thank the many people who generously volunteered their time over the past year in preparation for this conference. This includes members of the PME-NA Local Organizing Committee, the PME-NA Steering Community, UA Continuing and Professional Education, strand leaders, proposal authors and reviewers. We appreciate all of your hard work and dedication, and your commitment to ensuring a high-quality conference program. We also wish to thank the generous financial support of the University of Arizona (UA) College of Education, the UA Department of Mathematics, and Visit Tucson.

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PME-NA 38 Conference Co-Chair

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PME-NA 38 Conference Co-Chair

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Chapter 1

Plenary Papers

EN LA LUCHA/IN THE STRUGGLE: RESEARCHING TO MAKE A DIFFERENCE IN MATHEMATICS EDUCATION

Sandra Crespo
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In this plenary I reflect on the research I have shared over the past many years of participating and presenting in PME and PME-NA conferences to identify three intellectual divides that continually challenge and motivate my scholarship. I discuss how these intellectual borders (mathematics/education; expert/novice; research/teaching) create adversarial relationships and unwarranted hierarchies in our field and among ourselves. Although these divides have been with us for a long time, I contend that these are far more dangerous than we might realize. These divides prevent us individually and collectively from imagining new and creative solutions to the perennial question about how to improve the quality and equity of mathematics education. I propose a reflexive and collaborative approach to identifying and problematizing the intellectual divides that challenge our goals and commitments to pursuing research that makes a difference in the very communities that we seek to serve.

En esta plenaria reflexiono sobre las investigaciones que he compartido durante los muchos años que he participado y presentado en las conferecias de PME y PME-NA para identificar tres divisiones intelectuales que continuamente retan y motivan mi trabajo académico. Discuto cómo estas fronteras intelectuales (matemáticas/educación; expert(a)/novato(a); investigación/enseñanza) crean relaciones adversas y jerarquías injustificadas en nuestro campo y entre nosotros(as). A pesar de que estas divisiones hace mucho tiempo que están con nosotros(as), sostengo que éstas son mucho más peligrosas de lo que podríamos darnos cuenta. Estas divisiones nos impiden individual y colectivamente imaginar soluciones nuevas y creativas a la eterna pregunta de cómo mejorar la calidad y equidad de la educación matemática. Propongo un enfoque reflexivo y colaborativo para identificar y problematizar divisiones intelectuales que desafían nuestras metas y compromisos de hacer investigación para contribuir a las comunidades que queremos servir.

Keywords: Equity and Diversity, Policy Matters, Research Methods

Introduction

I became a member of this organization in the year 2000 when I attended and presented at my very first PME-NA conference in Tucson, AZ. I was an early career faculty then and had just become a new mom. I traveled with my 5-month old in tow and my mother who looked after baby while I was presenting and attending the conference. I have fond memories of Tucson and continue to have a feeling of being *con familia* within this organization. Little did I know then that I would be back in Tucson 16 years later as a plenary speaker at this very conference. PME-NA has been a nurturing community in which to share my works in progress, test out my ideas, and learn and grow as a scholar. I have attended almost every year since the year 2000, in fact I counted a total of 18 publications in PME and PME-NA proceedings.

It is with trepidation that I accepted the invitation to deliver this year's PME-NA plenary address. I admit that plenaries are not my thing. Delivering a conference plenary has never been one of my professional aspirations. I much prefer dialogical and more interactive presentations and workshops where I can engage with the audience rather than standing in a podium talking at the audience. When I asked the program committee why they had chosen me the response was that I am the poster child for this year's conference theme: *"Sin Fronteras! - Without Borders!"* as I am someone who is constantly crossing geographical, cultural, linguistic, and intellectual borders. I could see their point

and decided to accept the invitation. After all I am not one to back down from a challenge or from stretching myself beyond my comfort zone.

I am engaged in mathematics education research because I strive to contribute to improving mathematics education in ways that align with the goals and values of democratic and anti-oppressive education. I am especially interested in learning and teaching practices that redistribute power and challenge stereotypes and hierarchies in the mathematics classroom, and this has pushed me to see social interactions from multiple perspectives and theoretical lenses. I approach my work in collaboration with colleagues, schools, and teachers committed to social change. I do this work across three countries, Dominican Republic, Canada, and the U.S. Within mathematics education I straddle the worlds of elementary/secondary education, of formal/informal mathematics, of theory/practice and of equity/excellence debates and debacles. More importantly, I have learned to embrace the tension and burden of working within and across these many communities and boundaries.

While it is true that my work crosses boundaries this is not unique to my scholarship. I would argue that we are all in some way or another navigating multiple personal and professional communities that require us to negotiate interactions that challenge us and that nurture us. So I am here not to claim that I have something unique to share or to stake claim to a piece of intellectual property that is solely my own. To the contrary, the work I have done over the past 20+ years as a mathematics educator has been possible because it has taken a whole village of collaborators who have helped me to keep front and center my commitment to anti-oppressive education and to remain hopeful that as math educators we can make a difference. So my approach to this plenary is to reflect on the kinds of boundaries I have had to cross throughout my career, taking stock of the work I have presented and published at PME-NA, to make visible intellectual divides that I consider dangerous and worthy of bridging and eventually take down.

I use "in the struggle/en la lucha" in the title of this plenary to remind myself of Paulo Freire's (1970) pedagogy of hope in which he discusses our struggle as educators to work within the system that oppresses us and that we seek to change. I am also channeling bell hooks' (1990) idea of teaching to transgress where she calls on educators to challenge ourselves to find new ways of thinking about teaching and about learning so that our work "does not reinforce systems of domination, imperialism, racism, sexism, elitism." (p. xx). It is in that spirit, of dreaming big and dreaming the impossible that I then take the opportunity of this plenary to identify intellectual divides that continually push me and my scholarship and that unnecessarily drain our collective energy to address the problems facing public education today. I use this opportunity to reflect on my own work and how it has been challenged by pernicious intellectual divides that create adversarial relationship and unwarranted hierarchies in our field and among ourselves.

Fronteras Intelectuales and Dangerous Divides

Towards the middle of last century, in an influential lecture, C. P. Snow (1959) identified "two cultures" within academic circles that threatened the whole enterprise of the University as a place that values diversity of intellectual pursuits and epistemologies. A border crosser himself Snow spoke as a participant in both literary and scientific communities about the deep rooted divide between two fields —the literary intellectuals and the scientists—and how each exalted its own virtues by vilifying the other's values. He described them as two polar groups: the literary intellectuals at one pole and at the other the scientists. "Between the two a gulf of mutual incomprehension. They have a curious distorted image of each other." (Snow, 1959, p.4). Snow's characterization highlighted that the literary intellectuals value nuance, subtlety, depth, responsiveness and imagination, whereas scientists will talk about those qualities as touchy-feely and fuzzy-minded subjectivism. Similarly, the scientists value rationality, objectivity and functional prose

while literary scholars consider those qualities dull, literal minded, and lacking depth of understanding.

In "Disciplinary Cultures and Tribal Warfare," a chapter in her book "Scandalous Knowledge," Herrstein Smith (2006) explains the dangers of creating intellectual camps and hierarchies and revisits C. P. Snow's two cultures adding that the tendency to polarize, compare, and rank ourselves is part of what all social groups do, including academics and intellectuals. In academic circles this is known as the ideology of the two cultures and refers to our tendency to identify ourselves with one or more social groups (e.g., religious, ethnic, political, professional), to experience that identity through contrast and comparison to one or more other groups - or, in other words, to experience the world in terms of 'us' and 'them'. This is known as a tendency to self-standardize and other-pathologize, said another way "to see the practices, preferences and beliefs of one's own group as natural, sensible and mature and to see the divergent practices, preferences and beliefs of members of other groups, especially those considered as the 'other', as absurd, perverse, undeveloped or degenerate" (Herrstein Smith, 2006; p. 113). Another consideration is that this tendency to pathologize the other is self-perpetuating in that these are invoked and circulated as ideological narratives within and across various communities.

In mathematics education there are numerous intellectual divides to choose from (see Stinson & Bullock, 2012; Davis, 2004; Davis, Sumara, & Luce-Kapler, 2015). In the 80's the quantitative/qualitative debate took center stage as did the constructivism vs. social theories of learning. The 90's witnessed the cognition vs. communication, and acquisition vs. participation debates (Sfard, 1998), while the 2000's experienced the sociocultural vs. sociopolitical divide (Gutiérrez, 2013). These debates have been played out in the intellectual domain and among academics and eventually have slipped into the everyday conversations of schools and universities as ideological narratives that cast polar opposite characters (reform vs. traditional) battling out intellectual wars. Although these debates have faded they still frame current conversations and practices in mathematics education. Furthermore, they fall into the dualistic intellectual tradition that Snow (1959) characterized as the ideology of the two cultures and that Herrstein Smith (2006) describes in her writings as the tendency to self-standardize and other-pathologize.

I will focus here on three enduring divides that have not had as much play as those named above but are ever present in our everyday practices as mathematics educators and fuel an "us vs. them" mentality as described in the ideology of the two cultures. These are: a) Mathematics/Education, (b) Expert/Novice, and (c) Research/Teaching. I contend that these divides may seem innocuous but are nevertheless more dangerous than they appear to be. As I looked back across my PME and PME-NA publications with these three divides in mind, I could see how these have been and still continue to be a challenge in my own scholarship but also to our field more broadly. Although I could see all three divides in each of these articles, when I considered which divide was most foregrounded the following groupings emerged—7 articles foregrounding (a) [the mathematics/education divide], 5 of them foregrounding (b) [the expert/novice divide], and 6 articles foregrounding (c) [the research/teaching divide]. Rather than synthesizing the three groupings I use one representative article (see below) to springboard the discussion on each intellectual divide. I purposefully picked articles that are 6-7 years apart so that they represent broadly the scholarship that I have been engaged in over the many years I have been a part of the PME and PME-NA organization.

1. **Mathematics/Education** –Crespo, S. (2000). Learning mathematics while learning to teach: Mathematical insights prospective teachers experience when working with students. In M. L. Fernández (Ed). *Proceedings of the North American Chapter of the International Group for the Psychology of Mathematics Education, 22,* (vol. 2, pp. 549-554). Columbus, OH: ERIC Clearinghouse.

2. **Expert/Novice** –Crespo, S., Oslund, J. A., & Parks, A. (2007) Studying elementary preservice teachers' learning of mathematics teaching: preliminary insights. (pp. 975-982). In *Lamberg, T., & Wiest, L. R. (Eds.). Proceedings of the North American Chapter of the International Group for the Psychology of Mathematics Education, Stateline (Lake Tahoe), NV: University of Nevada, Reno.*

3. **Research/Teaching** –Crespo, S. (2013). Getting smarter together about complex instruction in the mathematics classroom. *PME Newsletter (March-April 2013)*, pp. 4-5, Psychology of Mathematics Education (PME International).

The Mathematics/Education Divide

Looking back at my very first PME-NA presentation I can see the mathematics/education divide prominently highlighting the separation between where and how teacher candidates can learn mathematics in their teacher preparation programs. I experienced this divide both in my own undergraduate education as I traveled from one side of campus, where I was studying mathematics and physics to the other side of campus where I was taking education classes. This structural divide continues to persist and is very present in my own practice as a mathematics teacher educator. The very structure of teacher preparation programs in general continues to reaffirm the mathematics/education divide by locating the learning of mathematics content in designated math courses and separating it from the learning of teaching methods contained in education courses. Embedded within the structure is also the assumption that learning to teach entails learning the content first and the teaching methods second (rather than concurrently).

In *Learning mathematics while learning to teach: Mathematical insights prospective teachers experience when working with students* (Crespo, 2000), I argued that prospective teachers engage in mathematical inquiry within their education courses and in particular when working directly with students. I provided three examples—posing tasks, analyzing students' work, and providing mathematical explanations—where teacher candidates could gain mathematical insights while learning educational methods and theories. This surely is no longer a controversial point, but at the time mathematics educators were just beginning to consider Ma's (1999) work and Ball and Bass' (2000) work describing the profound understanding of mathematics entailed in the work of elementary mathematics teaching. The push back from mathematics educators who dug their feet firmly into the mathematics side of the divide was intense, making anything that they did not recognize as mathematical sound crazy or simply stupid. Therefore, the process of selecting examples that were recognizable as mathematical by those holding dominant perspectives about mathematics was a challenge.

Let me provide a few illustrations. In Crespo (2000) I included several examples to illustrate the ways in which mathematical questions and insights arise when prospective teachers work on teacher preparation course projects that have them exploring mathematics with students. In one example I shared how, when interviewing a 2nd Grader about her strategies for sharing cookies among different number of people, a prospective teacher found her student conjecturing that if the number of cookies was even, it could be shared evenly among people, and that if the number of cookies was odd, it could not. The young student concluded this after having shared several even numbers of cookies, such as sharing 30 cookies among 3 and then 5 people. In this situation the prospective teacher found herself in a position of exploring this student's conjecture by offering her several more examples to have the student test her conjecture and see whether or not it does or does not work for other cases.

In another example a prospective teacher had adapted a mathematics problem (Watson, 1988) we had explored in our university class to try it out with fifth graders in her field placement. This problem read:

> Three tired and hungry monsters went to sleep with a bag of cookies. One monster woke up and ate *1/3* of the cookies, then went back to sleep. Later a second monster woke up and ate *1/3* of the remaining cookies, then went back to sleep. Finally, the third monster woke up and ate *1/3* of the remaining cookies. When she was finished there were 8 cookies left. How many cookies were in the bag originally?

The prospective teacher chose to rescale the problem by changing the fractional number in the problem from *1/3* to **1/2**. By doing so, she made an interesting discovery, that is, that her students were able to arrive at the correct answer by using a restrictive solution method that in fact does not work for the original version of the problem. Students had approached the problem by multiplying the left over cookies by 2 (8x2x2x2), basically doubling the left over cookies three times. Yet, even though this method works for halves, it yields an incorrect answer for thirds, fourths, and any other fractional part. This unexpected outcome launched the prospective teacher into her own mathematical investigation into the reasons for how and why such a minor numerical change could alter the nature of the original problem (Crespo, 2000).

I have made similar and related arguments about mathematics as a practice that occurs and is learned everywhere not solely inside mathematics classrooms and most definitely not solely in coursework offered in mathematics departments. I recognize the history of why and how disciplinary knowledge broke off and was elevated from the everyday knowledge and practices and the privileges that this affords to those of us in the field of mathematics education. However, to me mathematics is a human practice that belongs to all of us not solely to mathematicians (Bishop, 1990). Hence throughout my career I have argued that it is especially important for prospective teachers to consider their teaching as a site for mathematical inquiry and for problem posing with their students and to find ways to explore the mathematics that students learn in their communities and in out of school contexts. I have continued to address the mathematics/education divide in multiple ways and especially as I have increasingly foregrounded educational equity within the curriculum and pedagogy of the mathematics education courses for future elementary and secondary mathematics teachers. If concerns and push back about "where is the mathematics?" or "how is this mathematics?" were raised with regards to learning mathematics through learning mathematics pedagogy, the push back to infusing educational equity in the teaching of mathematics has been even that more forceful.

The divide between mathematics and education continues to be reflected in the intellectual but also in the physical divide found on most University campuses. This divide contributes to the lack of coherence and continuity in the curriculum and pedagogy of teacher preparation (Feiman-Nemser, 2001). Mathematics courses are offered in mathematics departments, taught by instructors who do not address questions that concern educators. Education courses in turn are offered in colleges of education and are typically focused on educational issues without attending to specific content issues. The mathematics methods course is also influenced by this divide. Instructors of these courses often assume that teacher candidates have to "unlearn" oppressive approaches to the teaching and learning of mathematics that they have picked up in the math courses they have taken. The rift between mathematics educators who work in colleges of natural science and mathematics educators who work in colleges of education is very palpable at my current institution and I suspect across many other institutions as well.

As a mathematics educator who has colleagues in the college of natural science and in the college of education I am constantly challenged by both sides to see their perspective while neither side seems to see their own biases and entrenched ideologies. One side asks and insists on raising the question of "where is the math" whenever the conversation is focused on educational issues that transcend the narrow particulars of the discipline of mathematics as constructed and practiced by research mathematicians. I constantly hear the "where is the math" question raised in faculty

meetings, in students' comprehensive exams, in dissertations, and in colloquia. My education colleagues, on the other hand, ask and insist on raising questions about whether mathematics as a discipline can be trusted to embrace democratic ideals when so much of what is wrong and objectionable about today's public schooling can be attributed to the way mathematics is used to exclude and deny access to college to a large majority of non-white students, not to mention the oppressive ways in which mathematics continues to be taught and learned in schools.

To be clear, I consider the mathematics/education divide as dangerous because it shapes interactions among ourselves with colleagues on our campuses and members of various other communities. It instantiates the tendency to self-standardize and other-pathologize discussed earlier. It forcefully comes into play when faculty is engaged in doctoral admissions or discussing prospective colleagues who have or do not have a so called "strong" mathematical background or do not have a so called "substantial" classroom teaching experience. With each side digging their heels more deeply into their own camp they continue to reproduce their perspectives and pathologize the other. The danger lies in how this divide breeds toxic and deficit discourses within our own academic communities which not surprisingly is expressed outwardly through our research onto the very communities we are hoping to help (Shields, Bishop, & Mazawi, 2005). This intellectual divide becomes normalized and replicated in our teacher prepraration programs and travels to our partner schools. It undermines our goals to make mathematics a subject that many and more diverse groups of students engage with and enjoy, and a subject that supports the democratic values and ideals of public education. Not challenging this divide propagates the ideology that one field of study is more important than the other. It generates categories of students which are liberally applied to elementary prospective teachers and breeds the dominant narrative about elementary teacher candidates' "lack of knowledge" of mathematics. This issue speaks to the next divide — the expert/novice divide—which I discuss next.

The Expert/Novice Divide

Another divide always present in mathematics education is the categorization of experts and novices. I consider this to be another dangerous divide because the experts become the norm by which everyone else is judged and evaluated. It creates a hierarchy and a social reward system that promotes a rush to mastery which undermine and shortchanges the process of learning. Additionally, if the category of expert is associated with natural talent as it is often the case for mathematics and for teaching, gaining such expertise becomes unattainable for novices—let those be elementary age students or teacher candidates in undergraduate mathematics content or methods courses. Worse still, it suggests that only a few can ever be experts in the teaching and learning of mathematics.

In *Studying elementary preservice teachers' learning of mathematics teaching: Preliminary insights* (Crespo, Oslund, & Parks, 2007), I worked to conceptualize a study that explored how prospective teachers learn to enact the practices of posing, interpreting, and responding (PIR project) during teacher preparation courses and experiences (Crespo, 2006). In that PME-NA presentation I argued that prospective teachers were most likely learning mathematics teaching practices that had not yet been documented in the mathematics education literature because the dominant research frames and tools were focused on a very narrow set of desirable teaching practices. If the window for what constitutes an expert performance is narrowly defined, then the bulk of what can and will be observed would be classified as not meeting expert quality, and by default they become novice performances or worse considered as examples of not very good teaching.

In the 2007 PME-NA research presentation (and at a later PME-NA presentation in Crespo et al., 2009) I discussed how and why we decided to revise our initial assumptions about expert/novice enactments of teaching practice. As a member of another research project, the TNE project (Battista et al., 2007), I was able to use similar research tools in order to explore the relation between mathematics knowledge for teaching (MKT) and PIR practices (see Table 1). Working on both these

projects at the same time allowed me to see quite a few strange results that called into question assumptions about what experts and novices do/don't know and can/cannot do in their teaching of mathematics. Results from the TNE-Math surveys for example which were administered concurrently to prospective teachers at different stages in the program (studying math content and study math methods) had us looking at a number of very strange results such as a decline in mathematics knowledge for teaching (MKT) as prospective teachers transitioned from learning about content to learning about teaching practice.

Table 1: TNE and PIR teaching scenarios focused on two-digit subtraction

TNE Project – MKT Scenario	PIR Project – Teaching Practice Scenario
W1. *Imagine that one of your students shows you the following strategy for subtracting whole numbers.* $$\begin{array}{r} 37 \\ \underline{-19} \\ -\ 2 \\ \underline{20} \\ 18 \end{array}$$ **W1a**. *Do you think that this strategy will work for any two whole numbers?* Yes No I don't know **W1b.** *How do you think the student would use this strategy in the problem below?* $$\begin{array}{r} 423 \\ \underline{-167} \end{array}$$	**PIR2a.** Imagine you are teaching a lesson about two-digit subtraction and you ask the class to explore different ways to solve the following subtraction. The students look puzzled. What do you imagine <u>saying</u> and <u>doing</u> next? $$\begin{array}{r} 37 \\ \underline{-19} \end{array}$$ **PIR2b.** After giving students some time to work on the task you call on their attention and ask for volunteers to share their strategies. Imagine that one of the students shows the following strategy. What can you imagine <u>saying</u> and <u>doing</u>? Say a bit about what you would want to accomplish by saying and doing so. $$\begin{array}{r} 37 \\ \underline{-19} \\ -\ 2 \\ \underline{20} \\ 18 \end{array}$$

Another curious result was uncovered when the PIR team compared the prospective teachers' MKT and PIR responses to tasks such as those in Table 1. In his 2009 PME-NA presentation Brakoniecki (2009), then a graduate research assistant to both projects, reported on prospective teachers who had participated in both the TNE and PIR projects. He showcased three prospective teachers who had correctly addressed the MKT question about generalizing a student subtraction algorithm using negative numbers (see Table 2). All three teacher candidates showed that they could apply the alternative algorithm to a new example. However, their instructional responses to the PIR teaching scenario were all very different (see Table 2) and raised all sorts of questions for the PIR team about the relationship between MKT and PIR practices. So here we have three novices, Dean, Becky, and Lisa (all pseudonyms), who demonstrate that they can do the mathematics that is required to assess the validity and generalizability of an alternative computation algorithm that a student may offer in their classroom, but each of them responds quite differently to a hypothetical teaching scenario. Becky disapproves and does not seem to appreciate the value of this algorithm, Dean seems willing to accept students' algorithms as long as they can show and explain their work, and Lisa makes connections between the standard and alternative algorithms as she expresses her view that there are "more than one way to solve a problem."

Table 2: Three prospective teachers' responses to MKT task and PIR teaching scenario

Responses to MKT task	Dean Response to PIR task	Becky Response to PIR task	Lisa Response to PIR task
Correct Responses to TNE task $\begin{array}{r} 423 \\ -167 \\ \hline -4 \\ -40 \\ 300 \\ \hline 256 \end{array}$	I would ask the student to re-write the problem and show each step they took to get to their answer. I would want the students to learn the importance of showing their work and how they can use it to retrace their steps in a problem	I do not like this way – Math for higher on is going to be a lot harder if they learn this now.	This strategy can work. The student knows that we start in the ones column. 7-9= -2. The tens column is also correct, as 30-10=20. Now what the student did was combine -2 and 20, to get 18. We got the same answer. I would want to let the class know that there is more than one way to solve a problem, and it is important to remember that subtraction of multiple digit numbers involves multiple subtractions, depending on how many places are in the number.

So what is a mathematics educator to do with these prospective teachers' responses, classify them as high MKT but then low (Becky), medium (Dean), and high (Lisa) with respect to their instructional practice? What are we to do with prospective teachers like Becky in our teacher preparation courses? Fail them and tell them they are not qualified to teach students? We seem to be willing to do so when they do not know the mathematics and not so willing to take such a stance when they do not know teaching practice. These initial insights made it clear to us that without reframing our assumptions about expert and novice performances of PIR practices we would continue to recreate and reinforce the same type of instruments and make the same kinds of claims about prospective elementary teachers. This would mean and we would continue to propagate the circular and dead-end deficit discourse about students and their teachers (Comber & Kamler, 2004). In Crespo, Oslund, and Parks (2007) we shared our revised definitions which then led us to design new kinds of teaching scenario instruments, ones that invited teacher candidates to provide multiple not just one response to the teaching scenarios, and ones that invited a more dialogical representation of their practice (see Crespo, Oslund, & Parks, 2011).

In the PIR project we were then able to document more of prospective teachers' strengths (could do and were able to do) than deficits. More importantly, it led us to propose another type of teaching scenario tasks that positioned prospective teachers as creators (not just as reproducers) of teaching practice. In this new type of teaching scenario prompt prospective teachers represented a whole class mathematical discussion in the form of a classroom dialogue. I argue that these kinds of dialogical scenarios elicit different kinds of representations from prospective teachers that make visible more of the complex and nuanced ways in which they imagine mathematics teaching practice. Unlike much of the research on prospective and practicing teachers of elementary school mathematics, my PIR project documented many ways in which prospective teachers take up the student-centered and equity-oriented pedagogies they are studying during teacher preparation. I argued that by researching

dialogical representations of mathematics teaching researchers and teacher educators can learn more about how prospective teachers transform what they are studying in teacher preparation courses into purposeful and principled teaching actions. This new insight would not have been possible without challenging and questioning the expert/novice divide that is so engrained within mathematics education's research/teaching practices, which is another divide I discuss next.

The Research/Teaching Divide

The research/teaching divide has been in the education research landscape for a long time as educational research was initially conceived as research **on** teaching and not **with** or **by** teachers. The animosity and distrust between teachers and researchers in the past and still in the present is reminiscent of Snow's (1959) characterization of the two cultures and it can be related to the longstanding divide between the theoretical and the practical. Researchers characterize teaching as resisting change and teachers characterize educational research as irrelevant to their problems of practice. The research/teaching divide became even more heated when some educational researchers proposed the notion of the teacher as researcher, which raised all sorts of debates, push back, and controversy (Cochran-Smith & Lyttle, 1990; 1999). As someone who studies her own teaching practice and who collaborates with teachers and students in the research process I have had to negotiate this divide and address questions about whether my scholarship counts as research or whether my research has made any impact in the everyday practice of teachers. These are questions rooted in the process of self-standardizing and other-pathologizing that I alluded to earlier. The tendency to vilify other perspectives rather than embrace the diversity in our field is very much alive and well in our own academic backyards.

The research/teaching divide has always puzzled me. As a teacher I have always considered myself a researcher of the mathematics I was teaching and of my students' learning, simply stated I considered myself a student of my students' mathematical thinking and learning. Therefore, I find the divide between education practitioners and researchers to be unhelpful and unnecessarily elitist. As a doctoral student I wrote a comprehensive exam paper titled *"What does research got to do with teaching?"* where I explored the contentious relationship between research and teaching and argued that the two had more things alike than things that were different. To me learning, teaching and researching are similar practices rooted in people's desire to inquire and understand what they do not know. Hence, research is no more than another learning practice that has been uprooted from the everyday practices of people and their communities (this is a similar point to the one I made earlier in relation to the mathematics and education divide).

As a researcher interested in educational experiences that are empowering and transformational for students and their teachers I see the boundary between teaching and researching as an unproductive divide. In my work, teaching involves research and research involves teaching, the two are deeply intertwined. In *Getting smarter together about complex instruction in the mathematics classroom* (Crespo 2013), I describe an example in my scholarship where research and teaching seamlessly collaborate to advance the goal of promoting equity in the mathematics classroom. Complex instruction (CI) is a collaborative teaching method that addresses inequitable teaching and learning. Applying the theory of status generalization to classroom interactions Elizabeth Cohen (1994) interpreted students' unequal participation in the classroom as a problem of unequal status. Unequal status breeds competitive behavior which in turn undermine everyone's learning. Status issues are rooted in societal expectations of competence for students who fit and do not fit the dominant culture's views about who is and not intellectually capable. In the mathematics classroom status issues come into play when students from non-dominant groups participate (or not) in learning activities. Rather than seeing students who under participate in the classroom as either disengaged or unmotivated, Cohen (1994) saw these students as systematically excluded from learning

opportunities not only by their teacher but also their peers, but more importantly by the classroom structures which endorsed rather than disrupt competitive forms of interactions among students.

But complex instruction seeks to not only understand unequal participation in the classroom, it seeks to engineer instructional structures and practices that could disrupt unequal peer interactions in the classroom and to promote a more collaborative learning environment. Rather than setting up the classroom as a competitive space for learning where some students rise to the top and some sink to the bottom, complex instruction sets up the classroom for collaboration and as a place where everyone is expected to succeed and to contribute to a greater understanding than it would be possible by one person alone. In a complex instruction classroom, no one is seen as more or less smart. Instead everyone's capacities, abilities, and experiences are acknowledged, valued, and nurtured as resources in the classroom.

Consistent with CI's theory about collaborative participatory learning —that no one is as smart as all of us together—my complex instruction colleagues and I have engaged in this work in ways that require and value each other's perspectives. We realize that simply talking about these issues and becoming aware of them is not enough. This work entails inviting practicing and prospective teachers to work with us on these ideas in the context of learning about lesson studies, which is unsurprisingly also a collaborative approach to teachers' professional learning. We design together complex instruction math lessons and investigate together questions about students' access, participation, and learning in collaborative mathematics lessons (see Crespo & Featherstone, 2012; and Featherstone et al., 2011). This has created a collaborative network of researchers and practitioners with a common goal and who share teaching and research insights across institutional settings using all sorts of communication outlets including social media, teacher blogs, research and practitioner journal articles, book chapters and books, workshops, talleres, and community forums.

En La Lucha/In the Struggle—Mathematics Educators Sin Fronteras

Returning now to the theme of this conference "Sin Fronteras/Without Borders" and how it might be possible to value and embrace diversity of perspectives in light of the issues I have raised here about the intellectual divides we manage to erect in the process of rationalizing and justifying our work as mathematics educators. Here I conclude with two approaches I have taken to counter my own tendency to self-standardize and other-pathologize by pursuing instead a more reflexive and collaborative mathematics education scholarship. A reflexive approach to mathematics education entails holding the mirror back to ourselves to identify ways in which we are complicit in the very things we criticize and seek to change. A reflexive researcher bluntly asks themselves whether their research is making things better or worse (Kleinsasser, 2010). In this case, consider how it is that we create intellectual divides with our own scholarship and practices. As I consider, for example, the extent to which my research reflects my commitments to anti-oppressive mathematics education, I have to wonder how to best represent these commitments through my research methods and practices, and whether my choices and approaches are making things better or worse.

For example, one important commitment I made early on in my career was to write and speak in ways that are accessible, inviting, and free of academic jargon inasmuch as that is possible. This was partially rooted in my own experiences as a non-native speaker of English and the challenges of reading academic papers in a non-native language. Also as a teacher of mathematics I worked hard to demystify the aura of super human intellect that is associated with the very compressed shorthand of mathematical symbolism that keeps so many students in the dark and excluded from using and conversing in mathematics. More importantly, I am continually reminded to question my motives and my hopes for the educational research I choose to pursue by the words of Elliot (1989) one of the authors I read in grad school.

Rather than playing the role of theoretical handmaiden of practitioners by helping them clarify, test, develop, and disseminate the ideas which underpin their practices, academics tend to behave like terrorists. We take an idea which underpins teachers' practices, distort it through translation into academic jargon, and thereby "highjack" it from its practical context and the web of interlocking ideas which operate in that context. (Elliot, 1989; p. 7)

Yet as I hold on to this commitment I also consider the critiques other scholars raise about taking what seems to be a reductionist and simplistic route to explaining complex educational issues. In their view, such an approach to scholarship feeds into rather than challenge the distrust people have of academics and anything that sounds too intellectual or overly complex whether those ideas come from science or the humanities (e.g., Davis et al., 2014). I also understand that our words are critical and that how we name and talk about people, communities, and students matter and shape our thinking and practices. Therefore, I also participate in discussions that seek to clarify, object, and subvert particular terms and language commonly used in research and in practice, especially language that is offensive and degrading to the very students and communities that need us the most.

The point here is that I have come to accept that there is inherent tension and contradictions within the work we do as researchers in mathematics education and appreciate Elbow's (1983; 2000) notion of embracing contraries as a way to see beyond our tendency to polarize and take sides without fully understanding and considering opposing views. Sfard's (1998) discussion of two metaphors for learning (as acquisition and participation) also takes a similar stance about opposing and contradictory perspectives. I have tried out Elbow's ideas in a recent editorial (Crespo, 2016a) for the *Mathematics Teacher Educator* journal which I am currently serving as editor to promote a more educative rather than adversarial approach to reviewing manuscript submissions to the journal. I also explored Elbow's embracing of contraries in a recent publication (Crespo, 2016b) focusing on the challenge to disrupt our tendency to polarize mathematics teaching practice when selecting and using video representations of mathematics teaching. This is an issue that the National Council of Teachers of Mathematics (NCTM) Research Committee (2016) recently discussed and identified as a pernicious storyline that circulate and influence the public perception about mathematics education.

Collaborative research is another way in which I have chosen to pursue research in mathematics education. This is one approach that discourages me from building intellectual divides. I have come to the point of realizing that educational problems are much too big for any one of us to take on and solve by ourselves and that it will take literally a whole village of committed mathematics educators to make the kinds of changes we are all striving to make. All this within a world of higher education and academia that is driven by competitive policies and reward systems. Although this can create hostile working environments for faculty, it is worth investing in developing collaborative networks with colleagues. Operating under the tenants of complex instruction that together we can learn more than individually, and that each collaborator needs to be willing to learn from each other's perspectives, I continually renew my belief and commitment in collaborative mathematics education research. And as I alluded to earlier, my work is only possible by collaborating with colleagues from all walks of life that are committed to social change.

In addition to the example I offered earlier with my complex instruction colleagues with whom I wrote the book *"Smarter Together,"* (Featherstone et al., 2011) I have also collaborated with another network of educators committed to identifying and challenging oppressive forms of mathematics education research and to making our field more inclusive of diverse perspectives and practices (see Herbel-Eisenmann et al., 2013). Another more recent collaboration is a book of cases for mathematics teacher educators (White, Crespo, & Civil, 2016) which includes a collection of 19 cases from different authors highlighting dilemmas they experienced while teaching about inequities in mathematics education in the contexts of content and methods courses and professional development contexts. Each case includes commentaries from three different authors. Altogether the

perspectives of over 80 mathematics educators are included in this book. The conversations that we have had and that will continue to have around these cases are very exciting to me and gives me hope that together we can and will make a difference in shaping the future of mathematics education research. I am also hopeful that the future generation of mathematics educators will engage with diverse perspectives by embracing contraries, said another way, by building bridges rather than walls.

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EXAMINING INDIVIDUAL AND COLLECTIVE LEVEL MATHEMATICAL PROGRESS

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A challenge in mathematics education research is to coordinate different analyses to develop a more comprehensive account of teaching and learning. I contribute to these efforts by expanding the constructs in Cobb and Yackel's (1996) interpretive framework that allow for coordinating social and individual perspectives. This expansion involves four different constructs: disciplinary practices, classroom mathematical practices, individual participation in mathematical activity, and mathematical conceptions that individuals bring to bear in their mathematical work. I illustrate these four constructs for making sense of students' mathematical progress using data from an undergraduate mathematics course in differential equations.

Un reto en la investigación en educación matemática es la coordinación de diferentes análisis para desarrollar una descripción más amplia de la enseñanza y el aprendizaje. Contribuyo a estos esfuerzos mediante la ampliación de los constructos del marco interpretativo de Cobb y Yackel (1996), los cuales permiten la coordinación de las perspectivas social e individual. Esta ampliación involucra cuatro constructos diferentes: las prácticas disciplinares, las prácticas matemáticas del aula, la participación individual en la actividad matemática y las concepciones matemáticas que las personas utilizan en su trabajo matemático. Ejemplifico estos cuatro constructos para dar sentido al progreso matemático de los estudiantes con datos de un curso universitario de ecuaciones diferenciales.

Keywords: Research Methods, Classroom Discourse, Cognition

Recent work in mathematics education research has sought to integrate different theoretical perspectives to develop a more comprehensive account of teaching and learning (Bikner-Ahsbahs & Prediger, 2014; Cobb, 2007; Hershkowitz, Tabach, Rasmussen, & Dreyfus, 2014; Prediger, Bikner-Ahsbahs, & Arzarello, 2008; Rasmussen, Wawro, & Zandieh, 2015; Saxe et al., 2009). An early effort at integrating different theoretical perspectives is Cobb and Yackel's (1996) emergent perspective and accompanying interpretive framework. In this paper I expand the interpretive framework for coordinating social and individual perspectives by offering a set of constructs to examine the mathematical progress of both the collective and the individual. I illustrate these constructs by conducting four parallel analyses and make initial steps toward coordinating across the analyses.

The emergent perspective is a version of social constructivism that coordinates the individual cognitive perspective of constructivism (von Glasersfeld, 1995) and the sociocultural perspective based on symbolic interactionism (Blumer, 1969). A primary assumption from this point of view is that mathematical development is a process of active individual construction and a process of mathematical enculturation (Cobb & Yackel, 1996). The interpretive framework, shown in Figure 1, lays out the constructs in the emergent perspective. The significance of accounting for both individual and collective activity is highlighted by Saxe (2002), who points out that, "individual and collective activities are reciprocally related. Individual activities are constitutive of collective practices. At the same time, the joint activity of the collective gives shape and purpose to individuals' goal-directed activities" (p. 276-277).

My and my colleagues' prior work with the interpretative framework (e.g., Rasmussen, Zandieh, & Wawro, 2009; Stephan & Rasmussen, 2002; Yackel, Rasmussen, & King, 2000) has raised our awareness of the opportunity (and occasional need) to extend the constructs in the interpretative

framework. In particular, in Rasmussen, Wawro, and Zandieh (2015), we expand the ways to analyze individual and collective mathematical progress. We use the phrase "mathematical progress" instead of "learning" as an umbrella term that admits analyses of collective mathematical development and individual meanings and activity. That is, while it might make sense to speak of individual student learning, it makes less sense to speak of collective learning because this incorrectly implies a deterministic, one size fits all approach. The phrase mathematical progress, on the other hand, offers a way to address both the collective and the individual without suggesting a deterministic stance toward the collective.

Social Perspective	Individual Perspective
Classroom social norms	Beliefs about own role, others' roles, and the general nature of mathematical activity
Sociomathematical norms	Mathematical beliefs and values
Classroom mathematical practices	Mathematical conceptions and activity

Figure 1. The interpretive framework**.**

On the bottom left hand side of the interpretive framework (Figure 1), the construct of classroom mathematical practices is a way to conceptualize the collective mathematical progress of the local classroom community. In particular, such an analysis answers the question: What are the normative ways of reasoning that emerge in a particular classroom? Such normative ways of reasoning are said to be reflexively related to individual students' mathematical conceptions and activity. In prior work that has used the interpretive framework, individual conceptions and activity has been treated as a single construct that frames the ways that individual students participate in classroom mathematical practices (e.g., Bowers, Cobb, & McClain, 1999; Cobb, 1999; Stephan, Cobb, & Gravemeijer, 2003). Such a framing of the individual is, in our view, compatible with what Sfard (1998) refers to as the "participation metaphor" for learning.

In an effort to be more inclusive of a cognitive framing that would posit particular ways that students think about an idea, Rasmussen, Wawro, & Zandieh (2015) split the bottom right hand cell into two constructs, one for individual participation in mathematical activity and one for mathematical conceptions that individual students bring to bear in their mathematical work. With these two constructs for individual progress one now can ask the following two questions: How do individual students contribute to mathematical progress that occurs across small group and whole class settings? And what mathematical meanings do individual students develop and bring to bear in their mathematical work?

Work at the undergraduate level has also highlighted the fact that, in comparison to K-12 students, university mathematics and science majors are more intensely and explicitly participating in the discipline of mathematics. However, the notion of a classroom mathematical practice was never intended to capture the ways in which the emergent, normative ways of reasoning relate to various disciplinary practices (Stephan & Cobb, 2003). In order to more fully account for what often occurs at the undergraduate level, we expand the interpretive framework to explicate how the classroom collective activity reflects and constitutes more general disciplinary practices. Thus there is an additional cell to the bottom left row of the interpretive framework, disciplinary practices. One can now answer the following two questions about collective mathematical progress: What is the mathematical progress of the classroom community in terms of the disciplinary practices of mathematics? And what are the normative ways of reasoning that emerge in a particular classroom?

To summarize, Figure 2 shows the expansion of the bottom row of the interpretive framework, which now entails four different constructs: disciplinary practices, classroom mathematical practices, individual participation in mathematical activity, and mathematical meanings.

Social Perspective		Individual Perspective	
Classroom social norms		Beliefs about own role, others' roles, and the general nature of mathematical activity	
Sociomathematical norms		Mathematical beliefs and values	
Disciplinary practices	Classroom mathematical practices	Participation in mathematical activity	Mathematical meanings

Figure 2. Expanded interpretive framework.

The left hand side of the bottom row comprises two different constructs for examining the mathematical progress of the classroom community, while the right hand side comprises two different constructs for examining the mathematical progress of individuals. The contribution that this expansion makes is in providing researchers with a more comprehensive means of bringing together analyses from social and individual perspectives. In particular, the expanded interpretive framework enables a researcher to answer the questions listed in Figure 3.

Disciplinary practices	Classroom mathematical practices	Participation in mathematical activity	Mathematical meanings
What is the collective mathematical progress in terms of the disciplinary practices?	What are the normative ways of reasoning that emerge in a particular classroom?	How do individual students contribute to the collective mathematical progress?	What meanings do individual students develop and bring to bear in their work?

Figure 3. Four constructs for analyzing mathematical progress and respective research questions.

Theoretical and Methodological Background

Classroom mathematical practices

Classroom mathematical practices refer to the normative ways of reasoning that emerge as learners solve problems, explain their thinking, represent their ideas, etc. By normative I mean that there is empirical evidence that an idea or way of reasoning functions as if it is a mathematical truth in the classroom. This means that particular ideas or ways of reasoning are functioning in classroom discourse as if everyone has similar understandings, even though individual differences in understanding may exist. The empirical evidence needed to document normative ways of reasoning is garnered using the approach developed by Rasmussen and Stephan (2008) and furthered by Cole et al. (2012). This approach, which we refer to as the documenting collective activity method, applies Toulmin's (1958) argumentation scheme to document the mathematical progress using three well-developed criteria, all of which involve tracing over time how ideas are used by students. In brief, central to Toulmin's scheme is the core of an argument, which consists of a Claim, Data to support that Claim, and a Warrant that explains the relevance of the Data to the Claim.

Disciplinary practices

Disciplinary practices refer to the ways in which mathematicians typically go about their professional practice. The following disciplinary practices are among those core to the activity of professional mathematicians: defining, algorithmatizing, symbolizing, and theoremizing (Rasmussen, Zandieh, King, & Teppo, 2005). Not all classroom mathematical practices are easily or sensibly characterized in terms of a disciplinary practice. This is because classroom mathematical practices capture the emergent and potentially idiosyncratic collective mathematical progress, whereas a

disciplinary practice analysis seeks to analyze collective progress as reflecting and embodying core disciplinary practices. In this report I focus on algorithmatizing, the practice of creating and using algorithms. Our method for documenting disciplinary practices builds on prior work that has examined theoremizing, symbolizing, and defining (Rasmussen, Zandieh, King, & Teppo, 2005; Rasmussen, Wawro, & Zandieh, 2015; Zandieh & Rasmussen, 2010).

Mathematical meanings

As students solve problems, explain their thinking, represent their ideas, and make sense of others' ideas, they necessarily bring forth various meanings of the ideas being discussed and potentially modify these meanings (Thompson, 2013). Our analysis of individual student meanings makes use of analyses from prior work that have characterized different ways that students think about the relevant mathematical ideas (e.g., Carlson, Jacobs, Coe, Larsen, & Hu, 2002; Habre, 2000; Harel & Dubinsky, 1992; Thompson, 1994; Trigueros, 2001; Rasmussen, 2001; Zandieh, 2000).

Participation in mathematical activity

This analysis draws on recent work by Krummheuer (2007, 2011), who characterizes individual learning as participation within a mathematics classroom using the constructs of production roles and recipient roles. In the production framing, individual speakers take on various roles, which are dependent on the originality of the content and form of the utterance. The title of author is given when a speaker is responsible for both the content and formulation of an utterance. The title of relayer is assigned when a speaker is not responsible for the originality of either the content nor the formulation of an utterance. A ghostee takes part of the content of a previous utterance and attempts to express a new idea, and a spokesman is one who attempts to express the content of a previous utterance in his/her own words. Within the recipient framing of learning-as-participation, Krummheuer (2011) defines four roles: conversation partner, co-hearer, over-hearer, and eavesdropper. A conversation partner is the listener to whom the speaker seems to allocate the subsequent talking turn. Listeners who are also directly addressed but do not seem to be treated as the next speaker are called co-hearers. Those who seem tolerated by the speaker but do not participate in the conversation are over-hearers, and listeners deliberately excluded by the speaker from conversation are eavesdroppers.

Setting and Participants

I illustrate the four constructs and address the respective research questions from Figure 3 using data from a semester-long classroom teaching experiment (Cobb, 2000) in differential equations conducted at a medium sized public university in the Midwestern United States. I selected a 10-minute small group episode from the second day of class based on its potential to illustrate all four constructs. There were four students in this group, Liz, Deb, Jeff, and Joe (all names are pseudonyms).

There were 29 students in the class. Class met four days per week for 50-minute class sessions for a total of 15 weeks. The classroom had movable small desks that allowed for both lecture and small group work. The classroom teaching experiment was part of a larger design based research project that explored ways of building on students' current ways of reasoning to develop more formal and conventional ways of reasoning (Rasmussen & Kwon, 2007). A goal of the project was to explore the adaptation of the instructional design theory of Realistic Mathematics Education (RME) to the undergraduate level. Central to RME is the design of instructional sequences that challenge learners to organize key subject matter at one level to produce new understanding at a higher level (Freudenthal, 1991). In this process, graphs, algorithms, and definitions become useful tools when students build them from the bottom up through a process of suitably guided reinvention (e.g., Rasmussen & Blumenfeld, 2007; Rasmussen & Marrongelle, 2006; Rasmussen, Zandieh, King, &

Teppo, 2005).

Results and Discussion

As previously stated, the analysis comes from video recorded work of a small group of four students, Liz, Deb, Jeff, and Joe, on the second day of class. Just prior to the small group work students completed the following task: The previous problem dealt with a complex situation with two interacting species. To develop the ideas and tools that we will need to further analyze complex situations like these, we will simplify the situation by making the following assumptions:

- There is only one species
- The species have been in the lake for some time before what we are calling time t = 0
- The resources (food, land, water, etc.) are unlimited
- The species reproduces continuously

Given these assumptions for a certain lake with fish, sketch three different population versus time graphs (one starting at P = 10, one starting at P = 20, and the third starting at P = 30).

This task was relatively straightforward for students and brought forth an imagery of exponential growth and the graphs they sketched were consistent with this imagery. The instructor then used their graphs as an opportunity to introduce the rate of change equation $dP/dt = 3P$ as a differential equation that was consistent with their graphs. In particular, as P values increase, so does the slope of the graph of P vs. t.

The follow up task, which students worked on for approximately 10 minutes, however, was much more cognitively demanding for students.

Consider the following rate of change equation, where P(t) is the number of rabbits at time t (in years): dP/dt = 3P(t) or in shorthand notation dP/dt = 3P. Suppose that at time t = 0 we have 10 rabbits (think of this as scaled, so we might actually have 1000 or 10,000 rabbits). Figure out a way to use this rate of change equation to approximate the future number of rabbits.

At t = 0.5 and t = 1.
At t = 0.25, t = 0.5, t = 0.75, and t = 1

Be prepared to share the reasoning behind your approach with the rest of the class. Organize your results in tabular and graphical form.

Recall that this is only the second day of class and students have not been introduced to any analytical, numerical, or graphical techniques for analyzing differential equations. In a related analysis, Tabach, Rasmussen, Hershkowitz, and Dreyfus (2015) provide the following a priori analysis of the knowledge elements that we expect students to construct when solving this task:

- Csy – establishing connection between P and dP/dt (if you know P you can find dP/dt)
- Cpit – population iteration (given P and dP/dt at a moment in time allows one to find P at a later time)
- Crit – rate of change iteration (applying Csy at that later time one can find the corresponding dP/dt)
- Cit – Cpit and Crit can be combined into a repeating loop.

To illustrate how these knowledge elements play out in student discourse, consider the following except from Liz which occurred near the end of the 10-minute small group work:

> *Liz*: What I understand is that we found our rate of change initially at time zero and that we are using that to find out what our population is after half a year. If we are expected to grow by 30 rabbits in a year then, in a half a year we grow by 15 rabbits. So we'll have 15, I mean 25 because 15 plus 10 is 25. Then you start over again, so it's kind of like our new initial population. We could label it time equals zero if we wanted to.

An example of Csy occurs when Liz says, "grow by 30 rabbits in a year" because in order to get the value of 30, Liz had to use the initial population value of 10, plug this into the rate of change equation to get 30. Cpit is illustrated by Liz when she says, "in a half year we grow by 15 rabbits. So we'll have 15, I mean 25 because 15 plus 10 is 25." That is, she uses what she knows about the population at the initial time and her knowledge of dP/dt at the initial time to compute how many rabbits there will be half a year later. Finally, when Liz says, "The you start all over again," she demonstrates an understanding that Cpit and Crit can be combined into a repeating loop to compute the population after another half year.

As a reminder, the primary research goal here is to demonstrate an approach for coordinating collective and individual analyses to gain greater explanatory and descriptive power, with the intention to better understand the individual and collective meaning making processes. I will therefore begin with analyzing the collective small group mathematical progress using the previously mentioned documenting collective activity method.

Small group collective mathematical progress

Using the documenting collective activity method I identified the following three ideas that functioned as if shared in this particular small group: dP/dt can be determined from P values (Csy), a value for dP/dt refers to the amount of change over 1 year, Cpit and Crit can be combined into a repeating loop. All three of these findings made use of the second criteria, namely that what was originally a Claim in one argument later functions as Data in a subsequent argument (Rasmussen & Stephan, 2008). In other words, an idea that initially required some form of justification is later used a means to justify new claims. Figure 4 shows the Toulmin analysis for first argument and Figure 5 shows the Toulmin analysis for the fifth argument made in this particular small group.
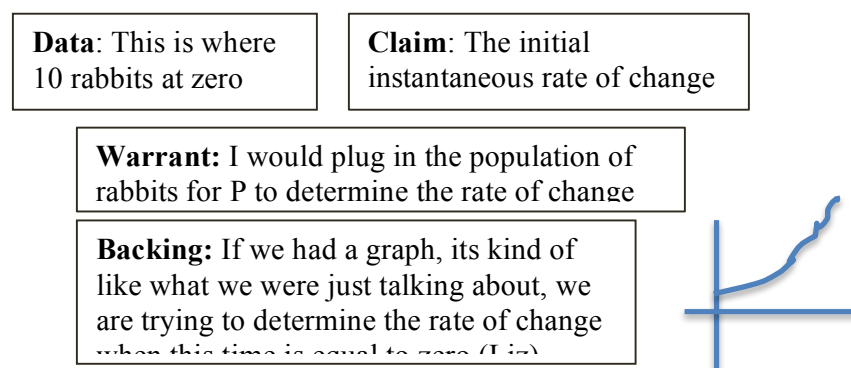

Figure 4. Toulmin analysis for Argument 1.

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

Data: We have the 30. Three times ten would give us our rate of change. Say 0.5 years	**Claim**: which will give us what, the new amount of rabbits plus the old amount

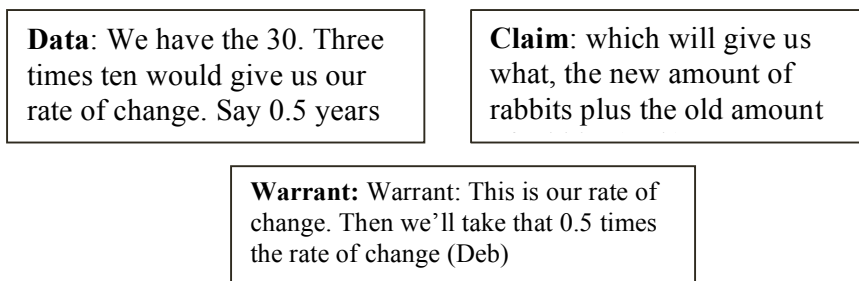| **Warrant:** Warrant: This is our rate of change. Then we'll take that 0.5 times the rate of change (Deb) |
|---|

Figure 5. Toulmin analysis for Arguments 5.

We see here in the first argument that Liz makes the Claim that the initial instantaneous rate of change is 30 (note that this is Csy) and then in the fifth argument Deb uses as a fact that the initial rate of change is 30 as Data to support a new claim. Thus per the second criteria it is concluded that one can determine dP/dt from P values functions as if shared in this small group. Full consideration of the data indicate that Liz, Deb, and Jeff (but not Joe) made individual progress compatible with the collective mathematical progress. That is, when a researcher determines that an idea functions as if shared, it does not mean that everyone shares exactly the same way of thinking.

I next step back from the specifics of the Toulmin analysis across the 10-minute episode to highlight overall trends in the discourse. In terms of talk turns, Liz spoke 26 times, Deb spoke 18 times, Jeff 13 times, and Joe 8 times. Thus Liz and Deb were the primary contributors, with Jeff often highlighting a final answer. Overall there were 14 different arguments (à la Toulmin) that consisted of at least Data and Claim. The following table shows the distribution of contributions (some contributions co-constructed).

Table 1: Toulmin argument contributions by student

	Liz	Deb	Jeff	Joe
Data	6	5	1	4
Claim	5	5	5	2
Warrant	2	5	1	0
Backing	2	1	0	0

In light of the collective small group mathematical progress, I next begin to address the following coordination questions: What meanings for dP/dt emerged and who expressed these meanings? What part did these meanings play in the collective mathematical progress? What roles did Liz, Deb, Joe, and Jeff play in all of this? In what ways did students' mathematical work reflect disciplinary practices?

Engaging different meanings for dP/dt

Across the 10-minute episode I identified seven different meanings for dP/dt used by one or more of the fours students. These seven meanings and who within the 10-minute episode engaged these meanings are: as steepness (Liz), ratio (Liz and Jeff), population length (Liz and Deb), tool (Liz), function (Deb), proportion (Deb), and fraction (Jeff). Thus, not only did Liz and Deb have more talk turns, they also engaged more and different meanings for dP/dt compared to Jeff and Joe. I next illustrate how students engaged these meanings, but due to space constraints a complete detailing is not possible.

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

17 Liz: So if we have that [initial rate of change is 30], the question is how can we use that to help us figure out the population after a half unit elapsed?

22 Deb: You said the population is 10 right [Liz: Um hmm]. So three times ten would give us our rate of change. Say 0.5 years passes, this is our rate of change. Then we'll take that 0.5 times the rate of change which will give us what [slight pause looks up to Jeff and Joe], the new amount of rabbits plus the old amount of rabbits.

In line 17 we see Liz wonder out loud how knowing that 30 is the initial rate of change can be used to achieve their goal of determining the population a half year later. That is, Liz would like to somehow engage rate of change as a tool to do work for them. Shortly thereafter Deb takes Liz up on how they might use rate of change as a tool and suggests that they could take the 30 and multiply it by 0.5. It is not entirely clear the meaning that Deb is making use of here, but one possibility is that she is engaging in a form of proportional reasoning. Continuing their discussion, we see an important conceptual advance.

25 Liz: So the old amount of rabbits is 10.

26 Deb: Am I making sense?

27 Jeff: I think so, so that would be 25, is that what you're saying?

28: Liz: Okay I think I get what you're saying. Ok, so like we're at time zero and we have 10 rabbits, and supposedly the rate of change, well not supposedly, we're saying that the rate of change is 30 [Jeff: yeah for the] at time zero. So its going to grow at a rate of, I don't know if I'm going to say this right, at 30 rabbits per year? [looks up at Deb]

In line 27 we see Jeff contributing to the discussion by highlighting final computations. Thus, while Jeff is not necessarily leading the intellectual work, he is following along and adding to the discussion. Then in line 28 we see Liz make an interpretation for 30, the initial rate of change, that later serves her and her group well. In particular, she interprets 30 as the amount of rabbits that will accrue over a one year time increment. I refer to such a meaning as a "population length." This is similar to how Thompson documented early meanings of rate as a "speed length" where a student thinks of say, 60 mph, as going 60 miles in one hour. Deb next picks up on what Liz says and then in line 32 Liz returns to how one might use rate of change as a tool for figuring out the number of rabbits a half year later.

29 Deb: Right. [Liz: Ok] So we'll have 30 more rabbits.

32 Liz: And so we're really not figuring out the rate of change we figuring…Well this is the rate of change and we're using the rate of change to figure out the number of rabbits we are going to increase by in half a year.

As the students continue, we see Deb leading the intellectual work of figuring out how to use the 30 to achieve their goal.

38 Deb: This is what I did. First I looked at the fact that this is a rate of change equation. So this is telling me how many rabbits are being produced every year. So If I know 3 times the original population is produced every year, then I have 3 times 10 is produced every year. But I want to know how many is produced in 0.5 years. So I know how many rabbits are produced per year, so if I multiply that by 0.5 then I'll know how many more rabbits have been produced. So I take that new number that I get and add it to the old population.

43 Jeff: I think you can go dp/dt=30, actually your dt will be 0.5, and then you add that to the old and then you do it again for the next one.

In line 38 Deb expresses three different meanings for rate of change. She starts off by saying that the rate of change equation tells one how many rabbits are produced every year. This is similar to a

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function meaning for rate of change and relates closely to Csy. Given an input you get an output. And the meaning of the output in this case is a population length ("so I know how many rabbits are produced per year"). Deb then engages in some proportional reasoning to determine how many more rabbits there will be in a half year. In line 43 Jeff shows that he is following the discussion and seems to treat dP/dt as a fraction. Soon thereafter Liz recapitulates their line of reasoning as follows, engaging both meaning of rate of change as population length and rate of change as proportion.

> *48 Liz*: Okay, so basically, I get you up into the point where you say you want to put in, what I understand is that we found our rate of change initially at time zero and I understand using that to find out what our population is after half a year. If we are expected to grow by 30 rabbits in a year then, in a half a year we grow by 15 rabbits. So we'll have 15

I now turn to reflecting on the roles these various meanings played in the collective mathematical progress. As we saw, there is a shift in the meaning of dP/dt - from steepness to a "population length" (clearly for Liz and likely for Deb). This shift coincided with "a value for dP/dt refers to the amount of change over 1 year" functioning as if shared AND the initial articulation of how to find the estimate for the population at t = 0.5. In relation to other work, the principle of a form-function-shift (Saxe, 2002) of notations in use is particularly suitable for analyzing the interplay between tool use (in this case dP/dt = 3P) and conceptual development. In particular, the form-function-shift describes the interplay between cultural forms (external representations) and the meanings that develop for structuring and accomplishing specific goals, not unlike what we saw happening with the individual meaning and collective production of meaning.

What roles did Liz, Deb, Jeff, and Joe play in the collective mathematical progress?

Drawing on Krummheuer (2007, 2011), I characterize student participation in the collective mathematical progress in terms of production roles (author, relayer, ghostee, spokesman) and recipient roles (conversation partner, co-hearer, overhearer, eavesdropper). Previously I specified the number of talk turns for each student: Liz 26; Deb 18; Jeff 13; Joe 8. The raw count of co-author shows that there was fairly even distribution (Liz 6/14; Deb 5/14; Jeff 6/14; Joe 4/14). However, a more nuanced look however reveals important differences: Joe offered 2 incorrect arguments, Jeff often revoiced (with and without reformulation), Liz and Deb did the main intellectual lifting (as was evident in the excerpts). For example, Liz was primary author (core of argument) for Csy and as Spokesman for meaning of dP/dt as population length. Deb, on the other hand, was the primary author for Cit The following excerpt provides a snapshot illustration of how the entire 10-minute episode was coded.

> *26 Deb*: [articulates the main iteration idea but without a numerical result – omitted here for space considerations] Am I making sense?
> *27 Jeff*: I think so, so that would be 25, is that what you're saying?
> *28: Liz*: Okay I think I get what you're saying. Ok, so like we're at time zero and we have 10 rabbits, and supposedly the rate of change, well not supposedly, we're saying that the rate of change is 30 [Jeff: yeah for the] at time zero. So its going to grow at a rate of, I don't know if I'm going to say this right, at 30 rabbits per year? [looks up at Deb]

In line 25 Jeff functions as a relayer as he was not responsible for either the content or the formulation of the idea. In line 28 we see Liz function as spokesman for this is the first time anyone has engaged the meaning of rate of change as population length. As such Liz is responsible for both the content and the formulation.

Regarding recipient design roles, Liz, Deb, and Jeff were for the most part conversation partners and co-hearers. Joe was mostly a co-hearer and at times an over-hearer. While the constructs of production and recipient roles were useful in distinguishing individual differences, I found them to be

insufficient to account for the different ways these four students participated in mathematical discourse. In particular, my analysis suggested a third role – that of facilitator roles. More specifically, I identified four different ways in which these students facilitated the flow of ideas in their small group. These four roles are:

- Focuser is assigned when a speaker directs attention to a particular mathematical issue
- Elicitor is given when a speaker attempts to bring out another's idea
- Checker is one who seeks agreement or sensibility of an utterance
- Summarizer pulls ideas together

For example, consider the following excerpts:

17 Liz: So if we have that [initial rate of change is 30], the question is how can we use that to help us figure out the population after a half unit elapsed? [32 sec pause, everyone looking down at their papers and making marks]
18 Jeff: So I was just going to say how would we work time into the equation to get the next, uh, population or change in population?
40 Liz: Yeah I get it, do you guys get what Deb is saying?
41 Jeff: Yeah you get 25 and then you get 55.
46 Liz: And the reason for putting in the new population would be what?
48 Liz: Okay, so basically, I get you up into the point where you say you want to put in, what I understand is that …..
53 Deb: Everybody agree?

In line 17 Liz functions as a focuser when she directs her group's attention to how they can use the initial rate of change as a tool for figuring out the population after a half year. In line 18 Jeff also serves the role of focuser when he directs the group's attention to how time gets integrated into their work. In line 40 Liz acts a checker when she queries the group to see if everyone gets what Deb is saying. Similarly Deb acts as a checker in line 53. In line 46 Liz functions as elicitor when she requests the rationale for carrying out a particular mathematical computation. Finally, in line 48 Liz pulls the ideas together and thus functions as a summarizer.

To conclude this section, I reflect on the ways in which students' mathematical work reflects the disciplinary practice of creating and using algorithms, or algorithmatizing.

The disciplinary practice of algorithmatizing

In the analysis previously presented we see the group of four students engaging in the first stage of creating an algorithm. These first steps lay the relational foundation for how to use P values and dP/dt values to approximate a future population value. An expert will recognize students' work as Euler's method, although the students do not as of yet know that what they have produced is in fact related to Euler's method. In the subsequent whole class discussion the different groups in the class discussed their work and together the class created the following algorithm: $P_{next} = P_{now} + \left(\frac{dP}{dt}\right)_{now} \times \Delta t$. The instructor then explained to the class that this algorithm is conventionally know as Euler's method and is an example of a numerical approximation. In subsequent classes students used this algorithm both with and without contexts and investigated the relationship between approximate solutions and exact solutions that are concave up, concave down, and that have a constant rate of change. Students also investigated how approximation graphs with different step sizes compared to each other and even different ways to improve Euler's method. Such mathematical progress reflects the disciplinary practice of creating and using algorithms, or what we refer to algorithmatizing.

More specifically, students' creation of the Euler method algorithm involved the following:

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engaging in goal directed activity, isolating attributes, forming quantities, creating relationships between quantities, and expressing these relationships symbolically. For example, Liz helped her group focus on a specific goal directed activity when she asked her group mates, "So if we have that [initial rate of change is 30], the question is how can we use that to help us figure out the population after a half unit elapsed?" This led to their group to think about 30, the initial rate of change, as the amount of rabbits that would be added in one year. Previously I referred to this meaning as a population length, which is an example of isolating an attribute and forming a quantity for this attribute. The population length was further refined when the group related this quantity to a time interval of a half year. The relationship between change and in time and population length was then further quantified. As pointed to previously, these relationships then formed the basis for Cpit and Crit functioning as if shared in this small group. Expressing these relationships symbolically occurred after the 10-minute small group work analyzed here.

Conclusion

In this section I first consider implications for practice and then implications for research. Regarding instructional design considerations, the analysis presented here raises the possibility of including in the student materials questions that focus student attention on the attributes that Deb and her group found particularly useful. Questions such as the following might be woven into the student materials: What is the initial rate of change? What does this value mean to you? How can you use the 30 to figure out the population after half unit of time? Of course this would have to be done in a way that does not take away from the challenge and cognitive demand of the task. Alternatively, such questions could be folded into instructor resource materials that support mathematics faculty in implementing inquiry-oriented curriculum. Indeed, efforts are underway by Estrella Johnson, Karen Keene, and Christine Larson to create such materials for differential equations, linear algebra, and abstract algebra (see http://times.math.vt.edu/).

Another instructional implication that this analysis raises is the how to help promote productive interactions between small group members. In this particular class the small group analysed worked extremely well together, even on the second day of class. This was largely good fortune. So then what might an instructor do to facilitate more productive interactions in small groups that do not function as well as Liz, Deb, Joe, and Jeff?

I now turn to discussing some implications for research. In addition to using various combinations of the four constructs to more fully account for students' mathematical progress, there exist multiple ways in which coordination across the four constructs is possible. For instance, one could choose an individual student within the classroom community and trace his/her utterances for the ways in which they contributed to the emergence of various normative ways of reasoning and/or disciplinary practices. Alternatively, when considering a normative way of reasoning, a researcher could investigate who the various individual students are that are offering the claims, data, warrants, and backing in the Toulmin analysis used to document normative ways of reasoning. How do those contributions coordinate with those students' production roles within the individual participation construct? For instance, does a student ever utilize an utterance that a different student authored as data for a new claim that he is authoring, and in what ways may that capture or be distinct from other students' individual mathematical meanings? One might also imagine ways to coordinate across the two individual constructs as well as across the two collective constructs. For example, how do patterns over time in how student participation in class sessions relate to growth in their mathematical meanings? Are different participation patterns correlated with different mathematical progress trajectories? In what ways are particular classroom mathematical practices consistent (or even inconsistent) with various disciplinary practices? Finally, future research could take up more directly the role of the teacher in relation to the four constructs.

I anticipate that future work will more carefully delineate methodological steps needed to carry out the various ways in which analyses using the different combinations of the four constructs can be coordinated. Indeed, this report is a first step in developing a more robust theoretical-methodological approach to analyzing individual and collective mathematical progress.

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TENDIENDO PUENTES ENTRE TEORIAS: ¿QUÉ PUEDE LOGRARSE?

BUILDING BRIDGES BETWEEN THEORIES: WHAT CAN BE ACHIEVED?

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El interés por entender el fenómeno del aprendizaje y la enseñanza de las matemáticas y por desarrollar nuevas y más efectivas metodologías de enseñanza del Cálculo Diferencial y del Álgebra Lineal me condujo a establecer diálogos entre la teoría APOE y otras teorías de la Educación Matemática. Esta iniciativa hizo posible entenderlas mejor. Se construyeron puentes entre ellas permitieron encontrar puntos de contacto, y expandir sus fronteras sin cambiar sus hipótesis fundamentales. En este trabajo se ilustra una reflexión sobre las posibles implicaciones del diálogo entre teorías y, a través de tres ejemplos, se muestra cómo puede promover experiencias enriquecedoras para los investigadores y propiciar la creatividad de los estudiantes. Se muestra también cómo es posible obtener resultados positivos en términos del aprendizaje de los alumnos y en el desarrollo de materiales didácticos efectivos.

Palabras clave: Pensamiento Matemático Avanzado, Teorías del Aprendizaje, Modelación

Hacer investigación en educación matemática implica el análisis de un fenómeno didáctico en términos de un punto de vista teórico específico. La elección debe hacerse entre una enorme cantidad de teorías que asumen perspectivas diversas frente a los complejos fenómenos del aprendizaje y la enseñanza de las matemáticas. Cada teoría se constituye en un campo delimitado por sus propias hipótesis, sus estructuras conceptuales y su metodología de trabajo desde el cual se definen y se abordan los problemas de interés. Así lo que aparenta ser un mismo fenómeno se define de distinta manera en cada teoría y las explicaciones de los mismos están acotadas por las limitaciones propias de cada una de ellas.

Recientemente ha surgido un interés creciente por tender puentes entre diversas teorías y analizar su posible complementariedad a través del estudio comparado de un mismo fenómeno o de su posible articulación en un nuevo y más extenso marco teórico (Artigue, Bosch & Gascón, 2011). Este interés puede ser teórico o puede resultar de una necesidad de análisis que surge en el contexto de una investigación, pero resulta problemático puesto que cada problema se construye en el marco conceptual de una teoría y con dicha teoría se trabaja y se interpretan los resultados utilizando herramientas específicas desarrolladas para ello (Trigueros, Bosch & Gascón, 2011). En este contexto surgen nuevas preguntas: ¿Por qué o para qué cruzar fronteras teóricas? ¿Cómo establecer un diálogo entre teorías que permita cruzar sus fronteras sin alterar los principios básicos de cada una de ellas? ¿Cómo construir puentes entre ellas o flexibilizar sus fronteras de manera que no se pierda la coherencia interna de cada una de ellas?

Intentaré, en lo que sigue, dar respuesta a este tipo de preguntas a través de los resultados de mi propia experiencia. Para ello discutiré algunos ejemplos específicos tomados de mi trabajo de investigación.

Cruzar fronteras entre teorías: Posibilidad de ampliación de dominios de aplicación

¿Cómo cruzar las fronteras de una teoría específica? ¿Qué se puede ganar con ello? Comenzaré por intentar dar una primera respuesta a estas preguntas. Un poco de mi historia personal puede ayudar a conformar algunas respuestas tentativas.

Mi trabajo de investigación en educación matemática ha utilizado como marco teórico la teoría APOE (Acción, Proceso, Objeto, Esquema) que es una teoría cognitiva que intenta explicar cómo los

estudiantes construyen los distintos conceptos matemáticos (Arnon et al. 2014). El interés personal por buscar nuevas metodologías de enseñanza, de acercar a los estudiantes a las matemáticas y de promover que aprendan y puedan utilizar ese conocimiento me condujo a explorar otras teorías y a buscar establecer un diálogo entre algunas de ellas y APOE. ¿Qué implica esta exploración? En mi experiencia la exploración se hizo desde la consideración de la posibilidad de analizar las teorías y pensar en las fronteras no como límites rígidos sino como algo flexible y dinámico. Esta idea me condujo a pensar cómo pueden complementarse las teorías y cómo se pueden enriquecer mutuamente cuando se establece entre ellas un diálogo que permite extender sus límites teóricos o metodológicos y ampliar las posibilidades de análisis de los fenómenos didácticos de interés.

En todos los ejemplos de diálogo que describiré a continuación una de las teorías involucradas es la teoría APOE. Comenzaré por describirla muy brevemente: La teoría APOE está basada en la epistemología de Piaget, en particular en el mecanismo de abstracción reflexiva. Su nombre es un acrónimo de los tipos de estructuras mentales que se supone construyen los estudiantes conforme realizan actividad matemática. Una Acción es una transformación de objetos construidos previamente siguiendo instrucciones explícitas que el estudiante considera como externas. Las acciones pueden interiorizarse, mediante la reflexión, en un Proceso que consiste en la posibilidad de describir, imaginar o llevar a cabo las transformaciones sin necesidad de hacer todos pasos explícitamente. Los procesos se pueden revertir o coordinar unos con otros para construir nuevos procesos. Cuando el estudiante es consciente de un proceso como una totalidad y puede realizar acciones sobre él, éste se encapsula en un Objeto que puede desencapsularse, en caso de necesidad, en el proceso que le dio origen. Los Esquemas se desarrollan mediante la construcción de relaciones entre acciones, procesos, objetos y esquemas construidos previamente y se considera, consciente o inconscientemente, como un marco coherente que se utiliza para resolver problemas matemáticos relacionados entre sí. Cuando el estudiante requiere hacer acciones sobre el esquema, éste puede tematizarse en un objeto.

En APOE se considera que la tendencia general del estudiante cuando trabaja con una serie de situaciones relacionadas con una noción matemática específica es diferente dependiendo del tipo de estructuras que muestra en su trabajo. Además de las estructuras teóricas descritas, la teoría APOE incluye un modelo que describe una forma posible de construir los conceptos matemáticos de interés en términos de las estructuras de la teoría. Este modelo, conocido como descomposición genética (DG) no es único y debe validarse a partir de resultados experimentales. Por ello la metodología de investigación de la teoría incluye ciclos de investigación en los que la DG se refina hasta que se considera estable.

La teoría APOE incluye también un ciclo de enseñanza: Actividades diseñadas de acuerdo a la DG, discusión en Clase y Ejercicios (ACE) para promover la construcción de los conceptos de interés en el aula. En él se utilizan actividades para la enseñanza diseñadas tomando la DG como base y que se reconsideran conforme ésta se refina.

Búsqueda de detalle en el análisis de la actividad matemática de los estudiantes

En el contexto de un proyecto sobre la construcción de los conceptos del Cálculo Diferencial e Integral de funciones de dos variables, conjuntamente con Rafael Martínez- Planell, estudiamos el aprendizaje de las funciones de dos variables (Trigueros & Martínez-Planell, 2010; Trigueros, M., & Martínez-Planell, 2012). El análisis de los datos obtenidos en el primer ciclo del proyecto puso de manifiesto la necesidad de profundizar tanto en el uso del lenguaje utilizado en la enseñanza como en la representación geométrica de estas funciones y su papel en la construcción de este concepto, para poder interpretar las respuestas de los estudiantes. Por ejemplo, ante la instrucción de esbozar la gráfica de la función $z = x^2 + y^2$ en R^3 cuando $z = 3$, muchos estudiantes respondían "*Es un cilindro*" y dibujaban gráficas como la que se muestra en la Figura 1.

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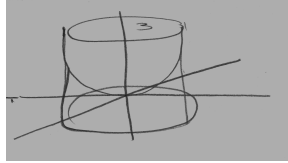
Figura 1. La gráfica de $3 = x^2 + y^2$ dibujada por muchos estudiantes.

Fue así que nos acercamos a una teoría cognitiva que detalla el papel de la relación entre distintas representaciones de un mismo concepto en el aprendizaje del mismo: La teoría de las Representaciones Semióticas (TRS) propuesta por Duval (2006). En ella se considera que el proceso de pensar matemáticamente requiere de la coordinación cognitiva de distintas representaciones mediante su comparación y análisis. Propone dos tipos de transformaciones de representaciones semióticas: tratamientos que consisten en cambios de representaciones que se encuentran en un mismo registro y conversiones que involucran cambios de registros de representación sin que se cambie el objeto denotado en ellas. Para discriminar los valores cognitivamente significativos de un objeto matemático mediante la comparación de representaciones similares es necesario hacer tratamientos, mientras que las conversiones permiten disociar el objeto representado del contenido de una representación particular e impedir que los diferentes registros queden en compartimentos ajenos para el aprendiz.

El análisis de las posibilidades de tender puentes entre las teorías APOE y TRS partió de la consideración del papel de las representaciones en la construcción de conocimiento en ambas teorías. En la TRS, los registros de representación y las transformaciones entre registros son centrales en la comprensión de los objetos matemáticos. En la teoría APOE las transformaciones entre representaciones y su relación están ligadas a distintas construcciones mentales que pueden ser descritas mediante los mecanismos de interiorización, coordinación, reversión o encapsulación. Así, es posible considerar que los tratamientos podrían describirse parcialmente en términos de acciones sobre un objeto en un registro de representación determinado y que la reflexión sobre ellas puede permitir destacar aspectos o propiedades significativas del objeto cuando estas acciones se interiorizan en procesos. La interacción con la TRS permite destacar el tipo y el papel de esas acciones al enfatizar la necesidad de discriminar entre las distintas características del objeto representado en el registro en el que se hacen las acciones. Consideremos, por ejemplo, la acción de sustituir la x por el número 1 en la función $z = x^2 + y^2$. Los estudiantes pueden reconocer a z como una función cuadrática en R^2. Mediante las acciones de sustituir distintos números en x pueden identificar una familia de funciones cuadráticas similares entre sí. Estas acciones pueden interiorizarse en el proceso de construcción de la familia de parábolas $z = a + y^2$. En este ejemplo, la TRS permite identificar el papel de esas acciones en la construcción de conocimiento sobre funciones que resulta fundamental cuando se consideran funciones de dos variables.

Por su parte, las conversiones pueden describirse en APOS como la interiorización del proceso que hace posible considerar las acciones de comparación e identificación de un objeto en registros diferentes de representación como una asociación; pero la interacción con la teoría de representaciones semióticas remarca la necesidad de incluir estos procesos en la DG por su importancia en términos cognitivos. Regresando al ejemplo anterior, la gráfica de las funciones $z = a + y^2$ en el plano permite subrayar aquellos aspectos de las gráficas como objetos que permanecen invariantes en los tratamientos y su relación con el parámetro en la representación analítica. Los procesos construidos mediante estas acciones pueden coordinarse con el proceso de representación de planos en R^3 en un nuevo proceso que permite representar las funciones cuadráticas sobre distintos planos fundamentales $x = a$ y en el proceso de construcción de la superficie correspondiente a la gráfica de la función. El puente entre las teorías permite, así, utilizar las nociones de tratamiento y conversión en la teoría APOE sin perder de vista su perspectiva genética y considerar la

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flexibilidad en el tránsito entre representaciones mediante la incorporación de especificidad en el análisis.

Tender un puente entre las dos teorías hizo posible un análisis más minucioso del trabajo de los estudiantes en tareas planteadas en una misma representación o que incluyen la conversión entre representaciones y las complejidades involucradas en él. Este análisis repercutió en un refinamiento de la DG. Permitió también mostrar que la generalización de la comprensión de las funciones de una variable a las funciones de dos variables no es directa, dado que los estudiantes no necesariamente perciben las analogías esperadas.

¿Cómo puede establecerse un diálogo fructífero entre teorías?

La posibilidad de coordinar teorías que examinan los fenómenos de la educación matemática desde muy distintas perspectivas puede parecer imposible. ¿Cómo tender puentes entre ellas? ¿Por dónde empezar a dialogar? En un esfuerzo por responder estas preguntas y tender puentes entre las fronteras de estas teorías, conjuntamente con Marianna Bosch y Josep Gascón, decidimos analizar las condiciones que debería satisfacer un diálogo entre teorías y aplicarlas a una conversación entre las teorías APOE y la Teoría Antropológica de lo Didáctico (TAD) (Trigueros, Bosch & Gascón, 2011).

El objeto primario de investigación de la TAD es la actividad matemática institucionalizada, es decir, cuestiona y modela los procesos de génesis y difusión intra-institucional e inter-institucional de las praxeologías (Chevallard, 1992). Éstas están compuestas por una parte práctica y una parte teórica. La primera puede describirse en términos de tareas y técnicas involucradas en la actividad y la segunda en términos de tecnologías y teorías que sustentan las tareas y las técnicas empleadas. El programa de investigación de la TAD requiere explicitar un modelo epistemológico (MER) de las matemáticas que sirve como referencia para analizar la actividad matemática institucionalizada y se basa en el proceso de transposición didáctica que establece las transformaciones que se pueden hacer sobre el conocimiento matemático para hacerlo "enseñable" en una institución específica. El MER se utiliza también para analizar y diseñar actividades de enseñanza y aprendizaje que permiten, a su vez, validarlo (Bosch & Gascón 2005). El análisis praxeológico de la actividad matemática en la institución permite entender las restricciones institucionales que limitan o favorecen ciertos tipos de actividad y analizar el papel del equilibrio en las prácticas matemáticas en la institución.

En el diálogo de este proyecto, la noción de praxeología de la TAD se usa como un marco de análisis reinterpretando las teorías como praxeologías de investigación con el fin de asegurar la consideración de todas sus componentes en el proceso de un diálogo fructífero. De esta manera, el dialogo se puede establecer desde los problemas que abordan (tareas), los instrumentos metodológicos que usan para abordar los problemas (técnicas y tecnologías) y las estructuras propias de las teorías.

Ambas teorías incorporan un análisis de la matemática escolar como parte fundamental de su acercamiento a cualquier problema didáctico e incluyen modelos alternativos para el desarrollo de conocimientos matemático como parte de su metodología: La descomposición genética (DG) en el caso de APOE y el modelo epistemológico de referencia (MER) en el caso de la TAD. Este hecho puede considerarse como un punto de contacto entre ambas teorías. La diferencia fundamental entre estos modelos radica en que la DG se utiliza para estudiar el desarrollo cognitivo de los individuos mientras que el MER se usa para estudiar las condiciones institucionales que permiten que las actividades matemáticas existan en una institución; la unidad de análisis es distinta y eso tiene implicaciones metodológicas importantes. Con el fin de llevar a cabo el diálogo, desarrollamos, en primer término, la noción de "estudiante genérico de una institución" que relaciona al sujeto epistémico de APOE con el "estudiante en una posición institucional" de la TAD. De esta manera se extiende la noción en ambas teorías: La dimensión institucional del análisis con TAD se puede trabajar a través de la noción del estudiante genérico de APOE y la descripción cognitiva de APOE

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puede integrarse a la de la TAD mediante el desarrollo de la posición del estudiante en una institución determinada.

El diálogo a partir de la componente técnico-tecnológica incluye las técnicas de investigación propias de cada teoría y los resultados que permiten, al interpretarse y generalizarse, acercarse a nuevos problemas. Este es un tipo de diálogo poco común en la educación matemática; en general no se discute cómo un resultado de una teoría se interpreta en términos de otra, ni se comparan resultados de distintas teorías. En el diálogo APOE-TAD observamos cómo cada una de las teorías se beneficia y amplía sus fronteras a partir de una reinterpretación de la metodología de la otra sin contradecir su lógica interna.

Las praxeologías en TAD hacen referencia a distintos niveles de análisis: puntuales (consisten en un tipo de tarea), locales (en el que varias tareas se agregan en una técnica) y así por agregaciones sucesivas se articulan las tareas hasta las regionales (se agregan distintas técnicas en una tecnología). La noción de tipos de concepción puede permitir una reformulación de este proceso de articulación de praxeologías mediante la distinción del tipo de técnicas que las componen como técnicas de tipo acción, proceso u objeto. Estos tipos de técnicas pertenecen a un continuo que describe el proceso institucional del desarrollo de las técnicas en la institución en la que se propone o se usa. Por ejemplo, al inicio de la preparatoria las derivadas de una función aparecen como una técnica acción constituida de gestos estereotipados basados en el cálculo de reglas memorizadas. Si los estudiantes son capaces de decidir entre dos reglas distintas a aplicar a la misma función en términos de su simplicidad o encontrar la antiderivada de alguna función, utilizan técnicas-proceso y cuando se cuestiona la aplicabilidad de la derivada a distintos tipos de funciones se usaría una técnica-objeto.

Por otra parte, la noción y los niveles de evolución de los esquemas en APOE puede relacionarse a la actividad que los sujetos llevan a cabo en la institución y podría ser utilizada para describir el desarrollo de las praxeologías de un sujeto en relación a las praxeologías en una institución. Regresando al caso del ejemplo de la derivada, podría considerarse que el nivel Intra- estaría asociado a la actividad matemática descrita mediante una colección de praxeologías aisladas o incompletas, como es el caso en que se presentan en la institución únicamente las reglas para construir gráficas de funciones utilizando las propiedades de la derivada como un algoritmo a seguir y no se requiere de justificación.

Por su parte la TAD contribuye a desarrollar en APOS la noción de relatividad institucional de la DG y así las herramientas de la TAD podrían apoyar el análisis de la DG en términos de sus posibilidades de existencia en una institución dada. En el caso de la derivada se podría ejemplificar por diferencias en las construcciones previstas para describir la forma en que se construye en la escuela secundaria o para quienes se forman como matemáticos en la universidad. En este caso las actividades para alumnos de secundaria no se referirían, por ejemplo, a las ecuaciones diferenciales, mientras ello sería indispensable en la formación de matemáticos.

El ciclo ACE (Actividades, discusión en Clase, Ejercicios) es un componente importante de la metodología de teoría APOE y está íntimamente relacionado con las estructuras conceptuales de esta teoría a través de la DG. Los seis momentos de estudio que describen los procesos de enseñanza y aprendizaje en términos de praxeologías didácticas en la TAD, podrían emplearse para analizar las actividades previstas para el ciclo ACE en términos praxeológicos para verificar si en la actividad prevista, como un todo, hay un equilibrio entre ellos que apoye la construcción del concepto de interés y permitirían contar con criterios de equilibrio y completitud desde el punto de vista institucional. En el ejemplo de la derivada podría darse el caso de que las actividades relacionadas con el momento del primer encuentro no incluyeran ninguna que permitiera a los estudiantes reflexionar sobre el objetivo de las acciones de cálculo de límites que permiten construir la definición del concepto, por lo que se requeriría del diseño de nuevas actividades.

Las aportaciones del diálogo a partir de las componentes teórica y técnico-tecnológica de APOE y TAD, tuvieron como resultado el desarrollo de nociones interesantes que surgen de un cambio de

focos de atención y de un análisis respetuoso y cuidadoso de las teorías; el diálogo puso en evidencia el alcance y las limitaciones de cada una de ellas y requirió de explicitar sus hipótesis implícitas mostrando que son más flexibles de lo que se podría haber considerado. Estas propuestas tienen el potencial de expandir las fronteras de cada una de las teorías sin violentar sus propuestas básicas. El diálogo partiendo de un problema formulado por una teoría quedó pendiente en esta parte del estudio y requiere de la reformulación del problema para hacer posible un análisis más global que permita poner en juego las nociones previamente presentadas. En otro proyecto se llevó a cabo parcialmente este diálogo y sus resultados se describen a continuación.

Diálogo a partir de un problema: Uso de algunos de los resultados anteriores
La investigación sobre la construcción de la noción de función de dos variables descrita anteriormente se continuó en dos ciclos adicionales en los que se incluyó la enseñanza mediante el ciclo ACE. Los resultados obtenidos permitieron refinar y validar la DG, así como diseñar un conjunto de actividades para su enseñanza. Con el objetivo de continuar con el diálogo entre las teorías APOE y TAD y de analizar el funcionamiento de las actividades en la enseñanza en una institución específica se decidió llevar a cabo una investigación utilizando las herramientas surgidas del diálogo como marco conceptual (Trigueros & Martínez- Planell, 2015).
Como primer paso se reformuló el problema de investigación de la siguiente manera: En una institución universitaria concreta ¿Qué características tienen las praxeologías que se utilizan en relación a interpretación de las gráficas de las funciones de dos variables? ¿Qué restricciones limitan que los estudiantes puedan recorrer los momentos didácticos y desarrollar las praxeologías adecuadas? ¿Qué se requiere en la realización de procesos didácticos para que las actividades diseñadas permitan el recorrido balanceado de los momentos de estudio y generen una praxeología matemática local ligada a dicha interpretación?
La investigación se inició con el uso de las herramientas aportadas en el diálogo a partir de las componentes teórica y tecnológico- técnica descritas anteriormente e incluyo varias fases: (a) Análisis de un texto ampliamente utilizado y observación de la forma en que el tema de interés se enseña en la institución concreta utilizando los momentos de estudio del ciclo ACE; (b) análisis de las actividades diseñadas mediante los momentos de estudio del ciclo ACE; (c) ciclos de rediseño de las actividades a partir de los resultados del análisis anterior y (d) evaluación de las actividades finales a través de los resultados obtenidos por los alumnos que las utilizaron.
Los resultados del análisis del texto y de la observación de la enseñanza evidenciaron un recorrido de los momentos de estudio sumamente pobre y desbalanceado. Los momentos de estudio relacionados al bloque práctico de las praxeologías se desarrollaban de forma superficial y muy limitada. La organización del estudio quedó limitada a unos cuantos ejemplos y explicaciones poco relacionados entre sí y, por tanto, resultó incompleta e inefectiva.
Esto permite explicar las dificultades de los estudiantes encontradas en el estudio con APOE presentado anteriormente. Si la organización del estudio no es coherente y no está balanceada, las posibilidades de aprendizaje quedan limitadas por la falta de oportunidades de reflexión.
Las actividades diseñadas para el ciclo ACE de APOE se agruparon en cuatro bloques relacionados respectivamente con la construcción de: (a) planos fundamentales y superficies; (b) cilindros; (c) gráficas y conceptos asociados a las funciones de dos variables y (d) mapas de contornos y familias de funciones. En todas ellas se enfatizaron las técnicas de construcción del espacio tridimensional y de las secciones trasversales, que mostraron ser fundamentales en los estudios realizados con la teoría APOS, para analizar y trazar gráficas.
Las primeras actividades del primer bloque pueden considerarse como pertenecientes al momento del primer encuentro del ciclo ACE y algunas correspondientes al momento de exploración. En ellas se inicia la construcción del espacio tridimensional, de curvas y planos fundamentales en R^3. A

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continuación se muestra un ejemplo de actividad correspondiente al momento del primer encuentro cuyo objetivo consiste en hacer las primeras acciones para construir el esquema de R^3:

En esta actividad, moverse "hacia adelante" o "hacia atrás" es moverse en dirección de x positivo o negativo, respectivamente; "hacia la derecha" o "hacia la izquierda" es en dirección de *y* positivo o negativo, respectivamente; "hacia arriba" o "hacia abajo" es en dirección z positivo o negativo, respectivamente: Halle las coordenadas del punto donde termina si comienza en el punto *A(1,2,3)* y se mueve 5 unidades hacia adelante, 4 unidades a la izquierda y 2 unidades hacia arriba.

El análisis realizado también evidenció que la mayoría de las actividades de los bloques (a) y (b) pueden considerarse como pertenecientes al momento exploratorio. Entre las actividades del primer bloque hay algunas que se enfocan en la interpretación de expresiones con variables libres y en tratamientos y conversiones entre registros de representación, mientras que en las del segundo bloque los estudiantes hacen acciones sobre superficies en el espacio tridimensional, en particular sobre superficies descritas mediante dos variables. Todas estas actividades promueven la reflexión sobre los procesos incluidos en el trazo de la gráfica de distintas funciones a partir de la acción de representación punto a punto. Se detectó un gran número de actividades perteneciente al momento exploratorio. Esto no es sorprendente dado el énfasis que la teoría APOE hace en brindar oportunidades al sujeto genérico de la institución de reflexión sobre sus acciones para interiorizarlas en procesos.

Por su parte, los bloques (c) y (d) enfatizan el momento del trabajo de la técnica mediante la coordinación del proceso de construcción anterior con los procesos de sección trasversal, e intersección para construir e interpretar gráficas de diversas funciones e incluyen algunas actividades que se puede relacionar con el bloque tecnológico- teórico de la praxeoogía. A continuación se presenta una actividad que combina partes correspondientes al momento de exploración y otras correspondientes al momento de técnico-tecnológico. Su objetivo es hacer reflexionar a los estudiantes sobre las acciones correspondientes a las intersecciones de subespacios en R^3, al tiempo que introduce una técnica que incluye las acciones para construir una superficie en el espacio, además de oportunidades de construcción del proceso asociado a ellas:

En este problema dibujaremos la gráfica de $x = 9 - z^2$ en el espacio tres dimensional. Esta gráfica consiste de todos los puntos en el conjunto $\{(x,y,z): x = 9 - z^2\}$. Para ello:

1. Dibuje en el espacio tridimensional los puntos donde el plano *y = 0* interseca la gráfica de *x* $= 9 - z^2$.
2. Dibuje en el espacio tridimensional los puntos donde el plano *y = 1* interseca la gráfica de *x* $= 9 - z^2$.
3. Dibuje en el espacio tridimensional los puntos donde el plano *y = -1* interseca la gráfica de *x* $= 9 - z^2$.
4. ¿Qué sucede cuando se le dan a y más y más valores positivos y negativos?
5. Dibuje la gráfica de $x = 9 - z^2$ en el espacio tridimensional.
6. Reflexione sobre lo que hizo en las tres actividades anteriores. ¿Cómo, en general, se traza la gráfica en tres dimensiones de una función si en su representación analítica solo aparecen dos variables?

Todos los bloques contienen actividades de reflexión sobre estos procesos que pretenden favorecer su encapsulación en objetos y una construcción coherente de un esquema para R^3. Estas actividades pertenecen al momento tecnológico- teórico.

En todos los bloques se encontraron tareas de verificación de la correspondencia entre gráficas trazadas y representaciones analíticas mediante el trazo de curvas que forman parte de la superficie trazada. Estas actividades se consideraron como parte del momento de evaluación. La que se muestra

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en la figura 2 pretende evaluar la construcción de los procesos de conversión del registro analítico al gráfico.

El conjunto de actividades no incluye específicamente la discusión de la teoría ya que juega un papel preponderante en la fase de discusión en grupo del ciclo ACE. Pero, se ofrecen oportunidades de justificación en las actividades que pueden considerarse como parte del momento de institucionalización.

En términos generales este análisis sugiere que el recorrido de los momentos de estudio está balanceado y que el uso de las actividades en el aula puede ser eficaz. Sin embargo, se presenta el reto de reducir el número de actividades asociadas al momento de exploración sin alterar las posibilidades de construcción de las estructuras previstas en la DG.

Las actividades se utilizaron en el aula sin la fase de discusión del ciclo ACE y se evaluaron a través de entrevistas a 9 estudiantes del grupo que las utilizó y de 6 de un grupo control. Los estudiantes fueron elegidos por sus maestros utilizando el mismo criterio de desempeño. El análisis de las entrevistas utilizando APOE mostró una clara diferencia entre los estudiantes de ambos grupos, lo que indica que el conjunto de actividades promueve una organización del proceso de estudio que tiene el potencial de estimular el aprendizaje de los estudiantes. Se puede concluir, a partir de estos resultados, que el uso de una praxeología didáctica en la que hay un balance entre los momentos de estudio contribuye a la construcción de las estructuras previstas en la DG. Este trabajo permitió mostrar también que existen puntos de contacto que permiten el acercamiento de las dos teorías.

Escriba la fórmula que corresponde al lado de cada una de las siguientes gráficas. Escoja entre: $z = x^2 seny;$ $z = x^2 - seny;$ $z = xsen(y^2);$ $z = x + sen(y^2)$. Use secciones para justificar plenamente su respuesta.
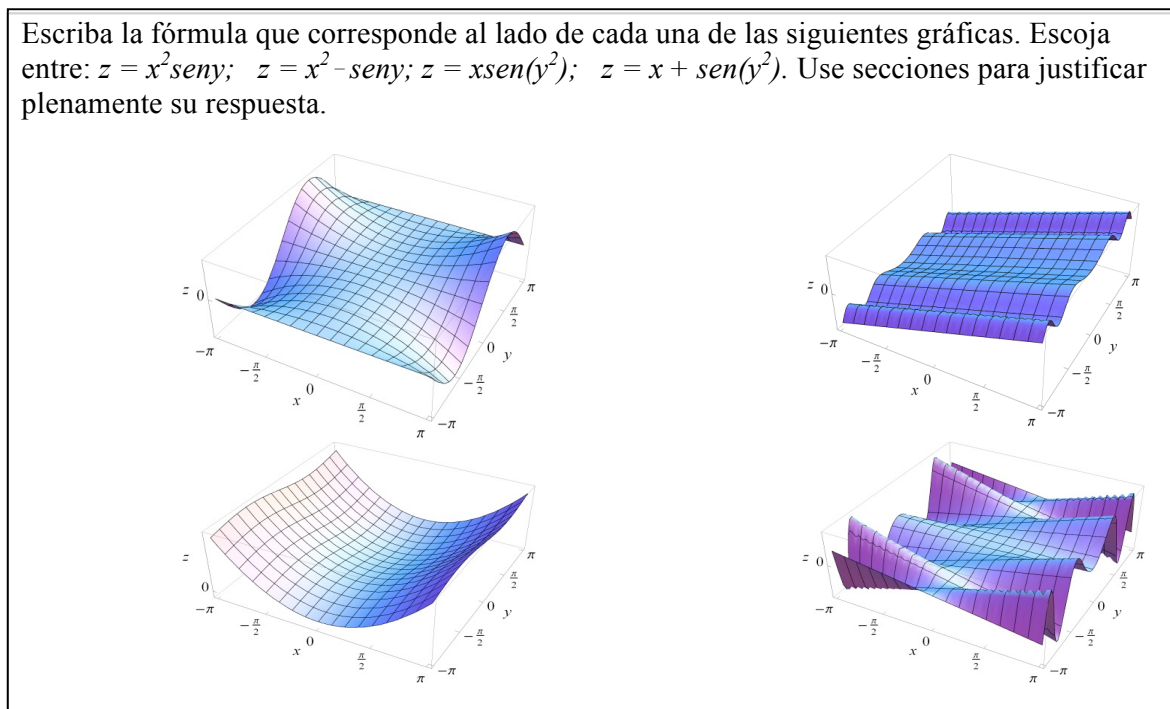
Figura 2. Actividad de conversión y justificación.

Cruzando fronteras metodológicas: Uso de la modelación en la enseñanza de las Ecuaciones Diferenciales y del Álgebra Lineal.

El álgebra lineal y las ecuaciones diferenciales constituyen dos cursos básicos de las matemáticas universitarias que coinciden en su gran aplicabilidad tanto en temas de las matemáticas mismas como en problemas de otras disciplinas. Los desarrollos recientes de la investigación en matemáticas y de la tecnología imponen, en principio, la búsqueda de nuevas estrategias para su enseñanza que no han

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sido tomadas en consideración por la comunidad universitaria en general. Los resultados de la investigación en educación matemática, por otra parte, ofrecen información sobre la forma en que los alumnos aprenden los conceptos de estas disciplinas y métodos de enseñanza que han probado ser efectivos, entre los que destaca el uso modelos (Por ejemplo, Rasmussen & Blumenfeld, 2007; English, Lesh, and Fennewald, 2008; Bas, Cetinkaya, and Kursat, 2009; Zandieh, & Rasmussen, 2010; Camacho, Perdomo, and Santos, 2012; Trigueros & Possani, 2013).

En un proyecto de investigación que se llevó a cabo durante cinco años, en la institución en la que trabajo, se intentó responder a las preguntas ¿Qué resultados se obtienen, en términos de aprendizaje, cuando se utilizan problemas de modelación en la enseñanza del álgebra lineal y de las ecuaciones diferenciales? ¿Qué aspectos del conocimiento de los estudiantes pueden recuperarse cuando se utilizan modelos en la clase?

Dado que las investigaciones con la teoría APOE no incluían hasta ese momento el uso de problemas de modelación en la enseñanza, se buscó la posibilidad de coordinar esta teoría con una teoría del ámbito de la modelación. Se seleccionó la teoría de Modelos y Modelación (TMM) (Lesh & Doerr, 2003) considerando dos criterios. Desde el punto de vista teórico, propone la posibilidad de construcción de conceptos matemáticos mediante el trabajo de los alumnos en la descripción del comportamiento de una situación en un contexto real o realista que les permita, por una parte utilizar su conocimiento previo y, por otra, construir nuevo conocimiento. Su metodología de enseñanza coincide con la de APOE en que la construcción de nuevo conocimiento se logra mediante trabajo en equipo y con discusiones en clase con el profesor. Por otra parte TMM postula un conjunto de principios que el problema planteado para la modelación debe satisfacer de modo que pueda ser aplicado exitosamente en el aula (Carlson, Larsen, & Lesh, 2003) que puede ser útil en el diseño de situaciones y que puede ayudar a extender la frontera de la teoría APOS de manera que la introducción de los problemas de modelación resulte natural en el ciclo ACE y que el desarrollo del conocimiento a partir de los conceptos previos utilizados por los estudiantes pueda ser descrito mediante una DG adecuada.

Desde el punto de vista teórico ambas teorías postulan formas de construcción del conocimiento. Es posible entonces proponer una posible coordinación entre ellas considerando que el trabajo sobre el problema original permite poner en juego un modelo matemático sobre el cual pueden hacerse dos tipos de actividad, una que permite analizar, desarrollar y validar el modelo matemático y otra en la que las acciones sobre el modelo matemático, libres o guiadas mediante actividades, pueden ser complementadas mediante actividades específicas diseñadas en términos de una DG. Estos dos tipos de actividad se pueden ir intercalando en ciclos l0,'9en los que la modelación da lugar a la necesidad de reflexión y a la construcción de las estructuras necesarias para el aprendizaje de los nuevos conceptos y éstos permiten mirar el trabajo sobre el modelo desde una perspectiva distinta que lo hace evolucionar. Desde el punto de vista metodológico la TMM puede utilizarse para validar el funcionamiento de la situación planteada en términos la promoción de nuevo conocimiento mientras que APOE puede utilizarse para analizar las construcciones de los alumnos y cómo podrían relacionarse con nuevas actividades que promuevan la construcción del conocimiento de interés pues, de acuerdo a esta teoría, el uso de conocimiento previo no es suficiente para garantizar la construcción de nuevos conocimientos. Los ciclos de enseñanza e investigación de APOE se insertan, además, de manera natural en los ciclos de modelación previstos por la TMM. Y así, ambas teorías pueden ampliar sus fronteras para plantear nuevas propuestas didácticas y de investigación. Con base en la coordinación de estas teorías se desarrolló un proyecto de uso de problemas de modelación en la enseñanza de las ecuaciones diferenciales y el álgebra lineal. Dos ejemplos ilustran los resultados que se pueden obtener de un diálogo de esta naturaleza, uno en el contexto de las ecuaciones diferenciales y otro en el de álgebra lineal.

Desarrollo de nuevas herramientas de análisis

En el primer caso (Trigueros, 2014) se planteó a los estudiantes el problema: *Una profesora, preocupada por la tendencia actual en la enseñanza y por el problema del manejo de información de los estudiantes, nos pide hacer un estudio sobre la memorización y el olvido a corto plazo. En particular le interesa un reporte en el que quede claro el tiempo que una persona puede retener información aprendida, el tiempo que tarda en olvidarla y cuáles son los factores de los que esto depende.*

En el ciclo que denominamos de exploración gráfica del problema a partir de la experiencia propia de los estudiantes y del uso de su esquema de función emerge un primer modelo gráfico conjuntamente con la definición de variables (Figura 3a). Aunque de inicio los estudiantes intentan encontrar una expresión analítica para la función, en sus diálogos evocan su esquema de derivada para analizar la gráfica propuesta:

> *Rosa:* Se puede usar la derivada para especificar las propiedades que debe tener la función que proponemos en la gráfica, pero… no tenemos, nadie tiene, la expresión analítica y yo no sé cómo obtenerla de las propiedades, más bien sé al revés, sacar las propiedades si sé la regla de la función.

El siguiente ciclo se identifica mediante la introducción de la variación a la solución del problema que permite sugerir un posible modelo que puede ilustrarse mediante este diálogo:

> *Fer*: Si suponemos que cada persona tiene un coeficiente de memoria y… lo que han aprendido crece de aquí a acá, pero va creciendo menos rápido por pedazos (dibuja una curva poligonal con segmentos de recta), y luego empieza a decrecer también primero rápido y luego no tanto hasta que se le olvida todo, o casi todo…
>
> *Juan*: …Debe crecer más despacio acá y si la persona tiene cierta capacidad de memoria, un coeficiente, *k*, para que primero crezca más y luego menos, podría ser algo como… (escribe $y'=k(y-T)$) porque así no seguiría creciendo sino va parando…

La discusión en grupo gira en torno a las ventajas de escribir una ecuación que incluya la variación. Después de un ciclo de actividades diseñadas con la DG en el que se introducen las ecuaciones diferenciales y las funciones implícitas, como respuesta a las dudas de los alumnos frente a una ecuación que no contiene la variable independiente, se regresa al modelo. Se presenta entonces un ciclo caracterizado por el análisis del modelo. En este ciclo emergen en equipos distintos dos métodos de análisis de la ecuación que no se habían introducido anteriormente: la aproximación de la función solución usando la idea de derivada como aproximación lineal y el uso de la gráfica de la derivada de la función contra la función misma (plano fase) o su descripción verbal, como se muestra respectivamente en los párrafos siguientes.

> *Fer*: Aquí (señala la ecuación). Si le ponemos valores para más fácil, (escribe: $y'=.5(50-y)$). Y si usamos lo de antes, la derivada como pendiente, (escribe $y' = y-yo/t-to$), entonces del tiempo cero al dos, *y* prima es…(escribe $y'=.5(50-y)=y/2$), entonces ... nos queda $y=25$, o sea que en el segundo tiempo se sabe 25 datos y la recta iría de 0 a 2 y de 0 a 25, con pendiente 12.5. Luego hacemos lo mismo pero ahora empezamos con 2 y hasta por ejemplo 4 y *y* vale 25 entonces (hace cálculos, no audible) sale *y* es 37.5 y así le seguimos…
>
> *Nadia*: …Estoy pensando en que si graficamos esa función de *y'*… pero como función de *y*, ¿qué significaría?... A ver… esa gráfica (Figura 3b) es una recta, con pendiente positiva. Dice que $y'=0$ si $yo/k+1=y$. Pero no debe ser porque ¿dónde queremos que haya un punto crítico? yo creo que en el *1* porque es cuando se aprende todo y luego cuando olvida empieza a bajar, pero esto está más arriba.
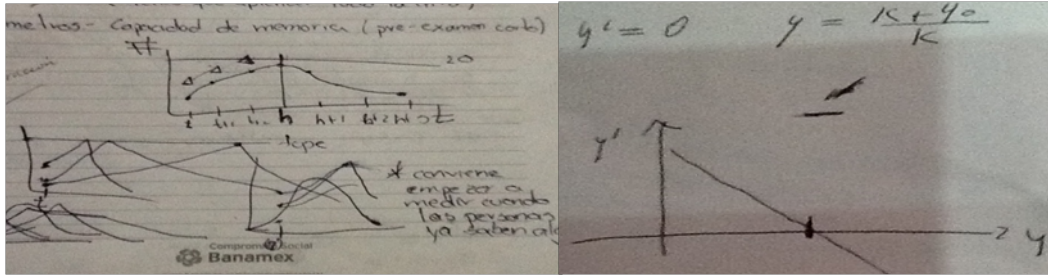
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Figura 3(a). Exploración.   **Figura 3(b).** Análisis gráfico del modelo.

La actividad continúa mediante otros ciclos de actividades diseñadas con DG y de trabajo en el modelo hasta terminar la actividad con un ciclo caracterizado por la búsqueda experimental del valor de los parámetros del modelo.

Destaca en esta experiencia el papel que juega la representación gráfica de la función como detonador de un cambio en la manera de abordar el problema y en la emergencia de nuevas herramientas conceptuales desarrolladas de manera independiente por los estudiantes.

Construcción independiente de conceptos

En el contexto del álgebra lineal se planteó a los estudiantes un modelo de empleo (Salgado & Trigueros, 2015): *En una economía en la que la fuerza de trabajo permanece constante, hay un cierto número de personas con empleo en cada período de tiempo y un cierto número de personas desempleadas en ese mismo periodo. Si conocemos la probabilidad de que una persona desempleada encuentre un trabajo en el siguiente periodo de tiempo y la probabilidad de que una persona empleada pierda su empleo ¿Cómo podemos describir la dinámica del empleo en el tiempo y cómo se comportará en el largo plazo?*

En este caso algunos estudiantes utilizaron sus conocimientos previos, tanto del curso de álgebra lineal, en el que se había trabajado anteriormente un modelo de poblaciones para introducir las ecuaciones en diferencia de una variable, como de los cursos de economía para plantear una ecuación para el modelo:

A_2: … en cualquier momento hay un cierto número de empleados y de desempleados, pero ese número puede cambiar en el siguiente periodo…

A_3: Es cierto, podemos pensar en el número de empleados en el siguiente periodo y esos deben ser los que todavía tienen empleo menos los que ahora están desempleados…

A_1:… en mi opinión, los empleados en ese nuevo periodo deben ser una proporción de las personas que tienen empleo y lo conservan más el número de personas que estaban desempleadas y consiguen empleo…

A_2: Si, sabemos que p son los desempleados que consiguen trabajo y q los empleados que siguen empleados, bueno, más bien as probabilidades (anotan $x_{t+1} = qx_t + py_t \rightarrow$ empleo)

A_3: Entonces el modelo es un sistema, hay otra ecuación igual para el desempleo (escriben $y_{t+1} = (1 - q)x_t + (1 - p)y_t \rightarrow$ desempleo)

El trabajo en la validación de la solución propuesta para la ecuación del modelo, obtenida de la generalización de la solución para el modelo de poblaciones mediante la introducción de la matriz del sistema, condujo a estos estudiantes a usar su conocimiento previo sobre matrices y exponenciales para hacer un proceso sobre la ecuación resultante (Figura 4).

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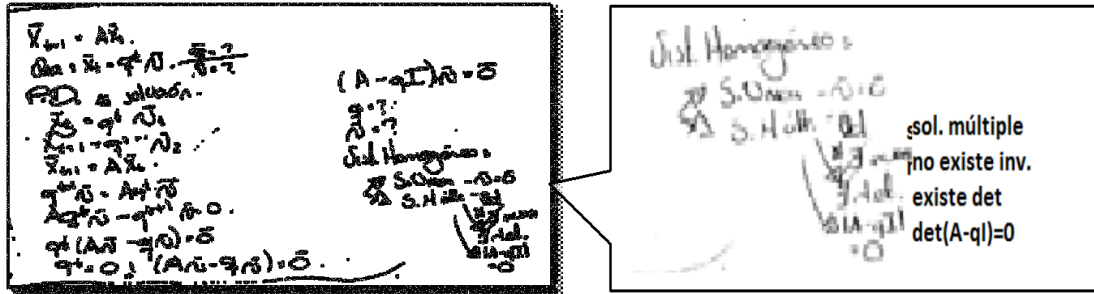
Figura 4. El uso del esquema conduce a la definición de los valores y vectores propios.

Su trabajo mostró evidencia de construcción de relaciones entre los conceptos desarrollados en el curso, lo que puede caracterizarse como la construcción de un esquema de conjunto solución de un sistema lineal de ecuaciones que incluye los conceptos de solución, de matriz, de independencia lineal, espacio nulo y determinante.

Este proceso los condujo a la definición de los conceptos de eigenvalores, eigenvectores y eigenespacios que no habían sido definidos con anterioridad. La reflexión de estos estudiantes sobre sus propias acciones permitió que construyeran estos conceptos como proceso. Con el trabajo en actividades diseñadas con la DG en las que se retomó y se institucionalizó su trabajo tuvieron oportunidad de construir estos conceptos como objetos.

Posteriormente calcularon los valores y vectores propios para valores específicos de los parámetros:

A_1: …para k_1 igual a uno los vectores tienen la forma $v = (2x_2/3, x_2)$ y x_2 es un parámetro; y para k_2 1/6, el vector es $v = (2x_2/3, x_2)$.

A_3: En ambos casos x_2 es arbitraria, o sea la familia de soluciones en el conjunto solución. ¿Es el parámetro el mismo en ambos casos?... No, entonces a uno le llamamos x_1…

A_2: Es cierto... entonces…solo para esos valores de k y esos valores de v, las funciones que propusimos son soluciones del sistema del modelo.

A_3: ¿Podemos usar un caso particular para cada familia de vectores?

A_1: No sé, pero en ese caso, un caso particular… para $k_1 = 1$, $v_1= (2/3,)$ y para $k_2 = 1/6$, $v_2= (2/3,)$ ¿estará bien?

A_3: Para cada k, los vectores generan una recta, pues solo hay una variable arbitraria en cada caso.

Después de un ciclo de actividades diseñadas con la DG para institucionalizar y brindar a todos los alumnos nuevas oportunidades de reflexión, durante la discusión los mismos estudiantes relacionaron estos conceptos con el problema a modelar.

A_3: … Encontramos que los eigenvalores eran $k_1 = 1$ y $k_2 = q - p$, o sea, diferentes, uno es independiente de las probabilidades dadas y el otro está relacionado con la diferencia de la probabilidad de que una persona continúe empleada y la de que un desempleado consiga trabajo…

A_2: Usamos esos valores y encontramos los eigenvectores, bueno, escogimos uno para cada k. Para $k_1=1$ encontramos $(p/(1-q), 1)$ así que está relacionado con la razón de la probabilidad de que una persona desempleada encuentre trabajo en el siguiente periodo y la de que un empleado pierda su trabajo en el siguiente periodo. Algo así como una razón de las probabilidades de cambiar de estado… para la otra k, $k_2 = q - p$ era independiente de los valores de las probabilidades, era $(-1, 1)$ y no pudimos explicar estos…también porque

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hicimos una gráfica (Figura 5) y el espacio propio correspondiente siempre tiene vectores con una componente negativa…

A_3: Entonces las constantes no tienen relación con las condiciones iniciales, aunque en el problema de población esas condiciones aparecían en la solución y discutimos su papel en términos de la familia de soluciones. Pero aquí ¿dónde están las condiciones iniciales del problema? ¿No juegan ningún papel?

La maestra regresó la pregunta en su trabajo en equipos y les pidió predecir el comportamiento a largo plazo. En el trabajo en un nuevo ciclo de actividades diseñadas con la DG los alumnos hicieron referencia al concepto de base:

A_3: Escogimos un vector particular para cada conjunto solución por separado, pero ¿podríamos tomar uno de cada familia para hacer una combinación lineal y generar tal vez R^2.

Esto puso en evidencia que los estudiantes construyeron relaciones entre su conocimiento previo y el nuevo.
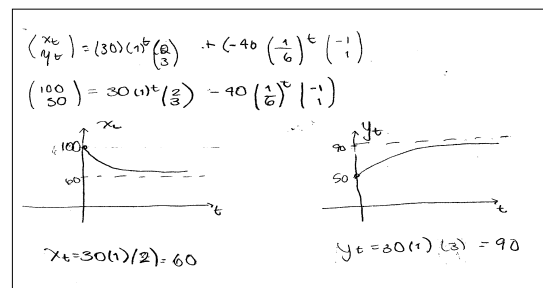
Figura 5. Comportamiento a largo plazo.

Los alumnos, en general, tuvieron dificultades para entender las funciones vectoriales resultantes, que no habían estudiado todavía. Algunos optaron por graficar para cada una de las componentes para encontrar el comportamiento a largo plazo y concluir a partir de ellas, que cada solución converge a una solución que corresponde al eigenespacio.

Destacan en esta experiencia la aparición de los conceptos de interés en el trabajo de los alumnos sin intervención alguna de la maestra y la construcción de un esquema por parte de algunos estudiantes en el que muestran haber construido relaciones entre los conceptos introducidos con anterioridad. Si bien en la entrevista posterior al curso se encontró que algunas dificultades persistieron en los casos de dimensión mayor, al menos 3 estudiantes mostraron una concepción objeto de los conceptos de interés.

Muchas investigaciones sobre el uso didáctico de la modelación insisten en la motivación que la solución de este tipo de problemas representa para el estudiante. Estas investigaciones lo confirman. Pero, demuestran además que cuando los estudiantes se involucran en problemas de interés y reflexionan libremente desarrollan nuevas estrategias que promueven la construcción de conceptos y de relaciones entre ellos. Los problemas de modelación pueden favorecer el aprendizaje de conceptos difíciles y abstractos y el nuevo conocimiento puede, a la vez, aplicarse a la solución de nuevos problemas. La ventaja de la coordinación de las dos teorías, por su parte, se hizo patente en la permeabilidad de sus fronteras y en la expansión de los problemas que cada una de ellas puede abordar.

Comentario final

Las experiencias de cruce de fronteras, diálogo y búsqueda de posibles maneras de coordinar teorías analizando cuidadosamente las hipótesis básicas y las posturas teóricas de cada una de ellas

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mostraron en cada caso posibilidades alentadoras de acercamiento sin confrontación y de encontrar aportaciones que las enriquecen mutuamente. Ponen en relieve, además, que mediante un acercamiento serio y abierto es posible establecer diálogos entre teorías aun en casos en que las teorías parecen lejanas entre sí.

Además del enriquecimiento teórico resultante de tender puentes entre distintas teorías, la aplicación de las herramientas surgidas de los diferentes diálogos hizo patente sus posibilidades de aplicación en el diseño de estrategias de enseñanza que favorecen el aprendizaje de los alumnos y la posibilidad de pensar en el trabajo propio de investigación de forma diferente.

Las fronteras teóricas pueden aparecer en un diálogo respetuoso y tolerante como fronteras flexibles y dinámicas que permiten la ampliación de su dominio de aplicación y el intercambio de ideas sin perder su propia identidad. Fronteras que se abren y dejan aparecer fenómenos que serían imposibles de ver si se hubiera utilizado una sola de ellas. El reto que enfrentamos como comunidad es acercarse al estudio de los fenómenos de la educación matemática construyendo puentes de manera creativa que permitan conservar las fronteras que dividen a las diferentes teorías pero al mismo tiempo las hagan porosas para que la disciplina se enriquezca a través de las relaciones de colaboración.

Interest in understanding mathematics teaching and learning phenomena, and to develop new and effective methodologies to teach Differential Calculus and Linear Algebra led me to look for ways to dialogue between APOS Theory and other mathematics education theories. This enterprise has facilitated a better understanding of them. Bridges between them have been constructed that enable to find possible points of contact and to expand their borders without changing their fundamental hypothesis. In this paper, a reflection on what a dialog between theories entails is illustrated and, through three examples, it shows how it promotes enriching experiences for researchers and stimulates students' creativity. Results also underline how positive outcomes in terms of students learning and in the development of effective teaching resources can be obtained.

Keywords: Advanced Mathematical Thinking, Learning Theory, Modeling

Doing research in mathematics education implies analyzing a didactical phenomenon in terms of a specific theoretical approach. Its selection must be made among an enormous number of theories that assume a diversity of perspectives in face of the complex mathematics learning and teaching phenomena. Each theory constitutes a field defined by its own hypothesis, conceptual structures and methodology. The research problems of interest, and the way they are approached, are defined in those terms. So, something that can be thought of, apparently, as the same phenomenon, is defined differently within each theory, and the explanations provided are delimited by each theory's own restrictions.

A growing interest has emerged recently to look for the construction of bridges between different theories and to analyze their possible complementarities through comparative studies of the same phenomenon or through their possible articulation in a new and broader theoretical framework (Artigue, Bosch & Gascón, 2011).. This interest may be theoretical or may result from an analysis need arising in the context of research, but is problematic since each problem is constructed in terms of the structure of the theory. Furthermore, it is studied and its results are interpreted in terms of specific tools developed for that purpose (Trigueros, Bosch & Gascón, 2011). New questions arise in

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this context: Why should we cross theoretical borders? How can a dialog between theories be established so that crossing their borders is possible without changing their basic principles? How can bridges be built from one to the other or how can their borders be made flexible so that the internal coherence of each theory is preserved?

In what follows I will try to respond to these kinds of questions through the results of my own experience. In order to do so, I will discuss some specific examples taken from my own research work.

Crossing borders between theories: Expandability of application domains.

How to cross the boundaries of a specific theory? What can be gained by doing so? I will start trying to give a first answer to these questions. A few words about my personal history can help shape some tentative answers.

My work on mathematics education research has used APOS (Action, Process, Object, Schema) theory as the theoretical framework. APOS theory is a cognitive theory that intends to explain how students construct different mathematical concepts (Arnon et al. 2014). My personal interest in looking for new teaching methodologies, to bring students to mathematics and to promote their learning and their possibilities to use their knowledge, led me to explore other theories and to seek to establish a dialogue between some of them and APOS. What does this exploration entail? In my experience the exploration was guided by considering the possibility to analyze the theories and think of their boundaries not as rigid limits, but as something flexible and dynamic. This idea brought me to consider how theories can complement each other and how they can enrich each other when a dialogue between them, that may extend their theoretical or methodological limits and expand their possibility of analysis of educational phenomena of interest, is established.

In all the dialogue examples that I will discuss, one of the theories involved is APOS theory. I will begin by giving a very brief description of this theory: APOS theory is based on Piaget's epistemology, in particular, on reflective abstraction mechanism. Its name is an acronym of the types of mental structures that are supposed to be constructed by students when they are involved in mathematical activities. An Action is a transformation of previously constructed objects by following explicit instructions that the student considers as something external. Actions may be interiorized, by means of reflection, into a Process which consists in the possibility to describe, imagine or perform those transformations without having to follow explicitly all the steps. Processes can be reversed or coordinated with other processes in order to construct new processes. When students are conscious of a process as a totality and can do actions on it, the process is encapsulated into an Object that may be de-encapsulated, when needed, in the process from which it originated. Schemas are developed by means of the construction of relations among actions, processes, objects and previously constructed schemas, and a schema is considered, consciously or unconsciously, as a coherent framework that can be used to solve mathematical problems related among them. When the student needs to do actions on a schema, it can be thematized into an object. APOS considers that the general tendency of students when they work on a series of situations related to a specific mathematical notion is different depending of the type of constructions they show in their work.

In addition to the theoretical structures described, APOS theory includes a model to describe a possible way of constructing the mathematical concepts of interest in terms of the structures of the theory. This model, know as genetic decomposition (GD), is not unique and most be validated in terms of experimental results. Therefore, the research methodology of the theory includes research cycles in which the GD is refined until it is considered to be stable.

APOS theory includes as well a teaching cycle: Activities designed with the GD as a basis, Class discussion and Exercises (ACE) to promote the construction of the intended concepts in the classroom. In it, teaching activities designed in accordance to the GD are used and they are reconsidered as the GD is refined.

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Searching for detail in the analysis of students mathematical activity

In the context of a project about the construction of the concepts of the Differential and Integral Calculus of functions of two variables, together with Rafael Martínez- Planell, we studied students' learning of functions of two variables (Trigueros & Martínez-Planell, 2010; Trigueros, M., & Martínez-Planell, 2012). The analysis of the data obtained in the first cycle of the project highlighted that in order to be able to interpret students' responses more detail was needed, both in how language is used in teaching, and in these functions' geometric representation and its role in the construction of this concept,. For example, when asked to sketch the graph of the function $z = x^2 + y^2$ in R^3 when $z = 3$ many students responded *"It is a cylinder"* and they drew graphs similar to that shown in Figure 1.
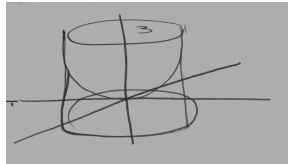
Figure 1. The graph of $3 = x^2 + y^2$ according to many students.

That was how we approached a cognitive theory detailing the role of the relation between different representations of the same concept in its learning: Semiotic Representations Theory (SRT) was proposed by Duval (2006). This theory considers that the mathematical thinking process requires the cognitive coordination of different representation by means of their comparison and analysis. It proposes two types of transformations of semiotic representations: treatments which consist in changes of representations that are in the same representation register and conversions that involve changes from one representation register to another without changing the represented object denoted in them. In order to discriminate those values of a mathematical object that are cognitively significant by means of comparisons of similar representations, it is necessary to do treatments, while conversions allow to separate the represented object from the particular representation's content and prevent that the different registers stay separated for the learner.

The analysis of the possibilities to construct bridges between these theories started from the consideration of the role of representations in the construction of knowledge in both theories. In SRT, representation registers and transformations between registers are central in understanding mathematical objects. In APOS theory transformations between representations and their relation are linked to different mental constructions that can be described in terms of interiorization, coordination, reversal or encapsulation mechanisms. It is possible to consider that treatments could be described partially in terms of actions on an object in a given representation register and that reflection on them can make possible to highlight significant aspects or properties of the object when those actions are interiorized into processes. The interaction with semiotic representations theory makes it possible to point out the type and role of those actions by emphasizing the need to discriminate among different characteristics of the represented object in the register where the actions are performed. Let's consider, for example, the action of substituting number 1 for x in the function $z = x^2 + y^2$. Students may recognize z as a quadratic function in R^2. When substitution actions are repeated using different numbers, students can identify a family of similar quadratic functions. These action can be interiorized in the process of construction of a family of parabolas $z = a + y^2$. In this example SRT allows the identification of the role of those actions in the construction of knowledge about functions that is fundamental when considering functions of two variables.

Meanwhile, conversions can be described in APOS as the interiorization of the process that makes it possible to consider actions of comparison and identification of an object in different representation registers as an association; but the interaction with SRT underlines the need to include those processes in the GD because of their importance in cognitive terms. Going back to the previous

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example, the graph of functions $z = a + y^2$ in the plane makes it possible to underline those aspects of the graphs as objects that remain invariant in treatments and their relation with the parameter in the analytic representation. The processes constructed by means of those actions can be coordinated with the representation of planes in R^3 in a new process of representation of the quadratic functions on different fundamental planes $x = a$, and in the process of construction of the surface corresponding to the graph of the function. The bridge between the theories enables to use the treatment and conversion notions in APOS theory without losing sight of its genetic perspective and to consider the flexibility in passing from one representation to other by incorporating specificity in the analysis.

Bridging between the two theories allowed a more thorough analysis of students' work on tasks set in the same representation or in those that need the conversion between representations, and the complexities involved in it. This analysis had repercussions on a refinement of the GD. It also allowed to show that generalization of the understanding of functions of one variable to functions of two variables is not direct, given that students do not necessarily perceive the expected analogies.

How can a fruitful dialogue between theories be conducted?

The possibility to coordinate theories that study the mathematics education phenomena from very different perspectives may seem impossible. How to bridge them? Where can a dialogue start from? In an effort to answer these questions and construct bridges between these theories, together with Marianna Bosch and Josep Gascón, we decided to analyze the conditions that a dialogue between theories must satisfy, and to apply them to a conversation between APOS theory and the Anthropological Theory of Didactics (ATD) (Trigueros, Bosch & Gascón, 2011).

The primary research object of ATD is the institutionalized mathematical activity; that is, this theory questions and models the intra-institutional and inter-institutional genesis and diffusion processes of praxeologies (Chevallard, 1992). Praxeologies are formed by a practical component and a theoretical component. The first one can be described in terms of the tasks and techniques involved in the activity, and the second one in terms of the technologies and theories that support the employed tasks and the techniques. ATD's research program requires making an epistemological model (EMR) of mathematics that is used as reference to analyze the institutionalized mathematical activity and is based explicitly on the didactic transposition process. This process proposes those transformations that can be performed on mathematical knowledge so that it can be taught in a specific institution. The EMR is also used to analyze and design teaching and learning activities which make it possible, simultaneously, to validate it (Bosch & Gascón 2005). The praxeologycal analysis of mathematical activity in the institutions allows understanding the institutional restrictions that limit or favor some types of activity, and analyze the role of equilibrium in mathematical practices in the institution.

In this project's dialogue, the notion of praxeology of ATD is used as a framework for analysis by reinterpreting theories as research praxeologies with the aim of making sure that all their components are taken into account in a fruitful dialogue. Thus, the dialogue can be conducted starting from the problems discussed (tasks), the methodological tools used to work on the problems (techniques and technologies) and the structures of the theories.

Both theories incorporate an analysis of school mathematics as a fundamental component of their approach to any didactical problem and include alternative models for the development of mathematical knowledge as part of their methodology: the GD in the case of APOS theory and the EMR in the case of ATD. This fact can be considered as a point of contact between both theories. The fundamental difference between these models is that the GD is used to study the cognitive development of individuals while the EMR is used to study the institutional conditions that make the existence of the mathematical activities possible in an institution; the analysis unit is different and that fact has important methodological implications. In order to carry out the dialogue, we first developed the notion of "generic student in an institution" which relates the epistemic subject of

APOS to the "student in an institutional position" of ATD. This way, the notion can be extended in both theories: the institutional dimension of ATD analysis can be worked through the notion of generic student of APOS and the cognitive description of APOS can be integrated into that of ATD by the consideration of the position of student at a specific institution.

The dialogue starting from the technical-technological component includes the research techniques of each theory and the results that allow to approach new problems when interpreted and generalized. This is an uncommon kind of dialogue in mathematics education; in general how a result of a theory can be interpreted in terms of other is not discussed and results of different theories are not compared. Through the APOS-TAD dialogue we observed how each theory can benefit and extend its boundaries starting from a reinterpretation of the other's methodology without contradicting its internal logic.

In ATD, praxeologies make reference to different levels of analysis: punctual (consists in one type of task), local (when several tasks are aggregated in a technique) and by making successive aggregations the tasks are articulated into regional praxeologies (aggregation of different techniques in a technology). The notion of different kinds of conceptions can allow a reformulation of this process of articulation of praxeologies by the distinction of the type of technique that compose them as action type techniques, process or objet. These types of techniques are part of a continuum that describes the institutional process of development of techniques in the institution where they are proposed or used. For example, at the beginning of high school derivatives of a function appear as an action technique constituted by stereotyped gestures based on the use of memorized rules. If students are capable to decide between two different rules that can be applied to the same function in terms of their simplicity or find the antiderivative of a function, they use process- techniques, and when they question the applicability of the derivative to different types of functions they would be using an object-technique.

The notion and the levels of development of schemas in APOS, on the other hand, can be related to the subjects' activity in the institution and could be used to describe the development of a subject's praxeologies related to an institution's praxeologies. Going back to the derivative example, it could be considered that the Intra- level would be associated to the mathematical activity described by a collection of isolated or incomplete praxeologies, as in the case where only the rules to sketch the graph of a function, based on the derivative properties, are presented in the institution as an algorithm for the student to follow without any justification.

Meanwhile, the ATD contributes to develop in APOS the notion of institutional relativity of the GD, and thus, the ATD tools could support the analysis of the GD in terms of its possibilities of existence in a given institution. In the case of the derivative, it could be exemplified by the differences in the expected constructions needed to describe the way in which the derivative is constructed in high school or in an undergraduate program in mathematics. In this case the activities for high school would not refer, for example, to differential equations, while this would be indispensable en the case of the mathematics program at the university.

The ACE cycle (Activities, Class discussion, Exercises) is an important component of the methodology of APOS theory and it is intimately related to the conceptual structures of this theory through the GD. The six moments of study describing the teaching and learning processes in terms of didactic praxeologies in ATD, could be used to analyze the activities anticipated for the ACE cycle in praxeological terms to verify if in the expected activity, as a whole, there is an equilibrium among them that supports the concept of interest's construction, and if they would enable disposing of equilibrium and completeness criteria from the institutional point of view. In the derivative example, it could happen that the activities related to the moment of the first encounter did not include any activity to help students reflect on the objective of the limit calculation actions needed to construct the definition of the concept, so it would be necessary to design new activities.

The contributions of the dialogue starting from the theoretical and technical- technological of APOS and ATD, had as a result the development of interesting notions that emerge from a change in attention focus and form a respectful and careful analysis of both theories. The dialogue evidenced the extent and the limitations of each theory and required making explicit their implicit hypothesis, showing that they are more flexible than what could be considered. These proposals have the potential to expand each theory's borders without violating their basic tenets. The dialogue starting from a problem formulated by one of the theories was not discussed in this part of the project and requires of the reformulation of the problem to make a more global analysis that will allow putting into play the previously presented notions. In another project this dialogue was partially carried out. Its results are described below.

Dialogue starting from a problem: using some results from the APOS-ATD dialogue
Research on the construction of the notion of function of two variables previously described was continued for two additional cycles where the teaching cycle was included. Results obtained were used to refine and validate the GD, and also to design a set of activities to be used in the teaching cycle. In order to continue the dialogue between APOS theory and ATD and with the goal of evaluating the designed activities in a specific institution, we decided to continue with a research using some of the tools developed in that dialogue as a conceptual framework (Trigueros & Martínez- Planell, 2015).

The first step in this research consisted in reformulating the research problem as follows: in a particular university as an institution, what are the characteristics of the praxeologies used in the interpretation of the graphs of two variables? What are the restrictions limiting students possibilities to go through the didactical moments and develop the associated praxeologies? What conditions are needed to warrant the development of a didactical process includes a balanced use of the study moments and generates a mathematical praxeology linked to that interpretation?

The research started with the use of the tools proposed in the dialogue starting from the theoretical and from the technological- technical components. It included several phases: (a) Analysis of a widely used textbook and observation of the way this topic is taught in the same institution using the moments of study of the ACE cycle; (b) analysis of the activities designed using the moments of study of the ACE cycle; (c) cycles to redesign the activities taking into account results from the previous analysis and (d) final evaluation of the final activities through the analysis of students results obtained by students who used them.

The results of the textbook analysis and the teaching observation evidenced a poor and unbalanced presence of the didactical moments. Those moments related to the practical part of the praxeologies were developed in a superficial and limited way. The study organization stayed limited to a few examples and explanations poorly related among them. So it proved to be incomplete and ineffective.

These results allow explaining the difficulties shown by students in the APOS study described above. If the study organization is not coherent and balanced, the learning opportunities remain limited due to the lack of reflection opportunities.

The designed activities for the cycle ACE of APOS theory were grouped in four sets related, respectively, with the construction of: (a) fundamental planes and surfaces; (b) cylinders; (c) graphs and concepts associated to functions of two variables and (d) contour maps and families of functions. In all of them, the construction of the tridimensional space and use of transversal sections techniques that were found to be fundamental in studies realized with APOS theory was emphasized in the analysis and sketching of graphs.

The first activities in set (a) can be considered as part of the exploration moment of the ACE cycle. In that set there are also activities corresponding to the exploration moment. In them, the construction of three-dimensional space, curves and fundamental planes in R[3] is introduced and

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explored. The following is an example of an activity corresponding to the first encounter moment. Its goal is to make the first actions on the intuitive notion of space to construct the R^3 schema:

In this activity to move "forward" or "backwards" corresponds to moving in the direction of positive x (the x coordinate increases, the y and z remain the same) or negative x (the x coordinate decreases, the y and z remain the same), respectively; "to the right" or "to the left" is moving in the positive or negative y direction, respectively; "up" or "down" is moving in the positive or negative z direction, respectively. Find the coordinates of the point where one ends if one starts at point $A(1, 2, 3)$ and moves 5 units forward, 4 units to the left, and 2 units up.

The analysis of results showed as well that most of the activities in sets (a) and (b) can be considered as part of the exploratory moment. Among the activities of the first set, there are some that focus on the interpretation of expressions with free variables and on treatments and conversions between representation registers, while in those of the second set students do actions on surfaces in tridimensional space, in particular, on surfaces that are described by expressions with two variables. All these activities promote reflection on the processes needed in sketching the graph of different functions starting from the action of point to point representation. A large number of activities were found to be part of the exploratory moment. This is not surprising given the emphasis that APOS theory makes in offering the generic subject in the institution opportunities to reflect on his or her actions to interiorize them into processes.

Meanwhile, sets (c) and (d) emphasize the work on the moment of introduction of the technique through the coordination of the above mentioned construction process and the transversal section and intersection processes to construct and interpret graphs of a diversity of functions. They also include some activities that can be considered to be part of the technological-theoretical part of the praxeology. The following activity combines parts corresponding to the exploration moment and parts corresponding to the technical-technological moment. Its objective is promoting students reflection on those actions corresponding to the intersections of subspaces of R^3, and, at the same time, the introduction of a technique that includes actions to construct a surface and also opportunities to interiorize them in the corresponding process:

In this problem the graph of $x = 9 - z^2$ will be drawn in three-dimensional space. This graph consists of all points in the set $\{(x,y,z): x = 9 - z^2\}$.

1. Draw in three-dimensional space all points where the plane y = 0 intersects the graph of $x = 9 - z^2$.
2. Draw in three-dimensional space all points where the plane y =1 intersects the graph of $x = 9 - z^2$.
3. Draw in three-dimensional space all points where the plane y = −1 intersects the graph of $x = 9 - z^2$.
4. What happens as y is given more and more positive and negative values?
5. Draw the graph of $x = 9 - z^2$ in three-dimensional space.
6. Reflect on what you did in the previous three activities. How, in general, are graphs in three dimensions sketched when the function's rule includes only two variables?

All the sets include activities that promote reflection on the introduced processes to favor its encapsulation in objects, and a coherent construction of a schema for R^3. These activities are also related to the technological- theoretical moment.

Tasks related to verification of the correspondence between sketched graphs and analytic representations, by using sketches of curves belonging to the sketched surface, were found in all the activity sets. These activities were considered as part of the moment of evaluation. The activity shown in figure 2 intends to evaluate the construction of the conversion from the analytical to the graphical registers.

The activities sets do not include specifically the discussion about the theory because it plays an important role in the Class Discussion phase of ACE cycle. But opportunities to justify answers, that can be considered as part of the institutionalization moment are included in the all the sets.

This analysis suggests, in general, that the study moments are balanced and that that the activities sets can be used effectively in the classroom. However, the challenge to reduce the number of activities associated to the exploratory moment, without altering the potential of the sets to promote the constructions predicted by the GD remains.

The activities were used in the classroom but the phase of Class discussion of the ACE cycle was not used. They were evaluated through interviews to 9 students in the group where they were used, and 6 students from a control group. Students were selected by their teachers using the same performance criterion. The analysis of the interviews was done by using APOS theory. It showed a clear difference between students of both groups favoring the experimental one. This can be taken as an indication that the activities sets promote an organization of the study process that has the potential to stimulate students' learning. It can be concluded from these results that the use of a didactic praxeology where the study moments are balanced contributes to the construction of the structures described in the GD. This study enabled us to show that points of contact exist that make the approach of the two theories possible.

Write the corresponding analytic expression beside each of the following graphs. Choose between: $z = x^2 seny;$ $z = x^2 - seny;$ $z = xsen(y^2);$ $z = x + sen(y^2)$. Use sections to clearly justify your answer.
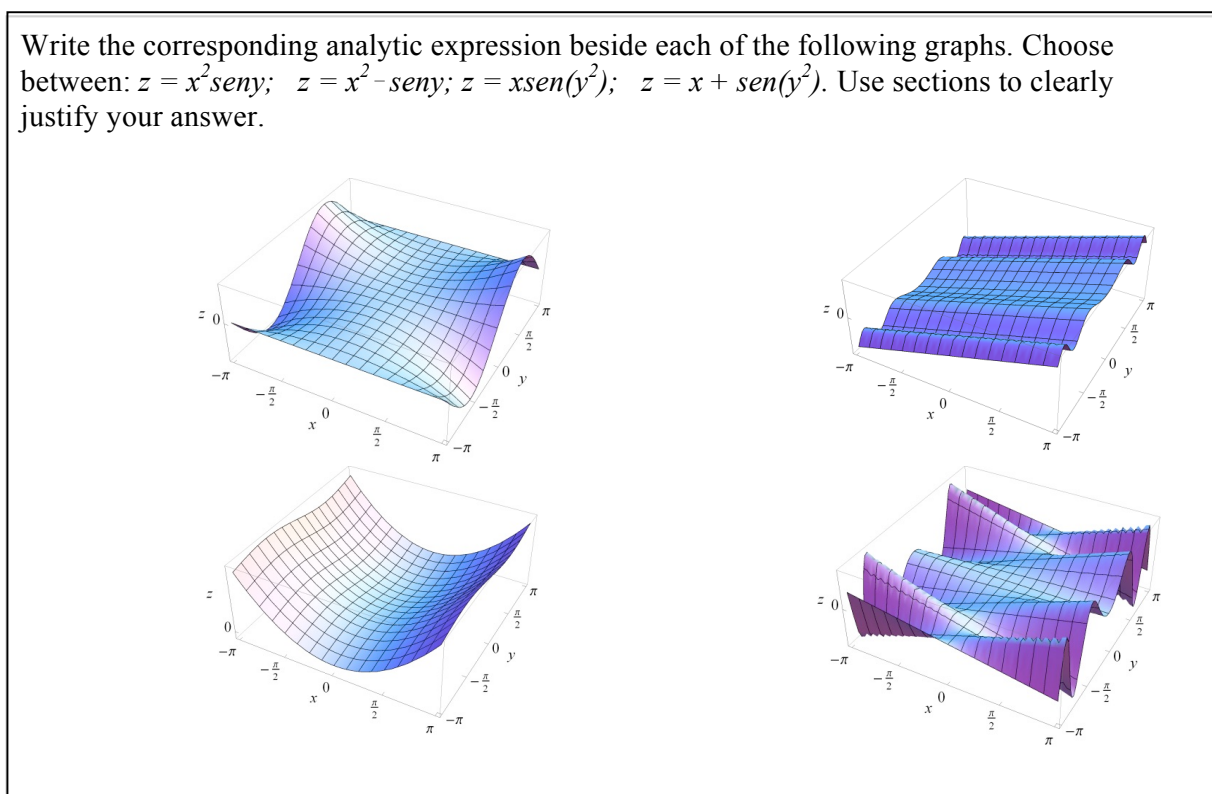
Figure 2. Conversion and justification activity.

Crossing methodological borders: Use of modeling in the teaching of Differential Equations and Linear Algebra

Linear Algebra and Differential equations are two basic courses in university mathematics that have a wide applicability both within mathematics themselves and in problems from other disciplines. Recent developments in mathematical research and in technology demand, at least in

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principle, the search of new teaching strategies. This claim has not been taken into account by the university community in general. Results from research in mathematics education, on the other hand, offer information about how students learn many of the concepts of this discipline, and teaching methods that have proven to be effective. Modeling is a prominent one (For example, Rasmussen & Blumenfeld, 2007; English,Lesh, and Fennewald, 2008; Bas, Cetinkaya, & Kursat, 2009;Zandieh, & Rasmussen, 2010; Camacho, Perdomo, and Santos, 2012;Trigueros & Possani, 2013).

In a research project that was developed during five years in the institution where I work, we tried to respond to the questions: What results about students learning can be obtained when modeling problems are used in the teaching of linear algebra and differential equations? What aspect of students' knowledge can be recovered when they work on modeling problems in the classroom?

Research studies conducted with APOS theory did not include, up to that moment, the use of modeling problems in teaching; we looked for the possibility to coordinate this theory with a theory related to modeling. Models and modeling theory (MMT) (Lesh & Doerr, 2003) was selected considering two criteria. From the theoretical point of view it proposes the construction of mathematical concepts by means of the work of students in the description of the behavior of a situation in a real or realistic context which enables them, on the one hand, to use their knowledge, and on the other hand, to construct new knowledge. Its teaching methodology coincides with that of APOS in that both of them use collaborative work of students in small group and whole group discussion with the teacher as means to foster students' construction of new knowledge. Additionally MMT postulates a set of principles that the problem selected to develop a model must satisfy so that it can be successfully applied in the classroom (Carlson, Larsen, & Lesh, 2003). They can be useful in the design of situations. It can also help in extending APOS theory boundaries so that the introduction of modeling problems can be naturally inserted into the ACE cycle, and so that the development of knowledge from previously constructed concepts used by students can be described in terms of an adequate GD.

From the theoretical point of view, both theories postulate ways of constructing knowledge. It is thus possible to propose a possible coordination between them by considering that work on the original problem makes it possible to bring into play a mathematical model in which two types of activities can be done: one that can be used to analyze, develop and validate the mathematical model, and another in which the independent or guided activity on the mathematical model can be complemented by specific activities designed in terms of a GD. These two types of activities can be interspersed in cycles so that modeling results in the need for reflection and in the construction of those structures needed to the learning of new concepts, and these result in the possibility to look at the work done on the model from a different perspective that can foster the model's development. From the methodological point of view, MMT can be used to validate the way the proposed modeling problem works in terms of promoting new knowledge, while APOS theory can be used to analyze students' constructions and how they could be related to new activities that may promote the construction of the concepts of interest, since according to this theory, the use of previous knowledge is not enough to guarantee the construction of new knowledge. Moreover, APOS' teaching and research cycles can be naturally inserted in the modeling cycles proposed by MMT, and so, both theories can expand their borders to pose new teaching and research proposals. By using the coordination of these theories, a research project was developed where modeling problems were used in the teaching of differential equations and linear algebra. Two examples illustrate results that can be obtained through this type of dialogue, one in the context of differential equations and another in the context of linear algebra.

Development of new analysis tools

In the first case (Trigueros, 2014), the problem posed to students was: *A teacher, worried by the actual tendency in teaching and by students' management of information problem, has asked us to*

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conduct a study about memorization and forgetfulness in the short run. In particular, she is interested in a report where the time that a person can keep the learnt information is made clear, as well as the time needed to forget it and what are the factors involved in those relations.

During a cycle of graphical exploration of the problem, as we called it, this exploration started from the students' own experience and the use of their schema for function. A first graphical model emerged together with the definition of variables (Figure 3a). Although at the beginning students tried to find an analytical expression for the function, in their dialogues they evoke their derivative schema in order to analyze the proposed graph:

> *Rosa:* The derivative can be used to specify the properties that the function we proposed must have, but… we don't have, nobody has the analytical expression and I don't know how to obtain it from the properties; I know for sure the other way around, to find the properties if I know the functions' rule.

The next cycle is identified by the introduction of variation into the solution of the problem. This enables students to suggest a possible model that can be illustrated by this dialogue:

> *Fer*: If we assume that each person can be assigned a memory coefficient and … what they have learnt grows from here to there but it is growing more slowly in each of these parts (he draws a polygonal curve with line segments), and then it starts decreasing, first quickly and then not so much until everything is forgotten, or almost everything …
>
> *Juan*: …It must grow more slowly here, and if the person has a certain memory capacity, a coefficient k, so that it first grows more and then less, it can be something like… (he writes $y'=k(y-T)$) because like this it cannot grow forever but will approach a point where it stops growing…

Group discussion is devoted to the advantages and disadvantages of writing an equation that includes variation. After a cycle where activities designed with the GD involving tasks for constructing differential equations and implicit functions are introduced, to respond to students' doubts when faced with an equation that does not contain the independent variable, students go back to work on the model. A cycle characterized by the analysis of the model's equation is identified. During this cycle two methods to analyze the proposed equation, that had not been previously introduced, emerge in different teams: the approximation of the solution function using the idea of derivative as a linear approximation, and the use of a graph of the derivative of the function against the function itself (phase plane) or its verbal description, as shown respectively in the next paragraphs:

> *Fer*: Here (showing the equation). If we use values to make it easier, (he writes: $y'=.5(50-y)$)… and if we use what we said before, the derivative as the slope, (he writes $y' = y-yo/t-to$), then from time zero to two, y prime is… …(he writes $y'=.5(50-y)=y/2$), then ... we have $y=25$, means that in the second time   he knows 25 data and the line would go from 0 to 2 and form 0 to 25, with slope 12.5. Then we do the same, but now we start with 2 and until, for example, 4 and y is 25, then (does calculations, not audible) we get y is 37.5 and we continue like that…
>
> *Nadia*: …I am thinking that if we graph that function y'… but as a function of $y,$ what would that mean?... Let's see … that graph (Figure 3b) is a line, its slope is positive. It says that $y'=0$ if $yo/k+1=y$. But it cannot be because where do we want a critical point? I think we want it in *1* because that means that everything has been learnt and then when she forgets it starts going down but this is up here.

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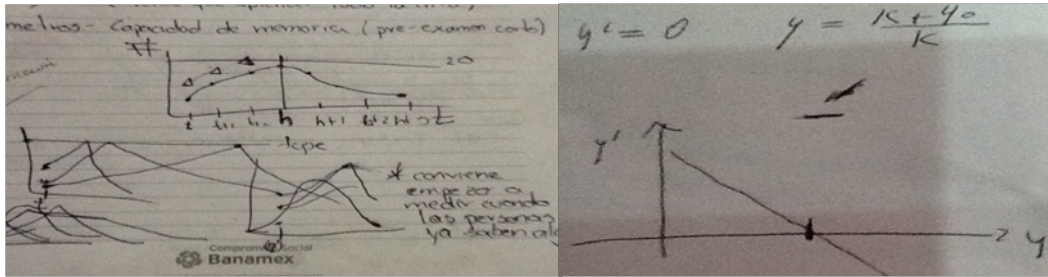
Figure 3(a). Exploration.   **Figure 3(b).** Graphical analysis of the model.

The activity continues with other cycles of activities designed with the GD and work on the model until the activity is finished with a cycle characterized by the experimental search of the model's parameters values.

In this experience it is worth highlighting the role played by the graph of the function as a trigger of a change in the way the problem is approached and in the emergence of new conceptual tools developed independently by students.

Independent constructions of concepts

In the context of linear algebra, a model for employment was presented to students (Salgado & Trigueros, 2015): *In an economy where the work force remains constant, there is a certain number of employed people in each period of time and a certain number of unemployed people in that same period. If we know the probability that an unemployed person finds a job in the next time period and the probability that an employed person looses it employment, how can we describe the dynamics of employment in time t? And, how will it behave in the long run?*

In this case some students used their previous knowledge, both from the linear algebra course, where they had previously worked on a population model to introduce difference equations in one variable, and from the economy courses to find an equation for the model:

A_2: … at each moment there is a certain number of employed and unemployed persons, but that number can change in the next time period …

A_3: It is true, we can think on the number of employed persons in the next period and that should be those who still are employed minus those that are now unemployed…

A_1: … in my opinion, the persons employed in that new period should be proportional to the number of persons that are employed and conserve it plus a proportion of those who were unemployed and get a job …

A_2: Yes, we know that p are the unemployed who find a job and q the employed that are still employed, well, I mean the probabilities (they write $x_{t+1} = qx_t + py_t \rightarrow$ employment)

A_3: Then the model is a system, there is another equal equation for unemployment (they write $y_{t+1} = (1 - q)x_t + (1 - p)y_t \rightarrow$ unemployment)

Students worked to validate a solution proposed for the model's equation, which was obtained as a generalization of the solution for the population model by introducing the system's matrix. During their work these students used their previous knowledge about matrices and exponentials to do a process on the resulting equation (Figure 4).

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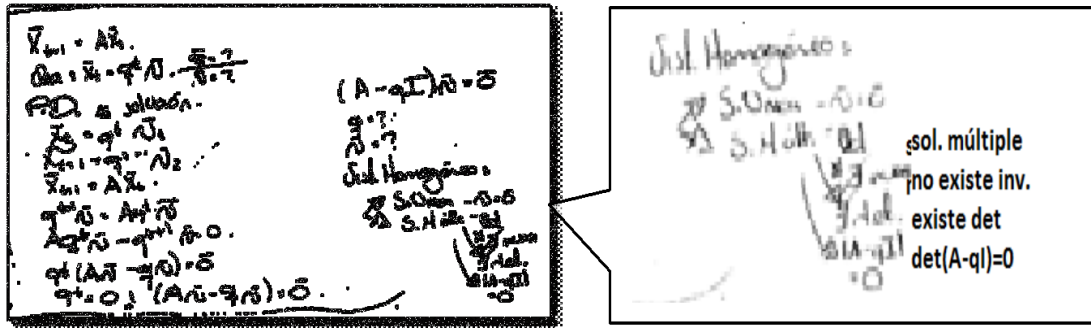
Figure 4. Using a schema, students find the definition of eigenvectors and eigenvalues.

Their work showed evidence that they had constructed relations among the concepts introduced in the course. This can be characterized as the construction of a schema for the solution set of a linear system of equations that includes the concepts of solution, matrix, linear Independence, null space and determinant.

This process led the students to the definition of the concepts of eigenvalues, eigenvectors, and eigenspaces that had not been introduced before. Students' reflection on their actions led them to construct these concepts as processes. Work with activities designed with the GD, where students work was reconsidered and institutionalized, gave them an opportunity to construct these concepts as objects.

Later, students calculated eigenvalues and eigenvectors using numbers for the parameters:

A_1: …for k_1 equal to one, the vectors have this form $v = (2x_2/3, x_2)$ and x_2 is a parameter; and for $k_2 = 1/6$, the vector is $v = (2x_2/3, x_2)$.

A_3: In both cases x_2 is arbitrary, that is, the family of solutions in the solution set. Is the parameter the same in both cases?... No, then let's call one x_1…

A_2: That's true... then…only for those values for k and those values for v, the proposed functions are solutions of the model's system.

A_3: Can we use a specific case for each family of vectors?

A_1: I don't know, but, in this case, a particular case … for $k_1 = 1$, $v_1 = (2/3,)$ and for $k_2 = 1/6$, $v_2 = (2/3,)$. Is this right?

A_3: For each k, the vectors span a line since there is only one arbitrary variable in each case.

After a cycle with activities designed with the GD intended to institutionalize and give students new reflection opportunities, during the whole-group discussion, these same students related these concepts with the modeling problem.

A_3: … We found that eigenvalues were $k_1 = 1$ y $k_2 = q - p$, that is, different, one is independent of the given probabilities and the other is related to the difference of the probability that a person continues being employed and that for an unemployed person to get a job …

A_2: We used those values and found the eigenvectors. Well, we chose one for each k. For $k_1 = 1$ we found $(p/(1-q), 1)$ so it is related to the rate of the probability that an unemployed person finds a job in the next period and that an employed person loses the job in the next period- Something like a rate of the probabilities of changing status. … for the other k, $k_2 = q - p$ it was independent of the values of the probabilities, it was $(-1, 1)$ and we were not able to explain this …also, because we drew a graph (Figure 5) and the corresponding eigenspace's vectors always have a negative component …

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A₃: Then the constants don't have a relation with initial conditions, although it was so in the population problem where those conditions appeared in the solution, and we discussed their role in terms of a family of solutions. But here, where are the initial conditions of the problem? Don't they play any role?

The teacher returned the question for students to work in teams and asked them to predict the behavior in the long run. During work on a new cycle of activities designed with the GD, students made reference to the concept of base:

A₃: We chose one particular vector for each solution set separately, but, could we take one of each family to write a linear combination and span, maybe, would be R^2…

This can be taken as evidence that these students constructed relations between their previous knowledge and the new one.
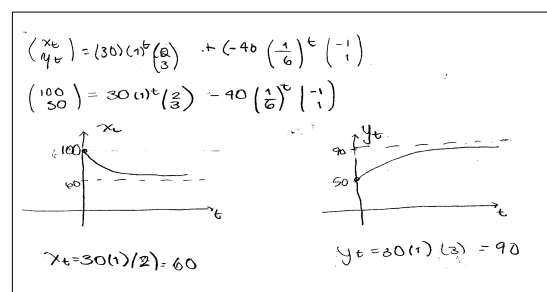
Figure 5. Long term behavior.

Students, in general showed difficulties to understand the resulting vector functions. They had not studied them yet. Some of them decided to graph each of the components to find conclusions from each of them, to find the long term behavior, and to conclude that it converges to a solution that corresponds to the eigenspace.

This experience put forward the emergence of the concepts of interest in the work of students, without any intervention from the teacher, and the construction of a schema by some students which shows that they had constructed relations among the concepts introduced before. Even though during the interviews conducted after the course it was found that some difficulties persisted in cases where dimensions were larger, at least three students showed evidence of having constructed on object conception of the concepts of interest.

Many research studies about the didactical use of modeling insist in the motivational aspect of these type of problems for students. These studies confirm this fact. However, they also demonstrate that when students are involved in interesting problems and are free to reflect, they can develop new strategies which promote the construction of concepts and of relations among them. Modeling problems can favor the learning of difficult and abstract concepts, and new knowledge can, in turn, be applied to the solution of new problems. The advantages of the coordination of the two theories, on the other hand, was made patent by the permeability of their boundaries and in the expansion of the problems that each of them can address.

Final Comments

The experiences of crossing borders, dialogue, and search for possible ways to coordinate theories by carefully analyzing their basic hypothesis and their theoretical positions showed, in each case, encouraging possibilities of approaching them without confrontation, and of finding contributions that mutually enrich them. They also underline that, a serious and open approach

enables the establishment of dialogues between theories even in those cases where the theories seem to be distant from each other.

Besides the theoretical enrichment resulting from constructing bridges between different theories, the application of the tools that emerged from each of the dialogues made clear their potential to be applied in the design of teaching strategies that can favor students' learning, and the possibility of thinking differently of our own research work.

The theoretical borders can appear in a respectful and tolerant dialogue as flexible and dynamic boundaries that make the enlargement of their application domain and the interchange of ideas possible, without losing their own identity. Borders that open themselves and let new phenomena appear. These phenomena would be impossible to perceive when only one theory is used. The challenge we face as a community is to approach the study of mathematical education phenomena by creatively building bridges so that even though borders between different theories remain, they can be made porous and let the discipline be enriched through collaborative relations.

Acknowledgments

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Chapter 2

Curriculum and Related Factors

IMPLEMENTATIONS OF CCSSM-ALIGNED LESSONS

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We analyzed 52 middle school mathematics lessons from multiple states and curriculum contexts to understand how teachers were enacting the CCSSM. The teachers stated that all of the lessons were CCSSM-aligned. We categorized curriculum materials according to two approaches, with one approach associated with curriculum programs funded by NSF and the other representing curriculum programs commercially produced, typically from a large publisher. We analyzed the nature of mathematical activity and level of interactions in the lessons. We found significant differences across curriculum approaches in the mathematical activity categories related to cognitive demand and in the level of interaction. The implications are that curriculum programs strongly mediated the enactment of the CCSSM.

Keywords: Curriculum Analysis, Instructional Activities and Practices, Standards, Middle School Education

The *Common Core State Standards for Mathematics* (CCSSM) (National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010) were initially adopted by 45 states plus the District of Columbia, and, despite the rollback in some states, the CCSSM or CCSSM-based standards are still in place in most states. Thus, there is a relatively common articulation of content and the progression of content across the grades in the CCSSM-adopting states. This provides researchers an opportunity to consider the impact of curriculum programs on how the CCSSM get taken up by schools and teachers. Furthermore, the CCSSM framers were agnostic with respect to curriculum and instructional approaches (McCallum, 2012), which provides potentially enhanced roles to curriculum materials as teachers work to interpret and implement the standards. Consequently, the widespread adoption of the CCSSM presents an opportunity and need for researchers to understand how districts and teachers interpret standards and to understand the role of curriculum materials in the process of interpreting and enacting those standards.

Teachers' initial interpretations of the CCSSM, when compared to prior state standards, were that the CCSSM required a greater emphasis on problem-solving, discovery, communication, and conceptually-driven instruction (Davis, Choppin, Roth McDuffie, & Drake, 2013; Choppin, Davis, Drake, & Roth McDuffie, 2013). Although teachers expressed a relatively strong view of the student-centered nature of the CCSSM, prior research based on teachers' enactments of similar recommendations in the National Council of Teachers of Mathematics (NCTM) Standards documents (NCTM 1989, 1991) shows that even the most reform-minded teachers did not implement the recommendations in ways beyond superficial features (Spillane & Zeuli, 1999). However, much of the research on the implementation of the recommendations in the NCTM Standards primarily occurred before curriculum programs based on those documents were developed and widely disseminated. In this study, we explore the association between district-adopted curriculum programs (e.g., *the designated curriculum* [Remillard & Heck, 2014]) and instruction that resulted when teachers used those programs to plan and enact lessons (e.g., *the enacted curriculum*). These lessons

– and the designated curriculum – were ostensibly aligned with the CCSSM (e.g., *the official curriculum*).

Framework

We frame our analysis by distinguishing between two broad approaches to curriculum and instruction. In describing each approach, we connect conceptions of curriculum design to conceptions of classroom instruction. In the first approach, we connect the notion of *curriculum as delivery mechanism* to that of *direct instruction*. In the *curriculum as delivery mechanism* approach, knowledge is detached from an authority or expert (i.e., textbook, teacher) and transmitted to novice learners (students), allowing those far from classrooms to exert control over content (Datnow & Park, 2009). Curriculum design is based on explaining and modeling concepts and procedures and presumes that learners have minimal understanding of the subject matter or intuitive understandings. Mastery focuses at the level of lesson or topic, with fluency expected on one topic before proceeding to the next. The treatment of language in curriculum materials from this approach mirrors the treatment of mathematics content. There is typically an emphasis on early formalization and precision, with little validation of less formal or everyday terminology, and terms are defined and explained before students have opportunities to explore the content. These conceptions of curriculum align with models of direct instruction, as defined by Munter, Stein, and Smith (in press). Munter and colleagues explain that direct instruction is dominated by teacher explanation and demonstration of procedures or definitions, which students then practice to develop accurate and fluent reproduction of those procedures or definitions.

In the second approach, we connect *curriculum as epistemic device* to *dialogic instruction*. In considering curriculum as epistemic device, the primary goal of curriculum is to provoke interactions that generate understanding. The role of tasks in curriculum materials is to elicit and progressively refine student thinking, individually *and* collectively, as contrasted with serving as a delivery mechanism for content. This conceptualization of curriculum design builds from a notion of text as *thinking device* that promotes dialogic interaction (Wertsch & Toma, 1995). A primary characteristic that shapes task affordances is the potential for heterogeneous approaches that vary in terms of their entry points and sophistication, or what has been called *low-threshold, high ceiling* tasks. This metaphor aligns with *dialogic instruction*, as described by Munter and colleagues (in press), which emphasizes students' collaborative work on challenging tasks and the positioning of students as co-participants in classroom discourse and as emerging mathematical authorities. These distinctions allow us to parsimoniously characterize distinct approaches evident in teachers' interpretations and enactments of the CCSSM, and to tie characteristics of curriculum to characteristics of instruction.

Curriculum Types

Building from the two approaches outlined above, we characterized curriculum programs into two types. Programs developed in ways aligned with the epistemic device approach, comprised exclusively of National Science Foundation (NSF)-funded curriculum programs, we labeled as ED Programs. These programs have some or all of the following characteristics:

- Problem contexts serve as the basis of exploration for multiple lessons.
- Students explore a problem and/or mathematical concept *before* concepts, procedures, and/or mathematical terms are formalized.
- Mathematical practices, particularly problem solving, reasoning, and argumentation, are considered essential in teaching and learning.
- Representational fluency is promoted through work with individual representations and the connections among representations.

- Grouped work is collaborative, used for high level tasks, and often involves a group product and/or presentation (Lappan & Phillips, 2009).

Programs aligned with the delivery mechanism approach, we labeled as DM Programs. These programs, comprised almost exclusively of publisher-produced programs, are geared primarily towards procedural fluency, with characteristics of direct instruction, including:

- Problems may be set in context, but the contexts vary with each problem are not focused on students reasoning analogically about the mathematics.
- Problem solving steps and procedure are described or provided through examples.
- Formal definitions are presented before students use terms or constructs associated with specific terms, and precise use of language and efficient procedures are considered essential to developing conceptual understanding.
- Group work and/or seat work are used primarily to practice problems demonstrated by the teacher (*cf*., Battista, 1999).

Methods

We analyzed 52 lessons from the video recordings of the lessons. The teachers stated that the materials and lessons were aligned with the CCSSM content and practice standards. We developed an observation tool designed to distinguish between direct and dialogic forms of classroom instruction. The tool was originally adapted from the instrument used in a large scale study (Tarr et al., 2008) to characterize the extent to which lessons aligned with what they termed *standards-based instruction*. To that end, the instrument emphasized conceptual understanding, multiple solutions and representations, and recognizing and building from student thinking. We ultimately transformed the instrument and associated analytic techniques by utilizing a modified time-sampling approach. We transcribed most of the whole class portions of the lessons and group work as the audio quality permitted. We parsed the transcripts into roughly two- to four-minute chunks delineated by participation structures and topical foci. We bounded the analytic segments first by participation structures (e.g., seat work, whole class discussion, group work), then by a combination of duration and topical focus, similar to what Mehan (1979) termed a *topically related set*. Given that lesson ratings were derived as a ratio of the number of lesson segments in which a code occurred divided by the total number of lesson segments, we wanted to maintain roughly similar time intervals for each lesson segment. Consequently, if a discussion around one problem extended beyond three or more minutes, we divided that discussion into multiple segments of roughly two minutes each. Similarly, if there were a series of rapid resolutions to a set of problems that were based on the same kind of mathematical activity, we combined these sets into one segment for analytic purposes. Further description of how we coded each segment is described below.

Data Sources

The data came from a larger NSF-funded study that explored teachers' perceptions of the CCSSM, the ways teachers were prepared to teach the CCSSM, how teachers drew upon curriculum materials to plan lessons they viewed as CCSSM-aligned, and how they enacted those lesson plans. We had comprehensive data sets for 52 teachers, including lesson observations from the 2013-2014 School Year. Of the 52, 21 of the lessons came from teachers using ED materials, and 31 came from teachers using DM materials. Sixteen of the ED lessons involved the second (CMP2) or third (CMP3) edition of Connected Mathematics Program, four of the lessons involved College Preparatory Mathematics (CPM), and one involved Core-Plus Mathematics. For the DM materials, 13 used Glencoe, five used *digits*, four used Prentice Hall, three used Math in Focus, and others used similarly organized curriculum programs. For 13 of the teachers using ED programs, their districts

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had recently adopted the programs specifically to address the CCSSM. Districts for seven of the other teachers had adopted programs before the CCSSM were adopted. Twenty-five of the 31 teachers using DM programs worked in districts that had recently adopted the programs specifically to address the CCSSM. Of those 25, eight used other materials regularly, including in the lessons we observed. Six of the DM teachers' districts had not adopted new materials; these teachers used a range of materials aligned with the DM approach.

Data Analysis

We coded lessons using three broad sets of analytic categories. We coded lessons using three broad sets of analytic categories. The categories were: *Nature of Mathematical Activity*, *Lesson Mode*, and *Elicitation and Presentation of Student Explanations*. The first set, *Nature of Mathematical Activity*, included five sub-categories, three based on levels of cognitive demand from the work of Stein and colleagues (Stein, Grover, & Henningsen, 1996), and two that were inductively developed as a result of the data analysis, as described below. The categories derived from the cognitive demand literature included: *recall, memorization, or basic application of definition or rule*; *procedural or computational routine*; and *procedure plus*. These three categories correspond to the first three cognitive demand categories of *memorization*, *procedures without connections*, and *procedures with connections*, respectively, but were revised to provide additional descriptive detail to facilitate coding and to reflect trends that emerged in our data. We developed two new categories to represent patterns identified in the mathematical activities observed in the lessons. The first, *interpreting or generating representations*, refers to tasks that involved creating or interpreting information a table, graph, equation, or other representation. This category involved mathematical activity that required students to translate information across types of representations or to extract and describe a pattern evident in a representation. As such, the activity extended beyond simple recall or application but did not necessarily involve a procedure. Consequently, we deemed it as higher cognitive demand than the *recall* category and aligned more with *procedure plus*, though it didn't involve a procedure. The second new category, *developing definitions or formulas*, refers to when tasks involved creating a definition or formula. This is different from when the teacher simply provided a definition, which was categorized as recall or memorization. We found this category most often occurred in geometry lessons, though Munter et al. (in press) emphasize the need for teachers and students to co-construct definitions across all strands when appropriate.

The Lessons Mode categories were used to make two distinctions. The first distinction was whether a segment predominantly occurred before students had an opportunity to work (alone or in groups) on a problem or occurred after students had an opportunity to work on a task. We considered cases when the teacher presented and explained examples in whole class discussions as having occurred before students had an opportunity to work on a problem, even if the teacher engaged in recitation-style interactions with students. The second distinction characterized the interactions within the segment. If the teacher strictly or primarily engaged in Initiate-Respond-Evaluate (IRE) interactions or spent the entire segment explaining the mathematics or the solutions to problems, then we characterized that as low-interactive. Segments were characterized as low-interactive if they were primarily comprised of a series of rapid evaluations of the accuracy of students' responses. A segment was considered high-interactive if the teacher elicited responses without engaging in immediate evaluation and if the teacher pressed students to explain their answers and to respond to the answers of other students. These two distinctions produced four modes, a pre- and post-work low interactive, and a pre- and post-work high interactive. We also included a fifth lesson mode, termed *directions or administrivia*, to code segments when the teacher was providing directions, taking attendance, attending to classroom rules, or the like. These lesson modes provided us a way to characterize the lessons as involving direct or dialogic instruction.

The third set of codes, Elicitation and Presentation of Student Explanations, included three categories that are relatively self-explanatory: teacher elicited student strategies or interpretations; teacher pressed students for steps and justification for steps; and students explained solution strategies.

We coded each segment with at most one code from the Mathematical Activity and Mode categories, and as many as appropriate from the Student Explanations categories. There were segments for which the nature of mathematical activity was not clear, so no code was applied. For each lesson, we divided the number of segments in which a code occurred by the total number of segments and then multiplied by eight in order to avoid having the ratings clump around one rating number, usually the lowest numbers in the scale given the relatively low occurrence of some codes. We added a one to each rating to avoid having values of zero, and rounded each rating to get an integer scale from one to nine. We then averaged the codes for each category across each curriculum type and used Excel to apply a two-tailed t-test of different samples of equal variance, with a significance level of 0.05.

Results

We found significant differences in the lesson segment codes across the ED and DM materials. In terms of the Nature of Mathematical Activity, the ED lessons had significantly fewer segments coded as *recall, etc.* ($p < 0.01$) or *procedural or computational routine* ($p < 0.01$) and significantly more segments coded as *interpreting or generating representations* ($p < 0.05$) and *procedure plus* ($p < 0.05$). Although there were relatively more ED than DM segments coded as *developing definitions or formulas*, the **difference was not significant** ($p < .225$). See Table 1 for a summary of the Mathematical Activity categories.

Table 1: Frequency and p-value for Mathematical Activity Categories

	Recall, memorization, or basic application of definition or rule	Interpreting or generating representations	Developing definition or formula	Procedural or computational routine	Procedure Plus
Frequency - ED	2.0	2.7	1.7	1.8	2.2
Frequency - DM	3.4	1.6	1.3	3.1	1.4
p-value	0.002	0.015	0.225	0.006	0.04

Similarly, in the Lesson Mode categories, there were significantly fewer ED segments coded as low-interactive / pre-work ($p < 0.01$) and significantly more ED segments coded as interactive (both pre- and post-work) ($p < 0.01$ in both cases). The only codes not significant were low-interactive post-work, though there were relatively fewer ED segments with this code ($p = 0.17$), and *directions or administrivia*, which occurred relatively equally across types $p = .33$). See Table 2 for a summary of the Lesson Mode results.

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Table 2: Frequency and p-value for Lesson Modes

	Presentation and explanation of task and mathematics in the task, primarily characterized by teacher explanation and IRE exchanges that evaluate students' understanding of facts and procedural knowledge (pre-work, non-interactive)	Teacher engages students in non-evaluative exchanges to establish the meaning of the problem context or to establish the mathematical focus of the task (pre-work, interactive)	Explanation of task and mathematics, typically after students had an opportunity to do work, primarily characterized by teacher explanation and IRE exchanges that evaluate students' accurate completion of the task (post-work, non-interactive)	Probing for, sharing of, or discussion of student strategies, characterized by non-evaluative exchanges in which the teacher elicits and probes student understanding of the task or strategies (post-work, interactive)
Frequency – ED	2.9	1.9	2.6	2.9
Frequency – DM	4.9	1.1	3.3	1.16
p-value	0.0004	0.002	0.17	0.0001

In the Elicitation and Presentation of Student Explanations categories, there were significantly more ED lesson segments coded as *teacher elicited student strategies or interpretations* ($p < 0.01$) and *teacher pressed students for steps and justification for steps* ($p < 0.01$). Though there were more ED segments coded as *students explained solution strategies*, the difference was not significant ($p = 0.15$).

Discussion

There was a strong association between curriculum types and the type of instruction, as characterized by the nature of mathematical activity, the level of interactivity, and emphasis on student explanations. That is, curricula deemed as aligning with curriculum as delivery mechanism were strongly associated with instructional forms that were predominantly encapsulated by Munter and colleagues' (in press) description of direct instruction, and curricula deemed as aligning with curriculum as epistemic device were strongly associated with instructional forms that were predominantly encapsulated by Munter and colleagues' description of dialogic instruction This finding is consistent with earlier case studies of eight teachers using two different curriculum programs, as part of the larger project. This occurred even though multiple curriculum programs, schools, and states were present in each of the Type 1 and Type 2 samples. Even though some of the differences may be due to underlying perceptions of the CCSSM (which we are exploring), the results lend credence to the notion that curriculum materials are strong mediating factors in enactments of the CCSSM, and more generally, that the designated curriculum is a strong mediating factor between the official curriculum and the enacted curriculum.

Although strong majorities of participants in the survey samples and interview samples from the larger study reported that the CCSSM required more communication, problem solving, exploration, and overall rigor than past standards (Davis, Choppin, Roth McDuffie, & Drake, 2013; Choppin, Davis, Drake, & Roth McDuffie, 2013), the lessons involving Type 2 materials typically lacked these features. Few teachers using the Type 2 programs expressed dissatisfaction with the fit between their materials and the CCSSM, suggesting that the designated curriculum served as the de facto representation of the official curriculum, and the teachers felt as long as they were using the materials, they were addressing the CCSSM. The results from the Type 2 teachers are consistent with results from lessons observed soon after the release of the NCTM Standards documents in which most teachers struggled to incorporate the recommendations in their lessons beyond surface features (Spillane & Zeuli, 1999), lessons conducted without the benefit of curriculum programs designed to comprehensively integrate the recommendations.

A plausible explanation for the differences in the observed instruction is that there are characteristics of the Type 1 materials that contribute to instruction rated as having higher cognitive demand mathematical activity, greater emphasis on interactivity, and a greater focus on student explanations. One possible explanation is that the materials convey underlying pedagogical messages, and the teachers create more opportunities for exploration and communication to follow what they perceive as the wishes of the curriculum designers. A second possible characteristic is the inclusion of task sequences in which students are first presented tasks "to which students do not have an immediate solution, but must wrestle with for a while without the teacher's interference" and then presented tasks that "help them become more competent with what they already know" (Munter et al., in press, p. xx). These design features are built into the Type 1 programs (Lappan & Phillips, 2009), and, even though there is typically wide variation in which these features are taken up by teachers (Tarr et al., 2008), there was nevertheless a stark overall contrast in our data between the Type 1 and Type 2 lessons, suggesting the presence of such features contributed to teachers' instructional decisions.

There were four categories that did not result in significant differences. The first, *developing definitions or formulas*, was most strongly associated with geometry lessons, of which there were few, and the lack of presence of this activity in other strands made the overall mean low across both types. The lack of significant difference in the post-work non-interactive Lesson Mode reflects the relatively even distribution in the Type 1 post-work lesson segments between interactive and non-interactive segments. That is, teachers using Type 1 materials engaged in IRE-style interactions and teacher-led explanations almost as much as they facilitated more interactions or emphasized student explanations. This compared to the Type 2 lessons, in which post-work lesson segments were rated as non-interactive fourteen times as much as they were rated interactive. Nevertheless, the Type 1 lessons included a similar quantity of non-interactive segments, so the difference between types was not significant. The third category whose difference was not significant was *directions or administrivia*, reflecting the relatively equal occurrence of this code across both types, suggesting that providing directions, taking attendance, managing behavior, and so forth is a staple of all lessons. The fourth category, *students explained solution strategies*, like the *developing definitions or formulas* code, did not occur frequently, with low means across both types. So, while teachers often elicited and probed students' explanations, there was either inadequate follow-up so that the student actually provided a full explanation, or the students were not able to provide a full explanation. Nevertheless, the low means (1.62 for ED, and 1.19 for DM, with 1 indicating no occurrence) across both types suggests that students still rarely have opportunities to provide comprehensive explanations for their solutions, approaches, or strategies.

The consistent presence of interactive forms of instruction in the ED lessons deviates in substantive ways from prior results of large-scale observations of middle school classrooms (Jacobs et al., 2006; Stigler & Hiebert, 1999), studies that largely took place before the widespread dissemination of NSF-funded materials. The results, however, are more in line with the findings of Tarr et al. (2008), whose study included NSF-funded programs. Our results, along with those of Tarr et al. (2008), suggest that NSF-funded programs can mediate longstanding lesson structures to make instruction more focused on student thinking and students' explanations.

Implications

An implication from our findings is that the choice of curriculum programs – whether by district, school, or teacher – is associated with instructional approach. An underlying question, then, is what these entities were responding to in their choice of curriculum programs, especially if the ostensible goal – to align instruction with the CCSSM – was the same across the contexts we studied. What messages or information were the schools and districts responding to, and what messages did they

want to send with their choice of programs? What does their choice of curriculum program say about the depth with which or the evidence with which decisions about curriculum programs were based?

A second implication is that interpretations of the official curriculum – the CCSSM – are heavily mediated by decisions and curricular choices at the local level. That is, the designated curriculum is potentially the strongest mediating factor in the ways that the CCSSM are being enacted. This suggests challenges for policy makers who hope to change classroom instruction without providing a stronger articulation of what classroom practices should look like or providing materials that have been developed with these practices in mind.

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WOO! AESTHETIC VARIATIONS OF THE "SAME" LESSON

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Efforts to enhance the aesthetic impact of mathematics lessons must account for the role of teachers in shaping the unfolding mathematical content of their enacted lessons. In this paper, we draw from Dietiker (2015) to describe differences in the mathematical stories of the enacted lessons of two veteran teachers teaching the same lesson. We identify connections between these differences and the variations in student experiences as illustrated by visible student reactions.

Keywords: Curriculum, Curriculum Analysis, Affect, Emotion, Beliefs, and Attitudes

How a story is told matters. For example, learning at the beginning of the play that Romeo and Juliet will die (e.g., in Shakespeare's play) offers an audience a very different aesthetic experience than when the storyteller chooses to withhold this information (e.g., in the 1968 movie). When mathematics lessons are interpreted as stories (Dietiker, 2013, 2015), the same dynamic can be recognized. The way in which information in a mathematical story is withheld and revealed can significantly impact the aesthetic experience of the students.

The aesthetic nature of mathematics teaching and learning has been largely ignored and needs more study. Although research has begun to understand the aesthetic dimensions of mathematical learning (e.g., Sinclair, 2004), little is known about how mathematics lessons can be crafted to take advantage of aesthetic opportunities. We use the term "craft" to describe the role of both designing curriculum and of teaching. Interpreting teaching as creative work is consistent with Brown (2009), who suggests that teaching with written curricula is similar to how musicians perform with sheet music. He argues that, "practitioners bring to life the composer's initial concept through a process of interpretation and adaptation… In both cases, no two renditions of practice are exactly alike" (Brown, 2009, p. 17).

In this paper, we demonstrate that in a mathematics classroom, how a story is told is just as important to its aesthetic impact as the story itself. That is, if we want to create opportunities for surprise or anticipation, we need to do more than design new mathematics curriculum and instead must consider the ways teachers enable and constrain how the content unfolds throughout the lesson. We present a case study of two veteran teachers teaching the same lesson from the same textbook with notably different student reactions to demonstrate the aesthetic variations of telling the "same" mathematical story. We also explain how these aesthetic differences resulted from the way in which the mathematical content unfolded throughout the lesson enactments.

Theoretical Framework

Mathematical lessons can be interpreted as mathematical stories (Dietiker, 2015). This interpretation specifically offers a way to recognize the logical and aesthetic dimensions of the unfolding mathematical content of a lesson. Not limited to contextual story problems, this framing foregrounds sequential changes that occur over time ("acts"), defined as the distinct portions of the mathematical lesson which can be identified by changes in the mathematical characters (the objects of the lesson, such as a quadratic function), actions (the acts by a student or teacher to manipulate an object), and/or settings (the representational "space" in which the mathematical characters and

actions are found). Just as literary stories are "told" by a narrator, mathematical stories are narrated by the utterances of the teachers and students.

In order to recognize the aesthetic dimensions of a mathematical story, we focus on the mathematical plot, which in literature is the tension felt by students between what is known and what is desired to be known in the story (Nodelman & Reimer, 2003). Analyzing the mathematical plot enables the investigation of how a mathematical sequence can generate suspense and surprise. Barthes' (1974) describes the transition from question to answer with codes: *question formulation*, *promise of an answer*, *snare* (misleading direction), *equivocation* (misleading ambiguity), *jamming* (the question is unanswerable), *suspended answer* (the delay of the answer), *partial answer* (progress), and *disclosure of the answer* (endorsing the answer). The transition from asking to answering a question forms a story arc. In mathematics classrooms, story arcs include questions asked by teachers, students, and curriculum. In addition, some story arcs may be based on implicit questions, raised by the goals of an activity but never stated.

Just as is the case with literary stories, the aesthetic value of a mathematical story varies by individual. That is, there is no "best" way a story can be told for everyone. However, literary theory suggests that there is a relationship between the form and function of narrative. It is thus an assumption of the mathematical story framework that distinguishing the different forms of the unfolding mathematical content throughout a lesson can identify how observed aesthetic reactions of students can be understood. For example, long story arcs containing nested shorter story arcs may give a larger purpose to the shorter questions and, thus, coherence for the lesson. In fact, without long story arcs, a long sequence of short story arcs may prevent a student/reader from recognizing a purpose for each question. Longer story arcs may also create opportunities for twists in the plot that lead to surprise and anticipation. In this study, we examine the relationship between the form of the mathematical plot and the evident aesthetic of an enacted lesson by asking the question, "How can enactments of the same written curriculum differ and what role do these differences play in the aesthetic reactions of the students?"

Methods

The current study compares two enactments by different algebra teachers based on the same textbook lesson. The goal of this research project is to learn more about the different ways that expert teachers, from diverse settings and communities, enact the "same" curriculum lessons in their respective classrooms. The lesson selected for this analysis focused on a method of solving quadratic equations by factoring using the zero product property.

In order to minimize the interference of classroom management issues on curriculum enactment, or unfamiliarity with the same written curriculum (Kysh, Dietiker, Sallee, Hamada, & Hoey, 2012), the teacher participants were required to have at least five years experience teaching mathematics, at least three years of which must have involved the selected written curriculum. Additionally, participants were selected only if they had a strong record of excellence in teaching (e.g. National Board certification, receiving a teaching award, a regular leader of professional development, etc.). The selected enactments were taught by two teachers in two different geographical regions (Mr. J and Ms. W) with 8 and 20 years of teaching, respectively, and 4 and 10 years experience, respectively, using the textbook.

The enactments were observed and videotaped in Spring, 2015, which included the video recording of the whole class as well as one focal student group within each class. Since a portion of each enacted lesson contained group work, the mathematical plots described in this paper represent the mathematical stories from the perspective of this selected group of students, including both whole class discussion and small group discourse.

To interpret the mathematical plots, the videos were transcribed and then analyzed to identify where acts start and stop by identifying when the mathematical characters, actions, and/or settings

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changed and the mathematical story advanced. Next, the researchers identified all of the mathematical questions formulated throughout the lesson, identifying questions that were asked by the teacher, student, or curriculum. Finally, each of the acts were coded using Barthes' codes (described earlier) by researchers in pairs who then came together as a whole group to resolve differences. The resulting mathematical plot diagrams for the two enactments are displayed in Figure 1 (for Ms. W's lesson) and Figure 2 (for Mr. J's lesson). In these diagrams, a shaded cell without a code means the question is still open. Note that the colors were used to highlight formulated questions that were common to both lessons. In addition, the width of the columns containing the acts do not signify increments of time. Rather, the columns represent the acts, which are portions of the lesson for which the story changes. These acts vary in elapsed time, which can be seen in the elapsed times provided in these diagrams.

Ms. W 8.2.2 S15

ACT	1	2	3	4	5	6	7	8	9	10	11	12	13
Beginning Time	01:36	05:37	08:26	12:09	13:48	15:36	21:05	23:39	25:33	27:55	28:41	35:22	38:05
	04:01	02:49	03:43	01:39	01:48	05:29	02:34	01:54	02:22	00:46	06:41	02:43	00:28

#	Question\Elapsed Time	
1	What is the zero product property?	15C
2	What characteristics does my sketch need to count as a correct match?	143 3 34
3	What information is sufficient to sketch a parabola?	163 63 3 3A
4	Does a sketch need to have axes?	2A
5	What should my coordinate axes look like?	24
6	How can a student sketch a specific parabola when only given the y-intercept?	16374 3A
7	What parabola is the teacher thinking of (with y-int at 2)?	1374 2A
8	Is the y-intercept enough information to sketch the teacher's parabola?	263 3A
9	Was the teacher's parabola sideways?	2A
10	How can a sketch a specific parabola when only given the two x-intercepts?	13 34 3 3A
11	What parabola is the teacher thinking of if the x-ints are (-3,0) and (-1,0)?	13A
12	Are the x-intercepts enough information to sketch the teacher's parabola?	163 3A
13	What does a parabola need to have at its ends?	13A
14	How can a student sketch a specific parabola when only given the two x-ints and the y-int?	134 3A
15	What parabola is the teacher thinking of if the x-ints are (-3,0) and (5,0) and y-int is (0,-15)?	134 A
16	Are the x-intercepts and y-intercept enough information to sketch the teacher's parabola?	134 3 A
17	What's wrong with my graph and how can I make it better?	234 3A
18	Why is the y-intercept not the vertex of this parabola?	13A
19	What is true about the value of y for all x-intercepts?	13A
20	What is true about the value of x for all y-intercepts?	13A
21	If x=0 at the y=intercept, what is the y-intercept of the graph at y=2x^2+5x-12?	13
22	Since the x-int occurs when y=0, what equation do you need to find the x-ints for y=2x^2+5x-12?	13463 A
23	Can you solve 2x^2+5x-12=0 for x? Why or why not?	13 3A
24	How can we solve the equation 2x^2+5x-12=0?	13 43C 5393 43
25	What is the solution to 2x^2+5x-12=0?	13 3A
26	Could there be more than one x-value that makes the equation true? Why or why not?	13A
27	What are the two numbers the students were given that have a product of 36?	13A
28	How can I determine the two numbers that have a product of 36?	163
29	What are the two numbers the students were given that have a product of 0?	13
30	How can I determine the two numbers that have a product of 0?	13 4
31	Can one of the factors be a set of numbers?	13A
32	What are the three numbers the students were given that have a product of 0?	13
33	How can I determine the three numbers that have a product of 0?	13A
34	What reasoning are you using as you consider this problem?	13A
35	What is a method, besides guess and check, to find the x-intercepts of a quadratic?	2E+05 34 5
36	How does the zero product property help us solve 0=2x^2+5x-12?	1493 4
37	Is there a product that equals zero?	13 4
38	How can I rewrite 2x^2+5x-12 as a product and what are the factors?	1B
39	What are two simpler equations that can help solve 2x^2+5x-12 = 0 for x?	1 34
40	What must be zero for the zero product property?	293 34 5
41	Why did I get the y-intercept when trying to solve this factored quadratic by plugging in zero for x?	823
42	What is the other answer, and how do you know?	13

Figure 1. Ms. W's mathematical plot, where 1: formulated question by teacher, researcher, or textbook, 2: formulated question by student, 3: progress by student, 4: progress by teacher or environment, 5: promise, 6: equivocation, 7: jamming, 8: proposal, 9: snare by student, A: disclosure by teacher, B: disclosure by student, C: suspended answer. Colors other than light grey indicate a question that was identified in both enactments.

Mr. J 8.2.2 S15	ACT	1	2	3	4	5	6	7	8	9	10	11	12	13
	Beginning Time	01:20	04:56	07:00	11:07	15:10	16:22	17:20	20:20	23:05	25:22	27:08	33:00	34:47
Question\Elapsed Time		03:36	02:04	04:07	04:03	01:12	00:58	03:00	02:45	02:17	01:46	05:52	01:47	05:51
1 What is the zero product property?		1C									3			A
2 What are roots?		1												A
3 How can I find quadratic roots?		1						43		34	3	34		A
4 How do you sketch a parabola with the y-intercept (0,-7) and a vertex elsewhere?		1B												
5 What does a parabola look like?		1	A											
6 How exact does a sketch need to be?		1	34	4										
7 Do all parabolas drawn with the same y-intercept look the same?		13												
8 How do I sketch a parabola with x-intercepts at (-1,0) and (5,0)?			1B											
9 How could we make the parabola look better?			13A											
10 How does finding the "middle" help sketch a parabola?			13A											
11 Does a parabola stop at the x-intercepts?			13											
12 Do all parabolas drawn with the same x-intercepts look the same?			143A											
13 How do I sketch a parabola when given 2 x-intercepts and the y-intercept?				13										3A
14 Do all parabolas drawn with the same x-intercepts and y-intercept look the same?				13A										
15 How many points are needed to draw a parabola?				13A										
16 Where is the line of symmetry on the graph?				13A										
17 What is the equation of the line of symmetry?				13A										
18 How could we get the y-value of the vertex of this parabola without an equation?				15										
19 What is true about the value of y for all x-intercepts?						193		4	A					
20 What is true about the value of x for all y-intercepts?						193	4		A					
21 Can y be equal to zero at the x-intercepts? Why or why not?						293		4	A					
22 What is the y-intercept of the graph y=2x^2+5x-12?						13			A					
23 How do you find the y-intercept of an equation and what is it?						13	643		A					
24 What is the y-intercept of y=3x+4?							13							
25 What is the y-intercept of y=4x^2-3x^3+12?							13							
26 What is the y-intercept of y=sqrt(x+4)-5 ?							1743							
27 Why is the y-intercept not -5?							134		A					
28 What did people falsely assume was the y-intercept of the equation y=(x+3)^2-1?							13A							
29 What equation would you solve to find the x-intercept of y=2x^2+5x-12?							13		A					
30 How do you solve 0=2x^2+5x -12?							23	7	34	3	3A			
31 Can we add x^2 and x terms together?							13A	23						
32 Can we divide the equation by something to help us solve it?								23	3A					
33 What is the definition of an x-intercept?								13A						
34 How do we solve a quadratic equation?									134	3	3A			
35 What is special about zero?										14A				
36 What is 5+0?										13A				
37 What is a number when its product with 5 is 0?										13A				
38 What is a number when its product with -7 is 0?										13A				
39 What is a number when its product with 1/4 is 0?										13A				
40 What is a number when its product with (5*6-4*3/4+6) is 0?										13A				
41 What is a number when its product with pi is 0?										13A				
42 What is a number when its product with epsilon is 0?										136				
43 If ab=0, what do you know about a and b?										13A				
44 How do we take 2x^2+5x-12 and rewrite it as the product of two expressions?										1343A				
45 What are the factors of 2x^2+5x-12?										143A				
46 What is the purpose of factoring?										1A				
47 What do we know about something when its product with (2x-3) is zero?										1364				
48 What do we know about something when its product with (x+4) is zero?										1364				
49 What is the solution to x+4=0 and 2x-3=0?										13A				
50 What other things might we find when we solve quadratics?										1				
51 What is the definition of a zero?											1A			
52 How do we use the intercepts to sketch the graph of y=2x^2+5x-12?														1434
53 Where is the vertex of y=2x^2+5x-12?														134
54 How do you find the mean of two numbers?														13

Figure 2. Mr. J's mathematical plot. See the caption in Figure 1 for the coding reference. Colors other than light grey indicate a question that was identified in both enactments.

Findings

On the surface, these lesson enactments appear very similar. Both enactments contained 13 acts and were approximately the same length (Mr. J: 39 minutes, Ms. W: 37 minutes). Both teachers focused their lessons on the same set of tasks in the algebra textbook, omitting the same task at the end of the lesson. Sixteen mathematical questions were common to both lessons and, with the exception of two questions, these were all introduced in the same order. A structural analysis of these diagrams can be found in Richman, Dietiker, and Brakoniecki (2016).

Despite these similarities, interesting differences in how the mathematical ideas unfolded across the lessons are evident. For example, Mr. J's lesson contained 54 story arcs while Ms. W's had 42. Ms. W's enactment contained a higher proportion of story arcs that remained open for multiple acts (52% compared to 30%) and yet her story arcs collectively demonstrate that her lesson contained two separate and disjointed activities while Mr. J's lesson had story arcs that unified the two activities. In addition, Mr. J's enactment had more story arcs open per act (an average of 9.4 story arcs open per act as compared to Ms. W's 6.8).

At this top level, it may be difficult to recognize how and why particular moments of a lesson occurred. Thus, comparisons of the two activities are next described to explain potential aesthetic differences for students in these two enactments.

Determining a Parabola

Both teachers began with the first task of the textbook lesson. The task prompted students to consider the number of points needed to determine a parabola. In both enactments, the same question (*What information is sufficient to sketch a parabola?*) arose, yet the way in which this question played out in the two classrooms was different. Ms. W's enactment enabled the question to arise early and remain open for Acts 1 through 5. In contrast, this question did not emerge in Mr. J's enactment until Act 3, after which it was answered quickly (see Figure 3). As with the plot diagrams in Figures 1 and 2, the width of each cell does not indicate the amount of time elapsed for the given act. Instead, the diagram is formatted to have the same overall width so that Acts 1-5 of Ms. W's lesson represents the same activity as Acts 1-3 of Mr. J's lesson.

What information is sufficient to sketch a parabola?

Ms. W's Lesson	1	2	3	4	5
Question #3	163	63	3		3A

Mr. J's Lesson	1		2		3
Question #15					13A

Figure 3. Story arcs representing how the question "What information is sufficient to sketch a parabola?" was raised and answered in two different enactments. See the caption of Figure 1 for the reference of numerical codes.

Ms. W. In Act 1, Ms. W revealed the *y*-intercept of an unknown parabola and asked students to sketch it. When students encountered difficulty, "What information is sufficient to sketch a parabola?" was implicitly raised (*formulated question,* code 1). The ambiguity on whether the information given was sufficient, was evident by students, "Wait, that's the only clue we get?" (*equivocation*, code 6). Another student notes, "It could be so many things!" (*progress*, code 3). Next, [Act 2] the teacher challenged the students to sketch a second parabola with only the *x*-intercepts (*equivocation*, code 6). Anticipation was evident in the celebration by two students when they guessed correctly with high fives and "Woo!" (*progress*, code 3). During the third challenge [Act 3], a student described his struggle to sketch a parabola with symmetry (*progress,* code 3). In Act 4, Ms. W asked this student to recount his progress to the class (*no new information*). In a discussion [Act 5], a student noted that with only one or two points the parabola could point up or down (*progress*, code 3). The answer to "What information is sufficient to sketch a parabola?" was finally disclosed when, in response to a student question, Ms. W revealed her parabolas (*disclosure,* code A). Ms. W then explained that the students who guessed the second parabola got lucky since the exact graph could only be determined with three points.

Mr. J. Mr. J prompted students to sketch a parabola when given the *y*-intercept [Act 1] and then asked students to compare with a peer. He repeated this process with two *x*-intercepts, and then *x*- and *y*-intercepts [Acts 2 and 3, respectively]. With this framing, the question of whether there was enough information to sketch a particular parabola was not even raised (which is why there are no codes for this story arc during Acts 1 and 2). Instead, the driving questions were "what parabola passes through ____?" and "are the parabolas the same?" As a consequence, the question of how many points are necessary to determine a parabola was not raised until the end of the activity, in Act 3, when Mr. J said "So, how many points are needed to draw a parabola?" (*formulated question,* code 1) to which the students responded "3" (*progress*, code 3). The teacher endorsed this response, disclosing the question (*disclosure*, code A).

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Identifying the Roots of a Parabola

Next, both teachers prompted student groups to work on the same task, asking students how they could use what they know about intercepts (e.g., for the y-intercept, $x = 0$) to solve a quadratic equation for its roots. In both enactments, the same question based on this task arose, yet its resolution was again strikingly different (see Figure 4). In Ms. W's enactment, the question was considered in Acts 6 and 7, was interrupted by another activity, and then re-appeared in Acts 11 and 12. In contrast, in Mr. J's enactment, the question was raised in Act 7 and remained open until Act 11. Even more striking, however, were the differences in the way in which the question was answered over the course of this portion of the lesson, as is evident by the differences in the types and distribution of codes. These differences are described below.

How can we solve the equation $2x^2 + 5x - 12 = 0$?

Ms. W's Lesson	6	7	8	9	10	11	12
Question #24	13	43C				5393	43

Mr. J's Lesson	4	5	6	7	8	9	10	11	12
Question #30				23	7	34	3	3A	

Figure 4. Story arcs representing how the question "How can we solve the equation $2x^2 + 5x - 12 = 0$?" was raised and answered in two different enactments. See the caption of Figure 1 for the reference of numerical codes.

Ms. W. At the start of Act 6, Ms. W assigned the task (*formulated question,* code 1) and students worked in groups while the teacher circulated (*progress*, code 3). The focal group began to consider what values would make $2x^2 + 5x$ equal to 12. Ms. W called the class back together [Act 7] and indicated that students should be trying to solve $0 = 2x^2 + 5x - 12$ (*progress,* code 4). A student from the focal group shared that they were able to determine solutions $x = -4$ and $x = 1.5$ by guess and check (*progress*, code 3). Ms. W then shifted their attention (*suspended answer,* code C) to a game that appeared unrelated at the time [Acts 8, 9, and 10]. During this, she assigned volunteers "secret" numbers and asked them to reveal the product of the numbers. With the product, the rest of class had to guess what the secret numbers were. Through this activity, the students realized that when the product of two or more numbers is zero, at least one of the numbers must be zero. Ms. W then [Act 11] had her students return to the task, indicating that this property can help find the solution to $0 = 2x^2 + 5x - 12$ (*promise*, code 5). The focal group struggled; after factoring the expression (*progress*, code 3), they substituted 0 for x in the factored expression (*snare*, code 9). One student gasped audibly and exclaimed "Oh my god, guys!" upon recognizing that doing this yields the number -12, which is the y-intercept of the equation. The group then realized that the x-intercepts occur when y equals zero, (*progress,* code 3) and shifted to solving $0 = 2x^2 + 5x - 12$. They knew that something had to be done with a factored form, so they tried a mixture of manipulations to rewrite the quadratic expression to no avail. The teacher arrived at the group and the students asked for help [Act 12]. A student said she got the answer $x = 1.5$ by guessing. The teacher then reminded the students of the game from Acts 8, 9, and 10 (*progress*, code 4), after which one of the students connected this answer to the solution of $2x - 3 = 0$ (*progress,* code 3).

Mr. J. Mr. J began this activity by focusing students on the definitions [Acts 4 and 5] and misconceptions [Act 6] regarding x- and y-intercepts. Groups then collaborated on solving $0 = 2x^2 + 5x - 12$, asking each other questions about strategy [*formulated question* and *progress,* codes 2 and 3, Act 7]. Mr. J again called the students back together and indicated that this task should

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be challenging, indicating perhaps it is unsolvable [*jamming,* code 7, Act 8]. He then reviewed the definitions of *x*- and *y*-intercepts and to discuss how to find the *y*-intercept of $y = 2x^2 + 5x - 12$. Next [Act 9], he asked students to try different algebraic strategies in an attempt to solve $0 = 2x^2 + 5x - 12$ (*progress,* code 3). When students struggled, Mr. J had students focus on how strategies that work for other equations do not work for quadratics (*progress,* code 4). In Act 10, Mr. J displayed a sheet of missing factor multiplication problems whose products are all 0. Students responded that the missing factor had to be zero in these problems (*progress,* code 3). Mr. J returned the class to $0 = 2x^2 + 5x - 12$ and reviewed with the class how to factor the expression to be the product of two binomials $(x + 4)(2x - 3)$, and then use the zero product property to figure out the roots. He then reveals the answers to be $(-4, 0)$ or $(1.5, 0)$ [*disclosure,* code A, Act 11]. Mr. J then had his students copy definitions into their notebooks for roots, zeros, *x*-intercepts, and the zero product property [no new progress, Act 12].

Differences in Storytelling

Collectively, these two episodes, which were based closely on the same written tasks, demonstrate that teachers influence the way that mathematical content unfolds throughout their lessons. Together with the students, the teachers craft a mathematical story that has aesthetic dimensions. Despite having so many similarities at a lesson level, the structures of the plot diagrams of these two mathematical stories are strikingly different.

As described in this paper, Ms. W's framing of the parabola sketching activity as a prolonged challenge led to several important aesthetic and mathematical differences. The question "What information is sufficient to sketch a parabola?" was introduced at the start and, as a result, this question was the driving mathematical focus of the activity. That is, the challenge to figure out the teacher's parabola gave purpose to the mathematical inquiry. Thus, the mathematical story in Ms. W's class included mathematical revelations of why only having the *y*-intercept or only having the two *x*-intercepts are insufficient. That different parabolas could result when given insufficient information was part of Mr. J's lesson, however, why that is the case was not. In the aesthetic dimensions of these enactments, Ms. W's framing enabled the students to be surprised by the lack of information and curious about the result (by later asking for the answer). This approach engaged students and resulted in visible celebration. In contrast, since Mr. J did not pose this question until after the parabola sketching activity, his students (while attentive and cooperative through the activity) barely participated in addressing the question of the number of points that are necessary to determine a parabola when finally asked.

When looking at the plot diagrams for the question "How can we solve the equation $0 = 2x^2 + 5x - 12$?" (Figure 4), we see a different structure than the plot diagrams during the parabola sketching activity (Figure 3). In Ms. W's classroom, the question was under consideration until it became temporarily suspended when attention was shifted to a seemingly unrelated game. Upon completion of this exploration, Ms. W's released her class to reconsider the question again, but now taking into consideration some takeaways from the zero product property game. It was through this open exploration that the students in the group experimented with different strategies and were able to have a visible aesthetic reaction to finding out that substituting in zero for *x* in the factored form of an equation still yields the *y*-intercept of the equation. In contrast, in Mr. J's classroom, the development of the mathematical ideas related to solving this question primarily flowed through Mr. J. He focused students' attention toward the problem for exploration, recapped what they already knew and don't know, helped discover a relevant property, and showed how this property could be used to solve the original problem. When this question was under consideration, the students in Ms. W's classroom were what could be called the mathematical actors, while in Mr. J's classroom, the

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teacher was the leading actor of the mathematical story. This could explain the students' aesthetic reaction to their mathematical discovery in Ms. W's classroom, and the apparent absence of one in Mr. J's classroom.

While the analyses of the plot diagrams of these individual questions help highlight possible influences on the aesthetic reactions and opportunities for students, analyses of the plot diagrams of the entire lesson yield other insights. Just as Ms. W's overarching question of "What information is sufficient to sketch a parabola?" offered coherence and purpose to the parabola sketching activity, Mr. J's whole mathematical story included multiple overarching questions that together gave purpose and meaning for the entire lesson (particularly "How can I find the roots of a parabola" and "How can I sketch a parabola with the x-intercepts and y-intercept?"). Although there was no visible aesthetic effect to this crafting of the content, we suspect that there was coherence by the end of the lesson that enabled students to recognize that both episodes together were needed to achieve the lesson goal; namely, to sketch a parabola by its intercepts.

Discussion

If mathematics educators want to change the aesthetic nature of mathematics classrooms, can we just change the curriculum? Our answer is no. As shown in this study, the curriculum is not the only factor that determines the aesthetic impact of the enactment. When mathematics lessons are interpreted as mathematical stories and are compared in this way, differences in the experiences of students can not only be noticed but also understood. This recognition enables future research to focus on the storytelling; that is, to explore how mathematics teachers can craft lessons that can offer engaging learning experiences for students.

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THE DEFINITIONS OF SPATIAL QUANTITIES IN ELEMENTARY CURRICULUM MATERIALS

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Research indicates that U.S. students have misunderstandings in measurement, and one reason is the highly procedural focus of elementary curricula. Because the definitions of quantities are basic conceptual content, we examined definitional-type expressions (related to formal mathematical definitions) for spatial quantities (length, area and volume) within and across elementary curricula with special attention to the inclusion of units, quantification, and exhausting space. We found inconsistencies in the ways these conceptual aspects were presented. In some cases, the inclusion of particular conceptual content depended on the type of quantity, (e.g. "exhausting space" was strongly emphasized in area measure, but less so in length and volume).

Keywords: Curriculum, Elementary School Education, Measurement

Background

Definitions hold a central place in mathematics. They are distinguished from theorems because definitions cannot be proven true or false, and they are different from axioms because definitions can be contested (Kobiela & Lehrer, 2015). *The Common Core State Standards for Mathematics* indicates in the *Attend to Precision* standard that students should, "…use clear definitions in discussion with others and in their own reasoning" (NGA-CCSSO, 2010, p. 7). The practice of defining is potentially rich as a classroom activity (Gilbertson, 2015); yet, many students encounter definitions as simply being presented by the teacher or their textbook. In some ways this approach is sensible, given the inherent messiness of involving students in the defining process to create precise and meaningful definitions which are both mathematically accurate and useful for testing whether a mathematical object fits the definition or not.

Complicating the issue is that some mathematical ideas span multiple grade levels, meaning that students may have multiple exposures to definitions of the same concept. This creates a dilemma particularly for textbook authors because they must maintain consistency, while using language that is age-appropriate and meaningful to students based on their mathematical experiences. The definition of a mathematical function, for example, might look quite different to a middle school student than a student in an abstract algebra course. *Consistency* is not meant to imply the definitions are identical, rather that the definitions convey similar meaning noting important characteristics of the object being defined. This raises the question as to how fundamental mathematical ideas are conveyed to students across grade levels, and the extent to which these are done consistently or inconsistently.

One approach to study this phenomenon would be to study how teachers define mathematical ideas in their own classrooms. We chose instead to study written curriculum materials because these materials comprise the main mediating tool between the ideas of the broader mathematical community and the ideas developed in classroom interactions. Although one limitation of studying written materials is the variation of implementation across classrooms (Tarr et al., 2008), written curriculum plays an important role in what teachers teach, the opportunities they present for

mathematical sense-making (Stein, Remillard, & Smith, 2007) and their profound impact on K-12 classroom instruction (McCrory, Francis, & Young, 2008).

Instead of studying definitions across content domains, we selected one particular topic and grade band with documented issues of student learning to investigate the extent to which definitions could possibly support or hinder opportunities for student understanding. We chose to study definitions of spatial measurement at the elementary level because educational researchers (e.g., Kamii & Kysh, 2006) and national assessments (e.g., NAEP) have indicated deficiencies in elementary students' conceptual understanding of spatial measurement such as confusing the meaning of area and perimeter (e.g., Bamberger & Oberdorf, 2010; Barrett & Clements, 2003; Woodward & Byrd, 1983) amongst many other issues. Additionally, the lack of clarity about the quantity to be measured and what constitutes a measure of that quantity significantly increases the challenges that students face in understanding the measurement process (Smith, Males, Dietiker, Lee, & Mosier, 2013). Curricular analyses of spatial measurement (e.g. Smith et al., 2013), have argued that deficiencies in written curriculum materials may be a contributing factor to these conceptual deficiencies. Since definitions are central to describing the meaning of terms, it raises the question as to how definitions can be presented to contribute in some way to students' conceptual issues.

Purpose and Research Questions

The purpose of this study is to describe opportunities for students to understand definitions of length, area, and volume in written curriculum materials, with specific attention to the consistencies and inconsistencies across grade levels and materials. We use the term, *definitional-type expression* [DTE] to designate any written description of a term that aims to support students in understanding a term's definition. These include all explicit definitions in the text along with other expressions that describe or delimit a quantity or its measure. In elementary textbooks in particular, it is not always the case that formal definitions appear as explicitly as one might see in higher grade levels (e.g. "The definition of a square is…"). Thus, our broader interpretation of definitions includes expressions which aim to define or support defining, but may not be explicitly indicated by the materials as a "definition." Specifically, for spatial measurement, we define DTEs as (a) expressions of meaning for length/area/volume as *spatial characteristics*, (b) expressions of meaning of length/area/volume as *measure* of characteristics, or (c) expressions which clarify the meanings of spatial characteristics or their measures (e.g., "area is not perimeter"). Framing our inquiry were the following questions:

- *What characteristics of DTEs are found in elementary curricula across spatial measurement topics, e.g., length, area, and volume?*
- *What similarities/differences exist across curricula and measures with respect to DTEs?*

Methods and Data Sources

Textbooks chosen for this study were from three elementary textbook series (Grades K-5): (a) The University of Chicago School Mathematics Project's *Everyday Mathematics* (2007)—henceforth *EM*, (b) Scott Foresman/Addison Wesley's *Mathematics*, Michigan edition (Charles, Crown, & Fennell, 2008)—*SFAW*, and (c) Saxon Publishers' *Saxon Math* (Larson, 2004)—*Saxon*. Selection of the textbooks was based on a large market share (e.g., *EM* and *SFAW*) and variance in features of textbooks (e.g., *EM* is Standards-based and *Saxon* is unique in its direct instruction approach). We explored DTEs for *length* in grades K through 3, *area* in K through 4, and *volume and capacity* in K through 5. The coding stopped at certain grades (e.g., length at grade 3) because the central processes, concepts, and tools for the measurement (e.g., length) had been presented by that grade level.

Four researchers divided into pairs with one pair examining DTEs of volume and another pair examining DTEs of area and length. After comparing and discussing characteristics of DTEs in

length, area, and volume, we agreed on a final list of emergent DTE characteristics: (a) dimensionality, (b) units, (c) continuous versus discrete materials, (d) exhausting the space, (e) specific shapes, (f) boundary, (g) quantification (e.g., counting, applying a formula), (h) tool use, and (i) space. Each pair analyzed each group of DTEs for the listed characteristics and resolved disagreements together or brought to the whole research team for resolution. Typically, the coded unit of curricular content was either a single full sentence, or a clause of a sentence.

Table 1: Analytical Framework for Coding Definitional-Type Expressions in Measurement

Aspects	Coding Questions
Dimensionality	Is dimensionality explicitly stated (i.e. terms such as "3-dimensional" appear in the text)? If yes, what dimensionality? (e.g. 1, 2 or 3 dimensions)
Units	Are units of measurement indicated (e.g. gallons, cm^2, feet)? If yes, which units?
Continuous vs. Discrete	What type of substance/object fills the space being measured (e.g., a continuous quantity of liquid or a discrete number of linking cubes)?
Exhausting the Space	Is there mention of the need to completely fill, cover or tile the space (e.g. cover the surface with square tiles without gaps and overlaps)?
Specific shape	Is the measure referenced for a specific shape or object (e.g. a rectangular solid, a triangle)?
Boundary	Is there mention of end-points or other enclosure of the space (e.g. a length measure defined as the distance between two points)?
Quantification	Is there mention of counting, layering, reading a tool or applying a formula (e.g. the volume of an object is the number of cubic units)?
Procedural tool use	Is a specific measurement method using a tool and/or unit indicated (e.g. using a ruler to measure length)?
Space	Is the word "space" explicitly mentioned?

While all these categories potentially contribute to students' understanding of measurement concepts via definitional expressions, we focus our attention on three categories that seem to be most relevant for students' meaningfully understanding measurement. Specifically, we focus on the categories: (b) *units,* (d) *exhausting the space*, and (g) *quantification.* These three categories are central to understanding quantities and their measures; as any measure of space involves a choice of a unit, and that the measure is a quantification of the units that completely exhaust the space. Results for this paper will focus primarily on these three categories. Table 2 (below) describes examples of coding for these categories.

Table 2: Analytical Framework Used to Code Aspects of Measurement in DTEs

Aspects	Questions and Examples
Units	Are units of measurement indicated? If yes, which units? Yes:... *measure in gallons...* (Volume: SFAW, gr. 5, p. SRB157) Yes:...*number of unit squares...* (Area: EM, gr. 4, p. 671 Yes:…*unit of measure will be a linking cube...* (Length: Saxon, gr. 1, 35-2-4) No: … *measure how much space there is…* (Length: EM, gr. 3, p. SRB2)
Exhausting the Space	Is there mention of the need to completely fill, cover or tile the space? Yes:...*fill the space taken up by the object...* (Volume: EM, gr. 4, p. 868) Yes:...*cover the surface without overlaps and/or gaps...* (Area: EM, gr. 4, p. 671) Yes:…*must be marked off "end to end," without leaving spaces...* (Length: EM, gr. 1, p. 282) No: …*To find the area, count the squares inside the shape.* (Area: SFAW, gr. 2, p. 351A)
Quantification	Is there mention of counting, layering, reading a tool or applying a formula (specification of procedure is not necessary)? Yes:… *by counting how many*… (Volume: SFAW, gr. 5, p. 610) Yes:…*count the number of tiles...* (Area: Saxon, gr. 3, 88-5) Yes:…*count the cubes*… (Length: SFAW, gr. K, p. 139) No: …*area is the measurement inside...* (Area: SFAW, gr. 4, p. 485)

Results

We provide percentage and frequency distributions for DTEs across curricula for length, area, and volume, and summarize findings for each quantity (e.g., length) regarding curricular attention to each characteristic (e.g., units). Note that DTEs could (and did) address multiple characteristics. We provide percentage distributions representing the proportion of DTEs that mention a characteristic across grades for each curriculum (i.e., *SFAW*, *Saxon*, and *EM*) and for each measurement (i.e., length, area and volume) in Figures 1, 2, and 3 below. Over 200 DTEs were found across the three curricula and three measures, with few for length (20), more for area (60) and even more for volume (131). Thus, students have more opportunities to learn definitions for spatial quantities as dimensionality increases, which may be a result of having more procedures as this occurs (e.g. for volume one can count cubes, using various multiplication formulas, and pour liquid in the case of liquid volume).

Across curricula and measures, many DTEs failed to include all three components in any one expression. Additionally, there was inconsistency within each measure and curriculum as to whether the inclusion of units, the inclusion of exhausting space, or the inclusion of quantification was necessary in defining the measure. This was generally true with the exception of *Saxon*. One exception occurred in their description of length measure, where all DTEs included reference to a unit and quantification. The other exception occurred in *Saxon's* exclusion of needing to specify exhausting space in length and volume.

This last result points to a more general issue of how some curricula placed increased emphasis on particular conceptual knowledge based on the measure. For example, the DTEs in *EM* placed stronger emphasis exhausting space in length and area (greater than 70% of DTEs in both cases), but de-emphasized this conceptual aspect in volume (less than 20% of DTEs). *SFAW* had variation as well in the exhausting space category, with nearly 70% of DTEs in area compared to less than 15% in volume. In the case of emphasizing a particular conceptual aspect in length, this may be attributed to the fact that length is typically emphasized in earlier grades before area and volume, meaning that the development of the concept is necessarily deeper for length compared to the other two measures.

It does however, raise the question as to whether students are able to sufficiently connect conceptual aspects across measures, namely that unitizing, quantification, and exhausting space are essential for all measures, not just the particular ones emphasized in the curriculum.
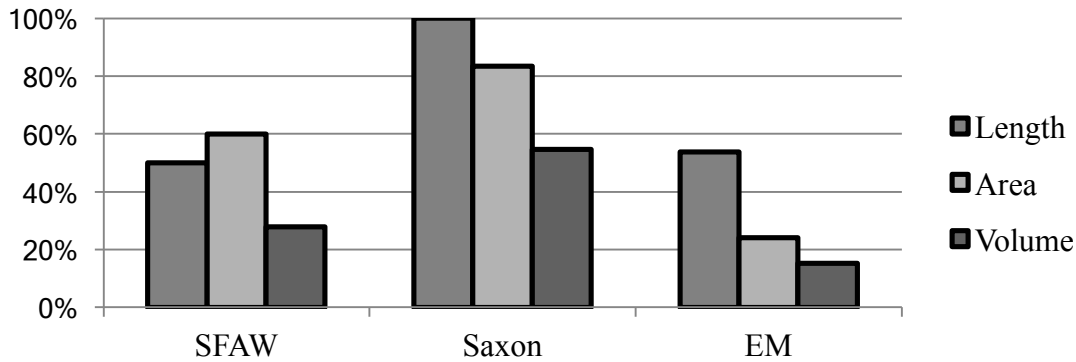
Figure 1. Percentage of DTEs that reference "Units" of measurement.
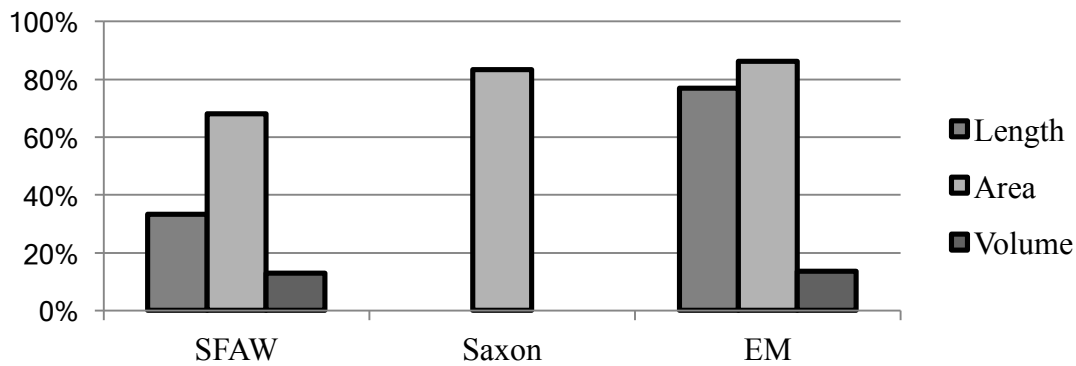
Figure 2. Percentage of DTEs coded as "Exhausting the Space".
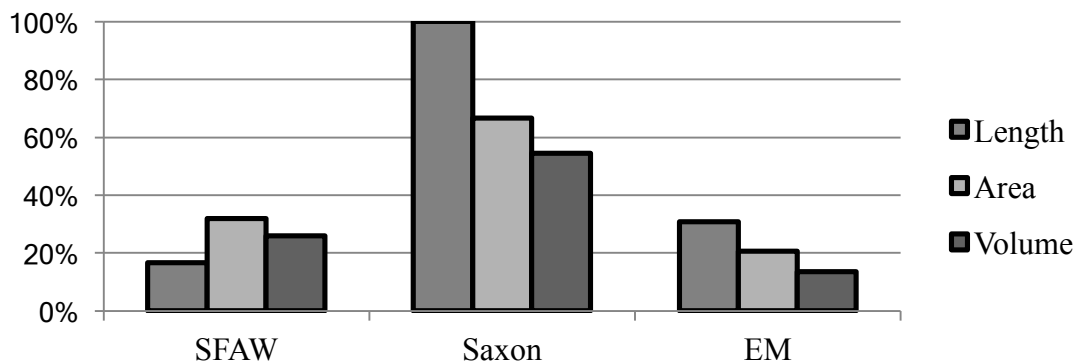
Figure 3: Percentage of DTEs coded for "Quantification".

Length

Across the three curricula, we found 20 DTEs in length measurement in grades K through 3: *SFAW* (6), *Saxon* (1), and *EM* (13). These DTEs appeared more in teacher pages and in the form of declarative statements about measurement (as opposed to questions or problems to pose to the students). The curricular emphasis was in Grades 2 and 3 because 75% of the DTEs appeared in

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these grades. Slightly more than half of these DTEs (11 out of 20) were associated with length measurement with specific units. Slightly over half of these DTEs were associated with specific units (11) or covering of space or distance (12).

Area

In area measurement we found 60 DTEs, with *Saxon* (6), *SFAW* (25), and *EM* (29). Of all DTEs, 45% mentioned units, 32% implicitly referenced quantification (e.g., counting), and 78% addressed exhausting space. *Saxon*'s area DTEs mentioned exhausting the space, where its length and volume DTEs did not (see Figure 2). We found a clear distinction in references to units between *EM* ("unit squares") and *SFAW* ("square units").

Volume

Across the three curricula, we coded 131 volume DTEs. Only four appeared in kindergarten and first grade, with a range of 21 to 46 DTEs appearing in Grades 2-5, with higher frequencies in higher grades. Of the 131 DTEs, 24% mentioned units, 22% mentioned quantification, and 12% mentioned exhausting space. One inconsistent characteristic common to all three curricula was in differentiating the meaning of "volume" and "capacity". In some cases, the inconsistency manifested as DTEs that viewed volume and capacity as essentially equivalent and in other places different. As one set of examples, *SFAW* stated in the student materials, "Capacity is the amount a container can hold" (Grade 2, p. 353; Grade 4, p. 609), and later in the teacher materials "Discuss the meaning of the word volume, explaining that it is the amount a container can hold" (Grade 2, p. 69B). Later on students are asked to, "Explain the difference between volume and capacity" (Grade 5, p. 618). Thus, while the terms are seen as essentially the same, it is not clear what the intended differences are in relation to these two terms.

Discussion

The motivation for this study was assessing the extent to which DTEs might contribute to students weak conceptual understanding of measurement. Different from definitions where being minimal is a requirement (Vinner, 1991), our analysis focused on conceptual aspects of definitions because we want to describe how curriculum materials provide a body of knowledge in developing concepts of length, area and volume. We found the lack of consistency in including or excluding aspects of measurement (e.g. units, quantification, exhausting space) in many DTEs, a pattern that may add to students' challenges in learning the meaning of spatial quantities and their measures. For example, if a procedure such as "counting squares" is given in one DTE, and is later omitted in another DTE this raises the question if the student associates a count of squares as a necessary aspect of the definition or simply one of many ways to understand the meaning of area.

Perhaps more troubling is that the inclusion or exclusion of certain conceptual aspects depend (to a great extent) on a particular measure. This may be an example of a lost opportunity for students to see the underlying conceptual structures that exist across measures, making them appear more different than they are similar. At the onset, we hoped to see consistency across particular types of DTEs, but our analysis has shown trends of inconsistency both within curriculum materials, and across measures.

The emphasis (or lack thereof) on certain aspects of measurement in these DTEs might have potential to shape the meaning students make from these definitional expressions. Textbooks have the benefit of providing opportunities to develop students' ideas over extended periods of time, so one might not expect any singular definition to sacrifice readability and meaningfulness to young students for the sake of mathematical precision. This being said, one would hope to see consistency across definitions in relation to their conceptual underpinnings, something we did not find as strongly as we initially expected.

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PLANNING WITH CURRICULUM MATERIALS: AN ANALYSIS OF TEACHERS' ATTENDING, INTERPRETING, AND RESPONDING

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Curriculum materials are integral to the teaching and learning of mathematics and have been described as having the most influence on what teachers actually plan for and enact in their classrooms. In this paper we describe how two teachers with varying years of experience, but similar experiences with curriculum materials, use curriculum materials in planning. Through a semi-structured think aloud interview, we describe what teachers attended to, how they interpreted what they attended to, and how they decided to respond to the curriculum materials when planning a hypothetical lesson on slope.

Keywords: Instructional Activities and Practices, Curriculum, Teacher Knowledge

Curriculum materials are integral to mathematics instruction. In fact, more than 80% of K-12 teachers use a textbook or curricular program for mathematics instruction (Banilower et al. 2013). Furthermore, curriculum materials have the most direct influence on what teachers actually plan for and enact in their classrooms (Brown & Edelson, 2003) and, although research does describe what teachers do with materials, we do not necessarily know the process by which teachers make decisions about what to do and how to do it (Stein, Remillard, & Smith, 2007).

In this paper, we consider how we might gain insight into the process by which teachers make decisions about the use of curriculum materials while planning. We draw on a framework called Curricular Noticing (Males, Earnest, Amador, & Dietiker, 2015) to describe what teachers attend to, their interpretations of what they attend to, and how they respond to (or make decisions) about how to use curriculum materials to plan lessons.

Background and Theoretical Framing

Phases of Curriculum Use: From Written to Intended

The influence of written curriculum on student learning is not straightforward. Figure 1 illustrates that before a written curriculum is experienced by students, it goes through a series of transformations; the first occurs when the teacher interacts with the written curriculum to produce the intended curriculum, or the teacher's plan. These phases are mediated by a variety of factors, such as teachers' knowledge and beliefs, orientation toward curriculum, classroom structures and norms, and organizational and policy contexts. Therefore, while the written curriculum materials themselves have an influence on student learning, this influence is not direct because the written curriculum is transformed by teachers before students access it. Although the enacted curriculum is the phase that researchers indicate most directly influences students mathematical experience and ultimately what they learn (Remillard & Heck, 2014), it is within the enacted curriculum or the "design-in-use" (Pepin, Gueudet, & Trouche, 2013) phase that teachers draw on the plan that they developed when interacting with the written curriculum.
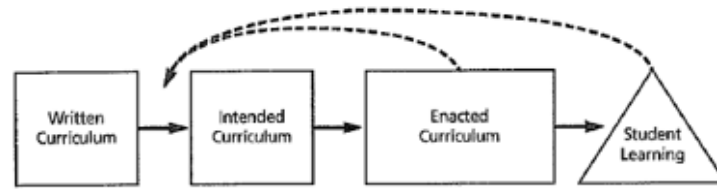
Figure 1: Temporal Phases of Curriculum Use.
(Adapted from Stein, Remillard, & Smith, 2007, p. 322)

In this paper, we focus on planning, specifically the transformation from written to intended curriculum, because this is one of the most critical activities in improving teaching (Morris, Hiebert, & Spitzer, 2009). Furthermore, understanding how teachers use curriculum in planning is imperative in understanding how curriculum influences student learning because it is within this transformation that important variations in written curriculum get introduced. These variations shape the opportunities and experiences students have and therefore, ultimately shape what students learn.

Teachers' Use of Curriculum Materials

Research on teachers' uses of curriculum materials has presented us with a foundation for describing what teachers do with materials. In the midst of planning and enacting instruction, teachers engage in a variety of activities with curriculum. Remillard (2005) describes the teacher-curriculum relationship as a dynamic transaction in which teachers participate with the materials. The socio-cultural conception of this relationship emphasizes the fact that both the teacher and the curriculum influence what and how curriculum materials are used. Using this conception, researchers have outlined ways in which teachers participate with curriculum, including the activities teachers engage in such as reading, evaluating, and adapting (Drake & Sherin, 2009) and what Brown (2009) describes as offloading, adapting, and improvising. When considering the transition between the written and intended curriculum, the role of the teacher is that of a designer (Brown, 2009). In the pursuit of instructional goals, teachers must perceive and interpret existing resources, evaluate the classroom environment, balance tradeoffs, and devise strategies. Thus, there is an integral relationship between agents (teachers) and tools (e.g., curriculum materials); curriculum materials play a role in affording and constraining teachers' actions. It is this design activity that is the focus of this paper.

Curricular Noticing

Curricular noticing is the process through which teachers make sense of the complexity of content and pedagogical opportunities in written or digital curricular materials (Males et al., 2015) and involves sets of skills that unfold in the following three phases: *Curricular Attending, Curricular Interpreting,* and *Curricular Responding*. See Figure 2.
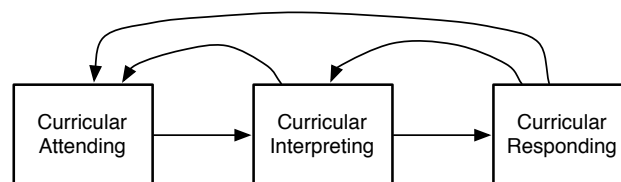
Figure 2: The Curricular Noticing Framework.

First, teachers attend to or perceive and recognize information on the page. Second, they interpret, or make sense of, what they attended to. Finally, teachers decide to respond in a particular way based on their interpretations of what they attended to. It is reasonable to assume that the more curricular opportunities noted by teachers in their curriculum materials, the more supported the

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teachers are to design and enact lessons that involve the types of mathematical experiences they want for their students. Although we describe a process that seems linear, we hypothesize that the process does not always occur in this linear fashion. For example, as a teacher interprets a portion of their textbook, this may trigger her desire to find an additional resource and therefore, she may attend anew before further interpretations or responses are made.

Research Focus and Questions

The focus of this study was to describe how two mathematics teachers with varying years of experience, but similar experiences with curriculum, use curriculum materials in planning. Specifically, we address the following research question: When planning a lesson with curriculum materials, what do two teachers with varying number of years of experience attend to, how do they interpret what they attend to, and what responses do they make (i.e., what do they decide to do and how do they decide to do it)?

Methods

Participants and Context

Participants. This paper focuses on two teachers, Aaron and Evan, teaching in two different high-needs urban high schools in the same district in a mid-western city. Both teachers were part of a cohort of Noyce Fellows. The Robert Noyce Teacher Scholarship Program, funded by the National Science Foundation (NSF), targets the needs of creating and retaining high-quality teachers by funding fellowships for qualified individuals to become or remain mathematics (and science) teachers and commit to teaching in high-need schools. NSF [DUE-1439867] funded 30 Master Teaching Fellows and 13 Teaching Fellows (preservice teachers) who participated in a 14-month Masters + Certification program. Aaron is one of seven Noyce Master Teaching Fellows who participated in this study. At the time of the study, Aaron, a National Professional Board Certified Teacher, had just completed 13 years of teaching. Evan was one of two Noyce Teaching Fellows who participated in this study. At the time of this study, Evan had completed two years of teaching.

These two teachers were purposefully chosen to examine the potential influence of experience on the ways in which curriculum materials are used during planning by keeping other characteristics similar. Although their number of years of teaching experienced differed significantly between the two teachers, both teachers have taught a variety of content at the high school level, ranging from classes for students who struggle to advanced/honors classes. Both had experience teaching slope, the focus of the lesson we gave them to use in planning. In addition, in the last two years, both teachers participated in a course focused on curricular issues, with a specific focus on learning opportunities in written curriculum materials, taught by the first author of this paper. Both teachers had never used the curriculum materials utilized in the staged planning as their primary course materials, but both had heard of the materials. Evan mentioned in the pre-lesson planning portion of the interview that he had seen them in a class taught by the first author, "but I haven't really studied them too much." Aaron had analyzed a portion of the materials (different from what we provided in this study) with a partner for a curriculum analysis assignment and has used activities similar to the activities in these materials in his classroom.

Context. The urban district serves approximately 40,000 students in a Midwestern city. There are six high schools and one alternative community school. The demographics for Aaron's and Evan's schools can be seen in Table 1.

Table 1: High School Demographics

School	Enrollment	% White	Free/Red Lunch	Special Needs	ELL students	Gifted
Aaron's School	1,750	46%	59%	17%	12%	17%
Evan's School	1,978	62%	53%	15%	6%	14%

Curricular Context. At the high school level, the textbook series in use by the district include those published by Pearson, Holt McDougal, McGraw Hill, and Wiley. Teachers teaching the same course throughout the district use the same book and are expected to follow generally the same progression of topics as guided by a list of course objectives written by district math curriculum specialists. Students are assessed using district common assessments written by curriculum specialists and district-selected teams of teachers. These assessments are written to align with the course objectives. In addition, during the semester in which this study took place, the state had recently adopted newly revised state college and career-ready standards for mathematics. Unlike previous versions of the standards, this new revision includes both a set of content standards, but also a set of processes.

Data Collection & Analysis

Data Collection Procedures. Three researchers (one mathematics teacher educator researcher, one mathematics education graduate student, and one undergraduate mathematics education preservice teacher) conducted semi-structured staged planning interviews with nine Noyce Fellows. Recently, staged planning interviews have been used by researchers (Mcduffie, 2015; Reinke & Hoe, 2011) to gain insight into teachers' use of curriculum materials. In a staged planning interview teachers are asked to produce a hypothetical lesson plan, meaning that the plan produced by teachers is not necessarily something that they plan to enact. The advantage to this is that teachers do not have to be tied to district policies and can feel free to use the curriculum materials as they wish. For the staged planning interview, we used a semi-structured think aloud interview protocol that included teachers planning a hypothetical lesson using as a resource Dietiker, Kysh, Sallee, and Hoey's (2013) *Algebra Core Connections* Lesson 2.1.2 How can I measure steepness? These interviews lasted approximately 50 to 120 minutes.

After explaining the procedures to teachers, including telling them what lesson they would be using from the materials, we gave them five minutes to familiarize themselves with the curriculum materials. We then provided teachers with a single-sided copy of both the teacher and student materials for Lesson 2.1.2. We asked teachers to imagine that these were newly adopted materials in the district and that they were planning a lesson that they would enact the next day. We asked them to develop a written plan even if they do not normally write out a plan. In addition, we asked teachers to indicate aspects of the materials that they were reading (highlight in yellow), planning to use as is (bracket in red), and adapt (bracket in green), and to continuously talk out loud as they did this. Once teachers completed the plan we asked follow-up questions, such as how they anticipated students would work through the lesson, what aspects of the materials helped them make decisions about how to plan the lesson, what, if any, portions of the materials did they skim or skip, what portions of the materials did they find most helpful, and how similar or different was what we did today from their typical planning process. During both the pre-lesson planning portion and the lesson planning portion, the researchers remained silent. All interviews were recorded using two video cameras, one focused on the teacher and one focused on the materials. All written materials, including the written or typed plan, any additional written work, and the student and teacher curriculum materials were collected.

Data Analysis. Written materials were electronically scanned in color and video was uploaded to a network server. Video and written materials were imported into MAXQDA12, a qualitative data analysis program. The first three authors independently coded each of the teachers' interviews in MAXQDA by selecting portions of the video in which teachers attended, interpreted, and responded. Attending codes were assigned when there was explicit evidence that the teacher saw something on the page as evidenced by reading aloud and/or highlighting the text. Interpreting codes were assigned when a teacher visibly or audibly attempted to comprehend or made sense of the content. Finally, responding was assigned when the teacher made a decision to do something, such as indicating an aspect of the lesson materials they would choose to use. During the analysis, an additional code was created to capture instances when the teacher described what they were doing.

To facilitate coding, each sentence of the teacher and student lesson materials was given a unique identifier made up of a number identifying the paragraph and a letter identifying the sentence within that paragraph (e.g., 01-A, 01-B, 02-C). These unique identifiers enabled the researchers to assign codes for teachers' attention to, interpretations of, and responses with respect to particular portions of both the teacher and student materials. After independently coding each of the interviews, coders met to discuss and come to consensus on the coding.

Results & Discussion

We first describe the process that each teacher followed to plan their lesson. The process by which the teachers engaged in the planning activity was not only visible, but both teachers verbally expressed how they planned to use the materials while planning the lesson. We follow this by describing how teachers responded to the materials by describing their plans. We then follow this with a description of what each teacher attended to and how they interpreted what they attended and, when possible, how this attention and interpretation connected to their responses.

Teachers Planning Processes

Although we did not explicitly ask teachers to describe the process that they planned to use to plan their lessons each teacher explicitly commented on how they were going to plan the lesson either from the start or within the first few minutes of beginning the planning portion interview.

Aaron. Aaron began by laying out the teacher and student materials side-by-side and stated "So I'm gonna start by just kind of reading through the teacher materials and possibly flipping through the student stuff as I read the teacher materials." Aaron did this until about 22 minutes into the interview at which point he decided to turn his attention solely to the teacher materials as he found it "a little confusing to translate from this [the teacher materials] to this [student materials] … So I'm thinking I'm gonna take the relevant information from here [teacher materials], thinking about it in terms of students."

Evan. Unlike Aaron, Evan did not at first place the student and teacher materials side-by-side. He placed the teacher materials in a pile in front of him with the student materials off to the side not in view of the camera. He then began to read through the teacher materials. Once Evan remembered he was supposed to be talking out loud as he planned, he provided insight into his process by stating

> I'm just reading what the lesson is all about and how the curriculum creators made this lesson. So, I'm just getting a sense of how I'm gonna start the lesson or how they're going to have me start the lesson and then just going from there. So, I'm just reading the information not making any decisions right now.

The notion of not making decisions early in the planning was true for both Aaron and Evan. Although both teachers made some decisions as they worked their way through the text about which problem they would use as is and which they might adapt, neither teacher began to write their plan

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until approximately 40 minutes into the interview. As both teachers wrote the plan, they attended again to the text in the teacher and student material at a quicker pace.

Responses to the Curriculum Materials: The Lesson Plan

These lesson plans produced by each teacher can be seen in Figure 3.

Aaron's Lesson Plan	**Evan's Lesson Plan**
- Objective: Understand slope - Intro: Students read questions - Do 2-11, 2-12, 2-13 - Discuss 2-13 (maybe) – Float: Struggling with 2-11, 2-12, 2-13 – Focus on graph - Do 2-14 as a team - Do 2-15 - Discuss 2-15 - Do 2-16 - Discuss 2-16 - What else could happen? - Check for Understanding 2-19 or 2-10 (similar to 2-14, 2-15) - Homework: Assign 2-21, 2-22, 2-24	- Objective: Introduction to slope - Materials: Student - Handout/Resource (Modified) - Students will be groups in teams of 3-4 at tables - Students read focus questions - Teams working on questions - Think, write, team share on 2-11 (3 min) - Think, write, team share on 2-12 (1-2 min) - Intro 2-13 as class, team work time - Monitor and push teams to read graph and not rely on table - Students work on modified 2-14 as team to think about growth/slope triangle for figure numbers Draw connection on part c compared with part a - Begin discussion of steepness and how line is steeper or less steep using arms - Do 2-15 as whole group. Emphasize vocab - Let teams work together on 2-16 focusing on steepness again using their vocab - Students work on 2-17 and discuss $A_y = 0$ & $A_x = 0$ - Discuss 2-18 with teams as error analysis to extension to idea of negative slope - Assign 2-19, 2-20 a & c, 2-21, 2-22, 2-23, 2-24, a teacher-created problem addressing negative slope

Figure 3: Final Lesson Plans

First, when looking at the lesson plans we see that Evan's has more detail than Aaron's. Although analyzing the level of detail is not a specific focus of this study it provides us with interesting methodological considerations as it makes it somewhat hard to analyze how Aaron interpreted some of the suggestions in the teacher's materials from reading his plan, particularly with respect to participation structures (i.e., teams, whole-class). That said, even Evan's more detailed plans leaves some of these questions unanswerable as he does not always include what groupings students will be working within, nor did he always verbalize this.

Second, we see a number of similarities in the plans. Both teachers decided to have students read the focus questions, do problems 2-11, 2-12, 2-13 as is (with no modification), discuss problem 2-15, and do problem 2-16, assign three of the same homework problems 2-21, 2-22, and 2-24, and focus students on reading graphs. One important point to make, however, is that although these decisions to have students engage with this content is the same it is unclear that the teachers expect the students to engage in the same way. For example, both teachers will use problem 2-15 and we know that Evan decided to complete this via a whole-group discussion, but it is not clear how Aaron will engage his students with this same problem.

Finally, we see some differences, beginning with the objectives. The objective as written in the teacher materials states

Students will gain an abstract understanding of slope as they discover that slope is the change in *y* (referred to as Δ*y*) divided by the change in *x* (referred to as Δ*x*) between any two points on a line.

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They will continue to connect growth and starting value to multiple representations of a linear function.

Both Aaron's and Evan's objectives are simplified with Evan opting to use the word "introduction" rather than "understand." In addition, Aaron does not have students share or discuss problems 2-11, 2-12, and is undecided about 2-13. He also chooses to have students do problem 2-14, but Evan decides to have his students do a modified version of the problem. We discuss this modification in the next section. Evan also decides to have his students complete problems 2-17 and 2-18, which Aaron does not, and give students additional homework problems including a teacher-generated problem that address the concept of negative slope. Although Aaron does not assign all of the additional homework problems Evan does, he does decide he will use one of these, either 2-19 or 2-20 as a check for understanding at the end of class because these problem are similar to two problems students had done in class.

Attention and Interpretation

Capturing teachers' attention proved to be somewhat difficult, particularly when teachers were not talking out loud and/or highlighting as they read. That said, we coded a teachers' attention whenever they highlighted a portion of the text or read the text out loud, meaning that it is possible that we missed aspects of the text to which a teacher attended. We coded teachers' interpretations whenever the teacher was attempting to make sense of the content. This included working out the mathematics in written form or discussing the mathematics or pedagogical structure or suggestions in the text.

Both Aaron and Evan attended to all text in the teacher edition, with the exception of the portion that outlined the mathematical practices, length of activity, core problems, and materials. Evan did not seem to attend to this. Interpretations made of the text in the teacher's guide were most often related to pedagogical suggestions and whether they were good or not. Both teachers also attended to all of the student text and although they did not seem to work through each problem, they tended to solve or work through the same problems. One such example was problem 2-14 in the student materials that asked students to examine a graph that depicts a line for a tile pattern and describe how the line was growing. As both teachers attended to this problem they worked out the slope and found that it was 27/3 or 9/1 and interpreted this problem to be a useful problem in helping student make connections to previous or future content such as the notion of a unit growth rate. This problem was an example of a problem in which both teachers took a few minutes to make sense of what the graph was representing. Although both teachers interpreted the graph in the problem to be a representation of some kind of tile pattern, Evan initially interpreted this to be connected to the earlier tile pattern in problem 2-11 creating a disconnect for him as the number of tiles did not seem to match. When Evan first attended to this problem he stated, "I'm getting stumped. Maybe I won't have my students do that or maybe I will if I can figure it out." After returning to the problem before writing his final plan he decides to add points onto the line because it is "tough" and mentions that he "doesn't think there is a connection [to problem 2-11]… maybe someone can tell me" His difficulty in interpreting this problem may have influenced his addition of the points, but it is not clear how this modification allowed him to see the connection he was looking for nor provided him with a reason to include the problem in his lesson plan.

Conclusion

This preliminary work has enabled us to gain insight into how teachers with different years of experience plan using the same curriculum materials. Not only could we identify similarities sand difference in their plans, but we were able to capture what these teachers attended to, how they interpreted what they attended to, and begin to explore how these interpretations influenced the intended curriculum, or their plan. We hope that future research can enhance the methods used in this

study to capture and report more details about the interactions between the three curricular noticing phases and expand this research determine how curricular noticing influences the enacted curriculum and eventually students' opportunities to learn.

Acknowledgement

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CONSIDERING COGNITIVE FACTORS IN INTEREST RESEARCH: CONTEXT PERSONALIZATION AND ILLUSTRATIONS IN MATH CURRICULA

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This paper examines two factors that have been shown in previous literature to enhance students' interest in learning mathematics – personalization of problems to students' interest areas, and the addition of visual representations such as decorative illustrations. In two studies taking place within an online curriculum for middle school mathematics, students receive problem types that vary with respect to these factors. Results show that while these interest-enhancing interventions may benefit students in the short-term as they solve modified problems, there is little evidence they allow students to understand key mathematics concepts more deeply.

Keywords: Middle School Education, Curriculum, Technology

Research has revealed how many students tend to disengage with mathematics over adolescence (Fredicks & Eccles, 2002; Frenzel, Gotez, Pekrun, & Watt, 2010), and increasingly have difficulty seeing the relevance of mathematics to their lives (McCoy, 2005). Accordingly, research on how interest can be activated and maintained in classrooms has become prevalent in educational psychology. Some interventions to enhance students' interest in curricular materials include adding colorful illustrations (Durik & Harackiewicz, 2007), personalizing instruction to students' out-of-school interests in topics like sports or music (Walkington, 2013), and giving learners choice in their learning activities (Patall, 2013; Potvin & Hasni, 2014). Visual representations that enhance interest may promote persistence and focus of attention, and sometimes they directly provide mathematical information to support students. Personalizing problems may also enhance interest, and allow learners to draw upon prior knowledge of concrete, relatable situations. While these interventions have shown promise for eliciting interest, consideration is not always given to the cognitive implications of the modifications. Specifically, features designed to enhance interest may distract learners from grappling with the mathematical concepts that should be the central focus, a phenomenon known as the *seductive details effect* (Harp & Mayer, 1998; Lehman, Schraw, McCrudden, & Hartley, 2007). Also, if learners become accustomed to these kind of interest-enhancing supports, they may struggle in situations where they must solve abstract mathematics problems. Research on desirable difficulties (Schmidt & Bjork, 1992) suggests that learners may benefit more in the long term from a lack of support in their learning environment, as this forces them to grapple with concepts and make important connections on their own. Further, research on simple symbols suggests that teaching concepts using abstract formalisms – rather than concrete applications – allows for better transfer of learning of the underlying mathematical ideas (Sloutsky, Kaminski, & Heckler, 2005).

Here we examine interest-enhancing interventions – personalization, choice, and the use of illustrations – in a curriculum for 6th grade math. We examine the short term effects of these modifications – whether the interest enhancement is supportive or seductive – and well as the long term effects on student learning – whether the interest enhancement is a crutch or a scaffold.

Literature Review

Visual Representations

Considerable research shows that learners benefit from visual representations (e.g., Mayer, 2009), and effectively using visual representations improves problem solving (Woodward et al.,

2012). Research into the effects of diagrams, one form of visual representations, in middle school mathematics includes topics such as the creation of schematic diagrams for spatially-oriented arithmetic word problems (Boonen et al., 2014), the use of diagrams in algebra word problems (Booth & Koedinger, 2012), and the use of diagrams in proportional reasoning tasks (Jitendra & Star, 2012). While there are some positive findings for diagrams, overall findings are mixed (Booth & Koedinger, 2012). More research is warranted to evaluate when, where, and how such diagrams will be most helpful. There are also mixed findings regarding the impact of decorative illustrations (Berends & van Lieshout, 2009; Jaeger & Wiley, 2014). Considerable support (Harp & Mayer, 1998; for a review, see Rey, 2012) has been found for the *coherence principle* (Mayer, 2009), which suggests that removing interesting but irrelevant information contained in purely decorative illustrations fosters learning. There are also mixed findings for decorative illustrations that contain no mathematical information, but that illustrate the context of a problem – they have been found to have no influence on problem solving (Dewolf, van Dooren, EvCimen, & Verschaffel, 2014) or be helpful (Elia & Philippou, 2004). Here, we use illustrations that have diagrammatic features that give mathematical information, as well as purely decorative illustrations that illustrate or do not illustrate the story context.

Personalization and Choice

We define personalization as an instructional approach that connects math tasks to students' out-of-school interests in broad topics such as sports, shopping, and video games (Walkington, 2013). Research on personalization in mathematics has yielded mixed findings – there is some evidence that personalization carefully accomplished through student interest interviews and open-ended surveys can promote achievement gains in algebra (Walkington, 2013), and that deeper and more authentic personalization can be more effective than shallow approaches in which a few words are simply swapped out of story problems and replaced with words related to an interest topic (Walkington & Bernacki, under review). Other research suggests that even very shallow attempts at personalization can promote student performance and learning (Anand & Ross, 1987; Cordova & Lepper, 1996). However, more recent studies have challenged whether personalization is even worth doing if there is not deep engagement with the actual quantitative knowledge that students actually use as they pursue their interests (e.g., Fancsali & Ritter, 2014). Finally, complementary research also shows that the facilitation of learner control or choice in a learning environment has the potential to enhance interest and motivation (Linnenbrink-Garcia et al., 2013; Patall, 2013; Potvin & Hasni, 2014), as well as learning (Cordova & Lepper, 1996).

Research Purpose

In prior work (Walkington, Cooper, & Howell, 2013; Walkington, Cooper, Nathan, & Alibali, 2015), we found that personalization and illustrations used in worksheets covering middle school mathematics concepts enhanced students' interest, but did not affect their performance. Here, we expand this research by examining several different types of illustrations and several different approaches to personalization in an online, adaptive mathematics curriculum, and examine the effect not only on short-term performance but on long-term learning. Our research questions are: How do different types of illustrations impact students' accuracy on the illustrated problems and their performance on a post-test without illustrations? How do different approaches to personalization impact students' accuracy on personalized problems and their performance on a non-personalized post-test?

Method

Both studies took place within the *Reasoning Mind* 6th grade curriculum. Reasoning Mind is a mathematics blended learning system developed by a nonprofit organization. Within this system,

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students use computers during their math class while their teacher conducts targeted interventions with students who are struggling. The student is immersed in a lesson environment that includes a tutor character, two other student characters, and a virtual blackboard. The student characters make common mistakes which the real student is asked to correct, they help the real student when he or she gets stuck, and they interact with the real student and the tutor in ways that are intended to promote beneficial mathematical attitudes and beliefs. Both studies took place in two small urban middle schools in Texas. Study 1 involved 265 6th grade students, while Study 2 involved 223 6th grade students (demographics in Table 1).

Table 1: School Demographics and Performance

Demographic Group	% in School A	% in School B
Hispanic/Latino	33.1	97.4
Asian	6.0	0
Black or African American	17.3	1.5
Native Hawaiian or other Pacific Islander	0.4	0
White	36.7	1.1
Two or more races	6.0	0
Economically disadvantaged	69.8	92.2
Limited English Proficient (LEP)	10.9	23.5
2014 STAAR Mathematics Passing Rate	30.2	24.3

Study 1: Visuals

Study 1 included 4 fractions problems in Lessons 88 and 89 of the curriculum. All problems gave a fractional measurement (e.g., 3/8 of a meter), and then described what part of a whole this measurement was (e.g., was 21/40 of the whole length). The students then had to solve for the whole. Students were randomly assigned to one of five conditions (see Figure 1): 1) a control condition with no illustrations for the 4 problems, 2) a condition with diagrammatic illustrations that contained mathematical information in the form of a number line (e.g., in Figure 1, the cord could be thought of as a number line with a certain amount of the whole line indicated by being "chewed") or a shaded area model (e.g., an illustration showing how much grass in a whole field had been mowed), 3) a condition with contextual illustrations that simply showed part of the story context and contained no mathematical information, 4) a condition with misleading diagrammatic illustrations that contained incorrect mathematical information (e.g., in Figure 2 the illustration makes it look like most of the cord is chewed, when the answer shows that only a small portion has been chewed), and 5) a condition with irrelevant illustrations that had nothing to do with the story context. The misleading illustration condition was added based on the observation that some of the illustrations already in the curriculum were actually misleading – for example, there would be a problem about a snake who had one third of his length wrapped around a pole, but the illustration would display a snake completely wrapped around a pole. The diagrams were designed to be *supportive* rather than *essential* (i.e., the problem could be solved without looking at them). This is common in math curricula, and it allowed for the problems to still be solvable in conditions where the visuals were purely decorative.

Study 2: Personalization & Choice

Study 2 involved two more extended problem scenarios involving rates in Lesson 103 of the Reasoning Mind curriculum. Students were randomly assigned to one of four conditions: 1) a control condition with the standard story problems already in the unit (shown in Figure 2), 2) a condition where the problem topic is modified and assigned based on students' highest reported interest across four personalized topics (sports, food, shopping, and video games) on an interest survey given to all conditions, 3) a condition where students are randomly assigned to one of the four personalized

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versions of the problem, and 4) a condition where the student is able to choose the problem topic from the four personalized topics before working on the problems.

Figure 1. Example of conditions in Study 1 (there was also a "No Illustration" condition).

Depending on condition, students were given one of five versions of the same problem. In each version, the numbers and question remained the same, but the topic was changed. The control problems are in Figure 2. The other four versions were personalized and changed the topic to sports, shopping, video games, or food. For the first problem (Figure 2, top), the personalization was shallow and involved swapping out "books" for another noun – footballs, lollipops, necklaces, and crystals. For the second problem (Figure 2, bottom), the personalization was deeper as more words were swapped – each of the locations was replaced with a setting that someone who engaged in the personalized topic would be interested in. For example, the video game variation discussed Kayla traveling to an Enchanted Forest, Dragon Cave, and Wizard's Tower while playing a video game. Readability factors were kept consistent among personalized variations and between personalized and control group problems, as was the presence and type of illustrations. For the first problem, all versions had two illustrations – one with an image of the object that had been swapped out for the pile of books (e.g., footballs), and one with a pile of dollar bills. For the second problem, all versions had the same diagram, but the images of each location were redrawn to match the locations given in the modified story (e.g., a dragon cave).

Let's use a table to investigate how the number of books I buy relates to the cost. I was at the bookstore the other day. I bought 1 book, which cost 4 dollars. Then I decided that I wanted two copies of the book. How much will two books cost?

Kayla walks at a speed of 50 meters per minute. It takes Kayla 10 minutes to walk the distance of 500 meters to school. After school, it takes her 20 minutes to walk the distance of 1000 meters to the library. After the library, Kayla walks to volleyball practice. The distance to volleyball practice is three times the distance she walks to school. If Kayla's speed did not change, and the distance increased by a factor of 3, then what do you think happened to the time?

Figure 2. Control (non-personalized) problems utilized in Study 2.

Data Analysis

In both studies, data were analyzed using linear regression models that predicted accuracy on each of the intervention problems and percentage of problems correct on the post test. The accuracy for individual problems was computed by assigning one point to each prompt the tutor asked students to type a numerical response into, and averaging performance across all such prompts. Predictors included Condition (4 or 5 levels), as well as controls for prior knowledge differences between conditions. We had slightly different prior knowledge data available in each study – in Study 1, we included performance on the previous unit test and previous lesson as controls. In Study 2, rather than prior test performance, we had available Guided Study Accuracy (an overall in-tutor measure of knowledge). The post-test measures for each study also varied slightly. In Study 1, there was a quiz immediately after the two intervention units that could be utilized. However, in Study 2 there was just a unit test available in the software that covered all the proportion lessons in the tutor, and no subsequent quiz. Although we used the unit test for long-term learning outcomes in Study 2, we also did supplementary analyses of individual items on the unit test that better aligned to the units where the intervention was placed. However, when analyses were done using individual items or sets of individual items, no additional significant effects were detected – thus we use the score on the entire test.

Results

Study 1: Visual Representations

Results showed that for the second problem, which is the one used as an example in Figure 1, having diagrammatic illustration significantly improved performance, compared to the other four conditions ($B = 14.97$, $p = 0.003$). However, there were no differences between the different conditions for models predicting performance on the other problems ($ps > 0.1$). The differing behavior of problem 2 was also shown by a problem by condition interaction when the data was analyzed with all the problems together. The illustrative diagrams were significantly more beneficial to performance on problem 2, compared to the other 3 problems ($Bs = 12.82, 13.95, 12.54$; $ps = 0.014, 0.007, 0.014$). Anecdotally, the diagrammatic illustration for Problem 2 may have been particularly effective because Pawthagoras is a fun and well-liked character within Reasoning Mind so students may be more likely to focus on the information in the diagram.

Also examined was performance on the post-quiz at the end of Lesson 89. Results showed that students who had received no visual images for the four problems while receiving their learning

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materials significantly outperformed the other conditions on the quiz ($B = 12.03$, $p = 0.030$). Thus while there was some evidence from one of the problems that diagrammatic illustrations allowed students to perform better in the short term, in the long term, not having any visual representations at all facilitated post-test performance.

Data were also available for students' response to the prompt "How much did you like this lesson?" on a 5 point scale (Terrible, Bad, OK, Good, Great) for lessons 88 and 89. Students' two ratings were averaged, and entered into a linear regression model predicting average lesson rating based on Condition as well as their ratings of the four prior lessons as controls. Data for 49 students were missing due to the student not answering the prompt for either of the lessons. Results showed that students who received illustrations that contained no mathematical information (i.e., Contextual Illustrations and Irrelevant Illustrations) rated that they liked the lessons significantly more than other students (B = 0.328, p = .0185). Thus illustrations added purely for decorative purposes seemed to enhance students' ratings of the lessons, although the effect was relatively modest (0.3 on a 5-point scale).

Study 2: Personalization & Choice

Results for performance on the different versions of the shallow personalized problem (Q1 in Figure 2) showed no significant differences by Condition (ps > 0.1). Results for the different versions of the deep personalized problem (Q2 in Figure 2) showed that students who received personalization by choosing immediately before the problem significantly outperformed the control condition on this problem ($B = 11.82$, $p = 0.005$), and students who received personalization based on a survey also outperformed the control ($B = 8.76$, $p = 0.035$). The benefit of receiving a personalized problem randomly assigned over the control condition did not reach significance ($p = 0.088$); no further contrasts were significant (ps > 0.1). Thus personalization, whether accomplished through choice or survey assignment, boosted students' performance relative to solving a non-personalized problem.

On the unit test, although students who received personalization and choice numerically scored highest, this difference did not near significance when compared to the control condition ($p = 0.305$). The only significant contrast for this model suggested that students who receive personalization and choice score significantly higher on the post-test than students who receive a random personalized problem ($B = 12.92$, $p = 0.033$). Thus there is further evidence that personalized versions of problems in the absence of some sort of intelligent selection system to assign them based on student interests are not particularly useful. However, there is little evidence to suggest that personalization acted as a crutch that hindered students' performance when later solving non-personalized problems.

Discussion & Conclusion

Study 1 gave fascinating results as to how visual representations interact with students' performance, attitudes, and long-term learning. Diagrams that contain mathematical information enhanced student performance for one of the problems, but only in the short term. This suggests that for some students, these representations may be a crutch when taking a post-assessment with no visuals. Purely decorative visuals were liked better by students, but there was no evidence they enhanced performance or learning. Finally, the absence of visuals altogether seemed to allow students to learn best from the materials. However, while learning is certainly important, having students enjoy working in their math curriculum, rather than find it tedious or boring, is an outcome that should not be dismissed as irrelevant. Study 2 suggests that personalization accomplished through intelligent selection of interest-based problems (either through a survey or learner choice) can enhance performance in the short term, but only for problems where the personalization is more deeply and thoughtfully accomplished. In the long term, receiving personalization did not show any advantage over receiving non-personalized versions of problems. Attitudinal measures were not

available in Study 2, so it would be interesting to observe in future work how different approaches to personalization impact lesson ratings.

Taken together, the results from both studies suggests that curriculum designers need to think critically about the outcomes they most value, and how those outcomes may be at odds with each other when considering interest-enhancing interventions. In some cases, interest enhancements that boost performance and interest in the short term as students are solving problems may not transfer to assessments of learning, and could even potentially harm learning compared to materials without the enhancements. Our results for visuals suggest that the absence of visuals may be a desirable difficulty that forces students to become accustomed to solving unadorned problems without visual supports or cues. Our results for personalization suggest that the addition of interest-based content is not seductive or distracting, and that it may help in the short term if it is well-matched to learners' actual preferences. However, the type of personalization we implemented here where words were simply swapped out of the stories may not be effective for promoting long-term learning. Instead, if long-term learning is the goal for personalization, research suggests that considering how students actually use quantities in their day-to-day life when pursuing their interests might be most effective (Walkington & Bernacki, 2015). Both studies also suggest that all approaches to interest-enhancement are not created equal – for visuals, whether the visual contains mathematical information is key; for personalization, the depth of the personalization and how problems are assigned to students are important. This suggests that when compiling data from multiple studies on interest enhancements, it is critical to pay close attention to how the enhancement was actually implemented in the curriculum.

Students' interest in learning mathematics can wane over the middle grades, and curriculum developers are increasingly drawn towards quick solutions to attempt increase student engagement with their materials. Many of these solutions can involve considerable cost to the curriculum developer (e.g., hiring an artist or writing multiple versions of each problem), thus it is important to consider how motivational enhancements impact students' understanding of mathematical ideas. Future research should delineate the most effective interest-enhancing supports for different profiles of learners, and for different mathematical content areas.

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SENSE-MAKING PRACTICES OF EXPERT AND NOVICE READERS

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Mathematics textbooks are a significant pedagogical tool, particularly in light of the growing interest in "flipped" classrooms. However, there has been little research on how mathematics students read and comprehend textbooks. This article uses the ideas of the implied reader, sense-making frames, and sense-making gaps to analyze students' reading of a section of a calculus textbook. In order distinguish potential weaknesses in students' content knowledge from their reading abilities, the article also compares students' reading strategies to the reading strategies of "expert" readers: professors in technical fields other than mathematics.

Keywords: Cognition, Post-Secondary Education

As "flipped" and blended classroom pedagogies become more widely used, students are increasingly expected to read and learn from various text materials (e.g., Staumsheim, 2013). However, research has suggested that mathematics students struggle to read their textbooks effectively (e.g., Shepherd, Selden, & Selden, 2012). Consequently, it is important for us to understand how students make sense of reading mathematics text materials.

As Osterholm (2008) noted, there is little research that describes how students read and comprehend mathematical texts. Shepherd, Selden, and Selden (2012) found that undergraduate calculus and precalculus students struggled to read their textbooks effectively. Shepherd and van de Sande (2014) compared the reading practices of first-year mathematics students, mathematics graduate students, and mathematicians, and characterized their reading strategies based on background knowledge, use of resources, and self-monitoring of comprehension.

We aim to expand the current research on how students read and understand mathematics textbooks by exploring the ways that students make sense of mathematical texts and to distinguish the role of content knowledge from the role of reading ability by comparing the students reading practices with those of "experts."

Theoretical Framework

Sense-making

In order to describe the aspects of the text that students focus on and the ways in which they interpret the text, we use the idea of a *conceptual frame*, which is "a mental structure that filters and structures an individual's perception of the world by causing aspects of a particular situation to be perceived and interpreted in a particular way" (Weinberg, Wiesner, & Fukawa-Connelly, 2014, p. 169). From this perspective, readers experience and seek to organize a collection of phenomena as they read texts; they use their knowledge and experience to create frames, and these frames then determine which phenomena are noticed and how they are interpreted.

Also central to the sense-making process are the ideas of gaps and bridges. Gaps are "questions that must be answered in order for the student to engage in or construct meaning for the mathematical situation or activity" (Weinberg, Wiesner, & Fukawa-Connelly, 2014, p. 170). A bridge is the answer that the student constructs. The ideas of frames, gaps, and bridges are complementary: the frame influences the nature of the gaps that arise, and, after constructing a bridge the student may notice different aspects of the text or interpret aspects of the text in a new way. Weinberg, Wiesner, and Fukawa-Connelly (2014) identified four types of frames:

- *Content*: Noticing mathematical aspects of the situation (e.g., symbols, definitions, facts, and concepts) and seeking to understand the meaning of the mathematical content or how to use it in an example that is being presented.
- *Communication*: Noticing the instructor's spoken, written, and gestural actions for organizing and presenting mathematical ideas and seeking to understand the ways the instructor is categorizing or connecting ideas, the ideas communicated by board layout, and the instructor's organizational cues.
- *Situating-mathematical purpose*: Noticing mathematical aspects of the situation and seeking to understand the usefulness or mathematical significance of the concept.
- *Situating-pedagogical purpose*: Noticing communicational aspects of the situation and seeking to understand how the instructor's pedagogical actions and decisions—such as choosing and ordering lecture content—are related to the meaning or significance of the mathematical ideas.

Implied and Empirical Readers

One of our goals is to identify the characteristics of the reader that might explain the differences in what people actually learn from reading a textbook. To do this, we use the idea of the *implied reader* of the text, which is "the embodiment of the behaviors, codes, and competencies that are required for an empirical reader to respond to the text in a way that is both meaningful and accurate" (Weinberg & Wiesner, 2011, p. 52). A *code* is "a way of ascribing meaning" to the symbols, words, and formatting (Weinberg & Wiesner, 2011, p. 53). *Competencies* are the "mathematical knowledge, skills, and understandings [that are required] to work within the established context" (Weinberg & Wiesner, 2011, p. 55). For example, in order to understand a description of the limit definition of the derivative, a reader would need to have some knowledge of functions, limits, rates of change, and their various representations.

Research Questions

Based on the goals and theoretical framework, our research questions are:

- What gaps do students encounter as they read a math textbook, and how do they bridge these gaps?
- What sense-making frames do students use while reading?
- How can mismatches between the implied reader and the actual reader explain the gaps that students encounter and the bridges they construct?
- How do the practices of "expert" readers compare to the students?

Methodology and Methods

Student Participants

All 22 students in the second author's second-semester calculus class were invited to participate in the study. Five students volunteered and participated in the interviews. All of the students had previously completed a standard first-semester calculus course in college or received AP credit for first-semester calculus.

The textbook used in the class—and the interviews—was *Calculus: Single Variable* (Hughes-Hallett et al., 2012). Throughout the semester, the students had been regularly asked to read sections of the textbook outside of class and complete various activities, such as annotating the textbook and writing summaries of textbook sections in groups. Thus, by the time the students participated in the interviews, they had considerable experience reading their textbook.

Faculty Participants

In order to distinguish the role of content knowledge from the role of reading ability, we compared students with "expert" readers. We thought of an "expert" reader as a person who was used to reading dense, technical articles—and had some background in the concepts of calculus—but was neither a mathematician nor an expert in the specific concepts in the texts.

To recruit "expert" readers, we contacted colleagues in the physics, chemistry, biology, economics, and computer science departments at our institution. All contacted professors agreed to participate: two from the physics department, and one each from chemistry, biology, economics, and computer science. With the exception of the physicists, the professors interviewed had not actively engaged with introductory calculus ideas for (at least) the past ten years. The physics professors had encountered introductory calculus concepts in their teaching but had not recently encountered the specific topics we used in the interviews.

Texts

Our goal in selecting excerpts of the textbook was to find sections for which students should have the necessary background knowledge and weren't purely procedural (i.e., only presenting a formula or a step-by-step procedure). We selected the excerpts from the "applications of integration" and "systems of differential equations" chapters that presented formulas as well as conceptual explanations for how these formulas were derived. In this report, we describe an interview based on sections from Hughes-Hallett et al. (2012) Chapter 8.2, *Applications* [of the integral] *to Geometry*": the introduction and the section "*Arc Length*".

Interview Methods

Each student participated in two interviews, approximately five and ten weeks into the (15-week) semester. Each professor participated in a single interview. The interviews were video-recorded to capture the interviewee's gestures and writing, and the audio was transcribed.

In order to elicit the interviewee's perspective and experience of reading the textbook, our methods involved interviewing participants to identify (1) how they perceived the situation; (2) the gaps they encountered; and (3) they way they drew upon their resources to bridge the gaps. We used a *message q/ing* protocol (Dervin, 1983), where participants were asked to read the text and stop at places where they had questions or were confused, and engage in discussion. When they were finished reading the text, the participants were asked to describe the main ideas of the section. Then, we interviewed the participants using an *abbreviated timeline* method (Dervin, 1983): we asked them to explain the meaning of each graph and/or formula, how the terms in the graphs and formulas had been derived, how the graphs and formulas were connected to each other, and why the text's explanation of the connections and derivations made sense.

Analytical Methods

Sense-making frames, gaps, and bridges. We viewed a gap as a question that a reader has while reading the text. However, a reader may not be consciously aware of the gap when it occurs, and is unlikely to consciously think of it in terms of an explicit question. Thus, gaps may be either recognized by the reader while reading the text or during the subsequent interview.

To identify gaps, we individually read each transcript line-by-line and identified collections of utterances that appeared to be evidence that the reader had had an explicit question about the text, had constructed a bridge, or where there were verbal hesitations or pauses. In order to identify the sense-making frame that the reader used, we identified the aspects of the text that were the focus of the explanation or question.

Implied reader. We used the concept of the implied reader as a tool to provide theoretically-grounded explanations for why a particular reader may succeed or fail to construct mathematically

correct conceptions from the text. We compared all of the interviews to identify places where at least one interviewee experienced a gap or described the concepts incorrectly. Then, we described the various behaviors, codes, and competencies that might be required to construct a mathematically accurate interpretation of these aspects of the text and used these descriptions as a working hypothesis for why some interviewees struggled while others did not.

Results

Types of Gaps

In many cases, the gaps that students and professors experienced appeared to be related to their lacking one or more of the aspects of the implied reader. Most of the were related to missing mathematical competencies: the meaning of derivatives of functions and approximating a derivative at a point, the meaning of Riemann sums and their connection to definite integrals, and the reasoning underlying the derivations of formulas. The interviewees typically used content sense-making frames when they identified these gaps. For example, when Frank read the text's explanation of arc length, he appeared to think of the Riemann sum as computing area under the curve shown in Figure 1 and described several options for the method that you could use to do this—specifically, using left-hand, right-hand, or trapezoid rules. Thus, he encountered a gap in which he appeared to ask "which type of Riemann sum is being used?" This gap may be attributable to Frank's lack of understanding of Riemann sums as a general computational tool that isn't tied to a specific geometric method:

A definite integral can be used to compute the *arc length*, or length, of a curve. To compute the length of the curve $y = f(x)$ from $x = a$ to $x = b$, where $a < b$, we divide the curve into small pieces, each one approximately straight.
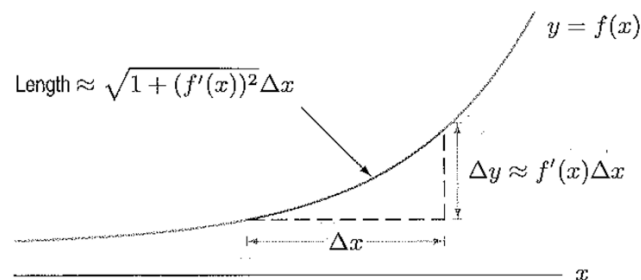
Figure 1. Excerpt from text showing construction of arc length approximation.

Frank: If you have, um, like a left or a right hand sum, where it's just a bunch of blocks, you know, making up the curve, um, it's kinda hard to imagine a curved line going through those blocks. But when you have a Riemann sum—more specifically, the Trapezoid Rule—it's a little easier, I guess, to envision, that like a bunch of, um, angled lines making up a curved line, as opposed to just a bunch of boxes making up a curved line.

Interviewer: Hmm. And you've mentioned the Trapezoid Rule. So how are they using the Trapezoid rule, where do you see that in their explanation?

Frank: Well it's… I'm not really seeing it, but I know that the Trapezoid Rule is really just a way of instead of drawing boxes, to calculate the area under a curve, you draw it, kind of triangles to represent the area under the curve. Draw a box, and then, like, a triangle on top of it connecting two points on the line, and if you were to look side by side, like a left and right hand sum, versus a trapezoid rule, um, to find the length of a curve like this, it would, uh,

look a little more natural, I guess. But, I don't know, that's just what I thought of when I saw that they were using Pythagorean Theorem to calculate the arc length.

Unlike the professors—who were usually able to identify the source of their gaps—the students' missing competencies resulted in their being unable to identify the origins of their gaps. For example, Frank appeared to believe that Riemann sums could only be used to find areas; this resulted in his gap being a question of *which* type of area approximation was being used, rather than what the Riemann sum represented; this led to a mathematically incorrect bridge.

Most professors and students experienced a gap when they encountered the formula $\Delta y \approx f'(x)\Delta x$, as shown in Figure 1. This gap appears to have originated due to their not possessing one or more required competencies related to understanding the derivative and how it is related to slopes of secant lines. For example, in the excerpt below, Professor D., a computer scientist, initially attempted to bridge this gap by drawing on his knowledge of other mathematical concepts (specifically, linear functions), but eventually felt that this bridge was insufficient:

> *Professor D.*: I'm still not sure why the change in *y* is roughly…. And it has to do with the linearity assumption, like piecewise linear sort of functions. And I just forget why.... You know, it's like *y=mx+b* kind of thing. So the derivative is like the *m* and the *b* is zero, because it's just a change from your point. So it's something like that, but that's just the only part that I'm like, not... I'm sure they explained that earlier in the book too.

Professor K., a physicist, experienced the same gap, but was unable to construct a bridge:

> *Professor K.*: So the one thing that I am puzzling to remember is why delta-*y* is *f*-prime *x* times delta *x*. Otherwise this all makes good sense.
> *Interviewer*: And so is it just the assertion that this is true, or how they...?
> *Professor K.*: No, how they use it is fine. This is all very clear. But this assertion... I don't remember.
> *Interviewer*: Is it just the notation, or is there something about the concept behind that?
> *Professor K.*: Yeah, the notation is fine, it's just remembering why *y* is going to be equal to the derivative of *x* times delta *x*.
> *Interviewer*: But then once you assume that's true, then everything else…?
> *Professor K.*: Everything else is fine, yeah.

Although most students appeared to experience the same competency-related gaps as the professors, they often did not recognize these gaps until the interviewer explicitly asked them about the section of the text. For example, Peter's initial hesitation in his response suggests that he did not experience a gap while he read; he recognized this gap during the interview, but was unable to construct a bridge and experienced a lack of understanding:

> *Interviewer*: Did it make sense when they said that the change in y was just f-prime-x times the change in x?
> *Peter*: Um… not entirely. I don't really know... Like yeah, they kinda just threw that in there without really explaining it.

In addition to experiencing gaps related to mathematical competencies, most of the interviewees also experienced gaps related to mathematical codes. For example, several students expressed confusion about the role the symbols *a* and *b* played in Figure 1. Among the professors, these appeared to primarily include (missing) codes related to mathematical terminology and notation. For example, Professor I., a biologist, experienced gaps when reading "integrand" and "elementary antiderivative."

Professor I.: Well I would need to go back up and look at some of these vocabs. But I guess an integrand is the solution of this integral problem. Um… but this elementary antiderivative. I guess to find an integral, you have to find the antiderivative? But I don't know what an elementary antiderivative is. Um… so I was a little bit worried about that.

Bridging and "Jumping" Gaps

As shown in several of the examples above, there were numerous instances where students and professors experienced gaps that they were unable to bridge; they continued reading by "jumping" the gap. However, the professors were much more likely to make the un-bridged gap an object of reflection, and to both acknowledge the lack of a bridge and make conjectures about what a bridge might look like or how the un-bridged gap might impact their understanding. For example, Professor I. made educated guesses about how "elementary antiderivatives" might be used, and Professor K. appeared to recognize how not knowing why $\Delta y = f'(x)\Delta x$ might impact her understanding. In contrast, Peter didn't recognize his own lack of understanding about this equation, and did not appear to reflect on how this unbridged gap might impact his interpretation of the text.

Sense-Making Frames

All of the students and professors tended to use a content sense-making frame for much of the reading. However, a few of the students and all professors used *multiple* sense-making frames while reading. For example, Professor K. used and coordinated content, communication, and situating-pedagogy frames to understand the derivation of arc length:

Interviewer: Okay. How would you describe the book's method for finding arc length?
Professor K.: How would I describe the method... Um... I'm not quite sure how to describe it, except that they're following the same steps that they suggested in their box. So the first step that they have here is that they show how you can find the length by breaking this up into small pieces. So in delta-*x* it's a two-dimensional function, and so delta-*x* and delta-*y*. So they show how you could approximate it. And then they take that into a summation over very small pieces. So you go from these delta-*y*'s to derivatives. So that gives me very small pieces. And then they take that sum, and they turn it into an integral. So they're following their own steps and laying out their procedure. And then they give me a nice boxed equation that I can just go to and work from.

In examining the details of the calculation, Professor K. employed a content frame. Her reference to following steps and laying out a procedure suggest a situating-pedagogical frame, and her discussion of the boxed equation suggested she was using a communication frame.

All of the professors regularly used multiple sense-making frames—in particular, situating-pedagogical frames—and this appeared to be associated with their ability to recognize more gaps, construct (mathematically correct) bridges, and to use various resources to create these bridges. Using these frames appeared to enable the professors to bridge content-related gaps, even when they lacked some of the competencies or codes of the implied reader. In contrast, the students relied almost exclusively on content sense-making frames.

Drawing on Other Knowledge

In addition to using multiple sense-making frames, professors tended to draw on informal knowledge and their own disciplinary knowledge to make sense of the mathematical ideas. For example, Professor D., who was a computer scientist, encountered gaps related to understand the limiting process of transforming a Riemann sum into a definite integral. He described his bridge as based on understanding the integral as a sum of discrete entities:

Professor D.: I guess I see an integral just being a sum of a bunch of discrete things, because I'm a computer scientist. I'm, you know, discrete math instead of continuous math. So I actually really see the thing above it. The summation of all of these little hypotenuses being summed together. So uh, this notion of like having a discrete sum, and then as you integrate—as you integrate from, you know, you integrate the number of sections, and those go to infinity and the size of the section goes to zero, like it's kind of the same thing. And it's just summation and integral are basically the same symbol in my head.

Professor M., a chemist, lacked some of the implied reader's competencies for limits and Riemann sums. Rather than draw on his disciplinary knowledge, he used informal knowledge to bridge gaps he encountered when trying to understand how to approximate volume of a sphere:

Professor M.: If we're approximating this volume with a, with um, with a bunch of little cubes, you know, um... And smoothing it out to make a round shape, then that would be maybe what this is trying to do. I have no idea if that's right. That's... That's kinda what I'm thinking about... So if we were doing this sphere, we're taking, and we're like breaking it into Lego blocks, those blocks are easy volumes to calculate, so we take those, add up all those blocks together, and then we have to... [makes a rounding shape with his hands] to smooth out the edges.

Discussion

There were numerous similarities between the "novice" and "expert" readers. Both groups primarily used content sense-making frames and experienced numerous gaps. Most professors interviewed were not familiar with the formal notions of Riemann sums or limits and frequently did not recognize terminology; this could be, in part, due to their not having actively thought about calculus concepts since they were students in college. Although the students had recently discussed ideas of Riemann sums and limits, they also often experienced gaps that appeared to be related to the same calculus concepts as the professors.

These gaps can be viewed as the result of discrepancies between the implied and empirical readers—that is, the students and professors lacked particular competencies and codes that were required to construct mathematically accurate interpretations of the text. For the students, lacking some of the competencies resulted in them mis-identifying the source of their gaps, being unable to bridge the gaps, or constructing bridges that were mathematically incorrect.

The professors were generally better at recognizing gaps while they were reading the text. In contrast, the students tended to only notice the gaps when they were asked to focus on a specific section of the text or asked directed questions. This suggests that the professors tended to employ behaviors that enabled them to monitor their own understanding as they read, whereas the students did this less frequently.

In contrast to the students, the professors were often able to construct mathematically accurate codes and competencies in order to bridge gaps related to their lack of knowledge of calculus concepts. They appeared to identify the sources of their gaps; they drew on their other mathematical knowledge to construct bridges; they drew on both informal knowledge and knowledge from their own discipline to construct bridges; and they employed multiple sense-making frames—in particular situating-pedagogy frames. These practices appeared to enable them to generate accurate interpretations of the text, to construct missing codes and competencies in some places, and to recognize the limitations of their interpretations.

Taken together these results suggest that constructing accurate interpretations of a mathematical text requires particular background knowledge and ways of interpreting the various symbols and technical terms. However, a lack of background knowledge can be overcome in several ways. In particular, the examples here highlight the importance of attending not just to the mathematical

content, but the way it is communicated and organized. This analysis suggests various ways of helping students structure their reading experience—such as asking questions that help students focus on non-content aspects of the text. It also suggests that students need help recognizing when they encounter a gap and developing strategies to bridge gaps. However, some of the practices of the "expert" readers, such as making connections with previous mathematical knowledge, might be more difficult to help students do.

These results build on the prior literature by beginning to disentangle the impact of content knowledge from reading practices. For example, Shepherd and van de Sande (2014) found that *mathematicians* employed many of the effective practices reported in this study. However, our results suggest that readers who are less familiar with the content (i.e., the professors) might read less effectively than people who are more familiar (i.e., the students who were currently taking calculus). Furthermore, using multiple sense-making frames and reading reflectively might enable readers to overcome deficiencies in background knowledge. These results highlight the importance of attending to both knowledge and reading practices when asking students to learn from reading mathematical texts.

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SECONDARY SCHOOL STUDENTS' CONCEPTIONS OF PROOF IN TRINIDAD AND TOBAGO

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This study examines students' conceptions of proof in Trinidad and Tobago. I conduct semi-structured interviews with 21 secondary school students, to investigate their opinions of (a) the purposes of proof in mathematics and (b) the type opportunities for proving in geometry in the Caribbean Secondary Education Certificate (CSEC) examinations. My analysis suggests that students identified the roles of verification, explanation, systemization, and appreciation within the work of mathematicians or school mathematics. The latter role suggests students' understanding of the intellectual need of proof in their mathematical learning. All 21 students considered the calculations with explanations questions in the CSEC examinations as informal opportunities to construct proofs. The development of these non-proof arguments has the potential to go beyond the borders of this reasoning and proof activity to evolve into opportunities for construction of proofs.

Keywords: Reasoning and Proof, Curriculum, High School Education

Introduction

The reformers of mathematics education in Trinidad and Tobago suggest that students should have more opportunities with proof in their secondary school mathematical experiences (Republic of Trinidad and Tobago, 2009). In Trinidad and Tobago, researchers are working to understand students' perspectives about what constitutes a proof and the roles of proof in school mathematics. Researchers are also interested in how the new reform-oriented mathematics curriculum, Caribbean Secondary Education Certificate (CSEC) examination materials, and teachers' instruction influence students' notions of proof in school mathematics. To date, there are no existing studies, which examine students' mathematical learning in the post-implementation period of the reform-based curriculum. Furthermore, the recent CSEC examinations offer more calculate and explain type of questions rather than directly asking students to prove (Caribbean Examination Council [CEC], 2014). As a result of these concerns, there exists a need to investigate students' perceptions of (a) the purpose of proof in school mathematics and (b) their opinions of the opportunities to do proofs in their textbooks and CSEC assessment materials.

Research Questions

The research questions, which drive my inquiry, are:

RQ.1 How do students in Trinidad and Tobago view the purposes of proof in mathematics and secondary school mathematics?

RQ.2 What are students' conceptions about the type opportunities for proving in geometry in the CSEC examinations?

Theoretical Perspectives

In this study I use a framework previously used by McCrone and Martin (2009) and Dreyfus and Hadas (1987) to help investigate students' conceptions of proof. This framework entitled: The Six Principles of Proof and Understanding describes the knowledge any person within an informed mathematics community should possess about the roles, structure, validity, and generality of a proof. In this brief report, I discuss the findings associated with the roles of proof in mathematics, school mathematics, and assessment materials.

Methodology

In this study, I conducted semi-structured interviews with 21 students (male or female who are 13 to 16 years old) in forms three (ages 13-14), four (ages 14-15), and five (ages 15-16) from three selected school sites. For each school, I randomly selected seven students from those who provided voluntary assent and parental consent to participate in the student interviews. Additionally, these students also participated in the proof-based lessons on Congruency of Triangles, I observed in another study examining the teaching of reasoning and proof. The selection of the seven students from each school was according to the grade level I observed at the respective school. Each interview lasted 45 minutes to one hour.

Results

According to students' responses, the roles of proof in mathematics included verification, explanation, and systemization. In Table 1, I present a summary of the meaning of each of these roles of proof students identified in mathematics. I also present counts of the number of students identifying each role. These counts do not indicate the number of times a student mentioned a role but the number of students who talked about a specific role.

Table 1: Roles of Proof in Mathematics

Role of proof	Description	No. of Students
Verification	To verify that a statement of conjecture is true	9
Explanation	To give insight into why a statement is true	15
Systemization	To build an axiomatic system of results	2

Representative Quotes of Students' Conceptions of the Roles of Proof in Mathematics

When I asked the question "In your opinion, why do mathematicians write proofs?" The following students, stated:

Ava: Mathematicians write proofs to explain. I mean that proof explains the things a lot easier for other people like other mathematicians and people who will read the proof see exactly how other mathematical concepts make up reasons supporting why the result is true rather than just showing it is true. (Explanation)

Ray: To back up themselves to show they are right and really this is important because of how mathematics topics are usually connected, he may need to verify that his Theorem is right so that he can use it to prove other results later on. (Verification & Systemization)

Roles of Proof in School Mathematics

In their responses, students identified two roles of proof in school mathematics. By school mathematics, I refer to the proof opportunities in school during the teaching of various secondary school mathematics topics. Students stated the following roles (a) promoting understanding and (b) appreciation of mathematics. Of the 21 students, 19 students talked about the promotion of understanding. According to the students, three main themes emerged in their discussions of this role of proof. Students claimed that through the promotion of understanding they can develop (a) insight into why a theorem is true, (b) knowledge of the utility of a proven result, and (c) habitual inclinations for their own proof writing practices.

Of the 21 students I interviewed, 5 students talked about the role of appreciation. This role is an interesting finding because it demonstrates a unique conception of proof held by students in Trinidad and Tobago. According to these students, the teaching of proof allows them to appreciate the usefulness of the underlying axioms and other mathematical results used to construct a proof. For example, when I asked, "In your opinion, could you provide reasons why you are taught proof in mathematics?" the following students explained:

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

Danni: I think Miss really shows us the proof even though it is not required, to help us appreciate the mathematics you are learning like where it came from and how to use it to solve problems.

Melissa: Well the teaching of proof helps us to really value the usefulness of mathematics when you see how Miss apply some other results we learned before to prove some result we are now learning.

Taylor: To help us understand and appreciate the mathematics we are doing cause if she does not show us the proof for a result we will just write it down and not really understand the purpose of it and how to connect it to other stuff and we would really be excited about it.

Overall, these students demonstrated through their opinions that the appreciation of mathematics is an important component to their learning in school. Each student articulated how the role of appreciation motivates an awareness of the usefulness of their pre-existing knowledge and the application of a new result. In particular, Taylor emphasized that her appreciation of a new result allows her to see the connections to previous knowledge. Melissa emphasized that through her teacher's application of previous knowledge when writing a proof, she valued the usefulness of her previous knowledge. As a result, Melissa noticed the links between her previous knowledge with the new knowledge she learned. Danni asserted that through the appreciation of a mathematical formula, he became aware of its' utility during problem solving.

Opportunities for Proof in CSEC Examination

In the introduction of this report, I highlighted the prevalence of calculate and explain type of questions in the CSEC mathematics examination. Given the high stakes of this exam in determining students' future engagement with higher-level mathematics, it is important to investigate whether students consider that the informal explanations required by these exercises, are potential opportunities for constructing a proof.

Of the 21 students in this study, all students claimed that response to this type of question would qualify as a proof. Students provided the following reasons, (a) a clear explanation of why your answer is true is a proof, (b) a proof is an explanation that supports your answer, and (c) an explanation that promotes insight into your thinking is a proof.

When I asked Sean the question, "Consider the question taken from the January 2014, CSEC Mathematics examination. The question has a pair of parallel lines with a transversal cutting across the two lines at an angle of 24^0. You are required to find the missing angles and provide reasons supporting your answer. In your opinion why or why not would you consider an answer to this question as a proof?" He responded:

Sean: Well thinking about it, the question specified that you must provide supporting reasons for your calculations. In this case, I really think they [the examiners] want you to give a clear explanation supporting why your answer came out to be that way. Well to me explaining why is a proof of your claim. I believe by explaining this the examiners will understand your thinking.

Sean reflected on the examiner's requirement of providing supporting reasons for calculations. When Sean stated: "I really think they [the examiners] want you to give a clear explanation supporting why your answer came out to be that way" he suggested that the examiners expected students to provide supporting reasons for the steps taken to calculate the unknown values. As Sean stated "to me explaining why is a proof of your claim" he voiced the opinion that providing an explanation of why a claim is valid qualifies as a proof. Sean explained further the necessity for providing a clear explanation. For example, when he stated, "I believe by explaining the examiners will understand your thinking" Sean suggested that the examiners will understand the line of

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

reasoning students use to compute the unknown values. This latter quote also demonstrated that Sean saw the intellectual need of explaining one's thinking when proving.

Conclusion

The students' conceptions of the role of appreciation of mathematics demonstrated their understanding of the intellectual necessity of the mathematical knowledge they acquire at school. Harel and Tall (1991) identified the intellectual necessity principle as a standard for pedagogy that involves presenting subject matter in a way that encourages learners to see its intellectual necessity in their mathematical experiences. Therefore, students seemed to understand the role of proof based on their teachers' construction of proof arguments. The habitual inclinations for students to write proofs that provide insight and allow all readers to follow the line of reasoning is important for student's future metacognitive development of proof writing skills. The informal calculate and explain opportunities in the CSEC examinations do provide opportunities for students to construct proof arguments. Although these questions do not explicitly state the word "Prove," they provide the necessary scaffolding to develop rationales for reasoning that could eventually go beyond the border of this activity and evolve into formal proof arguments.

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OPPORTUNITIES CREATED BY MISDIRECTION IN MATHEMATICS LESSONS

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This study provides evidence that enacted lessons based on written curriculum do not need to follow a direct curricular path from beginning to end. Deviations in this path, which we refer to as misdirection, can create opportunities for enhanced student interest in mathematical questions. We present three examples of misdirection from two enacted lessons and describe how they intensified student investment in the mathematics by creating contradictions that inspired students to ask their own mathematical questions. These examples can serve as models to teachers and curriculum writers who seek ways to motivate students to pursue mathematical understanding.

Keywords: Curriculum, Curriculum Analysis, Affect, Emotion, Beliefs, and Attitudes

It is often assumed that the role of curriculum is to provide students with a direct path from what they already know to what they need to know (Tyler, 1949). The recent focus on spatial metaphors of curriculum, such as a trajectory (e.g., Clements & Sarama, 2004; Gravemeijer, Bowers, & Stephan, 2003; Simon, 1995) is consistent with this assumption. This increasingly popular metaphor draws attention to an initial point, described as the "teachers' hypotheses about the students' mathematical knowledge" (Simon, 1995, p. 138), and a target (i.e., a learning goal). This perspective highlights the instantaneous decisions of direction that teachers make as they guide their classes along anticipated "learning routes" toward a target and adjust instruction in order "get back on track" when necessary (Gravemeijer et al., 2003, p. 55).

Yet, in our analyses of mathematics curricula as they unfold in classrooms, we regularly see "learning routes" that are not linear and that do not progress smoothly. In fact, we have found some paths that appear to change direction, double back on themselves, or skew away from the target. We refer to these indirect paths as instances of "misdirection." Rather than conceptualize these paths as a problem with curriculum design, we instead recognize misdirection as potentially beneficial and generative for learning. Understanding these moments of misdirection can inform the crafting of mathematics curricula (written and enacted) that stimulate student curiosity. In this paper, we describe moments in mathematics classrooms during which obstructions toward the mathematical goal of the lesson motivated students to become invested in the pursuit of mathematical understanding. Our goal is to use these analyses to complicate curricular assumptions and suggest a new curricular metaphor that offers insight into this complexity.

Theoretical Framework

The complex ways that information can unfold across a sequence have been studied in narrative by literary theorists since Aristotle. Conceptualizing mathematics lessons as *mathematical stories* allows us to capitalize on this work. With this framing, we gain new analytic tools to interpret the way mathematical ideas emerge and unfold across a temporal sequence from beginning to end (Dietiker, 2015). This framework is not a focus on contextual story problems, but interprets all of the mathematical content of a lesson as a sequence of revelations. Although mathematical stories can be recognized in written mathematics curriculum materials (i.e., the intended curriculum), our focus is on the mathematical stories that are found in the complex realm of the classroom (i.e., the enacted

curriculum). In this framework, the students and teachers are the actors in enacted mathematical stories.

Literary stories engage an audience by provoking questions and then slowly revealing the answers in ways that both inform and entertain (Nodelman & Reimer, 2003). Stories that contain twists in the plot misdirect a reader into answering questions incorrectly, setting up opportunities for surprise when their predictions are later shown to be false. Framing mathematics curriculum as a story enables an analyst to understand how misdirection can occur. This framing also helps to illuminate the logical and aesthetic roles that misdirection can play in the experience of the students.

To analyze the narrative structure in mathematical stories, we employ the hermeneutic codes developed by Roland Barthes (1974). These codes were designed to describe the way in which literary plots unfold. With these codes, the emergence and progress made for any mathematical questions of a lesson are tracked, allowing for the identification of misdirection that occurs along the way as well as the factors that enabled the misdirection. Three codes in particular relate to misdirection -- equivocation, snare and jamming.

An *equivocation* describes a moment in a story when a reader can identify, upon reflection, that she or he was misled through ambiguity. The narrator of the story does not overtly lie, as the information presented is technically accurate. However, the telling of the story allows or arranges for a misunderstanding to occur. A *snare* describes a moment in a story when a reader can later identify, upon reflection, that she or he was explicitly misled. While equivocations are lies by omission, snares are lies by commission. *Jamming* is the interruption of progress on a question in a way that suggestions the question cannot or will not be answered.

It is important to note that students, as well as teachers, can be responsible for misdirection in a mathematical story. This conflicts with the common assumption that students in the classroom are solely the audience, not the actors, in the story. While students can be both actor and audience, we analyze the plot of a mathematical story by interpreting the potential mathematical story available to a hypothetical, silent, intellectually engaged learner.

Methods

The data presented in this study are from observations of two algebra classrooms in different regions of the United States using the same inquiry-based algebra curriculum materials (CPM Educational Program). The teachers, whom we will call Ms. Becker and Ms. Wilson, have 20+ and 30+ years of teaching experience respectively, and each has been using CPM materials for at least 5 years. Ms. Becker teaches at a public middle school in the South and Ms. Wilson at a private high school in the Northeast. Each teacher was video and audio recorded while enacting lessons from the same written materials. The three selected lessons focused on: 1) solving systems of equations by substitution, 2) different representations of quadratic functions, and 3) the zero product property.

The videos were transcribed to capture the whole class dialogue as well as the dialogue from one group of students in each classroom as they worked together on assigned problems. From the transcripts, a team of four researchers worked in pairs to identify questions that were formulated by the teacher, students, or textbooks. The two pairs then came together to resolve discrepancies. Next, researchers used Barthes' codes to identify progress toward the answering of the questions from the formulation of each question to its disclosure. With this coding, instances of equivocations, snares, or jamming, defined above, were identified and compared in order to answer the question "How does misdirection occur in high school algebra classrooms and how can its role in the student's experience be explained?"

Findings

Our analysis of these lessons revealed examples of all three types of misdirection. Below, we provide an example of each form of misdirection and explain how each one created surprise when a

contradiction was revealed. We then show how each contradiction led students to ask their own questions about the mathematics, thus increasing their investment in the outcome of the lesson.

Equivocation

An example of equivocation occurred in the first part of a lesson in Ms. Wilson's class on identifying the roots of a quadratic equation using the zero product property. The goal of this part of this lesson was for students to understand that three non-collinear points are required to uniquely identify a parabola. In order to introduce this idea, Ms. Wilson engaged the students in three rounds of a "guess my parabola" game. In each round, the teacher thought of a parabola, gave them information about that parabola and challenged them to guess her parabola. In the first round she gave them the y-intercept, in the second round she gave the two x-intercepts and in the third round she gave the y and both x-intercepts. The equivocation in this sequence of activities was the assumption from the beginning of the game that it was reasonable for someone to determine her parabolas with only one or two points. This was misleading in the first two rounds because there were an infinite number of parabolas that fit these conditions.

Rather than simply revealing to the students at the beginning of the activity that knowing three points of a parabola would uniquely identify it, this teacher created the expectation that it might take fewer. Students were then surprised when they discovered that it is unlikely for them to guess her parabola with just one or two points. This surprise gave way to curiosity and then various levels of understanding as students realized why they were having so much difficulty. Student investment in the mathematical questions was evident by the energy with which they expressed their realizations and how hard they pressed the teacher for more clues, and eventually the answer.

Snare

A dramatic example of what can happen as the result of a snare occurred in another of Ms. Wilson's classes during a lesson on different representations of quadratic functions. Students were trying to graph the parabolic path of a water balloon from an equation. One student made a calculation error that resulted in a large negative value for y, suggesting that after the balloon was launched it was underground. In response to this snare, a second student addressed the contradiction between the expected y-value [i.e., a positive value] and the calculated y-value by creating an inventive interpretation of the situation in the context of the mistake. The absurdity of the story that resulted, involving a bird and a remote control plane, possibly convinced the student to re-evaluate his calculation and thus catch his error. The twist in this mathematical plot, this time a misleading error, inspired these students to construct additional explanatory stories and persevere to resolve the contradiction.

Jamming

One example of jamming occurred in Ms. Becker's lesson on the substitution method for solving systems of equations. At the beginning of this lesson, the class was attempting to solve a system of equations using the previously learned equal values method (i.e. solve each equation for the same variable, set them equal to each other, and solve the resulting one-variable equation). The new system was different than other systems students had previously solved using this method because isolating a variable in the second equation led to non-integer coefficients. This complication led students to assume the problem was unsolvable using the method that they knew–they were "jammed." Later, when Ms. Becker introduced substitution — the method that solves this system in a more efficient way — the students demonstrated clear relief. By introducing a problem that led to jamming, Ms. Becker enabled the students to become invested in finding an alternate way to solve the system.

Discussion

In this paper we used a mathematical story framework to illustrate the counterintuitive proposition that misdirection in the mathematics classroom can be beneficial. Thus, we propose that studying how misdirection can be used effectively in mathematics lessons will offer potential solutions to how curriculum might improve student attitudes toward learning mathematics.

Misdirection, however, is just one of many devices that storywriters use to create captivating narratives (e.g. foreshadowing, suspense, etc.). Framing mathematics lessons as stories provides mathematics educators access to this valuable set of tools. We encourage teachers, other curriculum designers, and mathematics education researchers to extend this work—to seek new ways that powerful storytelling techniques can be used to design lessons that capture student attention and inspire students to become invested in mathematical outcomes.

Acknowledgements

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VERIFICANDO LA AUTENTICIDAD DE PROBLEMAS CONTEXTUALIZADOS PROPUESTOS EN UN LIBRO DE GEOMETRÍA

VERIFYING THE AUTHENTICITY OF PROPOSED WORD PROBLEMS IN A BOOK OF GEOMETRY

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Palabras clave: Currículum, Geometría y Pensamiento Geométrico y Espacial

Los libros de texto de matemáticas son una herramienta fundamental para el docente ya que pretenden fijar, organizar y estructurar los conocimientos que son requeridos para familiarizarse con una disciplina o un campo de conocimiento (Alzate, 2000). En investigaciones pasadas, como la de Santanero (2011), se demostró que los problemas propuestos en libros de texto de secundaria no son auténticos, es decir, el contexto en el cual se desarrollan no es del todo real. Además, Santiago, Slisko y Nolasco (2011) afirman que los contextos artificiales generan actitudes negativas en los estudiantes. Por esta razón, analizamos 29 problemas de un libro de Geometría y Trigonometría de Nivel Medio Superior (Martínez S., Espidio M., Santiago G. y Álvarez R., 2013) para explorar si en esta asignatura y nivel se utilizan problemas auténticos. Nos basamos en la teoría de Palm (2006) la cual tiene 8 criterios que el problema debe cumplir, pero sólo consideramos 3 los cuales son:

- **Evento**. Este aspecto se refiere al evento descrito en la tarea. En la simulación de una situación del mundo real es un requisito previo que el acontecimiento descrito en la tarea de la escuela ha ocurrido o tiene una ocasión justa de ocurrir.
- **Pregunta**. Este aspecto refiere a la concordancia entre la asignación dada en la tarea escolar y en una situación extraescolar correspondiente. La pregunta en la tarea escolar es una que se pudo presentar realmente en el acontecimiento del mundo real descrito, es un requisito previo para que una situación del mundo real correspondiente exista.
- **Información/datos**. Este aspecto se refiere a la información y a los datos en la tarea e incluye valores, modelos y condiciones dadas.

Si el problema no cumple con uno de estos criterios se considera como no auténtico. Se seleccionaron los problemas que tuvieran una imagen para ayudar a entender el contexto descrito en el texto. Lo anterior fue también con la intención de analizar el modelo situacional propuesto por el autor, y nos preguntamos si tal imagen ayudaría a los estudiantes a entender el contexto del problema y a resolverlo de manera correcta.

La mayoría de los problemas analizados no cumplen con la teoría de Palm (2006). Los autores de este libro de texto no consideran la autenticidad de problemas ya que los contextos mostrados no cumplen con la teoría de Palm. Además las imágenes mostradas tampoco ayudan ya que, en algunos casos, no muestran las situaciones descritas.

Keywords: Curriculum, Geometry and Geometrical and Spatial Thinking

Mathematics textbooks are essential tools for teachers. They aim to establish, organize and structure knowledge that is required to become familiar with a discipline or field of knowledge (Alzate, 2000). In past investigations like Santanero (2011) we see that the problems posed to reaffirm knowledge are not authentic, that is, the context in which they develop is not entirely real. In addition, Velázquez, Slisko and Nolasco (2011) argue that artificial contexts generate negative attitudes in students. For this reason, we analyzed 29 problems from a high school level Geometry and Trigonometry textbook (Martínez, Espidio, Santiago, & Álvarez, 2013) to examine whether authentic problems were utilized in this subject and level. We rely on the theory of Palm (2006) which has eight criteria that the problem must meet, but we consider only three of which are:

- **Event**. This refers to the event described in the task. In simulating a real-world situation, it is a prerequisite that the event described in the task has occurred or has a fair chance to occur.
- **Question**. This aspect refers to the agreement between the assignment given in task and in a corresponding formal situation. The question in the task is one that might actually occur in the real world event described is a prerequisite for a situation that exists in the real world.
- **Information/data**. This refers to information and data in the task and includes values, models and given conditions.

If the problem does not meet one of these criteria it cannot be considered as authentic. The problems selected had an image to help understand the context described in the text. This is to analyze whether the author can give a good picture of the situation and thus help students to understand the context of the problem and can solve it correctly.

Most problems discussed do not meet the theory of Palm (2006). The authors of this textbook do not consider the authenticity of problems as the contexts shown do not meet Palm's criteria. Also displayed images do not help because, in some cases, they do not show the situations described.

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EXAMINING HOW MATHEMATICS CURRICULUM MATERIALS ENCOURAGE STUDENT PERSEVERANCE

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The first Common Core State Standard (CCSS) for Mathematical Practice requires students to "make sense of problems and persevere in solving them" (CCSS, 2010, p. 6). While engaging with mathematical tasks, perseverance requires tenacity, the use of good strategies, and the flexibility to alter those strategies when necessary (Middleton et al., 2015). These components of perseverance help students overcome inevitable learning obstacles as they struggle to make sense of important mathematical ideas that are not immediately apparent (Hiebert & Grouws, 2007).

Curriculum materials can play a substantial role in nurturing student perseverance by providing opportunities for initial and sustained engagement with challenging mathematical tasks. Perseverance is not a rigid character trait; instead, students can become more perseverant by engaging with difficult mathematics tasks (Bass & Ball, 2015). Since the textbook is often the chief object of students' engagement with mathematics (Reys, Reys, & Chavez, 2004), curricular tasks should include features that can cultivate student perseverance. Such features include low-floors/high-ceilings, collaboration opportunities, and perceptions of student autonomy.

This study's purpose is to consider how CCSS-aligned mathematics curriculum materials support perseverant learners. Two analogous Algebra 1 textbooks are examined for how they offer opportunities to foster perseverance via student tasks. These opportunities were measured by targeting specific curricular features of mathematical tasks. The analytic framework identified task features such as low-floor/high-ceiling (*does the task invite student engagement and/or require higher-level cognitive demands?*), collaborative learning (*does the task explicitly incorporate group work?*), and perceptions of autonomy (*does the task allow for student choice?*).

The results indicate that though each curricula self-reported CCSS-alignment, the texts varied greatly in perseverance development opportunities. These findings suggest that textbook tasks do not solely provide opportunities for perseverance development through student mathematical tasks and teachers may need to provide additional supports. Also, claims about CCSS-alignment may be focused on the content standards alone and more precise alignment guidelines for the Standards of Mathematical Practice are needed. Thus, curriculum developers may need to provide more explicit guidelines as to how mathematical tasks can cultivate perseverant learners.

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THE IMPORTANCE OF A QUANTITATIVE REASONING EDUCATION

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Jesse Wilkins (2000) stated that, in the 21st century, the constitution of a person's functional literacy must extend beyond reading and writing, and include the ability to cope with quantitative information. Quantitative reasoning, also known as quantitative literacy, can be defined as "the skill set necessary to process quantitative information and the capacity to critique, reflect upon, and apply quantitative information in making decisions" (Gaze et al., 2014, p. 3).

Traditionally, college students were required to take algebra-laden courses to meet the math core curriculum requirement. But, in the past decade institutions have been shifting the focus of introductory/general education math courses toward Quantitative Literacy/Reasoning, or QLR, as an alternative to the algebra sequence for Liberal Arts students (Gaze et al., 2014). With research indicating the need of a quantitative reasoning education for individuals to function within our society (Madison & Steen, 2003; Steen, 2001; Steen, 2004), are we just to assume the STEM students who are in the calculus pipeline are somehow indirectly obtaining these skills? The questions this study intends to address are:

1. Is there a difference in quantitative reasoning ability in Liberal Arts students taking quantitative reasoning and STEM students completing the algebra sequence?
2. To what extent does taking a quantitative reasoning course explain students' QR skills controlling for gender, race, degree major, and other math courses taken?

Participants were selected from one campus of a mid-size college in central Florida with an open enrollment policy and a very diverse student population. Two intact classes of Calculus 1 and two intact classes of Liberal Arts Math in the Fall 2015 were selected. The instrument used in this study was the Quantitative Literacy & Reasoning Assessment (QLRA), an instrument that is both valid and reliable for measurement of quantitative literacy (Gaze et al., 2014).

The mean score for the Calculus 1 students was 37.78% (n = 36), while the mean score for the Liberal Arts Math students was 26.84% (n = 49). A two-tailed t-test determined that there is a statistically significant difference in the averages on the assessment ($p < 0.0014$). Multiple regression was performed to determine the extent a quantitative reasoning course explains a person's ability to reason quantitatively, controlling for gender, race, years in school, degree major and other math courses taken. It was found that taking a quantitative reasoning course was not a significant factor, while gender was significant, with women having a 14% lower score than men. It is the researcher's belief that the student's attitude towards mathematics might have played a significant role in the findings, as many students in the Liberal Arts classes did not take the assessment seriously.

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LA FUNCIÓN LINEAL EN EL BACHILLERATO TECNOLÓGICO: UNA MIRADA DESDE SU IMPLEMENTACIÓN

LINEAR FUNCTIONS IN TECHNOLOGICAL HIGH SCHOOLS: A VIEW SINCE ITS IMPLEMENTATIONS

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Keywords: Currículum, Educación Media Superior, Conocimiento del Profesor

Por sus múltiples aplicaciones, el concepto de función es fundamental en el estudio de cualquier rama científica. En el caso específico de las funciones lineales, las crecientes investigaciones revelan que los investigadores están reorientando su atención no sólo a explorar sobre las dificultades que tienen los estudiantes para comprender a las funciones lineales, como lo señala Birgin (2012), sino que también están preocupados por las concepciones de los profesores (Lloyd y Wilson, 1993) y por el uso de materiales curriculares (Chavéz, Grouws, Tar, Ross y McNaught ,2009).

Esta investigación busca articularse a los estudios que sobre función lineal se están desarrollando en el área de la matemática educativa considerando un punto de vista curricular, de ahí que la pregunta de investigación indaga ¿Cuáles son las transformaciones o cambios que tres profesores de matemáticas de bachillerato tecnológico generan al enseñar función lineal en el curso de Pensamiento Algebraico y de Funciones?

El estudio utilizó como marco teórico el modelo curricular de Stein, Remillard y Smith (2007) que incluye cuatro componentes: currículum escrito, planeado, implementado y el aprendizaje de los estudiantes y sólo los tres primeros componentes son utilizados en la investigación, ya que el estudio se centra en el profesor. Se empleó el estudio de casos como método para la recolección de los datos, lo cual incluyó la revisión tanto del plan de estudios oficial, las planeaciones de clase y la videograbación de clases. Los resultados obtenidos revelan que los profesores realizan transformaciones tanto en el currículum planeado como en el currículum implementado. Las transformaciones son generadas principalmente por dos razones: La primera tiene que ver con que el plan de estudios oficial, no precisa qué de función lineal debe ser enseñado y la segunda, está relacionada con la formación académica de los 3 profesores, implicando así, que agreguen u omitan contenidos relacionados con la función lineal, o que modifiquen el concepto.

Keywords: Curriculum, High School Education, Teacher Knowledge

Due to its variety of applications, the concept of linear function is essential to study any of the science fields. In the particular case of linear functions, increasing research shows that researchers are redirecting their attention not only to the exploration of the difficulties students face to understand linear functions, as it is stated by Birgin (2012), but also to teachers' conceptions (Lloyd & Wilson, 1993) and the use of curricular materials (Chavéz, Grouws, Tar, Ross, & McNaught, 2009).

The aim of this research is to connect to the studies that have been developing on linear functions in mathematics education, taking into account the curricular point of view. Hence the question that this research study investigates is: What are the transformations or changes that three mathematics teachers in Technological High schools make when teaching linear function in the course called "Algebraic Thinking and Functions"?

The research used as a theoretical framework the curricular model of Stein, Remillard, and Smith (2007) that includes four components: written, planned and implemented curriculum as well as the students' learning, but only the first three components are used in this investigation, since this research is focused on the teacher. The case study was used as a method for data collection, which also included the review of the official school curriculum, lesson plans and class video recordings. The results obtained show that teachers make changes in the planned curriculum as well as in the implemented curriculum. These changes are made for two main reasons: The first one is related to the official school curriculum, which does not specify what must be taught about linear functions, and the second one is related to the academic training of the three teachers, which implies that they add or omit content related to the linear functions, or even change the concept.

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HOW WELL ALIGNED ARE COMMON CORE TEXTBOOKS TO LEARNING TRAJECTORIES IN GEOMETRY?

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Recently, many researchers have been paying much attention to Learning trajectories (LT) in mathematics education, especially in advancing recommendations for standards, curriculum, and instruction (Clements, Spitler, Lange & Wolfe, 2011; Wilson, Sztajn, Edgington & Myers, 2015). Since LT is about students' natural progressions in specific mathematical domain, researchers believe that it is more effective, efficient and generative for the students to have learning that consistent with such natural developmental progressions than learning that does not follow these paths (Clements & Sarama, 2004). One important component of LT is designing sequence of instructional tasks that correspond to the order of LT (Clements & Sarama, 2004). Clements (2007) developed Curriculum Research Framework (CRF) and stated that having tasks sequenced according to LT can avoid the fragmentation common in U.S. textbooks. Thus, it is beneficial to have mathematical tasks in textbooks aligned to LT. The purpose of this study is to examine alignment of mathematical tasks in Common Core textbooks to the specific LT. This study uses the Surveys of Enacted Curriculum (SEC) which was previously used in textbook and curriculum analysis (Porter et al., 2012; Polikoff, 2015). Among many topics in mathematics, we have selected area and volume lessons because geometry topics have connections to other areas of mathematics and to students' experiences with the physical world (Sarama & Clements, 2009).

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MATHEMATICS-CONCEPT AND SECOND-LANGUAGE DEVELOPMENT_VISUALLY

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The Pedagogical Challenge

How can a visual game improve mathematics and language learning for English-language learners (ELL)? This APS-funded exploratory study provides educators with the opportunity to learn ways to capitalize on the visual-learning features of ST MATH, an innovative game-based curriculum, to support mathematics and literacy development for second-language learners.

Conceptual and Theoretical Perspective

Cognitive and semiotic theories of mathematics learning contend that the human mind understands abstract mathematical ideas through concrete representations. Yet, a common approach to mathematics learning for second-language learners is to introduce mathematical concepts through words. Building on the success of Warren, Harris, and Miller (2014) in developing teaching methods that support mathematics and literacy development for Australian ELL students, this study hypothesizes that order of instruction – visual experience of concepts before verbal – favors both mathematics-concept and language learning.

Research Design, Data and Analysis

How can teachers use ST Math games to support students' mathematics-concept and second-language development? How does the order of ST Math Game instruction influence student learning and teachers' use of ST Math in the classroom? To investigate these questions the study proposes a cycle of instruction that includes individual, pairs, and whole-class discussion of the math concepts embedded in the game but varies the order of ST Math game instruction. Experimental teachers ($n = 2$) will use an ST Math game to *introduce* a new fraction concept whereas Control teachers ($n = 2$) will use the target ST Math game(s) for *practice and extension* of the chosen study concept. In the first instance, ST Math is a *concept development* tool, and in the second, an *assessment, practice, and elaboration* tool. Data include instructional and student interview videos; pre/post student fraction test, attitude survey, teacher surveys; fraction lesson plans; and researcher notes. We propose open coding and frameworks related to question types and teaching moves for analyzing videos of ST Math teaching and interviews with ELL students.

Summary of Findings

Preliminary findings of this ongoing study challenge existing theoretical and pedagogical boundaries in the fields of mathematics education and second-language learning. Early results favor teaching approaches that explore math concepts and foster language development by using a variety of concrete and virtual math representations in conjunction with – but prior to - oral language over teaching approaches that focus on repetition and acquisition of math vocabulary.

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THE IMPACT OF DIRECT INSTRUCTION ON MARGINALIZED, HIGHLY MOBILE STUDENTS

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Marginalized youth score significantly lower than other students on standardized tests with mathematics often acting as a gate-keeper (Schoenfeld, 2004). At the forefront of this concern is a debate over which curricular strategy is best for these students: direct instruction or a reform model of instruction. While many schools have adopted conceptual-based approaches, a gap in access to reform teaching exists between lower SES, and/or minority students and their higher SES peers (Desimone & Long, 2010). A better understanding of the various inequities connected with access to reform instruction is especially crucial for highly mobile youth, as high mobility can also hinder educational progress (Ream, 2003). Few studies have explored this intersection. Thus, this poster examines the learning experiences of one highly mobile, marginalized youth.

I adapted an analytic framework to foreground students' mathematical empowerment (Alsop, Bertelsen, & Holland, 2006). A key component of this framework is the importance of opportunity structures (i.e., curricular strategies). Using interpretative methods, I analyzed one student's recalled experiences of curricular structures in her classrooms, specifically direct or procedural forms of instruction. Data sources included eighteen study sessions, pre- and post-interviews, and school records. I coded talk related to pedagogy and mathematical learning in each interview, wrote analytic memos across the data, and sought to understand her mathematical empowerment as connected to the curricular structures she recalled experiencing.

In this poster, I present the mathematical learning experiences of Sofia, a 16-year-old highly mobile Mexican-American sophomore. Like many marginalized students, Sofia reported experiencing only the opportunity structure of direct mathematics instruction, which led her to develop an understanding of mathematics as a speed-based, procedural discipline. For example, drawing from her own experience, she explained that students often "have to *hurry up* and try to learn everything *fast* and get it all wrong just *to keep up* with the rest of the people" (line 670). When Sofia became highly-mobile, starting in eighth grade, this understanding of mathematics became especially problematic. She explained, "I learned one math, how to do it one way, and the other teacher [at the new school] was like 'that's not how you do it,'" which led Sofia to ask, "Which is the right work, you know?," (line 654). Sofia felt she had to learn all new procedures, essentially starting over again learning mathematics when she transferred schools. Further research is vital, to enhance our understanding of the interrelationship, and resulting outcomes, of school choice and curricular strategies marginalized, highly mobile youth experience.

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OPERATIONALIZING EDUCATIVE GUIDELINES FOR CHILDREN'S MATHEMATICAL THINKING IN ELEMENTARY MATHEMATICS CURRICULUM

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Ball and Cohen (1996) made the case for positioning curriculum materials as potential agents for impacting teacher learning as, in their view, curriculum materials are "concrete and daily", "scaled up", and have "reach" in the system (p. 6). Davis and Krajcik (2005) then put forth a set of high-level guidelines for the design of educative curriculum materials. Drake, Land, & Tyminski (2014) built upon the work of Ball and Cohen and Davis and Krajcik to address how teacher educators can support prospective teachers' (PTs) reading and use of educative curriculum materials in ways that support them in acquiring the knowledge needed for teaching. Because the educative guidelines set forth by Davis and Krajcik (2005) were not specific to mathematics, examination of how these guidelines might be evident or applied within mathematics curriculum is necessary in order to further research on supporting PTs' curriculum use. We focused on the first educative guideline only – "educative curriculum materials could help teachers learn how to anticipate and interpret what learners may think about or do in response to instructional activities" (Davis & Krajcik, 2005, p. 5).

Methodology and Results
We used the literature base to outline a preliminary frame for the first educative guideline in terms of mathematics curriculum. We used our framework to determine: 1) lesson features indicative of educative guideline one; 2) the manner in which they were educative; and 3) how these features might be modified to address other aspects of the guideline in three lessons. This process allowed us to operationalize the first educative guideline.

Discussion and Implications
Operationalizing Davis and Krajick's (2005) first guideline for mathematics allows the field to better understand and support PTs' reading of educative curriculum features. Specifically, our framework could be used as a tool for identifying how a feature is intended to be educative specific to mathematics. This identification could support PTs in orienting to curriculum materials and could support researchers in understanding PTs' reading of curriculum.

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CONFLICTING VISIONS OF GOOD TEACHING: NEW CURRICULUM, TEACHER BELIEFS, AND TEACHING PRACTICES

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Keywords: Curriculum, Teacher Beliefs, Elementary School Education

In this study, I examined the relationship between new curriculum, teacher beliefs, and teaching practices. Along with the development of a new mathematics curriculum at the national level, new Korean textbooks aim for students' conceptual understanding through engagement in mathematical processes such as reasoning, problem solving, and communication. However, no one can simply say that the curricular intentions could be translated into purposive change in instructional practices. The possibility of this discrepancy is related to teachers because the teachers are responsible for designing their lessons. Therefore, I attend to the participatory relationship between curriculum and a teacher focused on beliefs (Remilliard, 2005).

In particular, I investigated the gap between an intended and enacted curriculum with a particular focus on teacher beliefs and teaching practices. Teachers' beliefs should be worthy of notice, considering their influences on mediation of knowledge, pedagogical decisions and implementation, and teaching practices (Thompson, 1992). It is also necessary to consider the influence of teachers' varying contexts on teaching practices (Kennedy, 2010). These all contributed to developing the conceptual framework about the interactions among espoused and enacted beliefs, an intended and enacted curriculum, and teaching practices.

This study employed a case study design with three elementary teachers, using a survey, interviews, and classroom observations. For data analysis, I relied on the literature for five framing codes: nature of mathematics, mathematics understanding, mathematics learning, teacher's role, and use of curriculum. Based on these framing codes, I created focused codes through opening coding from each teacher's beliefs, teaching and curriculum. And then, with the focused codes, I analyzed their (in)consistencies along with why they occurred. The primary means for this analysis involved moving between conjectures and refutations.

This study has three preliminary findings. First, teachers had similar beliefs of mathematics understanding but different conceptions of mathematics learning. Second, the curricular intentions were not embodied in teaching practices when teachers failed to recognize the intentions in spite of positively evaluating and rigidly following textbooks. Third, teachers had challenges such as lack of knowledge, students' low academic level, and educational atmosphere overemphasizing test outcomes, which places them on conflicting visions of good teaching within or between their beliefs, the curriculum, and teaching practices. It is recommended for teachers to have an opportunity of learning to teach to lessen challenges of the high-quality instruction that the new curriculum intends.

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STUDENT TRACKING USING DIFFERENTIATED EXAMINATIONS

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Keywords: Assessment and Evaluation, High School Education, Equity and Diversity

Student tracking or ability grouping in mathematics has been of interest to researchers over several decades. Specific in areas of study have been with respect to student achievement (eg: Ireson, Hallam, Hack, Clark & Plewis, 2002), student self-esteem (eg: Van Houtte, Demanet, & Stevens, 2012), and equity (eg: Forgasz, 2010). In addition, mathematics has been identified as a gatekeeper (Sriraman, & Steinthorsdottir, 2007) for students entering post-secondary programs. Equity concerns have been raised over when and should students be tracked in mathematics (Chmielewski, 2014) and if tracking students is contributing to inequity of access to post-secondary programs. The poster describes teachers' use of differentiated examinations to support students in selecting a mathematics stream. The main research question was: Can differentiating summative unit examinations assist students in choosing the mathematics course (stream) that best suits their abilities and interests?

This project took place in a Western Canadian province where, in Grade 11, students can choose one of two streams of general mathematics courses: pre-calculus or foundations in mathematics. The teachers in this study integrated the two courses to begin the semester for the three units of overlapping content and provided students with differentiated unit examinations. The purpose of the differentiated examinations was to allow students to experience the type of summative assessment that was expected in each stream. The poster will illustrate the different kinds of questions that were included on each part of the examination.

Based on a student survey, post-secondary requirements over-rid all other potential influences on student choice of mathematics course. Students who were unsure of which course to take, generally appreciated the opportunity to experience the different types of assessment for each course. Students that had already made a firm decision as to which math course they needed to take felt slightly disadvantaged by having the first three units of the course not specifically tailored to their needs. Differentiating examinations can play a role in assisting students to choose an appropriate mathematics course for them; however, the examinations are not the most influential factor in student choice.

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EXAMINING "VOICE" IN REFORM REMEDIAL MATHEMATICS TEXTS: A COMPARATIVE LINGUISTICS ANALYSIS

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Keywords: Curriculum Analysis, Equity and Diversity, Post-Secondary Education

In this study, a critical discourse analysis of textual features was conducted to answer the question: In what ways do the linguistic and structural features of Statway and the University of Maryland's (UM) remedial mathematics curriculum speak to and position developmental mathematics students in the introductory units of study? Beth Herbel-Eisenmann's text analysis framework (2007) guided the analysis. Interpersonal, ideational, and textual functions were examined to identify the potential of the intended curricula (Halliday, 1978; Remillard, 2005). Word counts and word searches were used to identify imperatives, modality and pronouns within the curricula. Images, cohesiveness and nominalizations were also examined. The results of the analysis suggest that Statway has greater potential to position students as members of a mathematics community. This positioning may be further encouraged by the way students are intended to experience the curriculum, through group work and relationship building. In addition, the text features suggest Statway has the potential to promote student agency by engaging them in tasks that represent and model human beings who use mathematics to make sense of the world they live in, and by calling on the reader to do so as well. The cohesiveness of Statway implies that mathematics is a collection of interrelated concepts that are useful in making sense of and solving relevant issues.

UM's remedial curriculum could potentially send conflicting messages to students about their role as learners and the role mathematics plays in their daily lives. The use of predominantly exclusive imperatives and limited expression of human agency could potentially lead developmental mathematics students to conclude they are artificially being inducted into a mathematics community in order to follow expected rules. This perception may be reinforced through the student's experience of the curriculum, where they work independently through their materials in a computer lab. The lack of cohesiveness could potentially lead developmental students to infer that mathematics is a set of discrete facts which must be followed, but don't necessarily connect to their lives or their ability to make sense of the world around them. Findings and suggestions for improving both texts, as well as suggestions for future research will be discussed in this poster presentation.

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UNSPOKEN MESSAGES: AN ANALYSIS OF IMAGES USED IN THE GRADE 1-2 INTEGRATED KOREAN CURRICULUM

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In this study, we examined images used in a mathematics curriculum to understand how the curriculum is supporting students' learning. Textbooks as curriculum materials have been considered an influential factor for school mathematics because they offer a blueprint for teaching and learning of mathematics in classrooms. Recently Korea reformed mathematics textbooks that introduce storytelling including stories and images. Storytelling aims for effective mathematics instruction by providing meaningful mathematics contexts that connect mathematics with students' personalized situations (Davis-Dorsey et al., 1991). Moreover, given the diversity of students, it is necessary to give attention to the possibility that textbooks are oppressive to certain groups of students in terms of cultural identity (Kumashiro, 2008). In this study, we attend to the role of curriculum that shapes students' learning with a particular focus on the images attached to the stories used in the chapters in a reformed Korean mathematics curriculum for grade 1-2, which consists of 23 chapters.

The curriculum does not provide a written version of the story. Students have to rely on teachers' reading of the story and the image. Because of students' direct contact with the image rather than the story, we carefully examined the image used. We followed Dowling's (2002) classification of signifying modes, referring to "a form of the relationship between expression and content that is implicated in sign production" (p. 151), such as an *icon*, an *index*, or a *symbol*. An index is abstract enough so that it does not signify the reader's presence, but it allows the readers to immediately do mathematics with it. In addition, we examined the setting of the story. Dowling suggested a loose system for classifying the space for social practice in mathematics textbooks. This system includes domestic settings, work settings, school settings, and travel settings. We add here imaginary settings and literature settings.

From the analysis, we learned that seven chapters were indexical. Three indexical representations were similar in that they all presented geometric shapes. Four chapters arranged objects in rows so that students can start counting them. When it came to the settings, five chapters were imaginary settings, four were literature settings, three were school settings, three were domestic settings. Some chapters had two overlapping settings. People in all chapters, except for the three chapters with literature settings, had dark hair and black eyes, which are typical to Koreans. The three chapters using literature settings presented European people. The curriculum may want to consider using indexical representation with topics other than geometric shapes. In addition, the curriculum could consider diversifying ethnicity of people in the images.

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INCORPORATING MODELING INVESTIGATIONS TO ENHANCE TRADITIONAL ALGEBRA I MATERIALS

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Keywords: Algebra and Algebraic Thinking, Modeling, Teacher Beliefs

Given the vast array of resources now available to teachers, understanding how teachers select supplementary materials to enhance their existing programs can inform the work in teacher education. This case study looks at issues that arise when a teacher selects and incorporates lessons designed for student-centered learning into the school's program. The poster will share findings from the study on an eighth-grade teacher's incorporation of selected investigations, chosen from a curriculum designed to promote a modeling approach to algebra, to supplement lessons from the school's traditional Algebra I program. Data were collected from the selected modeling lessons, the planning and adaptations the teacher incorporated into the lessons prior to teaching them, questions the teacher asked to engage students in the investigations, and teacher reflections on the impact of the lessons. In particular, questions posed by the teacher during instruction of the selected investigations were analyzed relative to the modeling cycle as described in the high school modeling standard of the Common Core State Standards (NGAC & CCSSO, 2010) to provide insights into challenges faced by a teacher juggling curricular choices.

The algebra modeling curriculum was designed and developed to be a separate companion course to support ninth-grade students concurrently taking Algebra I (Olson et al., 2015). Technology provides the opportunity for interaction with dynamic representations of concepts during classroom instruction and can be integrated throughout the curriculum both for teacher and student use. The lessons, built around problems, emphasize the use of models, promote the investigation of open-ended questions, and provide multiple opportunities for students to develop concepts, generalizations, and skills. Students are given opportunities to model, represent, graph, write about, and discuss their strategies for investigating and solving problems as they begin to internalize algebraic ideas and develop an understanding of algebraic techniques.

Using problems selected from the modeling curriculum appealed to the study's middle school teacher as a way to supplement the school's newly adopted program that emphasized procedures and algorithms rather than conceptual understanding. Patterns emerged from preliminary analyses of teacher-adapted worksheets, discussion questions posed, and teacher reflections. The data suggest that while the investigations selected by the teacher were designed to emphasize problem solving and modeling, the teacher provided more guidance, often in the form of added structure, for students to understand problem contexts and to consider specific models. Constraints to keep pace with the required Algebra I textbook, a limited availability and use of technology, and no professional development support for the teacher are factors likely to have contributed to less emphasis on the modeling aspect of the lessons outside the intended use.

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THE TEXTBOOK EXPERIENCE: AN ANALYSIS OF CONTEXT EXPOSURE IN CORE-PLUS MATHEMATICS TEXTBOOKS

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Keywords: Curriculum, Curriculum Analysis, High School Education

The Core-Plus Mathematics Project [CPMP] is a National Science Foundation funded reform based curriculum. The CPMP embraces mathematical learning in context and has been shown to have comparable performance results for students in most areas when compared to a traditional high school mathematics sequence (Schoen & Hirsch, 2003). The overview section of the CPMP 2015 edition's teacher's guide, states that "making real-world contexts accessible to students promote[s] greater access and equity in mathematics classrooms." Expanding on the notion that context is important for accessibility, there is a benefit in examining the contexts from that perspective. Macintyre and Hamilton (2010) discovered that students can be aware of disparities and perceive inaccessibility not immediately apparent to the authors of the curriculum.

In an effort to analyze the accessibility of the CPMP curriculum, an analysis using a five-part "No Context," "Factual," "Topic," "Location," and "Character" framework, was conducted on all 804 "On Your Own" problems in Course 1B and Course 2A. The five-part framework was adapted from one used by Sleeter and Grant (2011) in their evaluation of textbooks across different disciplines. The rationale for using this framework was that each dimension has the potential to affect the accessibility of problems from the students' perspectives.

The analysis revealed that on the *context* dimension, 303 (38%), of the problems analyzed were contextually-based problems (CBPs) (i.e. problems placed in a real-world contextual scenario). On the *factual* dimension, 24% of the CBPs included cited references for the information presented. On the *topic* dimension, a wide variety of topics were included. On the *location* dimension, 75 CBPs mentioned the United States, and 40 of them were further categorized by region; 23 CBPs were placed in the Northeast or Midwest while only 6 mentioned the Southwest or Southeast. Finally, on the *character* dimension, the CBPs included more references to students than to professionals. Furthermore, 13 of the 22 CBPs involving individuals where race could be definitively determined, the subjects were Caucasian. In every instance where a historical figure was mentioned by name, the individual was a white male.

In summary, despite the emphasis on context to promote accessibility, the results of the analysis suggests that the features of the contexts used in the CPMP curriculum may not reflect all students' experiences. Gaining awareness of potential disproportionate representations of student experiences, including location and ethnicity, may guide future efforts to improve the accessibility for all students in future iterations of the CPMP curriculum.

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Chapter 3

Early Algebra, Algebra, and Number Concepts

Posters

BENEFITS OF ANALYZING CONTRASTING INTEGER PROBLEMS: THE CASE OF FOUR SECOND GRADERS

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In this study, we explore four, second graders' performances on integer addition problems before and after analyzing contrasting cases involving integers. The students, as part of a larger study, participated in a pretest, small group sessions, one short whole-class lesson on integer addition, and a posttest. Based on their integer mental models and scores on arithmetic and transfer problems, each student progressed, although in different ways. We use these instances and their interactions in their group sessions to describe their progressions.

Keywords: Number Concepts and Operations, Elementary School Education, Instructional Activities and Practices

Students start learning about negative numbers and interpreting integers in real-world situations (e.g. temperature, account balance) in sixth or seventh grade; yet, they learn whole number concepts starting in kindergarten (National Governor's Association Center for Best Practices & Council of Chief State School Officers, 2010). During this large time gap, students solidify ideas about numbers and addition based on whole number experiences, and when they encounter negative numbers, they try to make sense of them using this prior knowledge in various ways (e.g. treating negative numbers as positive numbers or zero) (Bofferding, 2014). Helping students bridge the gap between whole numbers and integers by providing them with helpful instructional experiences is important.

One possible way to bridge the gap is to shorten it by introducing negative integers earlier. There is recent evidence that young children are capable of learning about and using negative numbers even in the first grade (e.g., Bishop, Lamb, Philipp, Schappelle, & Whitacre, 2011; Bofferding, 2014; Behrend & Mohs, 2005/2006). Another possible way is to help students make connections between the two types of numbers. Findings by Gentner (2005) suggest there is evidence that learners, who have opportunities to compare two analogous cases, are more likely to succeed at a "relational transfer" than those who have not compared the cases (p. 252). However, prior investigations in students' learning from contrasting cases have mostly focused on older students (e.g. Rittle-Johnson & Star, 2011).

In a larger study, we explored the potential of having second graders analyze different contrasting cases to see if it could help them learn about negative integer addition. Through case studies of two pairs of students, we aim to address the following research questions:

1. How do four, second graders' integer mental models and performance on integer addition problems change after analyzing contrasting cases of integer addition and participating in one lesson on the topic?
2. In attempting to account for these changes, what will the students notice and describe during the analysis of the contrasting cases?

Theoretical Framework

Integer Mental Models

Bofferding (2014) categorized first graders' order and value mental models into initial, synthetic, and formal, along with two transitional levels, which exhibit parts of both. Students with initial integer mental models ignore negative signs, operate with negative numbers as if they were positive

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

numbers, or order them correctly but treat them as having positive values (Bofferding, 2014). As they transition away from an initial mental model, some students give new meaning to the negative sign and treat it as a minus (transition I mental model). Therefore, they sometimes treat negative numbers as zero (as if the number is subtracted from itself) and sometimes treat them as positive numbers (numbers that have not been taken away yet). This reflects a focus on the binary (or subtraction) meaning of the minus sign (Bofferding, 2014; Vlassis, 2004).

Students with synthetic integer mental models acknowledge that negative numbers are less than zero but consider larger negatives as "more" (Bofferding, 2014). This focus on absolute value may lead them to reverse the order of negatives. Before developing a formal mental model for integers, some students will both consider negative numbers with larger absolute values as greater than smaller ones and treat them as less, depending on the tasks (transition II mental model) (Bofferding, 2014). As students move toward having a formal mental model, they develop a better understanding of the unary meaning of the minus sign (Bofferding, 2014; Vlassis, 2004).

Contrasting Cases

The key differences among the integer schemas described above are the meanings given to the symbols involved (i.e., the negative sign) and the relations established between order and value. One method for students to notice important relations in problems is by analyzing contrasting cases (e.g., Rittle-Johnson & Star, 2011; 2009, 2007). For example, students who analyzed contrasting alternative solutions in algebraic equations showed better improvement in their procedural knowledge than students who analyzed solution methods presented sequentially (Rittle-Johnson & Star, 2007, 2009). In one case focused on algebraic manipulations, three different comparison groups compared equivalent problems, different solution methods, or different types of problems. Students who compared solution methods gained greater procedural flexibility and conceptual knowledge than students in the other two groups (Star & Rittle-Johnson, 2009). Durkin and Rittle-Johnson (2012) also found that providing students with opportunities to compare incorrect and correct answers can help students improve their understanding of new concepts. Further, others found that even students with low performance on a pretest who later compared correct and incorrect examples of equations improved more in conceptual knowledge on a posttest (Lange, Booth, & Newton, 2014). In this study, we investigate second graders' understanding of integer addition before and after they analyze contrasting cases on the topic.

Methods

Participants & Setting

The larger study, from which this data is drawn, took place at two public schools in a rural mid-western district where about 32.2% of students are English-language learners and 75.2% qualify for free or reduced-lunch. Students were recruited from the second grade population, and 109 of the original 110 participants completed all aspects of the study. After a pretest, students were assigned randomly to groups that analyzed different types of contrasts.

For this study, we chose 4 students to investigate more closely. Because students in the larger study made sizeable gains from pretest to posttest, we chose the two initial case study students to exemplify this change. First, we narrowed the pool to students who had gained at least 30% from pretest to posttest. This resulted in 9 students out of 36 from the intuitive contrast group, 13 students out of 35 students in the conflicting contrast group, and 6 students out of 36 students in the same type, sequential group. We chose to look at students in the intuitive and conflicting contrast groups since they had made slightly more gains than those in the sequential group. We then selected one student from each of the remaining two groups who had 60% or above correct on the session

questions and had a consistent partner across the small group sessions. We also included their consistent partners, who had similar pretest scores.

Design and Materials

    Pretest. The pretest included three ordering questions (two involved filling in missing numbers on a number path and number line; we ignored the third because students had trouble interpreting it). Students also solved sixteen temperature comparison questions (eight where they circled the hottest temperature and eight where they circled the coldest) that targeted their understanding of integer values and four directed magnitude questions. For the arithmetic questions, they solved four positive integer addition and subtraction questions and then fourteen integer addition questions. Finally, they solved two types of transfer problems: three integer addition problems with three addends and six missing-addend addition problems. Overall, the highest possible score for the pretest was 50.

    Small group sessions. After students were randomly assigned to contrast group, they participated in 2 small group sessions. During the small group sessions students analyzed sets of contrasting integer addition problems with accompanying illustrations (both solved correctly and incorrectly) and talked about what was similar and different in the problems and pictures. They also talked about why some of the answers were wrong and how they could use the pictures to solve the problems. The illustration contexts involved (a) a gingerbread man moving on a number path, (b) ants going above and below ground next to a vertical number line, (c) positive and negative chips, and (d) a folding number line (see Tsang, Blair, Bofferding, & Schwartz, 2015). For some of the contrasts, students had to write down their thoughts without discussion, and on others, students talked about their answers aloud. The contrasts the two groups analyzed are shown in Table 1. Each intuitive contrast and conflicting contrast group session took about 20 minutes. At the end of each session, students solved 6 related integer addition problems for a total of 12 problems by the end of both sessions.

Table 1: Contrasts that the Students Explored in their Groups

Illustration Context	Intuitive Contrast, Session 1	Conflicting Contrast, Session 1	Intuitive Contrast, Session 2	Conflicting Contrast, Session 2
Gingerbread	3+5 vs. -3+-5	3+5 vs. 3+-5	-2+5 vs. 2+-5	-2+-5 vs. -2+5
Ants	2+7 vs. -2+-7	2+7 vs. 2+-7	-3+6 vs. 3+-6	-3+-6 vs. -3+6
Chips	6+4 vs. -6+-4	6+4 vs. 6+ -4	-5+5 vs. 5+-5	-5+-5 vs. -5+5
Folding number lines	5+3 vs. -5+-3	5+3 vs. 5+-3	-7+2 vs. 7+-2	-7+-2 vs.-7+2

    Whole-class instruction. After all students completed the two sessions, they participated in a 30-minute, whole-class lesson on negative numbers and addition problems with integers. Students learned that adding a positive number to an integer corresponds to moving the gingerbread boy up on the number path, and adding a negative number corresponds to going down on the number path. For example, to solve -2 + 2, the gingerbread boy first moves down 2 (more in the negative direction) and then up 2 (more in the positive direction), ending up at 0, where he started. This led to the introduction of the term "zero pair." Students explored a few more zero pairs on the number path and then played a card game based on zero pairs with a partner. Students had a stack of -1 cards and a stack of +1 cards (much like with positive and negative chips) and a dice with negative and positive

numbers on it to play with. Students rolled the dice and collected the appropriate cards. Using the cards in their hand, their goal was to make zero pairs and be the first person to run out of cards.

Post-test. The posttest contained the same items as the pretest with a couple of additions. We added 3 questions asking students to determine which integer is closest to 10 and 3 more addition problems with 3-addend numbers where it was advantageous to make zero pairs. The highest possible score on the posttest was 56.

Analysis

To analyze the data, we first classified each students' integer mental models on the pretest and posttest based on Bofferding's (2014) framework. We investigated students' answers on the order and value questions and their use of the minus sign on both pretest and posttest. We also referred to explanations they provided in the groups sessions to clarify their mental models.

Then, we transcribed the group sessions for the case study students to identify their contributions to the discussions and identify how each student interpreted negative numbers in the various contrasts and which meaning of the minus sign they used for their interpretations. Finally, we calculated students' gains on the ordering, arithmetic, and transfer questions across the testing situations. We looked across each student's data from the pretest, sessions, and posttest to build a case of their understanding.

Results

Overview

The larger study analysis shows that there were no significant differences between groups (Bofferding, Farmer, Aqazade, & Dickman, 2016); similarly, in our four sampled cases, the students' gains from pretest to posttest were the same regardless of their group. In the intuitive contrast group, X04 started with 22% correct on the pretest overall, which increased to 78% correct on the posttest. On the other hand, W05, her partner, started with 12% correct on the pretest and only increased to 30% correct on the posttest. Students' gains from pretest to posttest in the conflicted contrast group were similar to those in the intuitive contrast group. In the conflicted contrast group, Z08 started with 40% correct on the pretest and went up to 98% correct on the posttest. However, Z10, his partner, started with 30% correct on the pretest and only went improved to 46% correct on the posttest.

Pretest Mental Models

To better distinguish students' performance, we determined their mental models for the order and value questions and then calculated their percent correct for the 14 integer addition problems (with two addends) (see Table 2). Each group had comparable students who made similar high gains in percent correct over the entire set of questions (X04 gained 56% and Z08 gained 58%) or low gains (W05 gained 18% and Z10 gained 16%).

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Table 2: Case Study Students' Integer Mental Models and Arithmetic Scores (Pre & Post)

ID	Group	Gender	Mental models Shift	Pretest	Posttest
X04	Intuitive contrast	F	T1-T2 (shift of 2)	7%	100%
W05	Intuitive contrast	F	I-I (shift of 0, but progress)	0%	29%
Z08	Conflicted contrast	M	S-F (shift of 2)	29%	100%
Z10	Conflicted contrast	F	I-I (shift of 0, but progress)	36%	64%

I = initial mental model; T1 = transition I mental model; S = synthetic mental model; T2 = transition II mental model; F = formal mental mode (see Bofferding, 2014)

W05 and Z10. In both groups, one of the pair started with an initial mental model. Both W05 and Z10 filled in only whole numbers on their empty number path and number line. Further, they determined which temperature was hottest and coldest based on absolute value. The main difference was that W05 answered "none" for some of the comparisons. Although they both started with initial mental models, Z10 got 36% of the arithmetic problems correct on the pretest. Looking at the problems she got correct, Z10 was able to solve the problems when she could use the negative sign as a subtraction sign. Therefore, she got -1 + 8 correct (can be solved as 8-1), 7 + -3, 5 + -2, 9 + -9, and -4 + 6. W05, on the other hand, answered all of the problems by ignoring the negative signs (e.g., 9 + -9 = 18, -6 + -4 = 10).

X04 and Z08. Both groups also had a student who demonstrated some use of the negative signs on the pretest. X04 filled in positive numbers on both sides of zero when filling in the number path (i.e., 11, 10, 9, 8, 7, 6, 5 4, 3, 2, 1, 0, 1, 2, 3, 4, 5). Further, she answered the temperature comparison questions by focusing on absolute value. All of these responses would suggest that she has an initial integer mental model. However, she frequently treated negative numbers as worth zero on the arithmetic problems. For example, she answered -5 + 5 = 5 and 0 + -9 = 0. Explaining how she solved -9 + 2, she said, "Minus nine is zero then plus two, so there would only be two left." Her treatment of the negative sign as attached the particular number and indicating that the number was worth zero, suggested that she had a Transition I integer mental model. Finally, Z08 had some knowledge of negatives on the pretest. He filled in negative numbers on his number path and number line, but also included -0 (e.g., -2, -1, -0, 0, 1, 2). When completing the temperature comparisons, he correctly determined that positive temperatures would be hotter than negative ones; however, when choosing which negative temperature was hotter, he chose the larger negative. Therefore, he demonstrated a synthetic integer mental model on the pretest.

Students' Performance in Small Group Sessions

W05 (initial mental model) and X04 (transition I mental model). The intuitive contrast group investigated addition problems with two positive versus two negative numbers in the first session and then compared adding both a positive to a negative number and a negative to a positive in the second session. Based on their conversation, X04 and W05 both noticed the negative sign across the sessions; however, they used the binary meaning of the minus sign or called it line. When asked about differences between two problems (5+3 = 8 vs.-5+-3 = -8) and their corresponding pictures in a folding number line context, X04 explained, "That one [is] normal (referring to the initial picture) but that one has a one and a minus (referring to the second picture)." W05 later continued, "These all

there have a minus." In general, W05 primarily focused on the minus sign and made more generic comments, such as indicating that the two problems were not the same rather than pointing to what exactly was not the same. On the other hand, X04 frequently talked about more details beyond just the minus signs. She was more likely to talk about all elements in the problems and to come up with reasons why the information in the pictures matched the problems. On the end-of-session questions, W05 got 6 out of 12 (50%) of the questions correct, and X04 got 12 out of 12 (100%) correct.

Z10 (initial mental model) and Z08 (synthetic mental model). The conflicted contrast group investigated addition problems with two positive numbers versus a positive number plus a negative number in the first session and then compared adding a negative number to a negative versus adding a positive number to a negative number in the second session. Their conversations in the sessions showed that they mostly used the unary meaning of the minus sign and sometimes the binary meaning of minus sign when interpreting negatives. Unsurprisingly, Z08 was more dominant in reasoning about negative integers compared to Z10 and sometimes helped Z08. This occurred when they compared -2 + -5 (where a gingerbread man started at 0, hopped to -2 then hopped -5 more to -7) versus -2 + 5 (where a gingerbread man started at 0, hopped to -2, then hopped 5 to 3). Z10 looked at the ending point on the pictures and said, "This one starts at -7, and this one starts at normal 3." Z08 clarified, "That one actually starts at 0." Z10 then saw that the answers to the problems corresponded to the ending points for the gingerbread man. Z10 tended to focus on the minus as meaning subtraction. When thinking about how to solve -5+5=0 with the positive and negative chips, Z10 explained, "This takes this one (referring to a circled group of -1 and 1) and this one takes this one; these are zeros." On the other hand, Z08 correctly identified negative signs and used them to interpret the ant's movement on the number line when explaining why the ant would go down for 2 + -7: "Cause of negative seven." On the end-of-session questions, Z10 got 3 out of 12 (25%) of the questions correct, and Z08 got 11 out of 12 (92%) correct.

Posttest Mental Models

W05 and Z10. On the posttest, W05 and Z10 from the two different groups still had initial mental models, but they had advanced from whole number to absolute value mental models. Both filled in the number path completely. However, they continued to choose hottest and coldest temperatures by their absolute value. In W05's first session she compared adding two positives versus adding two negatives, and she paid attention to the negative signs. On the posttest, instead of providing only positive answers, she answered all problems by adding the absolute values and making the answers negative. By overgeneralizing the rule that negative problems have negative answers, she only correctly solved arithmetic questions where this strategy led to a correct answer (see Table 3). She was also able to correctly answer -2 + __ = -8 from the transfer problems.

Z10's original session focused on adding two positives and adding a negative to a positive, and she continued to treat negative signs as subtraction signs. Therefore, she performed better on solving arithmetic and transfer problems with 3-addend numbers compared to W05. On the pretest, she solved 8 + -3 + 4 as 8 – (3+4) = 1; however, she correctly answered it on the posttest.

Table 3: Case Study Students' Performance on Arithmetic Problems by Strategies

	Add Absolute Value and Make Negative			Use Negative as Subtraction			Count from Negative		
Test Items	-2 + 0; 0 + -9; -6 + -4; -1 + -7			-4 + 6; -5 + 5; -1 + 8; 5 + -2; 7 + -3; 9 + -9			-3 + 1; -9 + 2; 6 + -8; 4 + -5		
	Pre (n=4)	Mid (n=3)	Post (n=4)	Pre (n=6)	Mid (n=4)	Post (n=6)	Pre (n=4)	Mid (n=2)	Post (n=4)
X04	0%	100%	100%	17%	100%	100%	0%	100%	100%
W05	0%	100%	100%	0%	0%	0%	0%	0%	0%
Z08	100%	100%	100%	0%	100%	100%	0%	50%	100%
Z10	0%	0%	25%	83%	0%	100%	0%	0%	50%

X04 and Z08. Both X04 and Z08 shifted two mental model levels up after posttest. Both correctly filled in the number path and could answer all "coldest" comparisons correctly. When asked to select the hottest temperature with only negative number choices (e.g. -6, -2, -3), X04 selected, "none". Therefore, she demonstrated a transition II integer mental model; whereas, Z08, who got all of the comparisons correct demonstrated a formal mental model. Overall, they both were able to build on their understanding of negatives after analyzing the contrasts and solved almost all of the arithmetic and transfer problems correctly (see Table 4) on the posttest.

Table 4: Case Study Students' Performance in Transfer Problems (Pre & Post)

	Addition with 3-addend numbers		Missing numbers	
ID	Pre (n=3)	Post (n=6)	Pre (n=6)	Post (n=6)
X04	33%	100%	17%	83%
W05	0%	0%	0%	17%
Z08	0%	100%	17%	100%
Z10	0%	100%	0%	17%

Conclusions and Further Study

Contrasting cases helped the students focus on relevant details and develop their understanding of negative numbers to varying degrees. By chance, the students with lower scores had been paired with higher performing peers. This pairing throughout the sessions allowed them to share information with their partner. However, they did not seem to benefit equally from the sessions. In line with their lower background knowledge, W05 and Z10 focused on one main idea from their sessions. W05 attended to using negative signs, and she provided negative answers for almost every integer problem on the posttest (instead of only providing positive answers). Her initial session focused on negative problems only having negative answers, which may have contributed to this tendency. Z10 continued to use the negative sign as subtraction. Interestingly, her initial session reinforced the idea that adding a negative is similar to subtraction. Therefore, she might have made more progress if she had made a different comparison first that made her question her interpretation of the minus sign.

Both X04 and Z08 demonstrated that they treated negative numbers different from positive numbers on the pretest, although Z08 was further along in his understanding. They both benefited

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greatly from their sessions, reasoning in more depth about the problems they were analyzing. Based on these cases, it is not clear if they would have had similar gains if they had been in the opposite group. One of our future goals is to develop cases of all students to determine if there are any trends in students who benefited more from their group versus those who made less progress. This could help teachers determine optimal contrasts for students.

The two students in the conflicting cases group made some gains from the end-of-session questions to the posttest. In the case of Z10, his percent correct on the problems that would be correct if you used the negative as subtraction (e.g., 7 + -3) went down after the sessions and then went back up by the posttest. Z08, made steady progress on the problems that are best solved by counting from negative. These results suggest that especially for that contrast group, the instruction helped them make sense of the problems they had analyzed. Further exploration of the larger dataset will clarify the extent to which this was true for the second graders overall.

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ALGEBRA ACHIEVEMENT GAPS: A COMPARATIVE STUDY ACROSS THE STATES OVER THE YEARS

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The purpose of this study was to examine eighth grade students' algebra achievement scores across various states, over years, and by students' demographics (including ethnicity/race, language spoken at home, SES factors, and academic resources). The questions guiding the study were, (a) What are the differences in eighth grade students' algebra achievement scores across sixteen states between the years 2005 to 2015? (b) Are there significant differences in eighth grade students' algebra achievement scores by student demographics? There were significant differences found in the students' algebra achievement scores based on different students' demographics. Further, the students' mean algebra scale scores continue to rise over the years.

Keywords: Algebra and Algebraic Thinking, Equity and Diversity, Policy Matters, and Middle School Education

Introduction

The move towards "Algebra for All" is based on the notion that success in Science, Technology, Engineering, and Mathematics (STEM) competencies in the 21st century requires fluency in algebraic reasoning and problem solving. Although there is a great need for more students to join the STEM fields, few students pursue these fields in higher education. Scholars have proposed that algebraic thinking should be introduced in the early grades to ease the transition to learning complex algebra (e.g., Blanton & Kaput, 2011; Cai & Knuth, 2011). Indeed, research has shown that it is possible for students in the earlier grades to engage in functional thinking (Warren, Cooper, & Lamb, 2006), and algebraic reasoning (Bjuland, 2012).

The motivation of this study, therefore, is to examine the algebra achievement of eighth- grade students in various states. This comparative study will illuminate the algebra knowledge that students enter high school with as they begin the algebra requirement or their development of early algebraic thinking. Further, it will also illustrate the improvements in early algebra learning over the years. The algebra strand of mathematics is important because of the varying content taught across and within the United States (Kieran, 2007). More importantly, the abstract nature of algebraic reasoning, the language of mathematics using symbols, and structural characteristics associated with learning algebra makes it a challenging strand of mathematics (Rakes, Valentine, McGatha, & Ronau, 2010). Further, considering the adoption of the Common Core State Standards (2010) across the United States, there is the need to examine the influence of this initiative in the algebra achievement of students before they join high school.

The questions guiding this study are, (a) What are the differences in eighth-grade students' algebra achievement scores across sixteen states between the years 2005 to 2015? (b) Are there significant differences in eighth-grade students' algebra achievement scores by ethnicity/race, language spoken at home, socio-economic status (SES) factors, and academic resources? This report contains information on algebra achievement across various states, over the years, and by students' backgrounds. To compare students' algebra scores, we selected states identified as benchmark states in previous cross-national studies (e.g. Trends in Mathematics and Science Studies-2011) and the ten states with available data in the High School Longitudinal Study (HSLS). Using the National Assessment of Educational Progress (NAEP) data from 2005 to 2015, we present the trends in students' achievement in sixteen states.

The findings from this study will be used to inform the next series of studies that will investigate the opportunities to learn algebra related to the differential algebra achievements across the United States. Further, the states in which students' algebra achievement is higher based on students' demographics, and the states that have narrower algebra achievement gaps provide information for policy and practice.

Theoretical Framework

Overall, students' mathematics achievement in the United States has shown a steady improvement over the years in the different grade levels (e.g., Hemphill & Vanneman, 2011; Vanneman, Hamilton, Baldwin Anderson, & Rahman, 2009). However, the variation in mathematics achievement across different groups of students across the United States is prevalent. The studies that focus on mathematics achievement gaps include those that examine gaps across races, socioeconomic status, gender, and factors explaining the gaps that are predominant (e.g., Lubienski, Robinson, Crane & Ganley, 2013; McGraw, Lubienski, & Strutchens, 2006; Lubienski, 2002). Lubienski's (2008) research commentary emphasized the need for more mathematics educators to focus on gap analysis to inform education policy, classroom practices, and research. Additionally, the commentary points to the need to focus on opportunity gaps related to the inequities found in mathematics achievement (Lubienski, 2008). Furthermore, gap analyses inform public opinions and deficit notions when carefully conducted using factors related to the disparities found (Lubienski, 2008). Some suggestions for useful analysis suggested in the commentary include the study of gaps found in different strands of mathematics, intersections of race and socioeconomic status that could include or exclude gender. The literature cited in the present study includes achievements gaps by race, socioeconomic status, and language spoken at home. However, the information from the studies shown is in general mathematics without providing knowledge of particular strands of mathematics.

Racial and Socioeconomic Differences in Mathematics Achievement

Mathematics achievement gaps between races and socioeconomic status have abounded over the years. A study done by Lubienski (2002) that examined achievement gaps by race and socioeconomic status indicated that there were significant achievement gaps by race among students from both high SES and low SES backgrounds. In particular, white students with the lowest SES scored significantly higher or at par when compared to the black students in the highest SES in fourth, eighth and twelfth grade in 1990 and 1996. These findings indicate that racial achievement differences cannot be assumed to be the same as the socioeconomic differences. Indeed, the white students considered as low socioeconomically perform higher than black students from low socioeconomic backgrounds. Similarly, a focus on the SES across races, considering the socioeconomic measure on eligibility for free and reduced lunch, showed that in the fourth and eighth grade, the achievement gaps persisted by race. Specifically, for the fourth- grade students there was a significant achievement gap between those that received reduced price lunch in 2004 and 2007 for the black/white comparison. Further, among the eighth-grade students there were narrower achievement gaps in 2007 for free and reduced lunch, indicating that in this SES level, there was an improvement gain in the black students' mathematics achievement (Vanneman et al., 2009).

A focus on the achievement differences in Hispanics and White students also shows that the achievement gaps are narrowing. In particular, between 1990 and 2009 there was an improvement in Hispanics mathematics achievement. However, across these two years the difference in the achievement gaps was not significant (Hemphill & Vanneman, 2011).

The patterns of achievement gaps in mathematics are not consistent across the United States. Although there was a steady gain in eighth-grade mathematics achievement scores over the years in 41 states, the achievement gaps in Arkansas, Texas, Colorado, and Oklahoma was smaller in 2007, when compared to the gap in 1990. Similarly, the gaps between the black and white students in the

fourth grade were narrower in 2007 than in 1990 in 46 states (Vanneman, et al., 2009). In sum, these findings show that the persistence of these achievement gaps across the grade levels indicate that there could be other factors contributing to these gaps.

Differences in Mathematics Achievement by Language Spoken at Home

Students' language background is related to mathematics achievement. Howie's (2005) study indicated that students who spoke the language of the test (e.g. English) more frequently at home had higher scores on the mathematics test. Additionally, white pupils' mathematics scores were significantly higher than other groups who are not speaking English at home (Howie, 2005). Likewise, Reardon and Galindo (2008) found that fifth-grade students from homes where English is not the predominant language scored lower than students from English-speaking homes. These earlier studies show that there is need to analyze further the mathematics achievement gap comparing students' who use other languages other than English at home and those who do not are changing over the years and across states. In so doing, the results could indicate that the interventions on English language acquisition are influencing mathematics achievement.

Algebra achievement

The algebra content of mathematics is a gatekeeper in the selection of further mathematics and future career of students (Kaput, 2000). Some studies have shown that differences in the algebra achievement are too often associated with demographic and personality variables of students such as ethnicity, socioeconomic status, teacher quality, and student attitudes (e.g., McCoy, 2005) and instructional strategies (e.g., Rakes et al., 2010). Spielhagen (2006) suggested that the reduction of the opportunity gap and improved achievement is possible with the introduction of more algebra courses in the earlier grades. Similarly, other scholars have suggested that learning algebra in the early grades provides a strong baseline for more complex algebraic thinking that students experience in high school and beyond (e.g., Blanton & Kaput, 2011; Carraher & Schliemann, 2007).

In sum, these findings show that the persistence of these achievement gaps across the grade levels suggest that factors contributing to this scenario. One of the factors could be the algebra knowledge that students have acquired before joining secondary school. The level of algebra knowledge which can be either an advantage for learning further algebra content or not succeeding or selecting higher-level algebra-related courses.

Methods

The NAEP Main Assessment

The National Assessment of Educational Progress (NAEP) data provides information across the United States and over the years for possible state-level comparisons. The state assessments of NAEP have been administered across the states in grades 4, 8, and 12 in various subject matters since 1990. This data provides representative samples of students' mathematics achievement in grades four and eight from each state every two years. For example, 136,900 eighth grade students took the assessment throughout the nation in 2015. The NAEP assessment items include a variety of formats, such as multiple-choice and open-ended questions (requiring short constructed-response, and extended-response). The test items were classified by mathematical complexity: low complexity, moderate complexity, and high complexity. Each assessment question was designed to measure one of the five mathematics strands: number properties and operations, measurement, geometry, data analysis/statistics, and algebra. The data used in this study were from the years between 2005-2015 main NAEP mathematics assessments, and the participants that were the focus of this study were the eighth-grade students.

Variables

The NAEP background questionnaire items that were selected for this analysis include: language other than English spoken at home, student-reported mother's education level, the number of books at home, lunch eligibility, and race/ethnicity. Specifically, a description of the independent variables including the categories are: (a) A language other than English spoken at home (never (1), one in a while (2), half of the time (3), and all or most of the time (4); (b) Lunch eligibility (not eligible, reduced price lunch, and free lunch); (c) Academic resources including Mother's education level (did not finish high school, graduated college) and Books at home (less than 10 books, more than 100 books), For the academic resources at home the highest and lowest ordered responses were used; and (d) Race/ethnicity allowing multiple responses (White, Black, and Hispanic).

Analysis

Descriptive statistics were run using the NAEP Data Explorer (NDE) tool to illustrate the gap in eighth-grade students' average algebra scores across contexts, races and SES in selected states and over the years. The NDE is an online data analysis program that provides detailed results' from NAEP's national and state assessments. An independent-samples *t*-test was conducted to analyze differences between average algebra scores of eighth-grade students (a) whose mothers did not finish high school and those whose mothers graduated from college, and (b) who have less than 10 books at home and those having more than 100 books at home. An analysis of variance (ANOVA) was conducted to examine how the average algebra scores differ among (a) three ethnicity groups, (b) lunch eligibility (including "not eligible, reduced price lunch and free lunch"), academic resources and (c) language other than English spoken at home. Other ANOVA tests were also performed to assess the differences in algebra scores by ethnicity, lunch eligibility, academic resources, and a language other than English spoken at home over the years from 2005 to 2015.

Results

The results of this study were from nationally representative samples of eighth grade students from public and private schools. The alpha level (.05) for the quantitative analyses indicated that there was a statistically significant difference between the independent variables. Due to space limitations, for descriptive statistics we only provide Figure 1 presenting average algebra scale scores for eighth graders by ethnicity.

Race/Ethnicity

Figure 1 presents the graphs of the average algebra scores of the students by ethnicity (White, Black, and Hispanic) in each state over the years from 2005 and 2015. At the national level, the achievement gap between White and Black students seemed to be greater than White and Hispanic students. Notably, there is a large achievement gap between White, Black, and Hispanic students over the years. In Minnesota and Ohio, the achievement gap between Hispanic and Black seemed to be closing toward 2015. Interestingly, in Colorado, the Black students' algebra scores appeared to be dramatically increasing between 2007 and 2009 followed by a substantial decrease between 2009 and 2011.

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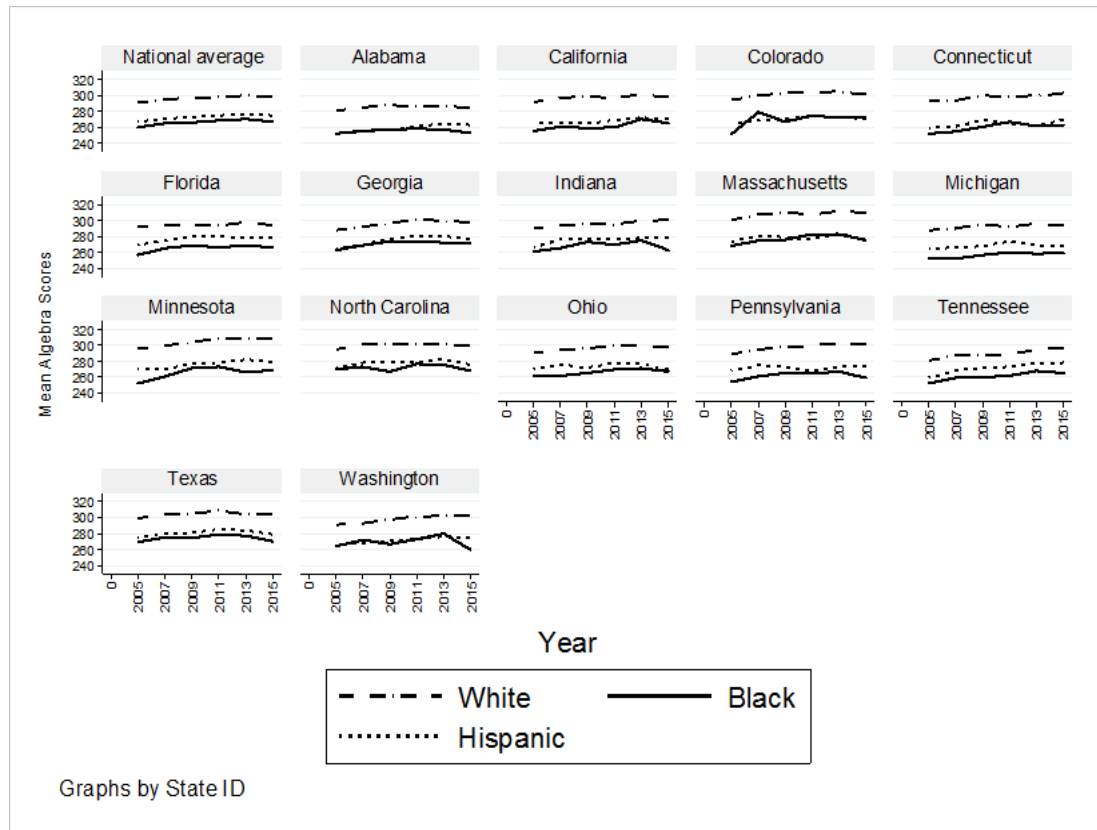
Figure 1. Algebra Achievement (Scores) by Race, 2005-2015.

A one-way ANOVA was conducted for the algebra score differences by race. The mean score for Black students was found to be lower than the mean scores for Hispanic and White students, and the mean score of white students was the highest. There were statistically significant difference in students' algebra scores between ethnicity groups, $F(2, 303) = 592.37$, $p <. 001$. A Post Hoc test result showed that there were significant differences between all race/ethnicity groups. A one-way ANOVA to determine differences in algebra scores by race/ethnicity across the years 2005, 2007, 2009, 2011, 2013, and 2015, was significant, $F(5, 300) = 2.98$, $p = .012$ for race/ethnicity over the years. Follow-up tests were conducted to evaluate differences among the means. The results indicated that the mean score of 2005 was significantly lower than the mean scores in the other assessment years. Additionally, the students' algebra scores were significantly higher in 2013 and 2011 than in the 2005 assessment year. The algebra scores by race/ethnicity increased from 2005 to 2013. However, the mean scores decreased from 2013 to 2015.

Lunch Eligibility
NAEP used student's lunch eligibility as a determinant of family income level. Students are eligible for free lunch if their family income is at or below 130 percent of the poverty level and are eligible for reduced-price lunches if their family income is between 130 percent and 185 percent of the poverty line. The findings show that there was a notable increase in the mean algebra scores for students who are not eligible for National School Lunch Program (NSLP) from 2005 to 2015 and also those eligible for free lunch. Massachusetts had the highest algebra scores in all their categories over the years when compared to the other states.

A one-way ANOVA was conducted to evaluate differences for the students' algebra scores by NSLP eligibility among the 16 states from 2005 to 2015. The dependent variable was lunch eligibility, including "not eligible, reduced price lunch, and free lunch". The ANOVA test was significant, $F(5, 285) = 5.217$, $p < .001$, for the eighth grade algebra scores by lunch eligibility over the years. A follow-up test result showed that the algebra score of 2005 was significantly lower than the score for all the other assessment years. Further, the students' algebra scores were significantly higher in 2015, 2013 and 2011 than in the 2005 assessment year.

The result also shows that the mean score of the students who were eligible for free lunch ($M = 270.06$, $SD = 6.86$) was lower than the mean scores of the students who are eligible for reduced lunch ($M = 280.75$, $SD = 6.84$) and the students who were not eligible ($M = 298.59$, $SD = 6.92$). There were significant differences in the students' assessment scores between groups, $F(2, 288) = 46.899$, $p < .001$.

Academic Resources at Home

Mother's Level of Education. An independent-samples t-test was conducted to evaluate the differences in the students' algebra scores by mother's education level, including two levels: did not finish high school and graduated from college. The test was significant, $t(202) = -29.66$, $p < .001$. The students whose mother graduated from college ($M = 299.12$, $SD = 7.35$) had higher scores than those whose mothers did not finish high school ($M = 270.65$, $SD = 6.32$). The 95% confidence interval for the difference in means was wide, ranging from 30.36 to 26.57. A one-way ANOVA of the algebra scores by mother's level of education showed no significant differences over the years.

Number of Books at Home. An independent-samples t-test was conducted to determine the differences in the students' algebra scores by the number of books at home over the years from 2005 to 2015 at two levels: less than 10 and more than 100. The test was significant, $t(202) = -41.56$, $p < .001$. Students who have more than 100 books at home ($M = 304$, $SD = 7.45$) had higher algebra scores than those have less than 10 books at home ($M = 264.25$, $SD = 6.14$). The 95% confidence interval for the difference in means ranged from 41.60 to 37.82. A one-way ANOVA of the algebra scores by the number of books at home showed no significant difference over the years.

Languages other than English Spoken at Home

A one-way ANOVA was conducted to determine the differences that exist in the eighth graders' algebra scores by languages other than English spoken at home from 2005 to 2015. There was a significant difference in algebra score by language spoken at home over the years, $F(5, 401) = 10.187$, $p < .001$. Particularly, the mean algebra scores increased over the years; the lowest mean score was in 2005, and the highest mean score was in 2015. Follow-up tests showed a significant difference that (a) 2005 eighth graders had a significantly lower score compared to the other assessment years, and (b) 2007 had a significantly lower score compared to 2013 and 2015.

We also compared students' algebra achievement by language spoken at home. The post-hoc test result showed that students who spoke English most of the time at home had a significantly higher algebra score than those who spoke English less often. Similarly, students who spoke English half of the time at home had a significantly higher score to those who rarely spoke English at home.

Discussion

The purpose of this study was to examine the achievement gap among students across the 16 states over the years from 2005 to 2015. There were differences in eighth-grade algebra achievement based on SES, academic resources, race/ethnicity, and languages other than English spoken at home, over the years. An earlier study that examined students' mathematics achievement in different grades had similar findings (Hemphill & Vanneman, 2011; Vanneman, et al., 2009). This present study's results also suggest that the eighth-grade algebra score in 2005 was significantly lower than the

algebra score in all the other assessment years. The improving algebra achievement scores over the years could be related to the adoption of the Common Core State Standards from 2010.

Although the students' average algebra scores continued to rise over the years, some significant differences persist. In particular, algebra scores differed significantly by race/ethnicity, SES, and languages other than English spoken at home. Specifically, the eighth- grade black students consistently posted significantly lower scores over the last ten years. Notably, in the last two years, ten of the selected states showed a drop in the black eighth-grade students' algebra achievement. The Hispanic students' algebra achievement dropped in six of the 16 states studied. In contrast, only five of the selected states showed a drop in the white students' algebra achievement. In general, there was an average drop in students' algebra achievement in the last two years. By highlighting the algebra achievement, it is evident that the Black and Hispanic students join high school with significantly lower algebra achievement scores. Similarly, students from families with lower income, and less academic resources, and those classified as having English as a second language join high school with lower algebra achievement. Indeed, other earlier studies also found that student algebra achievement scores were related to students' demographics, with notable achievement gaps across the different demographic groups (McCoy, 2005; Rakes et al., 2010). Thus, if students require knowledge of early algebra to be successful in further algebraic thinking learned in high school, it is evident that on average Black and Hispanic students could be joining high school with significant gaps needed for a successful transition to more complex algebraic reasoning.

The algebra achievement gaps should be further explored to show the related factors influencing these persistent patterns. Further, the opportunities that students have to learn algebra in high school could provide more information on the effect of these gaps in students' early algebra learning and future course taking patterns. Finally, the findings from this study suggest that early algebra achievement could be a factor related to the small representation of minority students in STEM fields.

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MORE AND LESS: LANGUAGE SUPPORTS FOR LEARNING NEGATIVE NUMBERS

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The language that students use with whole numbers can be insufficient when learning integers. This is often the case when children interpret addition as "getting more" or "going higher." In this study, we explore whether instruction on mapping directed magnitudes to operations helps 88 second graders and 70 fourth graders solve addition and subtraction problems with negative numbers. Further we explore to what extent having prior training with directed magnitude language (as opposed to just more and less language, without a direction specified) prepares students to benefit from the instruction. Our data shows that students, regardless of which language training they had, improved, and second graders, even with less initial knowledge, were able to make the same gains as fourth graders, suggesting that having initial exposure to negatives earlier could help students reach proficiency by the time the standards expect it.

Keywords: Number Concepts and Operations, Elementary School Education, Instructional Activities and Practices

Because subjects are usually taught separately in elementary school, there is often a boundary between language and mathematics instruction. Yet the National Council for Teachers of Mathematics (NCTM, 2000) standards call for students to "use the language of mathematics to express mathematical ideas precisely" (p. 60). Precise language can clarify ideas; whereas, vague language can create confusion. Unfortunately, the language that students use with whole numbers can be insufficient when they learn new numbers, such as integers. This is often the case when children interpret addition as "getting more" or "going higher." Consider, for example, the following second grade student (B05) explaining why -1 + 8 = -9: "Because negative one, and then eight *more* is negative nine." The student's statement includes language that highlights the operation (more = addition) and magnitude (8). However, "more" in integer operations has multiple meanings. The problem involves getting *more positive* (counting up the number sequence in the positive direction), an operation with a directed magnitude (adding positive 8), as opposed to getting *more negative*. Therefore, finding ways to help students interpret and apply directed magnitude language to comparisons and operations with negative integers is necessary as they transition from work with whole numbers to integers.

Theoretical Framework

Before the introduction of negative numbers, students learn that "more" corresponds to an increase in magnitude, which corresponds to counting up the number sequence (moving to the right on the number line), and maps onto addition (Case, 1996). With the inclusion of negative numbers, an additional level of specificity is needed. Students need to learn that an increase *in the positive direction* is getting "more positive", which corresponds to moving right on the number line and maps onto adding a positive number. Similarly, *an increase in the negative direction* is getting "more negative", which corresponds to moving left on the number line and maps onto adding a negative number. Similar relations exist for subtraction, except instead of an increase in a direction, there is a decrease in a direction (getting less positive or less negative).

Students' understanding of comparisons with more and less has been explored both from a language perspective and a mathematical perspective. Language studies indicate that, initially, comparisons with positive or unmarked adjectives are easier for students to answer than negative or marked ones. Therefore, determining which object is *higher* is easier than *lower* (Smith, Rattermann,

& Sera, 1988), and determining which set of objects or glass of liquid has *more* is easier than *less* (Palermo, 1973). However, there is still some debate as to whether this happens because positive adjectives are used more or because they are conceptually easier to understand (e.g., Ryalls, 2000).

Studies from a mathematical perspective focus on exploring students' responses to questions about which of two numbers is more. Students tend to categorize numbers as large and small. For positive number comparisons, when two numbers belong to different categories defined by the students (e.g., large and small), young children do better on the comparisons than when both numbers are part of the same category (e.g., two numbers they consider large) (Murray & Mayer, 1988). When comparing two negatives, experienced sixth graders are faster when the two numbers are farther apart than when they are closer. However, there are no differences when they compare a positive and negative number, suggesting they use a rule that positives are more than negatives regardless of how far apart the numbers are (Varma & Schwartz, 2011).

Aside from teaching students the rule that positives are greater than negatives, using questions that include a context could make the desired value or point of comparison more explicit. For example, if integer problems were presented in a golf context, negative numbers would be better than positive numbers; whereas, this would be reversed for many video games or board games, where the goal is to achieve a higher positive score. The desired value is also clearer with temperature (e.g., which temperature is hottest vs. coldest)? To get the same level of obviousness with numbers there would need to be a move away from the magnitude questions (where positive is the assumed reference category) to directed magnitude questions (where we explicitly say which type of value - positive or negative – is the reference category).

Knowing which direction "more" relates to (as opposed to assuming more always means more positive) is especially important when adding and subtracting negative integers. Students who have learned about negatives know that positive numbers are considered more than negative numbers and that the question, "Which is more?" is asked from a positive perspective. Therefore, they might be more receptive to a focus on the language connecting integer addition and subtraction to directed magnitudes. This would also align with the Common Core Standards for Mathematics' recommendation that integers be introduced in the sixth grade (National Governor's Association Center for Best Practices & Council of Chief State School Officers, 2010). On the other hand, students with little experience with negatives often determine which number is more based on absolute value (Bofferding, 2014). Given young students' willingness to consider "large negatives" as more, they might be more receptive to such a focus on language. Further, if current magnitude language, emphasized in whole number instruction, is limiting, it is possible that using directed magnitude language in early elementary would help students build more cohesive conceptions of "more" and "less".

Research Questions
Based on the issues described above, we explore whether instruction on mapping directed magnitudes to operations helps second and fourth graders solve addition and subtraction problems with negative numbers. Further we explore to what extent having prior training with directed magnitude language (as opposed to just *more* and *less* without a direction specified) prepares students for the instruction. Specifically, we investigate the following research question and sub-questions:
How does instruction mapping directed magnitude language to operations with integers affect students' learning of integer comparisons, addition, and subtraction?
  (a) To what extent does having a background understanding of language related to magnitude (e.g., more, lower) versus directed magnitude (e.g., more positive, less negative) influence learning?
  (b) How do gains differ for students in second grade versus fourth grade?

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Methods

Participants and Setting

This study took place at two public schools in the same rural Mid-western district. Students were recruited from all of the second and fourth grades, resulting in 88 second-grade, and 70 fourth-grade participants. The district reports 32.2% English-language learners and 75.2% as qualifying for free or reduced-lunch.

Design and Materials

The study involved a pretest, random assignment to language group (magnitude or directed magnitude), three small group sessions, individual language training, whole class instruction, and a posttest. We describe each of these below.

Pretest. Students individually completed a written pretest during a whole class session. Students not in the study had their tests returned to their teacher or shredded. The pretest consisted of filling in the missing numbers on a number path, value comparisons with different language (n=24), feeling-related questions (n=4), and integer addition and subtraction problems (n=13) (see Table 1 for examples).

Table 1: Example Items from the Pretest

Question Type	Examples
Integer Order	Fill in the missing numbers before 4 and after 6. [ ][ ][ ][ ][ ][ ][4][5][6][ ][ ][ ]
Value Comparisons	Circle the temperature that is… …hottest….most hot…least cold. / Circle the temperature that is… …coldest….most cold…least hot. $2\,^oF$ $3\,^oF$ $8\,^oF$ none / $2\,^oF$ $6\,^oF$ $4\,^oF$ none $0\,^oF$ $-6\,^oF$ $-1\,^oF$ none / $-1\,^oF$ $0\,^oF$ $-6\,^oF$ none $-4\,^oF$ $-2\,^oF$ $3\,^oF$ none / $3\,^oF$ $-4\,^oF$ $-2\,^oF$ none $5\,^oF$ $-10\,^oF$ $-8\,^oF$ none / $-8\,^oF$ $5\,^oF$ $-10\,^oF$ none $-6\,^oF$ $-2\,^oF$ $-3\,^oF$ none / $-3\,^oF$ $-6\,^oF$ $-2\,^oF$ none $-5\,^oF$ $-4\,^oF$ $-7\,^oF$ none / $-7\,^oF$ $-5\,^oF$ $-4\,^oF$ none
Feelings Comparisons	Circle the face that is **least sad**. / Draw a face that is **less happy** than this face.
Addition & Subtraction	4 + -5; -1 + 8; -6 + -4 / 1 – 4; -6 – -9; 5 – -3

Students were stratified within each grade level based on their performance on the pretest, and then students from each strata were randomly assigned to one of two language groups: a control group that used traditional magnitude language or an experimental group that used directed magnitude language.

Small group sessions. With 2-3 students from their group, students participated in three, 15-minute sessions. The initial session consisted of three activities. The first activity was a game where

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students were asked to find matching integers. Cards were arranged in an array, and students turned them over one at a time, seeking the matching value. Researchers asked questions, such as, "How are they (the numbers) the same?" or "How are they different?" in an effort to focus attention on the positive or negative values. During the second activity, students were told that negative numbers are used below zero. Students then explored integer order by filling in spaces on a number path marked only with zero. Using their completed number path as a reference, students played a card game for their third activity. Their goal was to collect and trade integer cards in an effort to get a card hand with three integers in a row (e.g., -2, -1, 0).

The second and third small group sessions both involved playing a game where students moved on a number path labeled -10 to 10. The language of the magnitude group remained consistent with the use of operating with magnitudes. Therefore, they moved "more or less" or "higher or lower" a certain amount; whereas, the directed magnitude group used language consistent with operating with directed magnitudes. They moved "more positive or less positive" and "less negative or more negative" a certain amount.

Training. During their training session, students completed four phrasing-type rounds, with up to twenty-eight comparisons per round. The control group received training on magnitude comparisons (phrasing: Which is more? Which is less? Which is higher? Which is lower?), and the experimental group received training on directed magnitude comparisons (phrasing: Which is more positive? Which is less positive? Which is more negative? Which is less negative?). Students continued only until they got seven consecutively correct for each phrasing type. They were then asked to order a set of non-consecutive integers according to the above-mentioned phrasing (e.g., from most positive to most negative). Both groups then sorted a set of integers into categories three times: high/low, less/more, and positive/negative piles.

Whole-class instruction. In each of the classrooms, two researchers presented a 30-minute lesson to all students using the directed magnitude language. To introduce the idea of moving on a continuum with directed magnitudes, the researchers showed an emotion-continuum of faces from very sad to very happy with a neutral face in the middle. They then read a story about a boy who got more happy, more sad, less happy, and less sad. Students helped show how the boy's mood changed on the emotion-continuum at the front of the room. After this introduction, students were told that addition means moving *more in a particular direction* (either more positive or more negative) and that subtraction means moving *less in a particular direction* (either less positive or less negative). The researchers then did a series of problems with the class to reinforce the mapping of the directed magnitude language onto the operations. Problems were designed as number strings (DiBrienza & Shevell, 1998; Lampert, Beasley, Ghousseini, Kazemi, & Franke, 2010), and students participated using their own number path and sticker to move as they solved the problems. Table 2 lists the typical order of the problems used.

Table 2: Problems Included in the Class Number Strings

1)	4 + 3	4)	2 – 6	7)	-1 – -7
2)	3 + -4	5)	2 – -6		Extra, if time:
3)	-3 + 4	6)	-6 – 2		-7 – -1; -5 + -3; -5 + 3

Posttest. The posttest was largely the same as the pretest, except for a few additions and changes. We only re-asked the most difficult temperature comparison questions from the pretest (n=12). We also changed the feeling comparisons to allow students to pick 1 of 5 faces (very happy, happy, neutral, sad, very sad). Finally, we added a small selection of transfer questions, such as solving for a missing addend, subtrahend, or minuend.

Analysis

We began analysis with a multivariate ANOVA on the difference in temperature comparison scores and the difference in arithmetic scores for grade and condition (including only items present on both pretest and posttest). Then, we conducted a 2 (second grade vs. fourth grade) x 2 (control vs. experimental) x 2 (pretest vs. posttest) x 2 (temperature comparisons vs. arithmetic) repeated measures ANOVA to learn more about potential differences. To better understand students' difficulties with the arithmetic problems, we further analyzed the results by looking at common responses students gave. For each of the arithmetic problems on the pretest and posttest, we calculated the percentage of students who provided a specific numerical response. For example, we looked at the percentage of second and fourth graders who, for -4 + -6, answered 10, -10, 2, -2, or 0. Further analysis of the temperature comparisons can be found elsewhere (Bofferding & Farmer, 2016).

Results

Gains from Pretest to Posttest

The results of the multivariate ANOVA were not significant for condition or grade. Therefore, any gains made were statistically similar for students regardless of whether they had the training with magnitude or directed magnitude language, and regardless of whether they were in second grade or fourth grade. Based on the repeated measures ANOVA, there was a significant interaction between test and item type, $F(1, 154) = 9.11$, $p=.003$. Students did significantly better on the temperature comparisons and arithmetic problems on the posttest than they did on the pretest. Further, on average, performance was significantly higher on the temperature comparisons than the arithmetic problems, $F(1, 154) = 83.27$, $p<.001$. Students had an average gain of 2.15 on the temperature comparisons versus 1.15 on the arithmetic problems. Fourth graders did start out performing significantly higher than second graders on both items, based on an item type by grade interaction, $F(1, 154) = 17.55$, $p<.001$; however, as mentioned previously both fourth and second graders made similar gains (see Table 3).

Table 3: Pretest, Posttest, and Gain Scores for Second and Fourth Graders

Grade	Temperature Comparisons Average Correct (SD)			Integer Arithmetic Problems Average Correct (SD)		
	Pretest n=12	Posttest n=12	Gain	Pretest n=13	Posttest n=13	Gain
2nd (n=88)	2.86 (2.97)	5.38 (3.66)	2.52	2.52 (1.62)	3.61 (2.41)	1.09
4th (n=70)	6.59 (3.54)	8.37 (3.77)	1.78	4.04 (2.06)	5.24 (2.74)	1.20

Although students as a group made significant gains, we investigated the gains further by running analyses for each grade separately. For fourth graders, there was no significant interaction between test and item type, so their gains from pretest to posttest are only significant when averaging over both item types, $F(1, 69) = 39.83$, $p<.001$. Averaging over both tests, they also did significantly better on the temperature comparisons than the arithmetic problems, $F(1, 69) = 66.58$, $p<.001$. For second graders, there was a significant interaction between test and item type, $F(1, 87) = 12.30$, $p=.001$, with performance on both the temperature comparisons and arithmetic problems increasing from the pretest to the posttest. These results suggest that in the overall analysis, the test by item interaction was largely driven by the second graders.

Arithmetic Problems Involving Negatives

Students' performance on the arithmetic problems involving negatives varied widely, and their gains on these items were not as large partly because many students got a problem correct on the pretest without knowing about negative numbers, and then got the same problem incorrect on the posttest because they were paying attention to the negative and trying to use it. For example, when solving -2 - -6 on the pretest, 31% of second graders correctly answered "4" as opposed to 16% of fourth graders. Several second graders who answered "4" also solved -4 - -3 = 1, as if all numbers were positive and 4 - -5 = 1, as if the problem was 5 - 4 = 1. Therefore, they likely got -2 - -6 correct because they solved the problem as 6-2=4. In contrast, on the posttest, B01 (who answered 0 on the pretest), explained why the answer was 4 as follows: "So there's a take away, and it's negative. You had to go farther from the negative because it's in the positive." This student had training with the directed magnitude language and used it to reason about the answer, correctly answering all but one of the arithmetic problems.

Both grade levels had 10%-16% of students gain on solving 1 - 4 and 6 - 8, suggesting they got better at counting into the negatives. However, 29% - 44% of students at each grade level still solved the problems as 4-1 or 8-6 on the posttest. Both grade levels showed a stronger focus on negatives on the posttest than on the pretest, even if they did not get correct answers. Second graders made a large gain on answering -6+-4=-10. On the pretest, 16% of them correctly answered this problem, while 34% of them answered "10", as if the numbers were positive. On the posttest, 38% of them correctly answered the problem, and their most prevalent incorrect response changed to "-2". Fourth graders made the largest gain on solving 5 + -2 = 3 (from 27% correct to 49% correct). On the pretest, 29% of them incorrectly answered "7", but on the posttest, their most common incorrect response was "-7".

Although there were several instances where the fourth graders made larger gains on certain problem types than the second graders, the second graders consistently performed better and made larger gains than fourth graders on problems where they had to subtract a negative number from a positive number (see Table 4). Especially for 5 – -3, students in both grades solved the problem as 5 – 3 = 2 on the pretest. Although the same percent of fourth graders still answered this way on the posttest, a lower percent of second graders did so.

Table 4: Percent of Students Providing Different Answers to Integer Questions

4 – -5	Second		Fourth		Change in %	
Answer	Pretest	Posttest	Pretest	Posttest	2nd Grade	4th Grade
Correct: 9	2%	15%	0%	6%	+13%	+6%
-9	6%	5%	10%	20%	-1%	+10%
1	25%	14%	10%	16%	-15%	+6%
-1	19%	32%	53%	46%	+13%	-7%
0	25%	17%	10%	3%	-8%	-7%
5 – -3	Second		Fourth		Change in %	
Answer	Pretest	Posttest	Pretest	Posttest	2nd Grade	4th Grade
Correct: 8	3%	15%	1%	0%	+12%	+9%
2	59%	47%	49%	49%	-12%	0%
-2	6%	13%	19%	14%	+7%	-5%
0	1%	2%	1%	1%	+1%	0%

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Conclusions & Implications

Overall, students in the study benefitted from instruction on mapping the language of directed magnitude onto the operations, regardless of which language they had training on. This suggests that the level of explicitness in the lesson was beneficial. Further, the differences between groups may have been reduced based on this common instruction. Therefore, future work should investigate students' knowledge just prior to the instruction to tease apart how much benefit they received from their training versus the instruction. Additional qualitative analysis will also help illuminate if students' language training played a role in how they answered and reasoned about the arithmetic problems, as was the case for student B01.

The higher performance on the temperature comparisons than the arithmetic problems makes sense given Case's (1996) framework. Based on his theory of number development, students need to understand the relations among number values before they can use this relation to add and subtract numbers. Further analysis of the ordering questions during the training session will clarify to what extent each student was able to map the directed magnitude language onto the integers.

The similar performance between second and fourth graders is interesting. On the one hand, the fourth graders performed higher than the second graders on the pretest. This is unsurprising as they had more time to be exposed to negative numbers and likely had more experience hearing about temperatures. However, second graders, even with less initial knowledge, were able to make the same gains as fourth graders! This suggests that the border established for when we introduce and/or teach negative integers should be reconsidered; second graders can make significant gains even with a relatively short intervention. If grade level doesn't matter, then having initial exposure to negative integers earlier could help students reach proficiency by the time the standards expect it.

Finally, second grade did better than fourth graders on problems, such as 4 - -5, which are traditionally the hardest (Wheeler, Nesher, Bell, & Gattegno, 1981). It is possible that they were more successful at using the language to make sense of the problems; this is an area for further exploration. However, it suggests that younger students might benefit more from earlier exposure to the cognitively conflicting problems like these.

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STUDENTS' CONCEPTIONS SUPPORTING THEIR SYMBOLIZATION AND MEANING OF FUNCTION RULES

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This paper explores the nature of students' quantitative reasoning and conceptions of functions supporting their ability to symbolize quadratic function rules, and the meanings students make of these rules. We analyzed middle school students' problem solving activity during a small group teaching experiment (n=6) emphasizing quadratic growth through covarying quantities. Results indicate four modes of reasoning supportive of students' symbolization of quadratic function rules: (a) correspondence, (b) variation and correspondence, (c) covariation, (d) flexible covariation and correspondence. We discuss implications for research on learning vis-à-vis students' representational fluency, as well as design principles to support quantitative reasoning.

Keywords: Algebra and Algebraic Thinking, Cognition, Middle School Education

Research Issues and Purpose

Students' understanding of functions is central to school algebra in which two perspectives on algebraic thinking are valued: a change and variation perspective, and a structural or symbolic perspective (Cai, Bie, & Moyer, 2010). An oft-cited goal of students' activity in algebra, and functional thinking particular, is to cultivate the ability to create, interpret, and connect numeric, graphic, symbolic, and verbal representations of functions (Kieran, 2007). Indeed, there is a large body of work addressing students' understanding of function vis-à-vis the study of students' abilities to translate between multiple representations, especially from symbolic to graphical representations of function (e.g., Leinhard, Zaslavky, & Stein, 1990). Such research often focuses on students' representational performances (e.g., abilities to connect multiple representations) to the detriment of understanding what representations *mean* to students and how this representational activity is supported or constrained by students' *conceptions* of mathematical ideas (Thompson, 1994b).

In this research we explore the interplay between students' representational abilities and their conceptions and meanings of functions. We focus on students' symbolization of quadratic functions of the form $y=ax^2$, a topic commonly taught using a "method of finite differences," with the meaning of *a* restricted to determining the steepness of the parabola (Ellis & Grinstead, 2008). In contrast to these common approaches, we ground our investigation of students' understandings of quadratic functions in a quantitatively rich context as a source of possible meaning for students' activity (Thompson, 1994a). We address three research questions:

- What is the nature of students' quantitative reasoning about quadratic growth situations?
- What ways of reasoning *support* students' abilities to symbolize quadratic function rules?
- How do students *make sense of* the symbolic quadratic function rules they write?

We first articulate three mutually supportive lenses that guided our interpretations of student thinking: quantitative reasoning, conceptions of function, and representational fluency.

Theoretical Framework and Background

Quantitative Reasoning and Conceptions of Functions

By quantities, we refer to measurable attributes of objects or phenomena (Smith & Thompson, 2007). A quantity is a mental concept, composed of one's conception of an object, a quality of the

object, an appropriate unit or dimension, and a process for assigning a value to the quality (Thompson, 1994a). Examples of attributes that can be conceived as quantities are height, length, and area. Students engage in quantitative reasoning when they operate with quantities and their relationships, conceiving new quantities in relation to one or more already conceived quantities. Comparing quantities multiplicatively is a critical aspect of quantitative reasoning.

A correspondence perspective of function identifies function relationships such as $y = ax^2$ as the fixed correspondence mapping between the members of two sets (Farenga & Ness, 2005; Smith, 2003). From this perspective, pairs of quantities are linked by a multiplicative relationship. Although this static view of function is privileged in school mathematics curricula, some researchers argue that a covariation approach to functional thinking is important in supporting a well-developed understanding of function as a dynamic relationship between quantities (e.g., Confrey & Smith, 1994; Thompson & Carlson, in press).

One stance on covariation involves the examination of a function in terms of coordinated changes of x- and y-values, in which students move operationally from y_m to y_{m+1} coordinating with movement from x_m to x_{m+1} (Confrey & Smith, 1994; Smith, 2003). This perspective requires that students understand quantities as having a sequence of values and relate the values in each sequence additively or multiplicatively. Thompson and Carlson (in press) instead emphasize the importance of helping students envision change through the notion of *continuous variation*. Students who can think about smooth continuous variation can imagine a variable's magnitude increasing in bits while simultaneously anticipating that within each bit, the value varies smoothly. From this stance, students engage in *covariational reasoning* when they can envision the values of two quantities, such as the height and the area of a growing rectangle, varying together (Thompson & Carlson, in press). In this study we will characterize students' reasoning as *covariational* when they attend to coordinated change across two or more quantities. This does not mean the students necessarily thought about continuous variation; in many cases, their understanding of variation was likely chunky, or even discrete. However, we characterize reasoning as covariational when students used language and gestures that suggested images of a rectangle's height values and area values simultaneously varying together.

Representational Fluency and Meaning Making

Zbiek, Heid, Blume, and Dick (2007) define representational fluency as "the ability to translate across representations, the ability to draw meaning about a mathematical entity from different representations of that mathematical entity, and the ability to generalize across different representations" (p. 1192). In this study we focus on both students' translations (i.e., the correct creation and interpretation of quadratic function rules from tables, words, or diagrams), and what these symbolizations *mean* to students. What one is able to "see" in a representation is supported and constrained by what one knows (Piaget, 2001). Thus we adopt a constructivist stance in building second-order models of students' mathematics (Steffe & Olive, 2010), making inferences about students' meaning making as opposed to first-order models of a researcher's meanings of representations.

Methods

Teaching Experiment

We conducted a 15-day videotaped teaching experiment (Steffe & Thompson, 2000) with 6 middle school students in an after school setting. The teacher-researcher (TR, Ellis) taught all teaching sessions, each lasting 1 hour. All sessions were transcribed and pseudonyms were assigned to all participants. One purpose of the small-scale teaching experiment is to gain direct experience with students' mathematical conceptions and the change in those conceptions over time (Simon,

1995). Our aim was to study the factors promoting students' algebraic generalizations as they explored rectangles that grew proportionally by maintaining the same height-to-length ratio. The teaching-experiment setting supported the development and testing of hypotheses about students' understanding in real time while engaging in teaching actions. Thus, the mathematical topics for the entire set of sessions were not pre-determined, but instead we created and revised new tasks on a daily basis in response to hypothesized second-order models of the students' mathematics.

Task design. All tasks were grounded in the growing rectangle context, in which the relationship between the height, h, and the area, A, can be expressed as $A = ah^2$. Students worked with computer simulations of the growing rectangles, drew their own rectangles, created tables representing the heights, lengths, and areas of growing rectangles, and created and justified algebraic rules comparing the rectangles' areas with their heights (Figure 1).

Task a. Here is a table for the height versus the AREA of a rectangle that is growing in proportion:

Height	Area
3	6.75
4	12
5	18.75
6	27
7	36.75
8	48
50	?
h	?

Task b. Here is a table for the height versus the AREA of a rectangle that is growing in proportion:

Height	Area
2	16
4	64
6	144
8	256
10	400
65	?
1/2	?
h	?

Figure 1. Sample far prediction and generalization tasks.

Data Sources and Analysis

Data for this study included video, transcripts, and PDFs of students' written work. We analyzed all student discourse and written work in which a student stated (written or spoken) a correct rule of the form $A = ah^2$. Data analysis focused on identifying the nature of students' quantitative reasoning (RQ1), ways of reasoning supportive of symbolizing function rules (RQ2), and students' meanings for the coefficient a in the function rule $A = ah^2$ (RQ3).

Two coders independently coded half of the data corpus in a first round of coding, then met to discuss findings and clarify questions. In a second round, one coder (Fonger) independently compiled all data from the first round of analysis and named major code categories and subcategories. In a third round of analysis Fonger re-analyzed all compiled data, one (sub)category at a time in a constant comparative fashion (Strauss & Corbin, 1990), referring to the original transcripts as needed. In a fourth round of analysis, Fonger engaged in axial coding (Straus, 1987) to discern relationships among codes. Coding was aimed at identifying how types of quantitative reasoning may have supported students' symbolization of rules and meanings of a. For a given episode it was possible for a student to demonstrate multiple forms of quantitative reasoning; all types were coded where appropriate, except when a student demonstrated evidence of covariation, in which case we did not also code variation. In cases in which students demonstrated flexibility across ways of reasoning, we coded multiple meanings of a.

Results

Students' Quantitative Reasoning

We characterized the nature of students' quantitative reasoning about functional growth situations (RQ1) into four types: (a) *Static Correspondence*, (b) *Variation*, (c) *Uncoordinated Variation*, and (d) *Covariation* (Figure 2).

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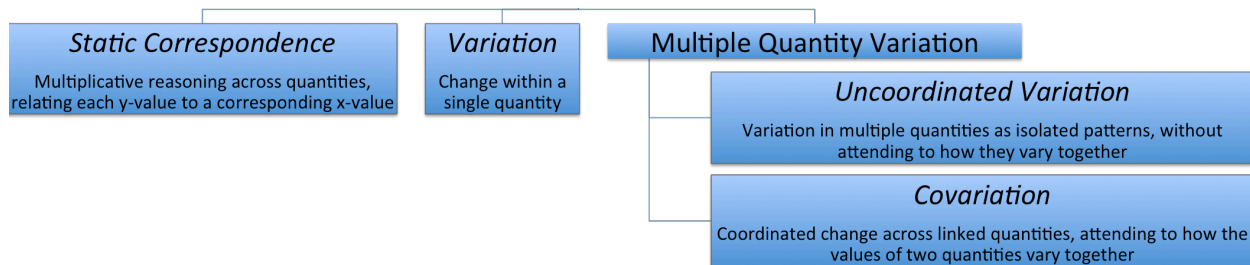
Figure 2. Quantitative Reasoning about Functional Growth Situations.

For *Static Correspondence*, students demonstrated evidence for reasoning multiplicatively across quantities, directly relating each *y* value to a corresponding *x* value. This includes both multiplicative relationships between height (*x*) and length (y_1) and height (*x*) and area (y_2); the first case was more common. For example, Daeshim said "height times 3 is length" and Bianca noted "length over height" as a static multiplicative relationship across linked quantities.

Students reasoned about *variation* when they reasoned about change within a single quantity, for instance, attending to growth in area without coordinating with growth in height. Typical ways in which students reasoned about *variation* in quantities was through attending to either differences in height or in length of the growing rectangle, but not to both. For example, for the rule $A = .75h^2$ Ally described .75 as "the rate of growth of the length." When students reasoned about variation in more than one quantity we categorized their thinking as either uncoordinated variation or covariation. *Uncoordinated variation* is typified by students' attention to variation in two or more quantities in a manner that does not coordinate simultaneous change, but rather treats the variation as isolated patterns or sequences (Figure 3).
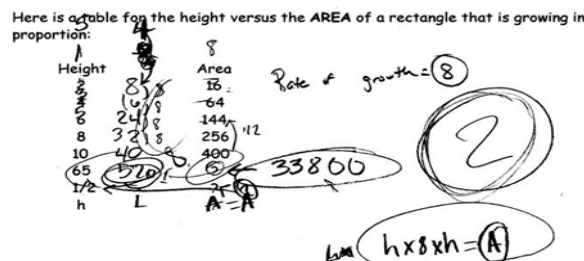
Figure 3. Ally's uncoordinated variation of length and height.

Ally's work in Figure 3 (for Task b) shows how she computed length values for each pair of height and area values. Ally attended to variation in L by computing successive differences in length (which the students called DiL). She also noted a pattern for the difference in height (which the students called DiH), writing, "you can't reduce regularly because it is going up by 2s." Thus Ally attended to DiL and DiH as isolated patterns.

Finally, we characterized students' quantitative reasoning as *covariational* when they attended to a coordinated change across more than one quantity, envisioning both quantities varying together. For example, Jim wrote the rule $A=4.5h^2$ and explained if you grew a rectangle "it would go over 4.5 for every time you go up the height 1."

Conceptual Supports for Students' Ability to Symbolize Function Rules

Students' symbolization of quadratic function rules of the form $A = ah^2$ was supported by four modes of reasoning (RQ2): (a) *Static Correspondence*, (b) *Variation and Static Correspondence* (when ΔH=1), (c) *Covariation*, and (d) *Flexible Covariation and Correspondence*. Each mode of reasoning is exemplified across two task types in Table 2 (save mode D, which combines modes A

and C). See Figure 1 for the two tasks (Tasks a and b), in which $\Delta H=1$ for Task a and $\Delta H=2$ for Task b. The examples in Table 2 are sample student responses to these tasks, and are paraphrased from our analyses of student thinking. These results reflect the relationships observed in students' quantitative reasoning.

Table 1: Conceptions of Functions Supportive of Symbolizing Function Rules

Mode of Reasoning	Example for $\Delta H=1$	Example for $\Delta H>1$
A. Static Correspondence. Student writes a rule by computing a ratio of two quantities, either (a) length to height, or (b) area to height^2.	$A=.75h^2$ because I take the ratio of length to height, 2.25/3, which is .75.	$A=4h^2$ because the ratio of the length to height, 8/2, is 4.
B. Variation <u>and</u> Static Correspondence. Student reasons about the difference in length (DiL), multiplicatively relates height and change in length (L=H*DiL), and writes the rule A=HL as A=H(DiL*H).	$A=.75h^2$ because .75 is the difference in length, so I can write L=.75*H, and A=LH so A=(.75H)(H)	n/a (i.e., mode of reasoning leads to incorrect rule)
C. Covariation. Student understands coordinated variation in two linked quantities and uses this to support writing a correspondence rule of the form $A = ah^2$. This includes change in length, change in height, and rate of growth of area.	$A=.75h^2$ because the length changes by .75 each time (the height changes by 1).	$A=4h^2$ because the length increases by 8 cm for every 2-cm increase in height.

There are two important cases to discuss regarding our results of RQ1 in relation to RQ2. First, in some cases, students' quantitative reasoning was evident in their thinking without necessarily supporting their ability to write a rule. For example, Bianca reflected on the rule $A=4.5h^2$ and questioned "What is the 4.5 that we came up with? What does that have to do with anything? Length over height!" Thus Bianca's reasoning about a correspondence relationship did not inform her writing of the rule (it was a retrospective connection she made). Second, some types of reasoning did not support students' ability to write correspondence rules. Recall Ally's reasoning on Task b (Figure 3). Notice that Ally wrote the rule $h \times 8 \times h = A$, which was informed by her understanding that $A=HL$ and $H=8L$ (mode B, *variation and static correspondence* for $\Delta H>1$). This example illustrates two findings: Ally's *variation and static correspondence* reasoning for $\Delta H>1$ led to an incorrect rule (see Table 2), and Ally's uncoordinated variation of DiL and DiH did not support writing a correct rule. Thus in what follows we focus only on the cases for which students' quantitative reasoning informed their symbolization of function rules. For brevity, we exemplify modes B and D.

In mode B, students coordinated their reasoning about variation in length with reasoning about a static relationship between height and length in order to re-write the area formula $A=HL$ in terms of these quantities. For example, on Task b, Tai found "the rate of growth of the length" to be .75 and explained "height times .75 equals the length … and then you take the length and you times it by the height." Thus Tai attended to variation in L to posit .75 as the rate of growth in the length, then stated a multiplicative correspondence relation between length and height ($H*.75=L$), and used the rule $A=LH$ to write $.75h^2$.

In mode D, students could think both covariationally and in terms of correspondence relations to symbolize function rules. Daeshim, Jim, Bianca, and Tai demonstrated flexibility in their conceptions of functions across the static correspondence view and the covariation view, which supported their abilities to symbolize correspondence rules. For instance, given a table of values in which the height increased by uniform increments of 3 cm, Jim found the rate of growth in length to be 6.75 cm for each 3-cm change in height. He wrote "$2.25h^2 = $ Area" and explained that he divided 6.75 cm by 3 cm "because that's what you're going up by each time." Jim's attention to the coordinated change in height and length supported his symbolization of the function rule. Jim also demonstrated flexibility

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in leveraging his correspondence thinking to write rules. In one example, he created a table for the height, length and area of a growing rectangle and quickly said "$4.5h^2$ … I just did the first two numbers and that's all I need to do." Jim computed the ratio of the original length to the original height, and thus a static multiplicative relationship informed his writing of the rule.

Students' Meanings of Symbolic Quadratic Function Rules

In light of the finding that students demonstrated great variation in their conceptions supporting their ability to symbolize quadratic function rules, it is not surprising that their meanings for a in the symbolic function rules they wrote of the form $A = ah^2$ varied as well. We discerned five types of meanings of a in students' symbolic rules $A=ah^2$ as shown in the rows of Table 3. We exemplify three themes that emerged across students' meanings next.

Table 2: Students' Meanings of Symbolic Function Rules

Meaning of a in $A = ah^2$	Nature of Quantitative Reasoning
A Quantity	$a = length$ when $height$ is 1
Ratio of Quantities	$a = (original\ length)/(original\ height)$; $a = length/height;$ or $a = area/(height^2)$
Variation in Quantities	$a = DiL;$ or $a = 1/DiH$
Ratio of Variation in a Quantity to Value or Quantity	$a = DiRoG/2$; $a=(DiL)/(original\ height);$ or $a = DiRoG/(2(original\ height)^2)$
Covariation of Quantities	$a =DiL/DiH$; $a = DiL$ when $DiH = 1$; $a = DiL/DiH$ when DiH>1; $a = DiRoG/2 = DiL$, $\Delta height = 1$; or $a = DiRoG/(2*DiH)$

Ratio of Quantities or a Quantity

Students made sense of the coefficient a both as a quantity and as a ratio of two quantities. For instance, Daeshim wrote a general expression nh^2, explaining n as "the length when the height is 1" (Task b). Bianca explained that the coefficient a could also represent "length over height." Students often wrote the static ratio of length to height in the general form, $a = (original\ length)/(original\ height)$. The parameter a could also represent a static ratio of area to height squared. For example, Tai wrote a symbolic rule $4.5h^2$ by computing "the area [288] divided by h squared [8^2=64] equals a number [4.5]". In this case, $a = area/(height^2)$.

Variation and/or Ratio of Variation and Quantity

As a second theme, students reasoned about the variation in length or in the difference in the rate of growth in area (which the students called the *DiRoG*), sometimes also computing a ratio with a fixed quantity. For instance, Ally made sense of a as "the rate of growth of the length" (Task b), or $a = DiL$. Students also attended to the DiRoG in order to determine the value of a. Bianca explained, "I just found the DiRoG and divided it by two and then I knew what number to multiply by," so $a = DiRoG/2$. Other students related variation in quantities to a fixed quantity. For example, Tai explained: "First you figure out the, um, length. And then you find out the rate of growth, and then you divide the rate of growth by the height, and then you put like whatever number, height squared." In this case, $a=(DiL)/(original\ height)$.

Covariation of Quantities

Students also conveyed meanings of a that were firmly grounded in a covariation perspective, both (a) as the ratio of the change in height (DiH) to the change in length (DiL), and (b) as a coordination of the difference in the rate of growth (DiRoG) with either the DiL or the DiH. As an example of the first meaning, Sarah explained, "the height squared times the DiRoG of the length equaled the area [$h^2*DiRoG_L=A$]. And then depending on how much it went up by, you would divide

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by that number [e.g., $h^2 * DiRoG_L/2 = A$]." In this case, $a = DiL$ when $DiH = 1$ and $a = DiL/DiH$ when $DiH > 1$. In the second case, students coordinated the difference in the rate of growth (DiRoG) with variation in height or length. For example, in one task, students had to relate the parameter a to the DiRoG. Both Joe and Tai made sense of a as the ratio of the difference in the rate of growth in area over 2 which equals the difference in the length. In this case Tai wrote DiRoG/2=DiL thus $a = DiRoG/2 = DiL, \Delta H = 1$. It is notable that in these cases the symbolic equation actually represents a covariational relationship between coordinated variation in the quantities length and height.

Discussion and Conclusion

This study addresses how students' ways of thinking may support their ability to create symbolic rules from numeric tables embedded in a quantitative context, and meaningfully interpret those rules from both the correspondence and covariational perspectives. Our results suggest that if students' function activity is grounded in a continuous quantitative context, such as the growing rectangle, and guided by a purposeful sequence of tasks focused on encouraging covariational reasoning, students may come to see function rules as representations of covariation. Such a stance can encourage a powerful and flexible understanding of function rules and serve as a productive foundation for further exploration in algebra.

Our findings also suggest that students need not necessarily be flexible in moving back and forth between correspondence and covariation views. Instead, students who develop a strong covariation view of functional relationships may come to make sense of algebraic symbolic rules as statements of covariation (e.g., seeing the parameter a as a relationship between two quantities that vary together). For example, recall Jim's reasoning; for him, the coefficient of 2.25 was a ratio of change in length to change in height. This suggests that a correspondence relation between area and height can emerge from attention to covariation in height and length, demonstrating the power of covariational reasoning in writing function rules (cf. Thompson & Carlson, in press).

Finally, when we consider the results of RQ1 and RQ2, an important finding emerges: students' conceptions of uncoordinated variation or variation alone were generally *not* supportive of their ability to symbolize function rules. This finding contributes to the literature on students' ways of thinking that might help explain representational disfluencies (i.e., unsuccessful translations from verbal descriptions or numeric tables to symbolic rules). On the other hand, we hypothesize that students' representational fluency in symbolizing function rules—especially the meanings they developed for the rule $A = ah^2$—seemed to be supported by (a) grounding their activity in a quantitatively rich task situation, (b) designing task sequences to encourage attention to the nature of how quantities change, (c) asking students to consider how quantities are related, and (d) prompting the generalization of those relationships by extending to far cases. Our approach to focus on the interplay between students' conceptions and meaning making from multiple representations of functions grounded in a quantitatively rich setting is a productive stance that warrants further exploration and research.

Acknowledgements

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DEVIN'S CONSTRUCTION OF A MULTIPLICATIVE DOUBLE COUNTING SCHEME: DUAL ANTICIPATION OF START AND STOP

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In this case study with Devin (pseudonym), which was part of a larger, constructivist teaching experiment with students identified as having learning difficulties in mathematics, we examine how a fourth grader constructed a dual anticipation involved in monitoring when to start and when to stop the simultaneous count of composite units (numbers larger than 1) in multiplicative tasks. We postulate that such a dual anticipation underlies the first, Multiplicative Double Counting (mDC) scheme (Tzur et al., 2013) that marks children's conceptual progress from additive to multiplicative reasoning. Data from three teaching episodes with Devin focus on his anticipation of the start/stop features of his double counting activity. We discuss theoretical implications of these findings in terms of similarity between the dual anticipation in additive and multiplicative reasoning, and practical implications in terms of task design and sequencing.

Keywords: Elementary School Education, Number Concepts and Operations

Introduction

Researchers have developed models for thinking about and promoting children's learning to reason multiplicatively (Park & Nunes, 2001; Sophian & Madrid, 2003; Steffe, 1992, 1994; Steffe & Cobb, 1994; Tzur et al., 2013). By and large, these models are rooted in studies of normally achieving peers' construction of particular multiplicative schemes. Currently, less is understood about how students identified as having learning difficulties (SLDs) construct understandings of multiplicative reasoning (Evans, 2007). To address this lacuna, our study addressed the problem: How may SLDs progress *within* the identified scheme of multiplicative Double Counting—the first in their transition from additive to multiplicative reasoning?

In particular, we examine a possible intermediate stage in SLDs' construction of the mDC scheme, which may be a corollary of the intermediate stage distinction of anticipating starts and stops in the additive operation of counting on (Tzur & Lambert, 2011). Specifically, they found that students who learn to count need to develop an anticipation of both a starting and stopping point for the double count needed. For example, a child who is adding 7+4 would anticipate both a starting point of the number after 7 and the need to double count each item added as being simultaneously a constituent of the sequence of added items (1-2-3-4) and of the total (8-9-10-11). That is, in counting on, the child's counting of the added items serves as a purposeful aid in knowing when to stop her activity. In this paper, we focus on a similar anticipation of knowing where to start and where to stop counting of composite units which can be inferred in SLDs' work.

Conceptual Framework

Our conceptual framework for this study consists of general and content-specific constructs. The general constructs are rooted in a constructivist perspective on knowing and learning (von Glasersfeld, 1995). We view one's transition from not knowing to knowing a new mathematical idea as a mental activity that an observer must infer from observable actions and language of learners. Specifically, we drew on the construct of scheme as a three-part mental structure (von Glasersfeld, 1995). The first part of a scheme is the recognition of a certain situation. Here, a learner uses assimilation to "recognize" the situation based on previously recorded, like experiences. This "recognition template" sets the person's goal. In turn, the goal triggers the second part of the scheme, a mental activity used to accomplish the goal that may or may not be coupled with observable

actions. The third part of the scheme is the result a learner expects to follow the activity based on previous, similar activities.

Building on this three-part notion of scheme and on Piaget's (2001) core notion of reflective abstraction, Simon, Tzur, Heinz and Kinzel (2004) articulated cognitive change as the mental process of reflection on activity-effect relationship. This mental process is postulated to take place as a learner compares between the expected result and actual effect of an activity, and through comparison across instances of similar activity-effect dyads. Learning is marked by a change in the relationship between an activity and newly noticed effects, which turn into a new anticipation that can be linked to different situations.

The content-specific constructs we used are based on Tzur et al.'s (2013) developmental framework of six schemes used by children to reason multiplicatively. They distinguished those schemes based on units and operations a child uses. Specifically, scheme for multiplicative reasoning are thought of in terms of units coordinating activities a child is inferred to be using (Norton, Boyce, Ulrich, & Phillips, 2015; Steffe, 1994; Steffe & Cobb, 1994). In this paper we focused on the first scheme, termed multiplicative Double Counting (mDC).

A child who has constructed the mDC scheme anticipates the effect of coordinating at least two levels of composite units (CU). For example, such a child anticipates that 12 could be composed by coordinating an operation in which, say, 4 units of 1 (e.g., 4 cubes per tower) are distributed over the items of another composite unit (e.g., 3 towers) (Tzur et al., 2013). Consider, for example, a child who is presented with a multiplicative situation such as, "There are six packs of gum, each with 4 pieces of gum; how many pieces of gum are there in all six packs together?" A student who can bring forth and use the mDC scheme might sequentially hold up six fingers while simultaneously distribute four pieces of gum across to each (e.g., the first is 4, the second is 8, etc.), and arrive at the answer of 24 pieces of gum.

Research has suggested several possible reasons that SLDs may not make adequate progress in mathematics, including reliance on counting based strategies longer than normally achieving peers (Geary & Hoard, 2003), memory issues (Raghubar, Barnes, & Hecht, 2010), and SLDs' lack of the developmental requisite of composite unit (Tzur, Xin, Si, Kenny, & Guebert, 2010) to construct multiplicative reasoning. However, it should be noted that different researchers define mathematical disabilities in various ways. In our study we drew on a definition that includes both students who may have "qualified" for specialized support given to those identified as having mathematics learning disability as well as students who are significantly behind in mathematics achievement (Mazzocco, 2007). Throughout the rest of the paper, we will use the term students with learning difficulties (SLD) to refer to students who are significantly behind in mathematics—whether or not they were identified by their school system as such.

Methodology

The case study on which we report in this paper was part of a larger constructivist teaching experiment (Cobb & Steffe, 1983) conducted with four 4th graders in a western US school as part of the first author's doctoral dissertation. The students were sampled for the instructional intervention based on underachievement measured on state mathematics assessments along with their classroom teacher recommendations. The two first authors conducted the video recorded teaching episodes with the group of students or with individual students. We focus in this paper on teaching episodes conducted with one student, Devin (pseudonym), twice a week (30-45 minutes each), from October through December of 2014.

To teach and study Devin's construction of the mDC scheme, in each episode the researcher-teachers engaged him in playing a version of the game, *Please Go and Bring for Me* (PGBM), which Tzur et al. (2013) described in detail. The PGBM game was designed to promote children's reflection on the units used in multiplicative reasoning, by asking them to build towers from a given number of

single cubes. The game is played in pairs (with either a peer or the teacher). The players take turns as either a "Sender" or a "Bringer." The Sender poses the task by asking the bringer to build and bring back several, same-size towers, one tower at a time. For example, the Sender may plan to ask the bringer to bring 3 towers with 5 cubes; she would first ask the bringer to construct and bring one tower of 5, then another tower of 5 cubes, etc. Once the bringer brought all towers to the Sender's satisfaction, the Sender asks the Bringer four questions (in our work – those were written on a poster to promote students' use of full sentences and explicit mention of units): (a) How many towers did you bring (emphasizes number of composite units)? (b) How many cubes are in each tower (emphasizes unit rate – number of 1s in each composite unit)? (c) How many cubes are in all the towers? (d) How did you figure this [total of 1s] out? Similarly, the poster included 'answer-starters' that enabled the bringer to express her answers as full sentences (e.g., "I brought __ towers"). Initially, the teacher constrained the game so Devin could only use particular numbers of cubes per tower (e.g., 2 or 5) and of towers in all (e.g., up to 6 towers). The teacher also asked Devin to use different numbers for each kind of unit (e.g., disallow bringing 5 towers of 5 cubes each).

Our line-by-line retrospective analysis of video records, transcripts, and researcher field notes taken during each episode focused on the third and fifth teaching episodes with Devin. The focus of the analysis was on transitions Devin might make within the multiplicative Double Counting (mDC) scheme, while 'zooming in' on the interplay between his ways of operating and the numbers chosen in each task. The two first authors conducted ongoing analysis following each episode. The entire team of authors then conducted the line-by-line analysis of the three segments presented in the next section.

Results

In this section we present and analyze three excerpts that demonstrate advances in Devin's construction of goal-directed activities of coordinated counting to solve multiplicative tasks in which we used what for him were harder numbers (e.g., 5 towers of 6 cubes each and 7 towers of 6 cubes each, symbolized as 5T6 and 7T6, respectively). We selected these data because, prior to Devin's work on these tasks, to solve PGBM tasks with "easier" numbers he could independently call up an anticipation suitable for figuring out the total of 1s in a compilation of CUs via a coordinated count of the dual accrual of 1s and CUs. However, he was yet to construct and independently use an anticipation of the need to monitor his simultaneous count of composite units to know where to stop the count when "harder" numbers were given.

Excerpt 1 presents how Nina (pseudonym, the teacher/researcher and second author of this paper) began the work with Devin in the typical way PGBM is played, particularly when towers are hidden, namely, asking him to repeat (and firmly establish) the number of towers and the number of cubes per tower. This common practice when playing the game can help eschew a claim that the child's difficulties result from short-term memory issues. Excerpt 1 shows Devin's facility in the anticipation required for the double counts in the accumulation of up to 5 towers, and his inability to anticipate where to stop in the accumulation of the unit rates.

Excerpt 1, keeping track of where to stop when counting the unit rate (student: Devin; task 5T6; date: October 15, 2014).
11:21 *Nina*: How many towers are there, Devin?
11:28 *Devin*: There are s…. There are … how many towers? Five towers.
11:30 *Nina*: How many cubes are in each tower?
11:32 *Devin*: There are six cubes in each tower.
11:34 *Nina*: [Covers the towers with a piece of paper after Devin has confirmed the number of towers and the number of cubes per tower.] How many cubes are there in all?

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11:45 *Devin*: [Under his breath.] Six; twelve; [He presses the thumb and index finger of his right hand on the table to indicate the first two towers, then shifts to counting six 1s with his left hand starting with his thumb.] 13, 14, 15, 16, 17, 18; [He presses the third finger of his right hand on the table and continues to incorrectly count five 1s using the fingers on his left hand.] 19, 20, 21, 22, 23; [He then presses the fourth finger of his right hand and continued to count five more 1s.] 24, 25, 26, 27, 28; [He then presses the fourth finger of his right hand and continues to count six more] 29, 30, 31, 32, 33, 34. [He raises his hand indicating that he has finished and is ready to give his answer.]

Devin's response to the question about how many cubes are there in all indicated an anticipation of the need to coordinate the counts of 1s and CUs. Specifically, he independently initiated a start of the counting from the second multiple of 6, indicating his coordination of the compilation of CUs and the unit rate. That is, with the first two easy (for him) numbers in the sequence of multiples of 6, he seemed to distinguish the number of 1s in each CU from the number of composite units for which he had been accounting so far. As he tracked the accumulation of cubes past these two easy numbers (6, 12), however, he shifted to counting only five (instead of six) items per composite unit while keeping track of six towers (instead of just five). That is, whereas with "easy" numbers the role of each unit in regulating Devin's activity was properly anticipated, with "hard" numbers his effort to focus on accrual of both types of unit took over these roles. In this sense, Devin provided an example of a student for whom a claim that the more difficult (5+n) numbers impact construction of mDC can be demonstrated.

Excerpt 2 provides data from an instructional prompt from Nina designed to capitalize on Devin's work (and errors). In this excerpt, Nina oriented Devin's reflection to the way in which he accurately used his left hand to keep track of six cubes.

Excerpt 2, teacher/researcher supporting tracking the unit rate (student: Devin; task 5T6; date: October 15, 2014).

12:42 *Devin*: There are 34 cubes altogether.

12:57 *Nina*: Now I want you to double check.

13:00 *Devin*: [Uncovers the cubes and starts counting at 12 cubes for two towers (confirming his mDC even when the cubes were available to him). Moves the first two towers and continues counting the rest of the towers by 1s, arriving at 30. Smiles and looks at Nina.] I got it wrong.

13:10 *Nina*: [Orienting Devin on his own tracking methods.] So, Devin; what I saw you doing, which I thought was so good, is this. [Nina holds her left hand over the table and begins to fold each finger down.] You had this hand and were going like this [Holds her left hand over the table and begins to fold each finger down.] Can you tell me what you were doing?

13:24 *Devin*: [Places both hands on the table and moves his right hand.] That [hand] was towers [moves his left hand] and this [hand] was [for] my cubes.

13:26 *Nina*: [Models Devin's error.] And what I saw you doing was something like this. [Folds over each of 5 fingers on her right hand]

13:34 *Devin*: Oh, it was 5 [Indicates possible recognition of his error of counting 5 cubes instead of 6 cubes.]

13:37 *Nina*: So you were doing… almost like… [Gestures at Devin to show his count.] Can you show me?

13:42 *Devin:* Um, oh so 5 towers [presses his right hand on the table] and cubes [presses his left hand on the table] so 6+6 is 12.

13:56 *Nina*: So how many towers is that?

13:57 *Devin*: [Demonstrates his ability to track towers.] Two.

13:58 *Nina*: Ok.

13:59 *Devin*: So my third tower is [presses each finger of his left hand on the table and counts his thumb twice for an accurate 6 count.] 13, 14, 15, 16, 17, 18.

14:06 *Nina*: Ok … [Does not indicate whether the response is correct or incorrect, but encourages him to continue.]

14:08 *Devin*: My fourth tower is: 19, 20, 21, 22, 23, 24.

14:13 *Nina*: Ok …

14:15 *Devin*: My fifth tower is: 25, 26, 27, 28, 29, 30.

14:20 *Nina*: [Compares Devin's answer to another student's response they both heard previously.] So you got 30, too.

Excerpt 2 shows that, with Nina's prompting, Devin appeared to recognize the inconsistency in his previous counting of the unit rate in this task (13:34). Furthermore, he was able to re-orient his attention to systematically monitoring the subsequent stops of each of the unit rate counts at 6. We assert that when Nina oriented Devin's reflection on his own tracking methods, she fostered his monitoring of his own goal-directed activity. Consequently, Devin could begin monitoring a stop at 6 for each tower as he monitored the accrual of the total number of cubes. A crucial point in the shift in Devin's counting activity is that, for each tower, he *first* stated its number in the sequence of accruing CUs and only then interjected the 1s that constituted that CU. This is a subtle but important difference from first counting the 1s and then raising a finger for the CU, in that the count of 1s after stating the CU's ordinal number may indicate distribution of those items into each of the CUs (as shown in the study by Clark and Kamii, 1996).

Excerpt 3 provides data from Devin's work on a task during an episode that took place three teaching episodes after the one presented in Excerpts 1 and 2. In Excerpt 3, Devin still needed prompting to regulate his stoppage of the count of the unit rate when solving mDC tasks with harder numbers. This provides additional evidence of the difficulty of developing the double anticipation (start, stop) required for an mDC when faced with larger numbers. Despite of Devin's acceptance of Nina's suggestion, and his subsequent ability to monitor the stops in the unit rate for 5T6 in Excerpt 2, Devin's anticipation of where to stop counting the unit rate remained inconsistent. This inconsistency suggests that, in the previous episodes, Devin's construction of the coordinated count was at the participatory (prompt-dependent) stage.

In Excerpt 3, Devin first determined the total number of cubes in 7 towers of 6 cubes in each tower (7T6). In this case, we purposely increased the difficulty of the numbers because both—the compilation of CU and the unit rates—exceeded the number of fingers (five) on each of his hands. Devin readily anticipated where to start both counts, but again was unable to regulate where to stop the unit rate count past the second tower.

Excerpt 3, anticipating the stop when tracking the unit rate (student: Devin; task 7T6; date: October 23, 2014).

15:24 *Devin*: [Raises the thumb and index finger of his right hand] So 6+6 is 12; [then] 13, 14, 15, 16, 17, 18 [Raises the middle finger of his right hand and continues counting.] 19, 20, 21, 22, 23, 24 [Raises the ring finger of his right hand and continues counting albeit with 5 counts this time.] 25, 26, 27, 28, 29, [Raises the pinkie finger of his right hand and continues counting 5 counts again.] 30, 31, 32, 33, 34 [He folds all fingers on his right hand and raises his thumb again. At this point, the thumbs of his right hand (towers) and left hand (cubes) are raised.] He counts six counts.] 34, 35, 36 37, 38, 39…. I got 39 but I got 40 before.

Excerpt 3 indicates that, in this task, Devin could independently initiate the coordinated count while beginning from the second (known) multiple of 6, just as he had in Excerpt 2. This independent initiation indicated his anticipated coordination of the compilation of CUs and the unit rate. That is, with the first two "easy" (for him) numbers in the sequence of multiples of 6, he seemed to clearly

distinguish and monitor the number of 1s in each CU and the number of CUs for which he had been accounting so far. As he tracked the accumulation of cubes for the fifth and sixth multiples of 6, however, he shifted to counting only five (instead of six) items per CU. At the seventh multiple, he recounted 34 (the last number he said when incorrectly counting the 6th multiple), and then accurately counted 6 counts of 1s. That is, whereas Devin's prompt-dependent anticipation of when to start/stop each count with "hard" numbers lagged behind his independent, correct use of this anticipation with "easy" numbers.

Discussion

In this paper we examined a difficulty SLDs may face when learning to anticipate both where to start and where to stop in each of the coordinated counts required in mDC. The difficulty to anticipate this dual monitoring illustrates a possible intermediate stage in the development of mDC, particularly by SLDs. mDC requires a coordination of more than one count and each of the coordinated counts requires a dual anticipation of where to start and where to stop. The analysis of Devin's work in this study amounts to suggesting that mDC requires the learner's (and teacher's) attention to both an anticipation of where to shift each count of a CU (unit rate, first anticipation) and of where to stop a count for the compilation of CUs (second anticipation). It is this dual anticipation that makes operating on harder numbers, for example, when CUs and/or 1s exceed the number of fingers on one hand, a challenging feat to overcome. We suggest that this difficult conceptual shift may be rooted in the need to count figural items standing for each type of unit through a 1-to-1 correspondence (e.g., finger and the unit for which it stands).

In addition, Devin was able to recall the number of towers and the number of cubes per tower, but not yet to anticipate the stops when operating on those units. Thus, his case indicated that besides memory issues, other, conceptually born factors may underlie challenges SLDs' face when constructing the mDC for any number. That is, Devin could clearly and independently remember and accurately continue a coordinated count from the 2nd multiple of six. With prompting, he could also distinguish the number of 1s in each CU and the number of CUs for which he had been accounting so far (2T6). However, Devin's action in the subsequent coordinated counts highlighted the ongoing difficulty of anticipating stops within the operation of a double count in the development of mDC. This finding seems consistent with Tzur and Lambert's (2011) identification of two intermediate sub-schemes in children's progress from counting all to counting on in additive situations. Specifically, it suggests a possible extension to include anticipation of the start and stop of two simultaneous counts in multiplicative situations.

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THE RELATIONSHIP BETWEEN FLEXIBILITY AND STUDENT PERFORMANCE ON OPEN NUMBER SENTENCES WITH INTEGERS

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To better understand the role that ways of reasoning play in students' success on integer addition and subtraction problems, we examined the relationship between students' flexible use of ways of reasoning and their performance on integers open number sentences. Within groups of students in 3 participant groups—39 2nd and 4th graders who had negative numbers in their numerical domains, 40 7th graders, and 40 successful 11th graders—we found that flexibility and success were positively related. That is, the more flexibly students invoked ways of reasoning, the greater their success. These findings indicate that rather than searching for one particular model or approach to teaching integer operations, teachers should support multiple ways of reasoning and discuss features of problems that might make one way of reasoning more productive than another.

Keywords: Number Concepts and Operations, Cognition, Learning Trajectories (or Progressions)

Students' lack of success operating on negative numbers is well documented in the literature (Kloosterman, 2014; Vlassis, 2002). Understanding students' thinking about integer addition and subtraction is important for teachers to better support students' learning and to promote their transition from arithmetic to algebra (Peled & Carraher, 2008). We see our research on the flexibility of students' ways of reasoning about integers as connecting to the PMENA theme of Questioning Borders in our examining the role of implicit borders among instructional approaches to teaching integers. Existing literature tends to be focused on identifying a single best model or instructional practice for supporting students' understanding of integers, that is, promoting a *border* around a single practice. Our findings indicate that an opening of borders (or promoting more than one way of reasoning) may be more productive for students than a closed-border approach (or reliance on a single way to learn to operate with integers).

Theoretical Framework and Literature Review

We approach our research from a children's mathematical thinking perspective. We consider it important to see mathematics through children's eyes to better understand the sense that they make. This perspective is based on principles that children have existing knowledge and experiences they bring with them into the classroom and upon which they continue to build (e.g., Carpenter et al., 1999). We take this view because our ultimate goal in our research is to find ways to better support children's learning of mathematics, and instruction that builds on students' ideas benefits both teachers and students and supports rich instructional environments (Sowder, 2007; Wilson & Berne, 1999). Moreover, we hope that by studying the variety of ways students reason about integer arithmetic we can broaden existing instructional approaches so that more students can successfully engage with this important mathematical topic.

Literature Related to the Teaching and Learning of Integers

To provide background for this paper, we examine the research on computational fluency, students' understanding of integers, and integer instruction. Much of the research related to students'

understanding of integers is focused on secondary students' difficulties with integer arithmetic (Gallardo, 2002; Kloosterman, 2014; Vlassis, 2008). Students also have difficulties solving algebraic equations, simplifying algebraic expressions, and comparing quantities that include negative integers (Christou & Vosniadou, 2012; Vlassis, 2002). However, researchers have found children to be capable of reasoning about integers in relatively sophisticated ways, even in the lower elementary grades (Behrend & Mohs, 2006; Bishop et al., 2014; Bofferding, 2014). Despite these findings, few researchers have focused on students' ways of reasoning about integers. Instead, in the majority of integer-related studies, researchers developed and tested a variety of approaches to the teaching of integers, using various models, tools, and contexts (see Liebeck, 1990; Linchevski & Williams, 1999; Stephan & Akyuz, 2012). In most articles related to integers instruction, a single model or instructional approach, such as movement on a number line or zero pairs, was proposed. Although some models showed promise in terms of student achievement, no model is perfect, and each one has both affordances and limitations. The lack of compelling results across interventions led us to consider whether multiple approaches might be warranted; thus, we shifted to the literature on flexibility, described in the next section.

Literature Related to Flexibility

Although the research related to integers instruction has been focused on identifying one way to support students, the research on flexibility heightened our curiosity about the role that flexibility might play in relation to students' success in solving integers problems. Star and Newton defined *flexibility* as "knowledge of multiple [strategies] as well as the ability and tendency to selectively choose the most appropriate ones for a given problem and a particular problem-solving goal" (2009, p. 558). In recent years, researchers have identified flexibility as a critical characteristic of one who has attained deep procedural knowledge (Star & Newton, 2009) and as a characteristic shared by expert mathematicians (Dowker, 1992; Star & Newton, 2009). Additionally, many researchers have studied the influence or development of flexibility in different content domains, ranging from 2-digit-addition and -subtraction to estimation, multistep linear equations, and proportional reasoning (e.g., Berk, Taber, Gorowara, & Poetzl, 2009; Blöte, Van der Burg, & Klein, 2001; Dowker, 1992; Star & Newton, 2009).

Given middle-school students' well-documented struggles with integers, we questioned the emphasis in the literature on finding a single, best model to support students' learning. Instead, we sought to determine the degree to which flexible use of ways of reasoning influenced students' performance in solving integer problems. We asked the following research questions: How flexible are students in using ways of reasoning when completing integer open number sentences, and to what degree is students' flexible use of ways of reasoning related to their success in this activity? In reviewing the literature, we did not find any study that systematically sampled students at different grade levels to document the flexibility in their reasoning about integer addition and subtraction. We here provide a cross-grades view of flexibility in students' ways of reasoning about integer addition and subtraction and then relate flexibility to performance. The findings reported here advance the field's understanding of students' flexibility and performance on integer addition and subtraction problems. This study contributes to the efforts of the mathematics education research community to support instruction that enables students to successfully transition from arithmetic to algebra.

Methods

Background and Participants

This study is part of a larger project in which our goal was to understand K–12 students' conceptions of integers and integer arithmetic. In this paper, we focus on flexibility in students' ways of reasoning—an aspect of integer understanding rarely mentioned in the literature yet which we

believe supports students to develop deeper conceptions of integers as well as more efficient computational strategies. Data for this study include clinical interviews with 160 students from 11 schools (3 elementary, 3 middle, 1 K–8, and 4 high schools) in the Western United States. During the 2010–2011 school year, these students participated in individual interviews about integer addition and subtraction. We conducted clinical interviews with students in Grades 2, 4, 7, and 11 (40 children from each grade level). Students in Grades 2 and 4 had yet to receive school-based integer instruction; students in Grade 7 had completed integer instruction; students in Grade 11 were enrolled in a precalculus or calculus course and, because of their course taking, were deemed to be successful high school mathematics students. Because our focus is on understanding students' productive ways of reasoning, we restrict our findings in this paper to students who had some knowledge of negative numbers: *2nd/4th with negatives* ($n = 39$), those 2nd- and 4th-grade students who provided evidence of at least limited knowledge of negative numbers, *7th-grade students*, and *successful 11th-grade students*. The *2nd/4th with negatives* group included 13 Grade 2 students and 26 Grade 4 students who, on the basis of responses to tasks posed at the beginning of the interview, provided evidence of having at least some knowledge of negative numbers.

Clinical Interview

The videotaped 60–90-minute clinical interviews (Ginsburg, 1997) were conducted at the students' school sites. Although we sought to understand and follow the child's thinking during the interviews, the interviews were standardized: All children were posed the same set of 47 tasks, except for those students who did not have negative integers in their numeric domains. The interview had four categories of tasks: introductory questions, open number sentences (e.g., -3 + ☐ = 6 and ☐ + 6 = 4, with unknown location varying), contextualized problems, and comparison problems. Findings shared herein are from responses to the open number sentences.

Coding and Analysis

The interviews were coded at the problem level for both correctness and the underlying way of reasoning the child used. The Ways of Reasoning coding scheme was developed and refined iteratively over a period of 2 years. We identified five broad categories we call *Ways of Reasoning* and a total of 41 subcodes that provide more detail as to the child's specific strategy or strategies. The five ways of reasoning are order-based, analogy-based, computational, formal, and developmental (see Table 1 below for definitions). Each response to the 25 open number sentences was assigned a way-of-reasoning code (some responses involved more than one way of reasoning and, thus, received multiple codes). For example, some students completed -3 + 6 = ☐ by counting up 6 units, by ones, from -3. This solution would be coded as an order-based way of reasoning because counting leverages the ordered and sequential nature of numbers. However, another student might complete the same sentence using analogy-based reasoning, explaining, "It's like I borrowed 3 dollars from my friend. It's like I owe him; that's minus 3. And I give him 3 from the 6 my mom gave me, and now I have 3." We would code this response as analogy-based reasoning because the student solved the problem by comparing negative integers to owing money. Of the 160 interviews 42 (or 26.25%) were double coded, and interrater agreement was 92% at the Ways of Reasoning level and 83% at the subcode level.

Table 1: Ways of Reasoning About Integer Addition and Subtraction

Ways of Reasoning	Definitions
Order-based	In this way of reasoning, one leverages the sequential and ordered nature of numbers to reason about a problem. Strategies include use of the number line with motion as well as counting forward or backward by 1s or another incrementing amount.
Analogy-based	This way of reasoning is characterized by relating numbers and, in particular, signed numbers, to another idea, concept, or object and reasoning about negative numbers on the basis of behaviors observed in this other concept. At times, signed numbers may be related to contexts (e.g., debt or digging holes). Analogy-based reasoning is often tied to ideas about cardinality and understanding a number as having magnitude.
Formal	In this way of reasoning, negative numbers are treated as formal objects that exist in a system and are subject to fundamental mathematical principles that govern behavior. Students may leverage the ideas of structural similarity, well-defined expressions, and fundamental mathematical principles.
Computational	In a computational way of reasoning, one uses a procedure, rule, fundamental mathematical principle, or calculation to arrive at an answer to a problem involving negative numbers either as part of the problem statement or as appearing in the solution set.
Developmental	This category of reasoning often reflects preliminary attempts to compute with signed numbers. An example of this category is a child's overgeneralization that addition always makes larger with the claim that a problem for which the sum is less than one of the addends (e.g., $6 + \square = 4$) has no answer. In this case, the domain of possible solutions appears to be restricted to natural numbers and the effect (or possible effect) of adding a negative number is not considered.

Measuring flexibility. *Flexibility* is a measure of the variety of ways of reasoning students use to solve integer-arithmetic tasks. Flexibility indicates whether a student primarily uses one way of reasoning or chooses different ways of reasoning depending on the affordances of the problem. We calculated a flexibility measure for each student by identifying the number of times each way of reasoning was used across all 25 open number sentences. Note that we did not include the developmental way of reasoning in this calculation because developmental approaches reflected preliminary attempts to compute and typically resulted in incorrect responses. We deemed a student to be proficient with a particular way of reasoning if that student used it three or more times during the interview (that is, used a given way of reasoning on at least 12% of the open number sentences). The number of ways of reasoning with which a student was proficient was our measure of flexibility. For example, if a student used order-based reasoning on 7 open number sentences, analogy-based reasoning on 2 open number sentences, formal reasoning on 3 open number sentences, and computational reasoning on 16 open number sentences, the student would be proficient with 3 ways of reasoning and receive a flexibility score of 3. We used this flexibility score to explore the relationship between flexibility and performance on the 25 open number sentences of students in the participant groups: 2nd/4th with, 7th-grade students, and 11th-grade students. In the next section, we report findings and provide case studies of three students to underpin the quantitative findings.

Findings

We found that flexibility is positively correlated with performance in our data within grades ($r = .347, .523, .429$ for 2/4, 7th, and 11th, respectively, two-tailed, all p-values < .05). In Table 2, we share frequency counts for flexibility scores of 0, 1, 2, 3 and 4 at each grade level. These frequency counts reflect the number of students who were proficient with 0, 1, 2, 3 or 4 ways of reasoning.

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More than half of the 2nd/4th graders with negatives used two ways of reasoning, and almost all (87%) used 1 or 2. The 7th graders had the greatest spread in flexibility; although almost half (45%) used 3 ways of reasoning, one fourth used 2 ways and one fourth used 4 ways of reasoning. Finally, more than half of the 11th graders (55%) used 3 ways of reasoning, and almost all (85%) used either 3 or 4. The findings that 11th graders were both the most flexible in their ways of reasoning and the most accurate indicate that particular ways of reasoning are not necessarily *replaced* by other, more sophisticated, ways of reasoning but, rather, that students who have access to and use multiple ways of reasoning are more successful than those who do not.

Table 2: Frequency Counts for Ways of Reasoning

Flexibility score	Grades 2/4 with negatives ($n = 39$)	Grade 7 ($n = 40$)	Grade 11 ($n = 40$)
0	2 (5%)	0 (0%)	0 (0%)
1	11 (28%)	3 (8%)	4 (10%)
2	23 (59%)	10 (25%)	2 (5%)
3	3 (8%)	18 (45%)	22 (55%)
4	0 (0%)	9 (23%)	12 (30%)

Cases

We provide cases of three 7th-grade students—Hannah, Sofia, and Maria (pseudonyms)—to exemplify the relationship between flexibility and accuracy. Hannah invoked order-based reasoning on more than 80% of all open number sentences, whereas Sofia primarily used a computational way of reasoning. Maria invoked every productive way of reasoning, each on one fourth to one half of the open number sentences.

The case of Hannah. Hannah completed about one third (32%) of the open number sentences correctly and used order-based reasoning almost exclusively. Although she could use this way of reasoning productively at times, more often than not her almost exclusive use of order-based reasoning hindered her success. Hannah tended to be able to productively use order by invoking the strategy *motion on a number line* for problems such as -3 + 6 = ☐ and -9 + ☐ = -4, for which the second addend (or, for subtraction problems, the subtrahend) is positive, correctly completing 73% of these types of number sentences. However, she also used the same order-based way of reasoning for almost every other problem and answered every one of them *incorrectly*. For example, on 6 – -2 = ☐, Hannah placed her pen at 6 on the number line and paused. She then moved her pen to the left 2 units and answered 4, explaining, "You go to positive 6 minus negative 2 equals 4." The interviewer asked for clarification and Hannah added, "I am just a little bit confused by this (points to the open number sentence), so I just looked at this (points to the number line). So then I just minused 2, and I got 4." The discussion continued.

Interviewer: Which part is a little confusing?
Hannah: Because it is a positive 6 minus a negative 2 (she points to the number line). I don't know if you go down here (moves her pen to the negative side of the number line), or, I don't know, but I think it's 4."
Interviewer: Was there something else (another answer) you were considering?
Hannah: Negative 4, but then I was like, "No. Because it [-4] is all the way down here (points to the negative side of the number line)."

Hannah appeared to implicitly recognize that the problem has a structure different from the other problems she had successfully solved, and she shared her confusion about how to adapt her use of motion on the number line when subtracting a negative number. Her second answer of -4 (also incorrect) appeared to reflect her attempt to account for subtracting -2 (rather than +2). Hannah's

almost exclusive use of order-based reasoning limited her options for solving problems. Because she appeared to have no other ways of reasoning to support her completion of the open number sentences, she was often unsuccessful, as were most others who exclusively relied on one way of reasoning. In the next case we describe Sofia, who, like Hannah, tended to rely primarily on one way of reasoning and also had limited success. Unlike Hannah, however, Sofia used primarily computational reasoning.

The case of Sofia. Sofia was more successful than Hannah, inasmuch as she answered about two thirds (64%) of the problems correctly. She used primarily computational reasoning. For example, on the problem $6 + \square = 4$, Sofia correctly answered -2, using a rule. In particular, she used what is often referred to as the *different-signs rule*, such that when adding a negative number and a positive number, one finds the difference of the absolute values of the numbers and appends the sign of the number that has the larger absolute value.

> *Interviewer*: So how come you can put -2?
> *Sofia*: Because [the sum] can still become a 4 if you subtract from a positive and a negative. You can still get a number that is positive because this one (points to 6) is bigger, and then the answer has to be positive. If [the positive number] is a lower number, like if a negative number is higher (points to -2), and this one (points to 6) is like 3, [the sum] becomes a negative."

Sofia solved this problem by applying a rule: She subtracted 2 from 6—what she described as "subtract[ing] from a positive and a negative"—and the difference of 4 takes the sign of the number with the larger absolute value. Her focus on primarily computational approaches likely explains why she also applied the different-signs rule on a problem for which the rule was not applicable. For example, on the problem $-8 - 3 = \square$, Sofia incorrectly answered -5, explaining that she subtracted 3 from 8 and, because the 8 was negative, her answer should also be negative. "When [the absolute value of] a negative number is bigger than a 3, like, a positive number (points to 3), if this one (points again to 3) is lower [than the absolute value of -8], then the answer becomes a negative sign." For this problem, Sofia inappropriately invoked the different-signs rule. Similarly, she invoked a rule for multiplying negative numbers to complete $-5 + -1 = \square$. Sofia explained, "I think it is 6 because if you add both negatives, then it becomes a 6, but if you see the signs negative and a negative, it becomes a positive number, so I say it's a 6." When asked why two negatives become a positive, Sofia responded, "Because when the signs are the same, it becomes a positive. Well, my teacher said that if … it was two negatives, then it [the result] would become a positive."

Like Hannah's exclusive focus on order-based reasoning, Sofia's focus on computational reasoning appeared to hamper her success. Looking across both cases, we see that the particular way of reasoning upon which students focused was less important than the fact that each student appeared to have one way of reasoning on which she relied, and being limited to a single way appeared to negatively influence success.

The case of Maria. In contrast to Hannah and Sofia, Maria completed every open number sentence correctly and used all four productive ways of reasoning. For example, she used order-based reasoning for $-3 + 6 = \square$ by "jumping to" 0. Maria explained her answer of 3 saying, "Half of 6 is 3, so then that would bring it to the 0. And 3 more would bring it to the 3. And that would equal 6." Maria decomposed 6 into 3 and 3 so she could jump to 0 using a partial sum. Her strategy of using a decade number of 0 supported her to count more efficiently than counting by ones. Maria then used analogy-based reasoning on the problem $-5 + -1 = \square$, by comparing negative numbers to "bad guys." For this problem she correctly answered -6, saying, "Since negative numbers are like bad guys, 5 bad guys met up with one more bad guy, so there were 6 bad guys total." In contrast, she correctly completed the number sentence $6 - -2 = \square$ using computational reasoning, explaining, "I changed the signs so it was plus-plus, and 6 + 2 is 8." Finally, Maria used formal reasoning for $6 + \square = 4$.

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She answered, "Negative 2. It's [the unknown] not going to be a positive because the sum is less than 6 and it [the operation] is addition. So it [the missing addend] has to be a negative number. So, -2." This response received a formal code because Maria used deductive reasoning (by noticing that if she added two addends and the sum was less than one of the addends, then the other addend must be a negative number) to determine that the sign of the number had to be negative.

Similar to others who used all the productive ways of reasoning, Maria not only flexibly invoked a wide range of strategies on the problems but also appeared to choose strategies that corresponded with the underlying structure of the problem, indicating that her (perhaps implicit) attention to the underlying structure evoked particular ways of reasoning. This attention appeared to be influential in her choice of strategy and, presumably, in her success.

Discussion

Our unique contribution to the research literature is documenting statistically significant and moderate-to-strong correlations within participant groups between flexibility and performance on integers open number sentences. Additionally, the cases of Hannah, Sofia, and Maria provide insight into how the degree of flexibility played out in relation to students' performance on the open number sentences. Although not causal, these findings indicate a need for students to have several ways of reasoning at their disposal. Further, the finding that 11th graders were both the most flexible in their ways of reasoning and the most accurate indicates that particular ways of reasoning are not necessarily *replaced* by other, more sophisticated, ways of reasoning but, rather, that a mark of expertise in operating with negative numbers is the flexible use of many ways of reasoning. These findings have implications for instruction, which we discuss below.

Implications and Contributions

In other work, we have found that many younger students used more than one productive way of reasoning about integer-related tasks prior to school-based instruction (Bishop et al., 2014). Thus, in any given classroom, teachers across grade levels may have different students approach the same problem in different ways. Teachers should be aware that they can leverage these different ways and that having multiple ways of reasoning appears to promote successful performance on these problems. That is, no single best model or way of reasoning that students successfully use across all problems exists, and attempting to teach students one all-encompassing way may have the unintended consequence of impeding students' success by limiting their flexibility. Rather than teaching any one particular strategy, teachers could instead discuss a variety of ways of reasoning and features of problems (signs of numbers, relative sizes of numbers in the problem, and operation) that might evoke one way of reasoning more than another.

Our findings align with the following famous quote: "I suppose it is tempting, if the only tool you have is a hammer, to treat everything as if it were a nail (Maslow, 1966, pp. 15–16). Those students who relied on a single tool (or way of reasoning) approached every problem with that tool, and they tended to be less successful than those who had a variety of tools from which to choose. This finding is particularly powerful in that the correlations between flexibility and accuracy held for every participant group—the more flexible students were, the more successful they were. Thus, the ways of reasoning may be thought of as a tool belt. We find that, in general, the more tools the better.

Acknowledgments

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DISTINGUISHING BETWEEN SCHEMES OF MATHEMATICAL EQUIVALENCE: JOE'S TRANSITION TO ANTICIPATORY QUANTITATIVE RELATIONAL EQUIVALENCE

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This study examined how a child constructed a scheme (abbreviated QRE) for producing mathematical equivalence via operations on composite units between two multiplicative situations consisting of singletons and composite units. Within the context of a teaching experiment, the work of one child, Joe, was analyzed over the course of 14 teaching episodes. Joe made the conceptual advance from a Relational Equivalence Scheme (RE) and absence of a Quantitative Relational Equivalence Scheme (QRE), through the participatory stage, to an anticipatory stage of a QRE scheme. Joe's progression distinguished between creating equivalence with an RE via operations on singletons and a QRE—a conceptual root for fundamental algebraic concepts such as the distributive property, solving linear equations, and a relational understanding of the equal sign.

Keywords: Learning Trajectories (or Progressions), Algebra and Algebraic Thinking

Introduction

This study examines the schemes of mathematical equivalence a child constructed to create equality between two multiplicative situations. Developing an understanding of mathematical equivalence is a foundational idea for algebra (Blanton et al., 2015; Knuth, Stephen, McNeil, & Alibali, 2006; McNeil, Fyfe, & Dunwiddie, 2015). Algebraic concepts such as solving linear equations and a relational understanding of the equal sign are rooted in children's conceptions of equivalence (Kieran, 1981; Knuth et al., 2006; McNeil et al., 2015). It is also well documented, however, children's struggles solving problems that require an understanding of equivalence (Falkner, Levi, & Carpenter, 1999; Kieran, 1981; Knuth et al., 2006; Stephens et al., 2013). This highlights the critical importance of learning what it means for children to come to understand equivalence and how this process occurs.

This study investigated schemes children construct for creating equivalence between two multiplicative compilations of abstract composite units (CUs) (Steffe & Cobb, 1998) (such as 7 baskets with 4 apples in each basket and 13 baskets with 4 apples in each basket). This requires operating on and coordinating two compilations and the difference between them. There are many ways to solve such tasks without concrete 1s present. For the purposes of this study, I will focus on two specific types of solutions that can occur when the CUs differ between the two compilations, but the unit rates are the same (13 CU of 4 and 7 CU of 4). For one solution the child may first find the total 1s in each compilation via multiplication (7•4=28, 13•4=52), subtract to find the difference in 1s (52-28=24), and then operate with the difference in 1s to create equivalence (52-12=28+12). Alternatively, the child may first produce the difference in CUs between the two compilations (13-7=6), and then operate with the difference in CUs to create equivalence (7+3=10, 13-3=10).

Multiplicative reasoning was chosen as the context for this study for two significant reasons. First, research on mathematical equivalence and children's understanding of the equal sign (Falkner et al., 1999; McNeil et al., 2015) has predominately focused on problems that require additive reasoning (e.g. 8+4=__+5). Results in the domain of additive reasoning have pointed to the possibility young children can develop a relational understanding of the equal sign (Baroody & Ginsburg, 1983; Jacobs, Franke, Carpenter, Levi, & Battey, 2007). However, the schemes of mathematical equivalence children develop have not been identified and it is also not known if equivalence schemes evolve as children's reasoning expands. The second reason the use of

multiplicative situations is so critical is that they allow for differentiation between children's schemes based on the units they operate on and with (Steffe & Cobb, 1988). A child's additive and multiplicative reasoning afford and constrain the algebraic schemes they construct (Steffe, Liss, & Lee, 2014). By including multiplicative situations, how children's constructions of equivalence schemes are afforded and constrained by their current additive and multiplicative reasoning can be studied.

Conceptual Framework

The cognitive foundation for this study was provided by the reflection on activity-effect relationship (*Ref*AER*) framework (Simon, Tzur, Heinz, & Kinzel, 2004; Tzur & Simon, 2004). *Ref*AER* is an elaboration of constructivist scheme theory (Piaget, 1985; von Glasersfeld, 1995) that describes how a learner forms a novel conception through two types of reflections on their mental activity (Simon et al., 2004). *Type-I reflections* are comprised of comparisons between the learner's goal and the effects (as noticed by them) of their mental activity. When newly noticed effects become associated with a particular (mental) activity or activity sequence, the learner can anticipate an effect from that activity and an activity-effect relationship (*AER*) is created. Each time a learner sets a goal, brings forth their mental activity and connects an effect with that activity, a record of experience is formed. *Type-II reflections* consist of comparisons across these records. Through such reflections, a learner can abstract invariants and develop the anticipation of when to call up an *AER*. The stage distinction (Tzur & Simon, 2004) is used to describe the stages that comprise the formation of a new conception. At the participatory stage, a learner has connected a particular mental activity or activity sequence with an effect and has created an *AER*. What they cannot do, until the anticipatory stage, is call up such an *AER* spontaneously and independently when the original activity is not available to them.

The content-specific constructs guiding this study were children's multiplicative reasoning and number schemes. The Explicitly Nested Number Sequence (ENS) (Steffe & Cobb, 1988) and the Generalized Number Sequence (GNS) (Steffe, 1994) were schemes drawn upon to describe children's operating on CUs. In each, the child conceptualizes smaller compilations as embedded within a larger compilation. For example, 7 CUs of 4 (7 baskets of 4 apples) and 6 CUs of 4 (6 baskets of 4 apples) are embedded within 13 CUs of 4 (13 baskets of 4 apples). Additionally, a child with GNS can operate with abstract iterable CUs and can anticipate the multiplicative structure (four 1s distributed over each of the 7 CU) prior to operating with it (Steffe, 1994). The multiplicative schemes included multiplicative double counting (mDC) (Tzur et al., 2013) and Unit Differentiation-and-Selection (UDS) (McClintock, Tzur, Xin, & Si, 2011).

Methodology

This study consisted of 14 teaching episodes taught by the author over 2 months with an 8th grader, Joe. It was conducted as part of a teaching experiment with two pairs of 7th and 8th grade students, designed to develop algebraic reasoning in middle school students. Joe was purposefully selected for this study because he was operating with at least an Explicitly Nested Number Sequence (ENS) (Steffe & Cobb, 1988). McClintock et al. (2011) identified operating with an ENS as key to children's construction of a UDS scheme.

The two primary tasks provided to Joe during the episodes were UDS tasks (McClintock et al., 2011) and QRE tasks. UDS tasks ask the learner to compare two compilations that differ by either the number of CU (3 CU of 5 singletons and 8 CU of 5 singletons) or by the size of the CU (3 CU of 5 singletons and 3 CU of 4 singletons). Once presented with the two compilations, the learner is then asked a succession of questions, "How are these collections similar? How are they different? Who has more cubes and how many more?" (McClintock et al., 2011, p. 166). QRE tasks can follow or be independent of UDS tasks. They also incorporate two compilations that can differ by the quantity or

size of the CU, but the central question asked to the learner is for them to make the two compilations the same. In some cases, the learner is also asked to build the compilations using Unifix cubes (3 CU of 5 = 3 towers with 5 cubes in each tower, designated $3T_5$) and to demonstrate their solution using the cubes.

Data analysis was conducted in two phases: during the planning and evaluations of teaching sessions and retrospective analysis. In the on-going analysis, critical events such as evidence of current conceptions and the stage of these conceptions were identified and discussed. Through this analysis, appropriate tasks and prompts were then selected for the next episode. The purpose of the tasks and prompts was either to foster development of new conceptions or to test the access a child had to their current conceptions. In the retrospective analysis, previously identified critical video segments were transcribed and analyzed. The children's written work was also analyzed. Of particular importance were segments that enabled inferences about Joe's operating and provided evidence of his transition between stages.

Results

In this section, data supporting Joe's initial relational equivalence (RE) scheme and lack of a quantitative relational equivalence (QRE) scheme are first discussed. Next, analysis of data is provided in support of the assertion that Joe progresses to a QRE scheme. Specifically, I claim Joe transitions into the participatory stage and then anticipatory stage of a QRE scheme.

Relational Equivalence Scheme

During our third session together, Joe and another child, Javier, were given a UDS task (McClintock et al., 2011) followed by a QRE task using the same compilations. Joe was given (for pretend) compilations consisting of 7 towers with 8 cubes in each tower ($7T_8$) and 9 towers with 8 cubes in each tower ($9T_8$), respectively. During the UDS portion of the task, he multiplicatively produced the two totals ($7 \cdot 8 = 56$, $9 \cdot 8 = 72$) and then operated on 1s ($72 - 56 = 16$) to find that Javier had 16 more cubes than he did. A QRE task then directly followed which asked Joe to create equality between the two compilations. Excerpt 1 contains his explanation of his solution to the QRE task (T=teacher-researcher, J=Joe).

Excerpt 1 (April 17, 2012)

J: I took eight from him and gave me 8.
T: Okay, you took eight from him and gave you eight. Why were they both equal –because they both equaled what?
J: Um. [Looks at his work for a few seconds and then starts to write.] I don't know.
T: That's okay. You just knew that they were 16 apart.
J: Yeah because if I subtracted it, then I could add it, then [inaudible].
T: Okay, so you don't really have to know what the number is. You just know it's equal.
J: [Writes 56 + 8 vertically and then 64 underneath.] It's 64.

The exchange provided evidence that Joe had an anticipatory RE scheme. Joe's goal was to create equivalent quantities of 1s by reducing the difference in 1s between the two compilations to zero. His activity sequence consisted of: select the CUs and unit rate in each compilation and produce two totals of 1s ($7 \cdot 8 = 56$, $8 \cdot 9 = 72$), subtract the smaller total from the larger total to produce the difference ($72 - 56 = 16$), divide the difference by two to produce half of the difference ($16/2 = 8$), dis-embed an amount of 1s equal to half the difference from the larger total and add it the smaller total. Joe relied on his part-to-whole reasoning and his operations with three levels of units (Steffe, 1994). He used part-to-whole reasoning to create a nested relationship between the two totals of 1s. Joe then operated with three levels of units as he transformed the totals. He operated on the three CUs (the

smaller total, the larger total, and the difference) where the smaller total and difference acted as a second level of units embedded in a third level of units (the larger total). Using his additive operations, he dis-embedded and re-embedded pieces of the difference as he saw fit to bring the two totals into balance.

Evidence of his solution method, designated Halve-the-Difference (HTD), was further supported by his written solution, "To make same u (you) need to take 8 from (Javier) and give them to me." Joe did not need to enact the transformations of the totals to verify that equality had been achieved. He did not produce the 64 until the researcher requested it. Joe anticipated that balance was achieved by his operating (re-distributing the difference to the totals) rather than by the end result of his operating (comparing the two new totals). Joe's RE scheme was at the anticipatory stage. He anticipated creating the embedded structure of 1s and the subsequent operations on 1s to produce equivalence without any prompting in a novel situation.

Fostering Construction of the Participatory Stage of a QRE Scheme

Joe created equivalence, but even when prompted, he did not balance the CUs. I hypothesized he lacked a QRE scheme. To promote Joe's construction of a QRE scheme, he needed to be oriented to operations on CUs. This required Joe to suspend producing the totals multiplicatively in the original compilations and operate with each of them as a collection of CUs. The following three episodes describe how Joe made this transition. In the first episode, our 6th episode together (April 26, 2012), Joe created equivalence via operations on CUs for the first time. The compilations for the task given were Javier scored 4 baskets each game for 5 games and Joe scored 4 baskets each game for 6 games. After finding who scored more baskets and by how many, Joe was given the QRE task, "Can you make them the same?"

Joe enlisted his RE scheme to produce the solution, "Take 2 baskets…" At this point, the teacher-researcher cut Joe off so he did not give the solution away to Javier. Making Joe wait for Javier prompted him to reflect on his operating and produce a second solution. Joe's initial utterance demonstrated he initially created equivalence via the HTD method. He took 2 baskets (half of the difference of 4 baskets) from the total of his compilation (24-2) and added 2 baskets to the total of Javier's compilation (20+2). But after the prompt, he balanced the two compilations and created equivalence by adding one more CU to the CUs in the smaller compilation (5 games with 4 baskets + 1 game with 4 baskets = 6 games with 4 baskets). Evidence of this was provided from his written work that included "6 – 5" and "add 1 game."

Despite Joe's subtraction of 5 from 6, I did not consider him to have purposefully returned to the compilations with the intention to find the difference in CU. Rather, I hypothesized he first reconstituted the difference in 1s as one CU (1 game of 4 baskets) via his segmenting operation (Steffe, 1992). This required Joe to return to the original compilations to retrieve the size of the CUs (4 baskets in each game). It was only after he had produced the CU of 1 game from the 1s in the difference (4 baskets) that he noticed it was the same as the difference in CUs between the original compilations. Joe had abstracted new activity through a Type-I reflection that could be used to produce equivalence, but it was tied to his original goal of operating on the difference as an embedded total of 1s. His current RE scheme still need to be executed before Joe could use CUs to produce equivalence. This indicated Joe still lacked a QRE scheme.

During session 7 on May 8, Joe continued to create equivalence via operating on CUs. I asked Joe to create equality between two compilations: 8 boxes of cookies with 6 cookies in each box (48 cookies) and 7 boxes of cookies with 6 cookies in each box (42 cookies). The purpose of the task was to test the participatory and anticipatory nature of his QRE scheme. If Joe had not moved to the participatory stage of QRE, the task was also designed to foster such a progression by offering him a chance to notice the difference in 1s (6 cookies) was also a difference of 1 CU. Joe produced equivalence by adding 6 to the smaller total of 42 cookies to create totals of 48 cookies. His initial

solution demonstrated that he did not have access to operations on CUs in anticipation and was not in the anticipatory stage of a QRE scheme. I then prompted Joe in two ways to orient him to the difference in the CUs.

I prompted Joe by asking him for an alternate solution. Joe again operated on 1s as he found and balanced the two totals by removing 3 cookies from the larger total and adding them to the smaller total (an HTD method). It was not until a second prompt that Joe operated on CUs. When I asked Javier to explain why Joe's method worked, it served as a prompt for Joe who immediately wrote down, "1 box to me." I hypothesized the prompt had re-focused Joe to the original compilations of the task and the relationship between the individual quantities across the two compilations. Excerpt 2 provides evidence of how Joe re-constituted his solution of adding 6 cookies to adding 1 box of cookies to balance the CUs (8 boxes of cookies per person).

Excerpt 2 (May 8, 2012)

J: You can also give me one more box.
T: What do you mean by, "give you another box"?
J: Well, he has 7 boxes. I have 7 and he has 8 boxes and each box has 6 in it, so it would
    give me one more box.

From his explanation, I inferred Joe was developing a new goal associated with a QRE scheme. Joe reflected on the relationship between the original compilations and the CUs (created through operations on 1s) used to create equivalence. I hypothesized that through a Type-II reflection, Joe was abstracting a new invariant: the embedded nature of the three compilations as collections of CUs. Moreover, Joe was transitioning into the participatory stage of a QRE scheme. My hypothesis was supported during the next task when Joe enlisted a method, designated Distribution-of-Embedded CUs (DECU), which used the embedded structure. The task asked, "Javier buys 7 boxes of cookies with 3 cookies in each box. Joe buys 9 boxes of cookies with 3 cookies in each box. Can you make them the equal?" The left side of Figure 1 shows Joe's initial computations of the totals. At that point, Joe paused for a moment, looked back at the statement of the problem, and then wrote, "add 2 more box(e)s to Javier."

Figure 1. Joe's written work demonstrating his transition to operations on CUs.

Initially calculating the totals served as an internal prompt for his activity associated with operations on CUs. I inferred that a reorganization of Joe's multiplicative and balancing schemes had occurred. Joe's new goal was to produce a difference in CUs and use those CUs to create equivalence. He progressed to a higher level of operating that included operations on CUs, but he could only access this goal through the original context (for him) of two embedded totals of 1s. From Joe's operating, I concluded he was in the participatory stage of a QRE scheme.

Anticipatory Stage of a QRE Scheme

During his 8th session on May 10, I tested Joe's reorganization of his QRE scheme. Joe continued to enlist his HTD method prior to operating on CUs, indicating he was not in the anticipatory stage of a QRE scheme. To help him gain access to his QRE scheme, I re-introduced Unifix cubes to use when he explained his solutions. This gave Joe an opportunity to physically see and mentally reflect on the relationship between the CUs in the original compilations and the

difference. Moreover, Joe could reflect on how balancing the number of CUs also produced equality of the total 1s because the CUs and total 1s were equal in the two compilations.

When solving a novel task at the beginning of the next episode, Joe independently and spontaneously enlisted operations on CU in anticipation. The task was "Javier buys 19 bags of candy with 6 pieces of candy in each bag. Joe buys 15 bags of candy with 6 pieces of candy in each bag. Make them so you have the same amounts." Joe was also given the constraint of written work was not allowed. This constraint created a new task for Joe in the sense that it required his QRE scheme. The size of the quantities coupled with the constraint introduced computational complexity for his RE scheme. Joe brought forth his QRE scheme and took 2 bags of candy away from Javier and added 2 bags of candy to his own collection leaving each compilation with 17 (19-2=17 and 15+2=17) bags of candy. Similar to his previous work with 1s, Joe did not produce the 17 bags of candy for each person. He did not have to because his method, designated Halve-The-Difference-CU (HTD-CU), had created equivalence without it.

Joe had balanced two compilations in a similar fashion with 1s when using his RE scheme. His equivalence scheme, in conjunction with his conceptualization of the difference as a qualifier of the nested relationship between the smaller and larger totals of 1s, enabled him to anticipate an even re-distribution of the difference brought the two compilations to balance. Now, as evidenced by the description of his HTD-CU solution in Excerpt 3, Joe had turned his focus to the CUs (4 bags) in the difference.

Excerpt 3 (May 15, 2012)

T: Yeah. It looked like at first you had plus two and minus two.
J: Yeah, I was thinking about bags.
T: You were thinking about bags.
J: Um-hum.
T: Tell me what you mean when you were thinking about bags. You took two bags from?
J: Javier, and [grabs two towers from his 15 towers of 6 and moves them to Javier's 19 towers of 6] I take these two. And then I give them to me. And then they'd be equal.

Joe enlisted his GNS (Steffe, 1994) as he operated with the four CUs (of size 6 pieces of candy) comprising the difference in a manner similar to his earlier operating with 1s. Joe produced units of units of units as he halved the difference into 2 groups of 2 bags with 6 pieces of candy in each bag. He then operated on the 2 units of units of units by dis-embedding them from the difference and then re-distributing them to the two compilations ($19T_6 - 2T_6$ and $15T_6 + 2T_6$) to create equivalence. His QRE scheme was now recursively being applied to the 2 CUs in the same way he had operated on 1s previously. Joe operated purposefully only on the CUs throughout the process and did not operate on the 1s in the compilations at all. He anticipated the *AER* (Tzur & Simon, 2004), connecting his mental activity of operating on the embedded the compilations as CUs to the effect of creating equivalence via operations on CUs.

In solving the task, Joe independently and spontaneously created equivalence via the activity sequence (a) differentiate the 1s and unit-rate from the CUs (b), select the CUs and operate on them to find their difference (using subtraction), (c) select the difference in CUs and operate on it to halve it (via division by 2), (d) subtract a quantity of CUs equal to half the difference from the CUs in the larger compilation and add the same quantity to the CUs of the smaller compilation. The key to getting Joe to demonstrate he had an anticipatory scheme was to create a need for him to use it in a novel task. The task was successful because of two aspects: the constraint of not allowing written work and the inclusion of large numbers that made mental computation difficult. Joe's solution provided evidence he was operating with an anticipatory QRE scheme.

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Discussion

This study provides three contributions. First, this study identifies two schemes for mathematical equivalence. RE and QRE both incorporate additive balancing operations that create equivalence between two multiplicative compilations. When operating with an RE scheme, the learner first multiplicatively produces the totals of 1s from each compilation. They then find the difference in 1s between the totals and create equivalence by operating on the totals with some or all of the 1s in the difference. For a learner with a QRE scheme, operations for creating equivalence with 1s are available, but they instead select CUs for operating. They enlist a method (such as DECU or HTD-CU) that requires operations with three levels of units. The learner produces a difference in CUs between the two compilations and creates equivalence via additive operations on CU. They may re-distribute the CUs in the difference or transform the CU in one of the original compilations to create equivalence. In either case, this study demonstrates how an anticipatory QRE scheme enables a child to relate multiplicative compilations via operations on CUs. Such operations can provide the conceptual roots for fundamental algebraic concepts such as the distributive property of multiplication over addition [$19 \cdot 6 - 15 \cdot 6 = (19-15) \cdot 6$] and solving linear equations ($8 \cdot 5 = \_\_ \cdot 10$).

Second, differentiating between RE and QRE can help teachers address why their students continue to struggle with understanding the equal sign as a relational symbol (Kieran, 1981). When examining children's understanding of mathematical equivalence, research has focused on children's conceptualization of one side of an equation as "the same as" the other side of the equation (Falkner et al., 1999; Knuth et al., 2006; Stephens et al., 2013). Moreover, because it has been shown that children can use their additive operations to make sense of the equivalence of expressions such as 4+5 and 6+3 (Baroody & Ginsburg, 1983; Stephens et al., 2013), researchers have posited that a relational understanding of the equal sign can be developed as early as first grade. However, this study suggests how children understand "same as" changes as their operations evolve. Children who have constructed a QRE scheme can make sense of equivalence in CUs or 1s, while children with an RE scheme need 1s to conceive of equivalence.

Finally, this study demonstrates the transition from a lack of a QRE scheme to the anticipatory stage of a QRE scheme. This transition is rooted in operations on CUs. The operations on 1s contained in the child's RE scheme must be "lifted" to a higher level where they become operations on collections of CUs. Joe solved the tasks with his current schemes, but through reflection, he made two new abstractions. First, when the compilations differ only in the quantity of CUs, he abstracted that the difference, as a number of CUs, could be operated with to create equivalence between the two compilations. His second abstraction was the embedded relationship between the compilations as collections of CUs. The key to the second abstraction made by Joe and his transition to QRE was his GNS allowed him to anticipate the multiplicative structure of each compilation prior to any operating.

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STUDENTS' REASONING AROUND THE FUNCTIONAL RELATIONSHIP

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Proportional reasoning is related to flexible use of the scalar and functional relationships that exist in proportional situations. More specifically, in regard to the functional relationship, students' understanding of the multiplicative comparison that exists between two quantities in a ratio is a key concept. We conducted student interviews with 12 high performing students to examine their conception of the functional relationship. Analyses provided initial evidence that the majority of students did not conceive of the multiplicative comparison when solving problems designed to press the functional relationship, indicating students' written work that makes use of the functional relationship should not imply understanding of the multiplicative comparison.

Keywords: Algebra and Algebraic Thinking, Cognition, Middle School Education

Introduction and Purpose

Students' ability to understand and apply proportional reasoning is critical to their future success in mathematics and science. Yet, there is evidence students are not developing proportional reasoning during their school experiences (Brahmia, Boudreaux, & Kanim, 2016: Cohen, Anat Ben, & Chayoth, 1999). One way to address this issue is for instruction to focus on developing understandings regarding mathematical relationships present in proportional situations, allowing for connections across topical borders into more sophisticated mathematics (e.g., rate of change, slope, and covariation) and other content areas (e.g., physics and chemistry) (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Teuscher & Reys, 2010).

One important aspect of the development of proportional reasoning is the understanding of the multiplicative comparison relationship that exists between two quantities in a ratio (Lobato, Ellis, & Charles, 2010; Tourniaire, & Pulos, 1985). The research focuses on students' conception(s) of the multiplicative comparison relationship (Lo, Watanabe, & Cai, 2004) and how they articulate their conception(s) of that relationship in a problem solving situation.

Theoretical Framework

We argue it is important for students to demonstrate the ability to flexibly and fluently make use of the scalar and functional relationships to solve problems (mathematics perspective), and at the same time they must possess understandings of these relationships through both a composed unit and multiplicative comparison conception (student cognition perspective).

Mathematics Perspective: Scalar and Functional Proportional Relationships

From a mathematics perspective proportional situations involve an equivalent relationship between ratios, such that $a/b = c/d$. There are two multiplicative relationships that can be found within proportions – scalar and functional (Tourniaire & Pulos, 1985). Imagine the situation "*Callie bought 7 cookies for $3. How many cookies can Callie buy for $12?*" as represented in the first row of Table 1. One can solve this problem by scaling up both elements of the original ratio by a factor of 4 to find 28 cookies for $12. This is referred to as the *scalar relationship* because we can scale up both components of the ratio by a common scale factor to create a new equivalent ratio. Alternatively, imagine the situation "*Callie bought 6 cookies for $2. How many cookies can Callie buy for $13?*" as represented in the second row of Table 1. Rather than scaling the original ratio by a factor of 6.5, one can use a simpler mathematical relationship expressing the number of cookies in

terms of the dollars (cookies are 3 times the dollars). This is the *functional relationship* because one variable is defined as a function of the other.

Table 1: Examples of contexts and student work around the scalar and functional relationships and students' related proportional reasoning conceptions

Mathematical Relationship			Students' Proportional Reasoning Conceptions
Scalar Relationship		Composed Unit	Students view the quantities in the ratio relationship as separate entities requiring coordination in conjunction with one another. For example, *"If I can buy 7 cookies for $3, then if I multiply the number of cookies times 4 to get 28, I have to multiply the number of dollars times 4 and get $12."*
Functional Relationship		Multiplicative Comparison	Students view one quantity in the ratio relationship in terms of the other quantity. For example, *"If I can buy 6 cookies for $2, then my cookies are always 3 times my number of dollars. Therefore, if I have $13, I know my cookies are 3 times more or 39 cookies."*

Missing-value proportion-based problems can be solved using both relationships. How one chooses to solve the problem typically depends on two factors. First, the context and the numbers therein may make the use of one relationship more efficient (Steinthorsdottir & Sriraman, 2009). Second, the choice is influenced by that particular student's proportional reasoning conception.

Student Conception: Composed Unit and Multiplicative Comparison

Students' conceptions of proportional relationships may parallel the scalar and functional relationships, but their use of a relationship does not necessarily mean a particular conception is present. Students may tend to see a given ratio as a *composed unit* involving the joining of two quantities in a ratio relationship (Lobato et al., 2010) which can then be scaled up or down to create equivalent ratios (see first row of table 1). Alternatively, students may tend to see one component of the ratio as a *multiplicative comparison* of the other component (see second row of table 1). Ideally students will possess both conceptions in order to flexibly operate with ratios in different contexts and situations.

Given the parallel nature of these relationships and conceptions, one may expect students to explain solution processes for scalar items through composed unit thinking and functional items through multiplicative comparison thinking. However, researchers have cautioned that students' solution processes that make use of the functional relationship may actually involve a composed unit conception (Lamon, 1993; Simon & Placa, 2012). For example, given the problem *Callie bought 6 cookies for $2. How many cookies can Callie buy for $13?*, a student may divide 6 cookies by $2 with a result of 3. How a student then expresses the meaning of '3' may indicate a focus on the composed unit or multiplicative comparison conception of the ratio relationship.

- Composed unit example: "Well 6 divided by 2 equals 3, so I knew $1 could buy 3 cookies. I had to multiply by the 3 cookies for a dollar by $13 to get $39." In this example, the student expresses the meaning of the '3' as 3 cookies and part of a composed unit, $1 for 3 cookies.
- Multiplicative comparison example: "Well 6 divided by 2 equals 3 that means the cookies are always 3 times the dollars. If I have $13, I multiply this by 3 to get the number of cookies." In this example, the student expresses the meaning of the '3' as a multiplicative comparison between cookies and dollars, 3 times more.

Students need both conceptions to fluently and flexibly proportionally reason. Yet there is little empirical evidence related to how students who make use of the functional relationship express their conception of this relationship. This study addresses the question: how do students articulate their

conception of the functional relationship when presented with problems that press the functional relationship?

Methods

We conducted interviews with 12 grade six students who exhibited high performance on an assessment of proportional reasoning to examine their conceptions of the functional relationship. The individual student interviews were conducted at one school immediately following the administration of the assessment. Three items that pressed the functional relationship were selected. Students were presented with their worked solution strategies for the three problems (bulleted below) and asked to describe their process. The interviews were video recorded and then transcribed for analysis.

- Marta found brownie deal with 3 brownies for $9. How many brownies can she buy with $12?
- Tomas found a hamburger deal with 4 hamburgers for $28. How much will 5 hamburgers cost?
- Mark found a hamburger deal with 8 hamburgers for $32. How much will it cost to buy 5 hamburgers?

Analysis

Thirty solution strategies from 12 students were available for analysis as not all students received all items in the interview process. The first round involved coding the written strategies as explicit, implicit, or indeterminate in regards to written evidence for use of a *composed unit understanding*. We opted to code based on evidence of composed unit understanding due to its relatively high frequency in our cursory examination of the strategies. An explicit code typically was applied to written use of the words 'per' or 'each'. An implicit code involved evidence of adding or multiplicative scaling ratios. All other work was coded as indeterminate. The second round of coding involved examining students' interview responses as confirms, indeterminate, or multiplicative comparison in regards to evidence for composed unit understanding. A code of confirms indicated verbal evidence of use of a composed unit strategy, typically involving the use of the words 'per' or 'each'. The indeterminate response could not be clearly coded as a scalar or functional strategy. The multiplicative comparison code involved evidence of one quantity being defined in terms of the other quantity multiplicatively, using the word 'times'.

Results

Of the 30 solution strategies, 27 were correct. The following categories (and counts) resulted from the coding process for the correct written work: explicit (7), implicit (6), and indeterminate (14) indication of composed unit understanding. We followed this by examining the type of thinking indicated in the interview response. The following categories emerged from the interviews: confirmation of a composed unit conception (23), indeterminate (3), and multiplicative comparison (1).

The number relationships were designed to press students to use multiplicative comparison understanding. While all students made use of the functional relationship, only one response out of 27 provided clear articulation of the multiplicative comparison between the quantities in the ratio and 23 of the 27 responses provided evidence of composed unit thinking. This indicates either the problems did not encourage multiplicative comparison conceptions or that the majority of the students did not possess this understanding. A second finding is written solution strategies often do not provide an explicit indication of whether students used composed unit or multiplicative comparison thinking.

Discussion

The results provide evidence that students' use of the functional relationship in their solution strategy does not indicate an understanding of the multiplicative comparison. Students typically made use of the functional relationship through division to generate a unit rate. These results provide empirical evidence to support previous statements in the literature (Lamon, 2005; Simon & Placa, 2012) regarding students' difficulty in understanding the functional relationship as a multiplicative comparison.

A small sample and the use of a discrete, easy to visualize context and missing value problem types may have impacted the findings. It is possible the context and/or problem type increased students' use of per-one or composed unit thinking; a different one may promote more multiplicative comparison thinking.

The primary implication is the need for intentional intervention to make explicit the functional relationship as a multiplicative comparison. Students' use of the functional relationship to generate a unit rate is often assumed to indicate students' understanding of the functional relationship as a multiplicative comparison. It is imperative that we build teachers' knowledge around these relationships and students' conceptions if we want students to develop a multi-faceted ability to proportionally reason.

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LEARNING ARITHMETIC BY PROGRAMMING ROBOTS

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With frequent predictions of upcoming technological and economic difficulties triggered by an impending shortage of information and communications technologies (ICT) professionals, the calls are growing stronger to include computer programming as a core element of school curriculum. These calls are bolstered by the suggestion that programming supports the development of thinking skills – which echoes a longstanding argument for teaching mathematics. Motivated by the parallel, we attempted to investigate some of the common ground between the learning to program and the development of core mathematical concepts. We photographed and video recorded children, aged 9–10, as they learned to build and program Lego Mindstorms^{TM} EV3 robots over four days. Our findings suggest that programming supports children's understandings of arithmetic and decimal numbers.

Keywords: Modeling, Number Concepts and Operations, Technology, Elementary School Education

Introduction

In recent years there has been a growing recognition that information and communications technologies (ICT) are a major contributor to innovation and economic growth. For instance, the Organization for Economic Cooperation and Development (OECD, 2013) considers computer programming a necessity for a highly skilled labour force. Shortages are already felt across the world impacting IT innovations and revenues (see Arellano, 2015; Clendenin, 2014).

Currently, a major push around the world is to include programming as a core part of school curriculum. In response, some educators and educational systems are shifting from teaching "how to use" software programs toward "how to code." These sorts of arguments for teaching computer programming parallel long-standing rationales for teaching mathematics. Similarly, many of the structures and strategies within programming bear strong resemblances to elements of mathematical concepts. We discuss a few of these resemblances in this paper, focusing on arithmetic.

Context

In this study we photographed and video recorded children as they learned to build and program Lego Mindstorms^{TM} EV3 robots over four half days. On Day 1, the children followed the instruction manual to build the robots. Day 2 children were taught to program their robots to "dance" and to trace a regular polygon. On Day 3 they were given the final challenge of building a robot that could find and douse a fire in any of four rooms in a building. Day 4 was the final challenge demonstration/competition.

The study's 22 participants were aged 9–10 in Pakan School at Whitefish Lake 128 First Nation in Northern Alberta. Data included video-recordings, GoPro digital images, field notes, and artifacts. Video analysis consisted of an iterative process for selecting videos and GoPro digital images that exemplified children's embodied spatial actions of mathematical thinking. A narrative developed through an iterative process of rereading the literature, reviewing the video and GoPro data, and rewriting. The instance that we use to focus our discussion was a trio of girls learning to program their robot to move a certain distance into the hallway to illustrate a developing understanding of number.

Findings

Arithmetic Topic 1 – Understanding Number

In the video[1]from Day 3, Krista (author) was helping the pink team program their robot to move into the building. The team members started out with a guess of 0.4 wheel rotations to move the robot into the first corridor of the building. After testing how far the robot moved and observing that the robot needed to move a considerably greater distance, Krista prompted the girls by asking what they should try next. Celina suggested they try 0.5. The small incremental change was still not enough, so Krista suggested they try 2. Two rotations moved the robot too far, which prompted the question, "What is between 0.5 and 2?" Celina responded "5." Krista drew a simple number line on the whiteboard and asked again, "What is between 0.5 and 2?" "Oh!" Celina exclaimed, "1.5." The number Celina chose was close to the number of rotations actually required, which indicated she understood the meaning of 1.5.

Analysis/Interpretation

In the exchange above, we take Celina's immediate and satisfactory response as evidence that invoking the number line appeared to provide an appropriate metaphor for helping Celina understand. Lakoff and Núñez's (2000) have noted that "Mathematical ideas…are often grounded in everyday experiences" (p. 29). For example, understandings of set theory are rooted in a "container" metaphor, which begins to develop as babies put things in their mouths and, later, drop objects into other objects. With regard to the concept of number, Lakoff and Nunez describe four grounding metaphors of arithmetic: *arithmetic as object collection*, *arithmetic as object construction*, the *measuring stick metaphor*, and *arithmetic as object along a path*.

The metaphor of *arithmetic as an object collection* is based on a one-one correspondence of numbers to physical objects. With this metaphor a greater size corresponds to a bigger number. For instance, 5 is *greater* than 3 because it forms a *bigger* collection. The metaphor of *arithmetic as object construction* is based on fitting objects/parts and arithmetic operations. In this instance, 5 is greater than 3 because an object comprising 5 units is larger than one comprising three. The *measuring stick* metaphor maps numbers onto distances, whereby 5 is greater than 3 because it is *longer*. The metaphor of a*rithmetic as an object along the path* is based on arithmetic as motion, by which 5 is greater than 3 because it entails moving further from a common starting point (i.e., zero).

In the following description, we summarize how the task of programming a robot to move into a room calls for all four of Lakoff and Núñez's representations of arithmetic. To begin, the metaphor of *arithmetic as an object collection* is used in most counting situations, whenever the forms being counting are perceived as discrete objects. It is called for in programming complicated actions into discrete steps or instructions. And more obscurely too, such conceptual moves as the discretizing of wheel turns, so that they can be counted and thus used as a tool in programming, might be argued to rely on this metaphor.

The *arithmetic as an object construction* might be encountered when programming a robot to move. Celina wanted a larger wheel rotation than 0.4, so she added an incremental amount of 0.1 wheel rotations to make 0.5 wheel rotations. Contrasted to the previous metaphor, in this instance, wheel turns are not perceived as discrete objects, but as parse-able continuities. Those parsed elements can then be assembled into an appropriate "object" to move the robot a precise distance.

The *measuring stick* metaphor also featured prominently in the children's programming, and was particularly prominent in in the frequent need to interpret wheel turns in terms of actual distances (e.g., when the phrase "1 wheel turn" was deployed not as a description of movement but was a reference to a distance of roughly 12 cm). In this instance, programming the code block requires understanding measurement of approximately 1.5 wheel turns.

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Programing the robot to move can draw upon the metaphor of *arithmetic as on object along the path*. In this case, starting place becomes a critical element that occurs where the robot enters the room and recurs in the opposite direction when the robot leaves.

To re-emphasize, each of Lakoff and Núñez's four metaphors of arithmetic are present in programming the robot to move a required distance in the room. The ability to identify to the particular metaphor(s) that a situation is calling for is a critically important teaching competence, as Krista demonstrated in the interaction with Celina. Re-interpreting that brief episode, Krista recognized that Celina was not interpreting number as a distance (i.e., she was not using a *measuring stick* metaphor), and thus reminded her of that metaphor by offering the image of a number line. No explanation other than an image of number that fitted the application at hand was required.

Closing remarks

In the episode reported, the tasks of programming robots required more than parsing complicated actions into singular direction; they entailed flexible engagement conceptual metaphors and mathematical models. Computer programming aligns closely with concepts and structures in mathematics and we suspect that it might provide other powerful instantiations for mathematical concepts that have not yet been noticed. That suggestion is perhaps not surprising, given the mathematical roots of computer programming. However, to our reading, it is not an aspect of programming that has garnered much consideration in either mathematics education or the technology education literature.

Given that likelihood, and in consideration of the fact that mathematics literacy, like competency with programming, is of growing relevance, the realization that engagement with emergent technologies can complement and co-amplify mathematics learning – while, perhaps, contributing to evolving understandings of what "basic" mathematics might be for our era.

With regard to important complementarities between learning mathematics and learning to code, the Lego Mindstorms[TM] EV3 robots and the associated programming language provide a powerful instance of "multiple solutions." They afford tremendous flexibility for accomplishing a range of tasks, from the trivial to the complex. None of the programming tasks set for the children in our study had pregiven or optimal "solutions." Despite that – or perhaps because of that -- the children were able to engage in manners that they could recognize as successful, even when "complete" solutions were not reached. With incremental tasks and iterative refinements, children were able to learn more sophisticated and efficient methods for programming the robot. It is not difficult to imagine a mathematics class with similar standards of success.

The results of this study underscore the importance of developing and implementing a computer programming curriculum in schools. Programming is an emergent literacy that can amplify other critical literacies, while affording access to a diverse range of cultural capitals. The reasons to teach programming go beyond the technical and economic; for us, they are fundamentally ethical.

Endnotes
[1]The video for understanding number is available at https://vimeo.com/144996708

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A FRAMEWORK FOR DESCRIBING CONCEPTIONS OF SLOPE

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This paper builds on the cannon of literature on the concept of slope to introduce a framework for describing conceptions of slope. The framework will allow us to consider how slope is viewed and used. It also provides researchers a way to classify student understanding of slope and educators a guide to help design slope tasks. The framework brings together ideas from APOS theory with research on slope conceptualizations to describe how students may develop an understanding of slope.

Keywords: Advanced Mathematical Thinking, Algebra and Algebraic Thinking

Theoretical Framing

APOS Theory (**A**ction, **P**rocess, **O**bject, **S**chema) is a cognitive theory, based on the ideas of Piaget, that considers how individuals may learn mathematical ideas (Arnon, et al., 2013). APOS is a well-known theory; it is now described briefly. In APOS, an **action** is a transformation of a previously constructed mathematical object that the individual perceives as external. It may be the rigid application of a sequence of steps that are explicitly available or the application of a memorized fact or property. When an action is repeated and reflected upon it may be interiorized into a **process**. A process is perceived as internal, and the individual may imagine, skip steps, and anticipate the result of a process without having to explicitly perform all steps. Processes may be coordinated with other processes and may also be reverted to the actions they came from as needed in a problem situation. As an individual needs to perform actions on a process he/she may gain consciousness of the process as an entity in itself. When the individual is able to apply or imagine applying actions to a process then it is said that the process has been encapsulated into an **object**. An object may be de-encapsulated into the process or processes it came from as needed in a problem situation. A **schema** for a specific mathematical notion is a coherent collection of actions, processes, objects and other schema, which are related to the given notion in the individual's mind. The schema is coherent in the sense that the individual can interrelate its different components and decide when a given problem situation falls within the scope of the schema. The progression from action to process to object often appears more like a dialectical rather than a linear progression. APOS Theory asserts that students' understandings of a particular mathematical concept as an action, a process, or an object influences their approaches in solving diverse mathematical tasks.

How Students Consider Slope: A Framework

How slope is used

In Figure 1, we provide descriptions for three ways students commonly use slope in mathematical classrooms to: (a) describe behavior, (b) measure steepness, and (c) determine relationships. For each of these uses a student may show either an action, process, or object conception of slope. By simultaneously considering how slope is viewed (i.e., as an action, process, or object) and for what purpose slope is used (i.e., to describe behavior, measure steepness, or determine relationships), the framework displayed in Figure 2 provides researchers a way to classify student understanding of slope and educators a guide to use when designing slope tasks. We will now consider how a "generic student" might consider slope as an action, process, or object.

Slope used to...

Describe Behavior	Measure Steepness	Determine Relationships
Example: Slope is used to describe that (and how quickly) the graph of a line or a table of output values increases or decreases.	*Example:* Slope is used as a means to measure the degree of tilt of a line in a real-world context to consider the pitch of a roof.	*Example:* Slope is a means to determine if two given lines are either parallel or perpendicular.

Figure 1. Diagram representing ways students commonly use slope**.**
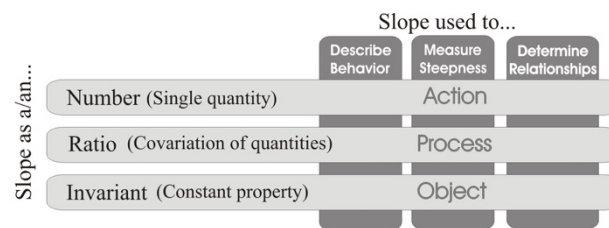
Figure 2. Diagram for how slope is viewed and used in light of APOS theory.

How slope is viewed

In light of APOS theory, we now look at how students view slope as: (a) a **number** (i.e., a single, calculated quantity); (b) a **ratio** that represents the covariation of two quantities; and (c) an **invariant,** that depends neither on a particular pair of points used for its computation within a given line nor on a particular line within the collection of all lines parallel to the given line. We claim that students who are limited to thinking of slope as a single quantity, a number, have an action conception of slope. We also claim that students who are able to think of slope in terms of covariation of two quantities must have at least a process conception of slope. As is detailed later in this paper, we argue that thinking of slope as an invariant implies that the student has an object conception of slope. In this section, we make reference (in italics) to 11 conceptualizations of slope described in Moore-Russo, Conner & Rugg (2011) and Stanton & Moore-Russo (2012): *physical property*, *algebraic ratio*, *geometric ratio*, *parametric coefficient*, *functional property, trigonometric conception*, *calculus conception*, *real world situation*, *determining property*, *behavior indicator*, and *linear constant*.

Slope is treated as a **number** when it is seen as a value that is calculated or determined either from a formula or by counting. A student who is limited to thinking of slope as a number has an action conception of slope and may falter when solving unfamiliar slope tasks (Nagle, Moore-Russo, Viglietti & Martin, 2013). For example, when given the graph of a line for problems related to measuring steepness, students often count vertically and then horizontally and then divide the two numbers to determine the line's slope without thinking about how the numerator changes with changes in the denominator (*geometric ratio*). It is often the case that students who are limited to this "count up, then count over" sequence struggle when they encounter decreasing lines, and some resort to moving from right to left to keep the "count up, then count over" steps intact. This action conception may also lead to the incorrect slope for a line graphed on a non-homogenous coordinate system where the vertical and horizontal units represent different quantities (Zaslavsky, Sela, &

Leron, 2002) or the incorrect assumption that lines on two non-homogenous coordinate systems are parallel because they appear to have the same slope or tilt (*determining property*). When dealing with the slope-intercept equation of a line, students often recognize that "the number in front of the x" is the line's slope (*parametric coefficient*); however, it also happens that students apply this procedure to all equations and erroneously report that the coefficient of the x is the slope even when a linear equation is written in standard form. Finally, it is also common for students to plug coordinate values either from two given points, or from a table, in the $m = \frac{y_2 - y_1}{x_2 - x_1}$ formula (*algebraic ratio*); although they often "forget if the y's or the x's are on top" since they are plugging into the formula without thought as to what the resulting value represents. In fact, for each of these examples, students may not have any notion of what the number produced for the slope actually means. The number is only the result of applying a formula or following a procedure blindly; hence, it is the result of an action.

When a student is able to move past blindly following procedures and comprehend slope as a **ratio** describing the covariation of two quantities, then that student has at least a process conception of slope. Students with a process conception of slope can move past blindly following procedures so that their engagement is purposeful. They are able to recognize patterns, do calculations with understanding, and think of slope as a ratio that represents the change in the dependent variable in light of the change in the independent variable. Individuals who are able to think of slope as the covariation of two quantities can relate the *geometric ratio* and *algebraic ratio* conceptualizations of slope, as outlined in Nagle and Moore-Russo (2013), by moving flexibly between the symbolic, numeric, and graphical representations as needed to think of slope as a ratio that is independent of representation. Moreover, they can coordinate processes of algebraic manipulation to relate the algebraic ratio, $m = \frac{y - y_1}{x - x_1}$, to $y = m(x - x_1) + y_1$ and then to $y = mx + b$ where $b = y_1 - mx_1$ thus connecting the *algebraic ratio* and *parametric coefficient* conceptualizations of slope. Also, they should be able to convert situations given verbally (e.g., word problems involving slopes of ramps) to other representations in ways that connect a *physical property* conceptualization with either the *geometric ratio* or *algebraic ratio* conceptualizations.

Thinking of slope as an *invariant* means students recognize that the slope of a line is the same regardless of which two points are used to compute it and regardless of which of a set of parallel lines is used to calculate it. It is important to stress that in our definition of *invariant* we require an understanding that goes beyond an action conception, so that these properties are not simply memorized facts but rather are obtained from doing actions on processes of geometric and algebraic ratios, and on a process of similar triangles. Recall that in APOS, doing actions on processes is needed to encapsulate a process into an object. The understanding of slope as a <u>constant</u> rate of change relates to the *linear constant* conceptualization of slope. This understanding of slope being related to the tangent of the angle and that all slope triangles are similar (and hence their corresponding angles are all congruent) ties into the *trigonometric* conception of slope.

Of course, students with a process or object conception may choose to apply an action; for example, they may choose to use the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$ if it happens to be a convenient or efficient means of determining slope. However, those students would not be limited to thinking of slope as a single quantity. For example, a student with a process conception of slope may identify the coefficient 2 in the linear equation $y = 2x + 1$ as the slope of the line, but may also recognize that the slope being equal to 2 means that for each unit of change in the horizontal (or input) value there is a change of 2 units in the vertical (or output) value (*functional property*). Similarly, a student who thinks of slope as an invariant and has an object conception of slope, will also be able to think of slope as a covariation of two quantities or as a single quantity as required by the problem situation, and will be able to apply the notion in different contexts (*real world situation*).

This framework brings together past research on slope with the ideas of APOS theory to provide new insight into students' reasoning about slope. This framework can be used by educators to design varied tasks and by researchers to classify students' conceptions of slope. It advances past research on slope by moving past classifying ways students interpret slope to describing a framework that can account for students' interpretations.

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STUDENT REASONING WITH FUNCTIONS: NEGOTIATING VISUAL AND ANALYTIC PRESENTATIONS

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This study provides an analysis of twelfth-grade students' reasoning about functions. It focuses on differences in performance and thinking about mathematical tasks depending on how functions are presented (visually or algebraically) and whether a task requires transformations from one mode of presentation to another. Using NAEP mathematics items that focus on functions as well as additional instruments to elicit student reasoning, the study examines the function performance of twelfth-grade students in Algebra 2, Pre-Calculus, and Calculus courses at a midwestern high school. Results are contextualized in relation to national data on function performance drawn from the Grade-12 mathematics NAEP. Findings offer insights into students' function concept and highlight opportunities to better support their efforts to navigate a range of function representations.

Keywords: Algebra and Algebraic Thinking, Curriculum, Assessment and Evaluation

Introduction

Although functions have long been presented most explicitly in algebra courses, the high school *Common Core State Standards for Mathematics* (CCSSM) (National Governors Association Center for Best Practices, Council of Chief State School Officers [NGA & CCSSO], 2010) present functions as their own conceptual category. (The other CCSSM categories are number and quantity, algebra, modeling, geometry, and statistics and probability.) This shift reflects the fact that functions are a dynamic concept that intersects with numerous other mathematical ideas, cutting across content traditionally divided into separate courses. For example, functions play an important role in statistics (e.g., linear regression) and geometry (e.g., area of a square as a function of the length of its side), and they are also prominent in real-world applications (e.g., the modeling of growth, decay, and motion).

Another important dimension of functions is that they can be represented in a number of ways (e.g., graphical, algebraic, tabular). The most significant and frequently discussed challenge presented by functions is this diversity of representations and the difficulty of moving between one form of representation to another (Eisenberg, 1991). Because functions have historically been stressed primarily in the context of algebra, students may have a limited concept image of functions. That is, the set of mental representations that they associate with functions does not reflect the range of representations of functions that they may encounter through their mathematical learning trajectory (Vinner & Dreyfus, 1989). This study examines students' performance on function items and their thinking about functions and their representations. These findings will help direct efforts in teacher professional development and pre-service teacher programs to maximize students' understanding of this crucial concept.

Background

For the Grade 12 math assessment, the National Assessment of Educational Progress (NAEP) reports overall results to represent nationwide math achievement. NAEP also offers results broken down by content strand (number and operation; geometry and measurement; algebra; and data analysis, probability, and statistics). NAEP does not, however, break these strands down into further subcategories or provide information about skills or concepts that extend over several content strands. This study builds on the researchers' past work to identify those NAEP items related to functions (across several content strands) and to analyze this nationally representative data (Pérez,

2014). The findings from that study provide context in the present study regarding what U.S. students know about functions in general as well as how they perform differently on items that elicit a visual or an analytic approach to handling functions.

Study Design

The study examines the function concept of a sample of twelfth-grade students from Algebra II, Pre-Calculus and Calculus courses at a midwestern high school. These students will respond to 6 function-related problems drawn from the 2005 and 2009 Grade 12 NAEP math assessments, a problem set that includes multiple-choice and short-answer questions. 6 function items will be selected from a body of items blind coded by math education professionals into three categories: (1) those items that engage analytic processing of information (e.g., the problem includes formulas and/or verbal description), (2) those items that engage visual processing (graphs and/or tables) of information, and (3) those items that require both visual and analytic processing to arrive at a solution. Table 1-3 below are examples in which the question presents a function analytic, visually, and both.

Table 1: Items 1 (Analytic)

Yvonne has studied the cost of tickets over time for her favorite sports team. She has created a model to predict the cost of a ticket in the future. Let C represent the cost of a ticket in dollars and y represent the number of years in the future. Her model is as follows: $C = 2.50\,y + 13$. Based on this model, how much will the cost of a ticket <u>increase</u> in two years?

    (A) $5 (B) $8 (C) $13 (D) $18 (E) $26

Source: U.S. Department of Education, Institute of Education Sciences, National Center for Education Statistics, (NAEP), 2005 Mathematics Assessment

Table 2: Item 2 (Visual)

x	y
-2	3
-1	0
0	-1
1	0
2	3
3	8

The table shows all the ordered pairs (x, y) that define a relation between the variables x and y. Is y a function of x? Give a reason for your answer.

Source: U.S. Department of Education, Institute of Education Sciences, National Center for Education Statistics, (NAEP), 2009 Mathematics Assessment

Table 3: Item 3 (Visual and Analytic)

A random sample of graduates from a particular college program reported their ages and incomes in response to a survey. Each point on the scatterplot represents the age and income of a different graduate. Of the following equations, which best fits the data?

    (A) $y = -1,000x + 15,000$ (B) $y = 1,000x$
    (C) $y = 1,000x + 15,000$ (D) $y = 10,000x$
    (E) $y = 10,000x + 15,000$
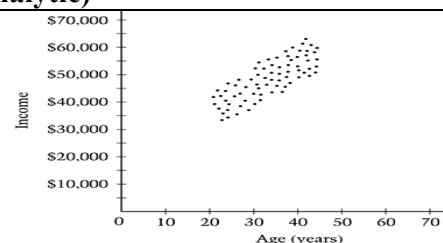
Source: U.S. Department of Education, Institute of Education Sciences, National Center for Education Statistics, (NAEP), 2009 Mathematics Assessment

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After completing the assessment, students will answer follow-up questions designed to elicit information about their reasoning, and selected students will be interviewed about their problem-solving process. Analysis of items and of interview responses will focus on identifying particular areas of challenge for students, whether in the presentation (visual versus analytic) or in the function concept that the item draws upon. The analysis will also examine the degree to which students' content knowledge parallels or differs from that of the nation's secondary students (as reported by NAEP).

Expected Findings

While the study is still in progress, for the presentation we will report concrete data on student performance and illustrate their thinking using brief excerpts from the assessment and related interviews. Preliminary findings from NAEP suggest that secondary students do relatively well on items that present functions algebraically and also offer algebraic solution choices. Similarly, they perform well on items that graphically present functions and offer graphical solution choices. By contrast, performance decreases significantly on those items that require students to engage both visual and analytic representations of a function to arrive at a correct solution. These outcomes are in line with previous research that has demonstrated a persistent gulf between students' visual concept images of functions and their analytic characterizations of functions (Eisenberg, 1991). Based on the NAEP results on function items, a similar pattern of performance is expected in the results from the local student sample. Nevertheless, the study also seeks to identify possible relationships between current course-taking (Algebra 2, Pre-Cal, or Calculus) and student thinking about functions.

The interviews in the study provide opportunities to better understand students' approach to the problems. Particular attention will be paid to the degree to which students express awareness of the interrelations between visual and analytic presentations of functions. This information will lead to insight on the areas in which students need additional support to develop the depth of understanding that will then allow them to successfully engage with mathematics as an integrated discipline.

Presentation Overview

Presenters will provide tables that describe: (a) nationwide NAEP student performance on analytic and visual function items as well as those that engage both representations and (b) local twelfth-grade students' performance on the same items (analyzed by course level). A brief discussion of the patterns in performance follows, and vignettes from interviews will illustrate observations about student reasoning with regard to functions. Implications of the pilot study for future research in student reasoning will be briefly discussed.

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ADDITIVE INVERSES: SECOND GRADERS' USE OF "ZERO PAIRS"

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Questioning boundaries between mathematics topics can illuminate ways to support students' developing mathematical knowledge. For example, algebra and negative integers intersect around the concept of additive inverses: $a + -a = 0$, which we call zero pairs. Two popular conceptual models can illustrate additive inverses (Wessman-Enzinger & Mooney, 2014). Movements along a number line draw on a translation conceptual model, where zero can be a relative position. The chip model, where positive chips (1) cancel out negative chips (-1), typifies a counterbalance conceptual model because although the quantities cancel out, they still remain; zero in this model results from having an equivalent number of positives and negatives. We explore the following questions:

1. How do second graders perform on zero pair problems before and after instruction?
2. What characteristics of zero pair problems do they notice?

This study involved 109 second graders and employed a pre-test, random assignment to condition, group sessions with test questions, instruction, post-test design. As part of their two group sessions, students analyzed problem contrasts involving a zero pair problem, illustrated using pictures of positive and negative chips with zero pairs circled. Following the group work, students learned about using a translation conceptual model to move up (+ positive) or down (+ negative) on a number path. The last few "zero pair" problems (e.g., $-2 + 2$), resulted in zero. Then, students played a card game involving a cancellation conceptual model where they had to make zero pairs to discard.

During small group sessions, students noticed that for $5 + -5 = 0$ and $5 + -4 = 1$, the chips not circled (or cancelled) were the answer. When looking at the work for $-5 + -5 = -10$ and $-5 + 5 = 0$, one student clarified, "This time they started on a -5, so on the number line you start on negative five, so $-5 + 5$, you get to zero." After the first part of the lesson, students came up with many zero pairs, such as "$4 + -4$" and "$-45 + 45 = 0$." When playing the card game, many students paired single -1 cards with +1 cards, and others paired multiple cards (e.g., paired -2 and 2).

When solving zero pair problems (e.g., $-5 + 5$, $9 + -9$), students averaged 15% correct on the pretest, 33% at the end of sessions, and 59% on the post-test after instruction. They also used this knowledge to solve the transfer problems on the posttest ($4 + -4 + 3$, $-5 + -1 + 6$, and $-2 + 5 + 2$). On average, students got 40% of these correct. Overall, students made reference to a translation and cancellation conceptual model, suggesting that the use of both conceptual models could be helpful in reaching more children. Analyzing contrasting cases proved to be a useful experience.

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LEVERAGING CONTRASTING CASES: INTEGER ADDITION WITH SECOND GRADERS

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Having students analyze contrasting cases can help them pay attention to similarities and differences in problems (e.g., Rittle-Johnson & Star, 2011). One area where this would be beneficial is with negative numbers, as students often treat them as positive (Bofferding, 2014). Therefore, we investigate to what extent second graders can improve their integer knowledge from analyzing varied contrasting cases and participating in short instruction.

The analysis focuses on 107 second graders. After a pretest, students were randomly assigned to one of three groups: one compared problems of the *same type* (e.g., 2+3 vs. 3+3 then -2+-3 vs. -3+-3); the other two either started with an *intuitive* contrast (e.g., 2+3 vs. -2+-3) or a *conflicting* contrast (e.g., 2+3 vs. 2+-3). After two comparing sessions and one lesson, the students took a posttest. The tests targeted students' understanding of integer order and values, integer addition (e.g., -5+5), and transfer problems with three addends or a missing addend.

During their 2 small-group sessions (20-40 minutes each), students discussed and wrote about how the contrasting problems were similar and different, why some answers were incorrect, and how to use the pictures to solve the problems. After each session they solved midtest items (problems similar to those they analyzed). Students also participated in a whole class 30-minute session where they learned that adding a positive corresponds to moving up on a number path and adding a negative corresponds to moving down and played a card game about making additive inverses.

A repeated measures ANOVA on the common items across testing times (-9+2, 9+-9, -1+-7) was not significant for the interaction between test and group. However, there was a significant effect for test, $F(2, 199) = 69.412$, $p = .001$; students did significantly better on the midtest questions compared to the pretest and on the posttest compared to the midtest questions. The repeated measures analysis for the items on pretest and posttest was also only significant for test, $F(1, 104) = 112.69$, $p = .000$, $\eta^2 = .520$. On the pretest, students got an average of 14% correct (range: 4%-24%); on the posttest, they averaged 32% correct (range: 14%-50%). As a whole, the second graders benefitted from analyzing contrasts, regardless of type. They then gained even more after explicit instruction and playing a game. The ease with which these diverse students made gains suggests we rethink the boundary of when integers and integer addition are introduced.

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COMPARING STUDENTS' COMPETENCE AND CONFIDENCE FOR THE CONCEPT OF SLOPE

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Confidence assessment evaluates students' confidence about their answers to items. It can be used to explore the strength of mathematical misconceptions. Hasan, Bagayoko and Kelley (1999) claimed that low confidence in a correct answer indicates guessing, which implies a lack of understanding. High confidence on a wrong answer indicates a misunderstanding. There have existed large arguments about gender difference in mathematics confidence (e.g. Hyde, 2005).

The slope of a line is a fundamental underpinning concept for students to understand mathematics (Carlson, Oehrtman, & Engelke, 2010). In precalculus, slope plays a key role in understanding the most basic rate of change of a function. In the calculus sequence, slope is used for the definition of a derivative and can be used to explain directional derivatives (McGee & Moore-Russo, 2015). Due to its diverse representations and conceptualizations in many different contexts, students' difficulties in slope have been observed in the literature throughout school math learning. Their difficulties in slope set up a barrier to their college-level math learning. To measure college students' slope competence and confidence, Ding (forthcoming) developed a 3-tier 27-item instrument on the slope of linear function (person reliability: 0.84 & 0.95; item reliability: 0.86 & 0.96 respectively). Along this line of research, this study is guided by two research questions: (1) How confident are college students about their slope understanding? (2) Is there any difference between females and males in confidence in slope understanding?

By the Rasch rating scale model, this study analyzed the self-reported responses of college students (n=93) in a 4-year pubic college on a 5-point scale (from *not confident* to *very confident)* 27-item slope confidence assessment instrument (Ding, forthcoming). The study found 33% of the sampled students were *very confident* in their slope understanding, while 25% of them were *not confident*. The items associated with low confidence indicated that students lacked competence in connecting multiple concepts. Students' slope competence and confidence were moderately related ($r = 0.55$). There was no significant difference between female and male students although females were slightly less confident than males ($p = 0.09$). The findings have implications for college mathematics and developmental mathematics teaching and learning.

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PRELIMINARY FINDINGS OF FIRST GRADE STUDENTS' DEVELOPMENT OF REVERSIBILITY

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We define *reversibility* as an individual's ability to rely on both inversion and compensation when solving problems. *Inversion* of an operation undoes the previous to return to the original. In contrast, *compensation* of an operation is a return to an equivalent state (Hackenberg, 2010). Hackenberg argues that an individual simply reversing one's scheme (inversion) without being aware of the need to coordinate operational structures (inversion and compensation) may not be engaging in reversibility. In this study, we focus on an analysis of preliminary qualitative data that suggest first grade students' engagement with piloted tasks may promote reversibility.

Methodology

Four, first grade children (Francis & Callie, Allen & Becka) from the northwestern United States were chosen as participants due to their below average Test of Early Mathematics Ability 3 (TEMA-3) score (Ginsberg & Baroody, 1983). Two teaching experiments (Steffe & Ulrich, 2014) were used to investigate students' cognitive reorganization when engaging with reversibility tasks. The 12 teaching experiment sessions included inversion and compensation tasks with counters to assess students' ability to count and group visual material.

Results/Conclusion

Callie struggled with the inversion and compensation tasks when concrete material was absent, but Francis could re-imagine patterned orientations and elicit her understanding of "doubles" providing her success when finding missing addends and compensating problems between tasks (e.g., $4 + 2 = 3 + 1 + 1$). Francis' reliance on patterned orientations also limited her ability to engage flexibly with groups. Francis cognitively reorganized her grouping thinking structures when tasks elicited "doubles" or "near doubles" (e.g., $4 + 4 = 5 + 3$). Allen and Becka solved the compensation tasks (e.g., $4 + 1 = 3 + ?$) in very different ways. Allen matched the counters on each side and then moved them to align with the original problem. Becka imagined the same subgroups on each mat with the same number and then described extra counters.

Preliminary analysis of these data suggest that students' abstract and flexible development of their grouping thinking structures provides effective engagement with inversion and compensation tasks. Implications of these findings provide educators with more insight as to intervene when promoting more sophisticated reasoning associated with reversibility.

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GETTING A GRASP ON MATHEMATICS MISCONCEPTIONS

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A crucial conceptual leap for students is to transition from grounded arithmetic problems to a more abstract world of algebra problems with variables (Heffernan & Koedinger, 1998). Students with prior misconceptions of mathematics solve fewer equations correctly and have difficulty learning new procedures and problems (Booth & Koedinger, 2008). It is therefore important to identify the type and source of common errors at a large scale. In the current study, 48 middle school students transformed and solved 15 linear equations with one variable using a new dynamic technology, Graspable Math (GM) (Ottmar & Landy, 2015) and then were asked to manually write their answers into the system. We found that, while many students were able to successfully use procedures to manipulate the symbols to arrive at the correct answer using GM, many students would then type in incorrect answers. In this study, we used error analysis to identify the most common error patterns and mal-rules that occurred after successful symbol manipulation. 4 categories of student errors emerged (Table 1). These errors can be used to inform future instructional interventions that target algebra readiness and student difficulties.

Table 1: Four Categories of Errors

Equation vs. Expression - When simplifying expressions with two unlike terms, students turned it into an equation, set the expression equal to zero, and then solved for x (ex. $2x-6 \rightarrow 0=2x-6 \rightarrow 6=2x \rightarrow x=3$)	**Everything to the Right -** Students moved everything to the right of the equals sign until a 0 was on the left and then simplified the right. Their answer was everything to the right of the equals sign (ex. $6=4x \rightarrow 0=4x-6$).
Combining Un-like Terms -Students simplified an equation resulting in two unlike terms (ex. $x=2t+10$). They then added together the 2 and 10, and entered $x=12t$.	**Hidden Negatives** - If there was a negative sign next to a term or number on the left, students would often ignore negative signs (ex. $-x=6 \rightarrow x=6$)

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PRECONCEPTIONS OF NEGATIVE NUMBER PRODUCTS AND QUOTIENTS

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Despite the regularity of rules for integer products and quotients, on a standardized test only about half of eighth-grade students accurately answered such problems (Ryan & Williams, 2007, p. 218). Ignoring negative signs or treating them as subtraction signs are typical student errors in integer arithmetic (Ryan & Williams, 2007; Vlassis, 2008). Although integer multiplication and division are difficult, with the exception of Sfard (2007), student thinking about this has been largely ignored as a research focus. Thus, to help future students improve learning outcomes, it behooves us to understand how students make sense of products with negative numbers prior to instruction (i.e., preconceptions). The term *Intelligent Overgeneralization* (*IG*, Ryan & Williams, 2007, p. 23) is used here to acknowledge inaccuracy of overgeneralization, yet recognize the intelligence of student sense-making from prior experience with non-negative numbers. This report seeks to contribute to student-centered views of transfer (Lobato, 2006) with the question: In what ways might conceptions students express when making sense of integer multiplication and division prior to instruction be IGs from prior instruction with non-negative numbers?

Fifth grade students from two schools (n=107) participated. Students were primarily European-American, 45% of whom were eligible for free or reduced lunch. Data come from two measures: (a) a written task in which students were asked to explain and draw what the problem "means" (e.g., ˉ2 × ˉ8) and (b) interview tasks with a subset of students. Analysis consisted of three phases with all coders confirming presence of each preconception. I report three of these preconceptions here. *Negative Sign is Like a Unit* revealed the idea that the negative sign is like other unit symbols that can be ignored while calculating, such as a dollar sign. *Synthesized Meanings of Opposites* refers to students who treated negative numbers both as opposite of positive and the inverse of multiplying. Students who *Coordinated Multiple Operations* saw a negative sign as a subtraction sign and used the order of operations to first multiply then subtract.

These results detail ways students may generalize from non-negative number instruction. Framing these preconceptions as IGs honors students' interpretations of structures in mathematics that common instructional practices may foster. The poster describes how each preconception relates to prior non-negative number instruction from which students could have intelligently overgeneralized and how such an IG might impact future learning.

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TEACHERS' COVARIATIONAL REASONING IN CONTEXT: RESPONSES TO A GRAPHING TASK

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In order to build and interpret graphs of functions, students must learn to coordinate changes in an independent variable with corresponding changes in a dependent variable. Covariational reasoning describes a hierarchy of mental actions involved in coordinating changes in two related variables (Carlson et al., 2002). Carlson et al. studied calculus students' responses to a graphing task in which the independent variable can be viewed as a proxy for a time variable. However, relatively little is currently known about how learners think covariationally about relationships in which neither variable changes linearly with time.

In Summer 2015, the authors taught a professional development course to middle and high school mathematics teachers in the Southwestern United States. Teachers spent class time working on activities meant to highlight key ideas in middle grades and secondary mathematics. During this course, teachers were asked to work individually on the following task:

> Arun gets on a Ferris wheel from a platform several meters above the ground. The Ferris wheel rotates and Arun makes several laps around it. A power line runs just above the Ferris wheel. Sketch a graph of the relationship between Arun's distance from the ground and his distance from the power line while he is on the Ferris wheel. Label and explain any key points or features in your graph.

Our data consisted of teachers' written work on this task. Approximately half of the teachers created graphs showing a non-linear relationship between the variables; most of these teachers drew either graphs of sinusoidal functions or graphs of functions that increased toward a maximum and then decreased, or vice versa. Approximately one fourth of the teachers drew graphs of decreasing linear functions, with most of these teachers appearing not to attend closely to the placement of endpoints. Of the remaining respondents, several sketched graphs that we categorized as "ambiguously linear": the graphs did not show visible variations in the rate of change of one variable with respect to the other, but also did not show clear evidence that the respondents understood that the function is linear. These responses suggest that the context of the problem, in which both variables change non-linearly with time, created resistance to recognizing the relationship between the two variables as linear.

Further investigation is needed to illuminate the types of reasoning by which learners arrive at the correct graph of the relationship between the two distances, and how the non-linear variation in each distance with time interferes with recognizing this relationship as linear. We hope to improve the quality of data by capturing video of teachers working on this and related tasks and asking them to explain the shapes of the graphs they create.

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INTEGRATION OF MATHEMATICS AND PROGRAMMING: EMPOWERING TEACHERS TO BRING COMPUTING INTO THE ALGEBRA CLASSROOM

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Computational thinking should have a central role in today's mathematics classrooms because this thinking is central to many growing fields such as computer science and bio-informatics (Next Generation Science Standards Lead States (NGSS), 2013). The reciprocal relationship between computational learning and mathematics understanding is an additional motivator for introducing computational thinking into mathematics classrooms (Weintrop et al., 2016). The potential of computational thinking to enhance mathematical reasoning makes integrating this twenty-first-century skill into algebra contexts especially compelling.

As part of a larger NSF-funded project on the integration of computer programming in a mathematics classroom, this study seeks (1) to enhance teachers' pedagogical content knowledge of functions and their computational thinking and (2) to support them in incorporating computer science in their teaching of functions. Success in algebra has been strongly linked to college access and attainment, and a deep understanding of functions—which are central to algebra and prominent across the mathematics curricula—has proven elusive for many students (Dubinsky & Wilson, 2013).

Twenty participant teachers from a medium-sized city in the Midwest will engage in a 10-day training to explore a unit that deepens function knowledge through computer modeling and programming. Observations of teachers' engagement with the training tasks will enable the team to position their programming knowledge on a spectrum running from novice to proficient and expert. For example, we ask teachers how proficient they are with designing function related problems. We anticipate that most participating algebra teachers will be at the novice level; to enable them to introduce their students to the modeling and programming experiences, we seek to move them to the level of "proficient." In addition to technical skill, the training seeks to increase teachers' self-efficacy in computational thinking to increase the teachers' motivation to implement the new approaches in their classrooms (Graham & Weiner, 1996). Based on an assessment after the training and analysis of the teachers' engagement in the training experiences, we will evaluate the degree to which it has shifted teachers' proficiency with computational thinking and teachers' self-efficacy.

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EL ESTUDIO DE LAS FUNCIONES LINEALES COMO UNA INTRODUCCIÓN DEL ÁLGEBRA EN EDUCACIÓN SECUNDARIA

THE STUDY OF LINEAR FUNCTIONS AS AN INTRODUCTION TO ALGEBRA AT GRADE 7

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Palabras clave: Álgebra y Pensamiento Algebraico, Modelación, Educación Secundaria

Respecto a los procesos de transición hacia el pensamiento algebraico, la investigación ha demostrado que la ruta de introducción determina las concepciones que se desarrollan en el estudio del álgebra. Específicamente, la introducción del álgebra vía el estudio de las funciones requiere el desarrollo de la noción de variable y de sus diferentes representaciones, mediante tablas, gráficas cartesianas y las expresiones algebraicas correspondientes (Sutherland, 2004). Si bien los resultados reportados de la implementación de propuestas didácticas con enfoque funcional son prometedores, se ha encontrado que la integración de las diferentes representaciones de las funciones es un asunto que requiere mayor investigación (Kieran, 2006).

El estudio que presentamos se centró en las dificultades y los avances de estudiantes de primer grado de secundaria al seguir una secuencia de problemas y actividades que, a partir del estudio de las funciones lineales, introdujo la simbolización algebraica como una más de las formas de representación de la variación, junto con la gráfica cartesiana y las tablas de valores (Sadovksy, 2005).

La secuencia fue tomada y adaptada del trabajo presentado por Bravo-Barletta y Matos (2005). En esta secuencia, los alumnos deben interpretar la información que surge de una lectura directa de las gráficas, así como obtener datos que requieren un análisis más profundo (de intervalos de crecimiento, máximos, mínimos, etc.). Se incluye además un trabajo de construcción de gráficas para modelar situaciones en contextos específicos.

Nuestra investigación se centró en problemas de fenómenos lineales, lo cual nos permitió establecer continuidades y contrastes entre el trabajo con las relaciones de proporcionalidad (estudiadas desde el nivel de primaria, en aritmética) y el trabajo con las funciones lineales (que en México se estudian en secundaria, con las herramientas del álgebra). Las expresiones algebraicas se introdujeron como una forma de representación que permite sintetizar el comportamiento de las relaciones relevantes del problema y hacer anticipaciones sobre ellas.

Esta secuencia se implementó con un grupo de 36 alumnos, de un poblado semi-rural de las afueras de la Ciudad de México. La secuencia consta de 12 problemas que se distribuyeron en 20 sesiones (de 50 minutos cada una). Se usó el video para registrar el trabajo colectivo, las intervenciones del profesor y sus interacciones con los estudiantes. El profesor a cargo de estas clases fue uno de los investigadores del equipo.

En el análisis se buscó estudiar la manera en que los estudiantes se apropian de las herramientas algebraicas para el estudio de las funciones lineales a través de esta secuencia específica. Entre los resultados encontrados están la construcción de representaciones "híbridas" por parte de los estudiantes, así como diversas dificultades en la coordinación de las distintas formas de representación de la variación, en particular al relacionar el "factor constante de proporcionalidad", la "pendiente de la recta" y la "razón de cambio constante".

Keywords: Algebra and Algebraic Thinking, Modeling, Middle School Education

Regarding the transition processes toward algebraic thinking, research has shown that the choice of the introductory route determines the conceptions that are being developed during the study of algebra. Specifically, the introduction of algebra through the study of functions requires the development of the notion of variable and its different representations, using tables, Cartesian graphs, and the corresponding algebraic expressions (Sutherland, 2004). Although reported results on the implementation of didactic proposals with a functional approach are promising, it has been found that the integration of the different representations of functions is a matter of further research (Kieran, 2006).

The research we are presenting was centered on the secondary First Grade (equivalent to Grade 7 in US and Canada) students' difficulties and advances while engaging in a sequence of problems and activities that, based on the study of linear functions, introduced algebraic symbolization as another form of representation of variation, joint with Cartesian graphs and charts (Sadovksy, 2005).

The implemented sequence was adapted from the work presented by Bravo-Barletta and Matos (2005). In this sequence students are requested to both interpret the information from a direct reading of the graphs and infer data that require a deeper analysis (such as growing intervals, and maximums and minimums values). Additionally, the sequence includes graph construction to model specific contexts.

Our research focused on lineal phenomena problems, allowing us to establish continuities and contrasts between the work with proportional relationships (studied at the elementary level, in arithmetic) and the work with linear functions (that in Mexico are studied at secondary level with the algebraic tools). The algebraic expressions were introduced as a type of representation that allows both synthetizing the significant relationships of the problems and making predictions about them.

This sequence was implemented in a 36-student classroom, from a semi-rural town in the surroundings of Mexico City. The sequence comprises 12 problems, distributed in 20 sessions (50 minutes each). Video was used to record the collective work, teacher's interventions and her interactions with the students. The teacher was a member of the research team.

During data analysis we sought to study the way students appropriate the algebraic tools for the study of linear functions through this specific sequence of tasks. Among the results we have found, there are the "hybrid" representations constructed by students and diverse difficulties in coordinating the several representation of the functional variation; particularly in relating the "proportional constant factor," the "slope of the straight line" and the "constant rate of change."

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ELEMENTARY STUDENTS' UNDERSTANDING OF EQUIVALENCE AND MULTIPLICATIVE REASONING

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Children with a relational (1+2 *is the same as* 3), as opposed to an operational (1+2 *makes* 3), conception of equivalence are more able to engage in algebraic tasks in later grades (Knuth et. al, 2006 etc.). Children's multiplicative reasoning (MR) also facilitates the development of algebraic reasoning in later grades (Hackenberg, 2013). However, there is little to no research available which examines how children's MR relates to their conceptions of equivalence. Therefore, this study examines whether elementary students' multiplicative unit coordination affects their scores on a conception of equivalence assessment.

Data came from a larger study on mathematical argumentation collected in May 2015 and includes 168 second and third grade students. Participants completed an assessment on equivalence (adapted from Rittle-Johnson et al, 2011) and MR via students' unit coordination (Kosko & Singh, in review). We used the MR assessment scores to categorize students as demonstrating one level of unit coordination (Tier-0, or pre-multiplicative), two levels of unit coordination (Tier-1, which includes skip-counting), and initial three levels of unit coordination (Tier-2, where students demonstrate disembedding by 1s). We used one-way ANOVA to examine if children at each level had different equivalence scores. The initial ANOVA violated the assumption of homogeneity of variance and normality. Therefore, we used Welch's ANOVA to account for this violation (Lix et al., 1996). The Welch's ANOVA test on levels of MR and equivalence test scores found statistically significant results, *Welch's F* $(2, 60.88) = 27.45$, $p=.000$. Post-hoc analysis found statistically significant differences between Tier-0 and 2 ($p=.000$) and Tier-1 and 2 ($p =.005$), but not between Tier-0 and 1 ($p = .318$).

These findings indicate that students at Tier 2 are significantly more likely to have a relational conception of equivalence than students at Tiers 0 or 1. One likely explanation is that students at MR Tier 2 can disembed with units (e.g., understanding 5x8 = 5x3+5x5), and this might be applied via the comparative strategies of students with relational conceptions. For example, solving 12+17 = 18+11, a student may disembed tens and ones on both sides to identify each expression as the same as 20 + 9 (or 29). Yet, this must be explored and confirmed by further study.

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ANALYZING GENERALIZATIONS THROUGH DISCOURSE

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Although generalizing is "intrinsic to mathematical activity and thinking" (Kaput, 1999, p. 137), students struggle to generalize, often make weak generalizations, and rarely justify their generalizations. Supporting generalizing in the mathematics classroom requires a better understanding of the source of students' generalizing. What are the mechanisms that encourage students' generalizing? This question is central to our research, which seeks to understand the nature of the discourse that supports students' generalizing. This study explored the ways in which a fourth grade class generalized and justified their generalizations about even and odd numbers during a lesson.

We conducted a multi-level analysis of the classroom discourse, identifying shifts and instances of re-centering. Based on the inferred goal of the speaker, each utterance was coded for purpose and technique for fulfilling that purpose (Gonzalez & DeJarnette, 2012). Additionally, we recorded generalizations (Ellis, 2011), and situated these generalizations in the discourse in order to describe the discursive moves that contributed to and resulted from the generalizations.

Two important shifts in the discourse occurred during the lesson. First, the mode of representation shifted from an array to a number expression—a shift in representation that indicates a shift in discourse. Second, after establishing "cubes and left over cubes" as a shared way to describe even and odd number generalizations, students shifted from using "left over/leftover" as a modifier or adjective to using it as a noun, an indication of an evolving understanding of the shared idea. Although five generalizations occurred during the lesson, they all occurred during the second half of the lesson, subsequent to the shifts in discourse. Thus, the results suggest that the shifts in discourse may function as generalizing-promoting actions (Ellis, 2011). We also identified particular discursive techniques, and suggest that future research might focus on these aspects of discourse to further explore generalizing-promoting actions.

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A TEACHER'S BOARD WORK AS A REFLECTION OF NOT ATTENDING TO STUDENT THINKING

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The content on the board (symbols, alphanumeric characters, and pictures) can provide evidence of the discourse, culture, and beliefs of the classroom participants to an outside observer (Stigler & Hiebert, 1998). Since the board is evidence of the classroom discourse, culture, and beliefs, it is our belief that we can study the work displayed on the board to analyze (in part) the type of teaching taking place in the classroom. In this study, we will analyze a middle school mathematics teacher's scheme for partitioning fractions on the board. Furthermore, we will examine the following questions: *How does the teacher's board work reflect the level of attention that she pays to student thinking, and how might this potentially influence student thinking about fractions*?

Data Sources and Methods

The data we collected for this study was gathered through the efforts of a large-scale professional development and research project focused on Arizona middle school teachers. The project focused on promoting the mathematical and pedagogical development of its participants.

Our subject, Ellie, is a 6th grade mathematics teacher. At the time of this study Ellie taught at a middle school that aligns with Common Core State Standards (National Governors Association, 2010). Ellie volunteered to have one of her class periods observed and filmed one to three times per week during her first year. These are the specific data in this study. These data were analyzed using a grounded theory approach (Corbin & Strauss, 2008) approach.

Conclusion

Based on her board work, Ellie didn't express attention to how students may have thought about fractions. There did not seem to be an explicit effort on her part to relate what students might have be thinking to what she presented at the board.

Ellie's board work consistently displayed fractions as merely procedures to engage in and commands to compute. While it is possible that Ellie might have held some productive meaning for fractions, her board work did not convey these meanings. Thus, we believe that she conveyed unproductive meanings for fractions to her students.

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LA ANTICIPACIÓN COMO RECURSO PARA INTERPRETAR A LAS MATRICES COMO TRANSFORMACIONES EN 2D

ANTICIPATION AS A RESOURCE FOR THE INTERPRETATION OF MATRICES AS 2D TRANSFORMATIONS

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Palabras clave: Álgebra y Pensamiento Algebraico, Cognición, Actividades y Prácticas De Enseñanza

La aproximación a la enseñanza del algebra lineal es frecuentemente formal, Dorier (2000) lo que hace que muchos de los conceptos de ella sean interpretados sólo desde sus definiciones, lo que obscurece las relaciones entre la expresión algebraica y la gráfica.

En este trabajo pretendemos proponer a los estudiantes de bachillerato y licenciatura actividades que les permitan observar las representaciones gráficas asociadas a las trasformaciones que dan origen las matrices en 2D de manera que articulen los registros de representación algebraico y gráfico, Duval (2006) usando como base de esta inspección ciertas matrices que tienen ceros en alguna de las diagonales.

En un primer momento trataríamos a las matrices de la siguiente manera:

Tabla 1: Tratamiento de las matrices de esta propuesta

Matriz como transformación lineal	Usando las propiedades de la linealidad	Matrices con diagonales cero como bases del espacio
$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$	$\alpha\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha x \\ -\alpha y \end{bmatrix}$	$\alpha\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \beta\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
Reflexión sobre el eje x	Contracción o dilatación de la matriz cuando α<0, 0≤α ≤1, α>1	Estas matrices son linealmente independientes, para α=β=0

A continuación, mostraríamos la forma cómo estos arreglos de vectores posibilitan expresar a todo punto del plano como un vector, es decir son una base para el espacio 2D. Esta aproximación no cubre todas las necesidades de explicación de cualquier tipo de matriz en 2D sobre puntos del plano, sin embargo permite imaginar que sucede cuando las matrices son usadas para conformar una base de este, incluso mostrar que el espacio puede ser generado por distintas bases, esto es, como distintos arreglos de vectores linealmente independientes.

Keywords: Algebra and Algebraic thinking, Cognition, Instructional Activities and Practices

The common teaching approach to Linear Algebra is frequently formal, Dorier (2000), this causes that many concepts are interpreted based only on their definitions and murking the relationship between the graphical and the algebraic expressions.

In this work we want suggest high school and college students some activities that will allow them to link the graphical representations associated with the transformations that lead (give rise) to the 2D matrices, so that they can articulate the registers of graphical and algebraic representations, Duval (2006), using as a base for this inspection matrices with zeros in one of the diagonals.

As a first step we would treat the matrices in the following way:

Table 1: Treatment of matrices with this proposal

Matrix as linear transformation	Using the properties of linearity	Matrices with zero diagonals as space bases.
$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$	$\alpha \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha x \\ -\alpha y \end{bmatrix}$	$\alpha \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \beta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
Reflection in the x-axis	Contraction or Dilation of the matrix when α<0, 0≤α≤1, α>1	These matrices are linearly independent, for α=β=0

Afterwards, we would show how these arrays of vectors make possible to express every point of the plane as a (single) vector, in other words, in fact these matrices are a base for the 2D space. This approximation does not include the explanation for every type of 2D matrix of the plane, however it allows us to imagine what happens when these matrices are used to make up a base of the plane even to show that the same space can be made up from different bases, that is to say, as different arrays of vectors linearly independent.

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USING STUDENTS' FUNDS OF KNOWLEDGE TO TEACH ALGEBRA

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Algebra is a gatekeeper to higher-level mathematics, with significant implications for equity in education and students' economic attainment (Moses & Cobb, 2001). Concepts from algebra are often not seen as being connected to students' worlds, including their home and community activities (Chazan, 1999). However, all students bring to the classroom mathematical funds of knowledge (Civil, 2007), ways of reasoning quantitatively from their home and community lives. In previous work, I found that students draw upon rich algebraic ways of reasoning when pursuing their out-of-school interests in areas like sports and video games (Walkington, Sherman, & Howell, 2014).

In the present studies, students author their own personalized "algebra stories." Here, *personalization* refers to the instructional approach of making connections between students' interests in topics like shopping, music, and social networking, and instructional content they will be learning in school. My overarching research question is: What are the affordances and constraints of personalized problem-posing as a scaffold for students' deep understanding of algebraic principles? I draw upon data from three studies where students pose, solve, and share personalized algebra problems related to their interests: single session interviews with pairs of 7th and 8th grade students; an extended teaching experiment of Algebra I students; and a large-scale intervention in intact 8th grade classrooms. These studies all took place at a high-poverty urban school with a significant proportion of students who spoke English as a second language.

Through a thematic analysis of videos, transcripts, and student work in these three research contexts, several key affordances and constraints were identified. First, personalized problem-posing sometimes involved compromising the *mathematical goals* of the lesson or activity. Specifically, when students' experiences did not fit well into the domain of linear functions, or when students had the tendency to fall back on more familiar everyday use of arithmetic, it was difficult to keep the activity focused on meeting grade-level standards. Second, the personalized problem-posing sometimes involved compromising the students' *funds of knowledge* of their interest area. Specifically, students would sometimes have to warp or misrepresent how they actually used quantities and change while pursuing their interests, in order to better fit into canonical school algebra norms. Despite these limitations, as a whole I found considerable and surprising affordances of students using their interests as a vehicle for understanding linear functions. Students generated a wide range of interest-based, mathematically-valid, content related to linear functions, and with practice found useful ways to draw upon their funds of knowledge as a resource in the mathematics classroom. In all three contexts, there was evidence of students' growth in their understanding of foundational ideas in algebra.

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Chapter 4

Geometry and Measurement

Research Reports

Brief Research Reports

Posters

VISUALIZACIÓN Y ANÁLISIS DE REPRESENTACIONES GEOMÉTRICAS

VISUALIZATION AND ANALYSIS OF GEOMETRIC REPRESENTATIONS

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En este artículo reportamos los resultados obtenidos al implementar actividades, cuyo foco es la visualización de representaciones geométricas, en ambientes de papel-y-lápiz y tecnológico con estudiantes[1] de grado 11. Esta investigación es de tipo cualitativo, y está apoyada en la teoría de representaciones y en la visualización de objetos geométricos con ayuda tecnológica. Discutimos cómo el uso de la tecnología favorece el aprendizaje de conceptos matemáticos a través del análisis de figuras geométricas estáticas y dinámicas. Nuestros resultados muestran que la tecnología favoreció la visualización de representaciones geométricas, durante la resolución de actividades, sin embargo, el papel-y-lápiz es fundamental para que emerjan conjeturas sobre el significado de conceptos abstractos relacionados con tales representaciones.

Palabras clave: Geometría y Pensamiento Geométrico y Espacial, Medición, Tecnología, Educación Secundaria.

Antecedentes y problema de investigación

Desde hace algunos años, la visualización matemática ha sido investigada como recurso para lograr el aprendizaje de conceptos matemáticos (e.g., Arcavi, 2003; Duval, 2003, 2006; Phillips, Norris, & Macnab, 2010; Presmeg, 2006; Zimmermann & Cunningham, 1991; entre otros). Los estudios de estos autores –sobre la visualización– han tenido el propósito de explorar distintas formas de "mirar" objetos matemáticos abstractos en ambientes de papel-y-lápiz y tecnológico, los cuales apoyan y median el surgimiento y desarrollo de conceptos matemáticos. Debido a las numerosas representaciones geométricas surgidas en la resolución de tareas de esta disciplina, al usar papel-y-lápiz y tecnología, se vuelve necesario identificar características y propiedades de éstas. En este artículo pretendemos responder la pregunta: ¿cómo influye la visualización en el aprendizaje de conceptos matemáticos en los ambientes de papel-y-lápiz y tecnológico?

Marco Conceptual

Esta investigación tiene como marco conceptual el relacionado con las aportaciones teóricas sobre la visualización matemática (e.g., Duval, 2003, 2006; Arcavi, 2003; Phillips et al., 2010; Zimmermann & Cunningham, 1991; entre otros). De acuerdo con el Diccionario de la Real Academia Española, el verbo "visualizar" puede ser entendido como sinónimo de "visualización" y se refiere a "formar en la mente una imagen visual de un concepto abstracto" (RAE, 2014). Diversas investigaciones en educación matemática han contribuido puntualmente respecto a lo que debe ser entendido por "visualización". He aquí tres de ellas: (i) "la visualización es la capacidad del individuo de producir una representación que, en ausencia de toda percepción visual de los objetos representados, por medio de la cual permite observarlos como si estuvieran realmente delante de los ojos" (Duval, 2003, p. 48); (ii) "la visualización es la capacidad, el proceso y el producto de la creación, la interpretación, el uso y la reflexión sobre figuras, imágenes, diagramas, en nuestra mente o sobre el papel con el propósito de representar y comunicar información, pensar y desarrollar ideas y avanzar en su comprensión" (Arcavi, 2003, p. 215); (iii) "la visualización matemática es el proceso de formación de imágenes mentales, usando papel-y-lápiz, o bien tecnología, y la utilización eficaz de dichas imágenes para el descubrimiento matemático y la comprensión de los objetos en estudio" (Zimmermann & Cunningham, 1991, p. 3). En estas y otras contribuciones sobre el significado de la

visualización matemática es patente el carácter cognitivo de ésta. A continuación, se describen algunos conceptos involucrados durante la visualización de representaciones geométricas en los ambientes de papel-y-lápiz y tecnológico.

Visualización en ambiente de papel-y-lápiz

La representación de objetos matemáticos involucra su análisis; el cual permite a los estudiantes acercarse a su significado institucional. Si las representaciones son figuras geométricas, entonces se debe identificar características de ellas, empleando el sentido de la vista. Sin embargo, independientemente de las concepciones, de la naturaleza y de la existencia de los conceptos matemáticos –de quienes aprenden o enseñan–, diversos autores (e.g., Arcavi, 2003; Duval, 2003, 2006; Phillips et al., 2010, entre otros) afirman que esa manera de construir conocimiento, a través de la visualización, debe efectuarse mediante la visión, la imaginación y la inteligencia. La visualización desencadena procesos automáticos no conscientes del sujeto; que dependen de todo eso que ha podido ser guardado en su memoria, los cuales permiten discriminar e identificar en menos de una décima de segundo los diversos elementos del campo de visión y sus relaciones (Duval, 2003).

Por medio de la visualización de representaciones, el individuo puede acceder a las propiedades, características y dar sentido al objeto matemático analizado (Phillips et al., 2010). La visualización matemática –en ocasiones– se emplea para describir representaciones; en otras, se usa para determinar cómo funciona cierta representación específica en la comprensión o resolución de problemas matemáticos; o bien, para definir la actividad cognitiva del sujeto cuando hace uso de representaciones. Las ideas surgidas de la visualización en matemáticas permite –a los estudiantes– enriquecer contenidos, cuya utilización resulta provechosa, tanto en las tareas de representación como en el manejo de conceptos de esta disciplina. De acuerdo con Phillips et al. (2010, p. 26), en la visualización se distinguen: (a) *objetos físicos* percibidos mediante el sentido de la vista (e.g., ilustraciones, animaciones, pantallas generadas por la computadora, etc.); (b) *objetos mentales* almacenados y procesados en la mente en forma de esquemas mentales, imágenes mentales, construcciones y representaciones mentales; (c) *funciones cognitivas* manifestadas en la percepción visual, la manipulación y transformación de las representaciones visuales en la mente, concretando los modos abstractos de pensamiento e imaginando hechos.

Phillips et al. (2010) enfatizan que estas distinciones son importantes para entender el contexto de la visualización y poder establecer aplicaciones eficaces de ésta en el salón de clase. Para que los estudiantes comprendan y utilicen formas pertinentes de representaciones geométricas, deben entender el significado de las rectas, puntas de flecha, marcadores de ángulos, números y variables, entre otras. Debido a que las representaciones no son fáciles de ser empleadas en la resolución de tareas, es necesario que los alumnos tengan conocimiento considerable de los símbolos y convenciones de ellas de modo que esas representaciones adquieran sentido para ellos (Phillips et al., 2010). La visualización involucra actividad cognitiva, como la interpretación y la abstracción de aquello que el objeto matemático representa (Duval, 2003; Arcavi, 2003; Hitt, 1995; Zimmermann & Cunningham, 1991; entre otros). Sin embargo, la imposibilidad de un acceso directo a los objetos matemáticos, fuera de toda representación, provoca en los estudiantes confusiones casi inevitables (Duval, 2003).

Visualización en ambiente tecnológico

En la actualidad, existe una gran cantidad de programas informáticos (e.g., GeoGebra, Maple, Mathematica, Cabri-Geometry, Matlab, entre otros) utilizados en la enseñanza de las matemáticas. Este tipo de herramientas favorece la visualización de representaciones de conceptos matemáticos, y se vuelve trascendental en educación matemática; en particular, el uso de algún Software de Geometría Dinámica (SGD) facilita la incorporación de la visualización en la enseñanza de conceptos de la geometría, ya que se cuenta con imágenes dinámicas, las cuales a través de las herramientas propias de los SGD se pueden medir, agregar trazos auxiliares o simplemente explorar

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alguna figura geométrica. Al hacer uso de representaciones dinámicas en la resolución de problemas geométricos, mediante la visualización, se facilita la manipulación, trazado o construcción de las figuras utilizadas y es posible modificarlas –si es necesario– en tiempo real (Hitt, 1995). Un atributo importante del SGD es su versatilidad de uso, el cual estimula el interés y la participación de los estudiantes cuando resuelven problemas geométricos. Diversos autores (e.g., Arcavi, 2003; Hitt, 1995; Phillips et al., 2010; Zimmermann & Cunningham, 1991, entre otros) mencionan que la visualización, empleando modelos dinámicos permite a los alumnos comprender conceptos o significados extraídos de las representaciones, y juega un rol importante en el desarrollo de su pensamiento analítico que, con frecuencia, no sucede en ambiente de papel-y-lápiz.

Metodología

El estudio es de tipo cualitativo. En él participaron 12 estudiantes mexicanos [grado 11], agrupados en parejas (seis Equipos). El trabajo estuvo dividido en dos fases: la primera se enfocó en la visualización de figuras geométricas usando papel-y-lápiz, con la finalidad de extraer las propiedades que, de acuerdo con el conocimiento previo de los alumnos, estaban implícitas en la figura; la segunda incluyó el análisis visual de la figura en ambiente dinámico (tecnológico). Mediante el uso del SGD (GeoGebra) los participantes debían validar las propiedades de las figuras visualizadas en papel-y-lápiz. Durante la resolución de las actividades en este ambiente, se buscó que los estudiantes conjeturaran el comportamiento general de las figuras cuando cambiaban algún parámetro de la misma, descubrieran invariantes, si los había, describieran y explicaran los objetos matemáticos visualizados.

La primera actividad fue retomada del artículo *A cognitive analysis of problems of comprehension in learning of mathematics* (Duval, 2006, p. 117). En esta actividad, el autor propone a los alumnos que a partir del esbozo de la gráfica (véase Figura 1) hallen la longitud del segmento $\overline{ED}$, tomando como datos fijos la longitud de los segmentos $\overline{AB} = \overline{DC} = 4$ cm y $\overline{BC} = 7$ cm. El autor reporta tres tipos de respuesta: 9% da 3 cm (repuesta matemática), 39.6% da 3.5 cm (medida directa del segmento) y 24.4% da otras respuestas, incluyendo la ausencia de ellas. Los resultados de los estudiantes tuvieron como referente el ambiente de papel-y-lápiz.
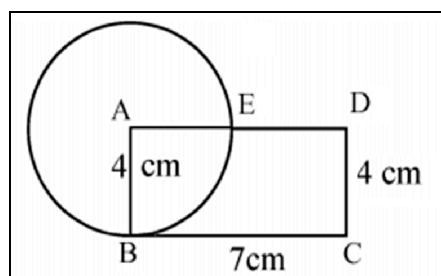
Figura 1. Cálculo de la longitud del segmento $\overline{ED}$, usando papel-y-lápiz.

La segunda actividad fue adaptada del libro *Plane and Solid Geometry* (Wentworth & Smith, 1913, p. 33). En nuestro estudio se propuso a los participantes un triángulo rectángulo ABC, que de acuerdo con sus características, permitiera a los alumnos identificar sus propiedades (implícitas y explícitas) con la finalidad de calcular su área. En la actividad, se pretende que los estudiantes identifiquen si existe alguna relación proporcional entre las áreas de los triángulos *ABC* y *XYZ*, este último formado a partir de los puntos medios del Δ*ABC* (véase Figura 2). Para determinar si existe (o no) cierta relación entre las áreas de los triángulos *ABC* y *XYZ*, se pidió a los alumnos que calcularan sus áreas; si existía alguna relación entre éstas, entonces los estudiantes debían explicar si tal relación podía (o no) mantenerse para cualquier triángulo independientemente de su tamaño o forma.

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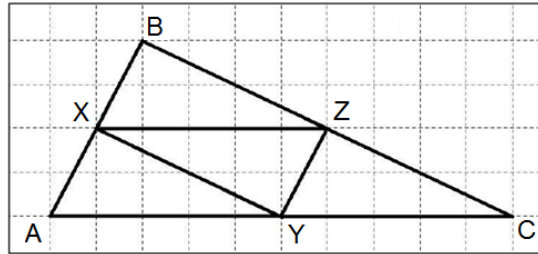
Figura 2. Identificación de la relación de áreas de triángulos, usando papel-y-lápiz.

El acopio de datos fue mediante el registro escrito de las respuestas de los estudiantes al resolver las actividades, las cuales fueron video-grabadas, además de preguntas del investigador –a los participantes– durante la implementación de éstas. Las actividades propuestas a los participantes debían ser abordadas usando papel-y-lápiz y tecnología (GeoGebra). En las mismas actividades se les solicitó que validaran sus resultados obtenidos en ambos ambientes.

Análisis de datos y discusión de resultados
El análisis de los datos recabados se basó en la interpretación y en el número de incidencias comunes, por parte de los alumnos, de sus soluciones en ambos ambientes, tomando en cuenta las video-grabaciones –de las discusiones [estudiante-estudiante y estudiante-profesor] durante la experimentación– y los registros escritos, una vez solucionado el problema. A continuación, analizamos y discutimos los resultados de las actividades implementadas.

Primera actividad
En la primera actividad –en ambiente de papel-y-lápiz– se obtuvieron los siguientes resultados: cinco equipos percibieron que el valor del segmento $\overline{ED}$ es 3.5 cm y sólo un equipo consideró las propiedades de la circunferencia, al darse cuenta de que el valor real del segmento $\overline{ED}$ es de 3 cm. De acuerdo con los resultados obtenidos, podemos afirmar que la manera de visualizar –por parte de los estudiantes– esta figura geométrica tiende a adquirir la forma de mayor simpleza (Duval, 2003), es decir, ellos perciben que los segmentos $\overline{AE}$ y $\overline{ED}$ son iguales. Esta afirmación está basada en que –la mayoría de ellos– identificó que el punto E (intersección de la circunferencia con el segmento $\overline{AB}$) se localizaba en el punto medio del segmento $\overline{AB}$. Un ejemplo de esta forma de visualización es la efectuada por el Equipo 1 (véase Figura 3), en la cual se muestra que la dificultad para visualizar de forma correcta radica en la manera de articular la información implícita y explícita dada (Arcavi, 2003; Duval, 2003, 2006). A partir de los resultados obtenidos, podemos asegurar que la visualización de la figura, por parte de los alumnos –en ambiente de papel-y-lápiz–, generó dificultades de percepción que impidieron lograr la visualización correcta del objeto geométrico mostrado (Duval, 2003, 2006).
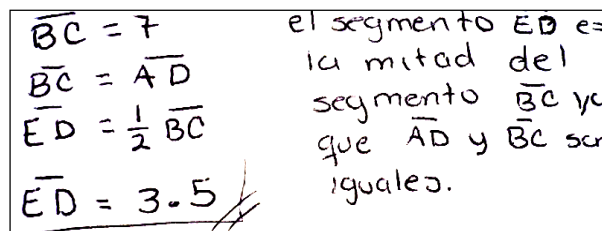
Figura 3. Influencia de la percepción visual en el cálculo de la longitud del segmento $\overline{ED}$, usando papel-y-lápiz.

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En la segunda parte de esta actividad (ambiente tecnológico), los participantes reprodujeron la Figura 1 empleando el SGD (GeoGebra) y, usando esta herramienta, calcularon la longitud del segmento $\overline{ED}$. Una vez resuelto el problema, usando el SGD, compararon el resultado obtenido con el software respecto del conseguido con papel-y-lápiz. Al hacer uso del software GeoGebra, el total de los equipos concluyó que la longitud del segmento $\overline{ED}$ es de 3 cm, debido a que E no está localizado en el punto medio del segmento $\overline{AD}$ (véase Figura 4).
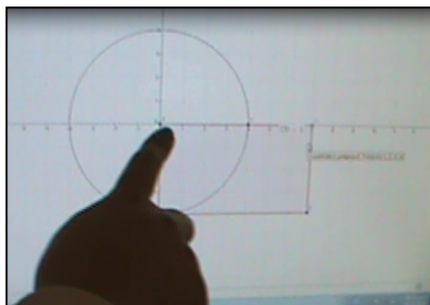
Figura 4. Influencia de la percepción visual en el cálculo de la longitud del segmento $\overline{ED}$, usando ambiente tecnológico.

A continuación, analizamos y discutimos un extracto de lo expuesto por el Equipo 4[2] (Estudiantes 4A y 4B) al contrastar los resultados surgidos en papel-y-lápiz, respecto de lo obtenido con ambiente tecnológico:

[1] *Profesor*: De acuerdo con la Figura 1 [*ambiente de papel-y-lápiz*] ¿cuál es la longitud del segmento $\overline{ED}$?

[2] *Estudiante 4A*: Como el segmento $\overline{BC}$ y el segmento $\overline{AD}$ son iguales. Entonces, si éste [punto E] está al centro [*punto medio del segmento $\overline{AB}$*] vale la mitad [3.5 cm].

[3] *Profesor*: Usando el software [*ambiente tecnológico*] ¿el resultado cambió o es el mismo?

[4] *Estudiante 4A*: Nos dimos cuenta de que el radio de la circunferencia es 4 cm y el   radio en todas las partes de la circunferencia siempre es el mismo, entonces mide 4 cm, aunque [*visualmente en la figura en papel-y-lápiz*] sea la mitad de $\overline{AD}$.

[5] *Profesor*: Finalmente, ¿cuál sería el valor [*la longitud*] de $\overline{ED}$?

[6] *Estudiante 4A*: Tres, ya que la distancia de $\overline{AE}$ es 4 cm y el segmento $\overline{AD}$ es 7 cm.

[7] *Profesor*: ¿De qué forma les ayudó el software en la visualización de la figura?

[8] *Estudiante 4A*: A simple vista se ve como si fuera la mitad [*ambiente de papel-y-lápiz*] y al ponerlo en la computadora [*GeoGebra*] se ven bien definidas las unidades de la figura [*perciben las propiedades del objeto matemático*].

[9] *Profesor*: ¿Qué pueden decir de la figura impresa [*ambiente de papel-y-lápiz*]?

[10] *Estudiante 4A*: Se tiene una figura que no es proporcional [*según sus características*].

Segunda actividad

En la segunda actividad (en ambiente de papel-y-lápiz), se pidió a los participantes que identificaran si existía alguna relación proporcional entre las áreas de los triángulos *ABC* y *XYZ*. Sólo el Equipo 6 tuvo dificultades en el cálculo de las áreas de los triángulos; ellos calcularon de manera errónea la altura del Δ*XYZ* (considerando la medida del segmento $\overline{YZ}$ como 2 cm), y concluyeron que las áreas de los triángulos *ABC* y *XYZ* eran distintas y no tenían relación alguna.

Cuando se preguntó a los demás estudiantes si la relación encontrada (de haberla obtenido) podía o no mantenerse para cualquier triángulo [rectángulo o no], dijeron lo siguiente: (a) Equipo 4: la relación no se mantiene "porque los ángulos no siempre son los mismos"; (b) Equipos 2, 3 y 5: la

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relación puede mantenerse "podría mantenerse [*esta relación*] siempre y cuando se cumpla la regla del punto medio en cada segmento o lado del triángulo" [*se refieren al ΔABC*] (Equipo 5); (c) Equipo 1: "se mantiene la relación porque los puntos que dividen a los lados del ΔABC son puntos medios"; (d) Equipo 6: "no existe relación alguna entre [las áreas de] los triángulos"; no argumentaron porqué.

En la segunda parte de esta actividad (ambiente tecnológico), los alumnos usaron el software GeoGebra para buscar las relaciones entre las áreas de los triángulos. Todos los equipos concluyeron que las áreas de los ΔXYZ, ΔAXY, ΔYZC y ΔXBZ eran iguales y que el área del ΔXYZ es la cuarta parte del área del ΔABC. En seguida, se les pidió a los participantes desplazar los puntos *A*, *B* y *C* (vértices del ΔABC), usando las herramientas del software con la finalidad de obtener nuevos triángulos, cuya longitud de sus lados fuera diferente entre sí e identificaran el porqué esta relación puede o no mantenerse para cualquier triángulo. He aquí un extracto de lo expuesto por el Equipo 5 (Estudiantes 5A y 5B):

[11] *Profesor:* Al desplazar los puntos *A*, *B* y *C*, ¿la relación [*se refiere a que el área del ΔXYZ es la cuarta parte de la del ΔABC*] se mantiene para cualquier triángulo?

[12] *Estudiante 5A*: Siempre que se respete el punto medio de cada uno de sus lados va a tener [*ΔABC*] cuatro triángulos inscritos con la misma forma del triángulo mayor y cada uno de ellos va a ser siempre la cuarta parte del área total (Figura 5).

[13] *Profesor*: ¿En qué momento se podría perder esta relación?

[14] *Estudiante 5A*: Cuando los puntos ya no sean los puntos medios [*se refiere a los puntos X, Y, Z*], ya que nos dimos cuenta [de] que al mover los puntos [*se refiere a los vértices A, B y C*] las áreas [*de los triángulos interiores*] seguían siendo iguales y que la suma nos daba la mayor [*área del ΔABC*] debido a que los puntos medios hacían esta proporción, pero sólo lo pudimos ver cuando hicimos diferentes casos con el programa.

Al hacer uso del software, todos los equipos lograron generalizar la relación de las áreas de los triángulos formados por puntos medios de cada lado para cualquier triángulo.
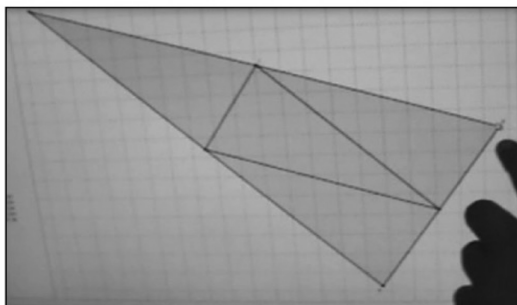
Figura 5. Generalización de la relación de áreas, usando tecnología.

Conclusiones

Respuesta a la pregunta planteada (c.f., p. 1 de este documento): ¿cómo influye la visualización en el aprendizaje de conceptos matemáticos cuando intervienen en ésta los ambientes de papel-y-lápiz y tecnológico? De acuerdo con las evidencias surgidas en la implementación de las actividades, podemos decir que la interpretación por parte de los estudiantes de la representación de los objetos geométricos, ya sea en el ambiente de papel-y-lápiz o en el tecnológico, depende de sus conocimientos previos y del contexto (trabajo en papel-y-lápiz o con tecnología; tipos de actividades; discusiones: estudiante-estudiante, estudiante-profesor), los cuales son fundamentales y les permiten interpretar y dar sentido a las representaciones, y por ende comprender al objeto matemático en cuestión.

A partir del reconocimiento de las propiedades vinculadas con las representaciones de figuras geométricas, los participantes lograron plantear conjeturas que, finalmente, les dieron pautas para definir conceptos ligados con la percepción visual del objeto matemático. La representación geométrica –en el ambiente de papel-y-lápiz– dio a los alumnos poca información sobre el objeto a visualizar, pues no lo pudieron manipular, ya que se trata de una figura estática. En cambio, al utilizar el software GeoGebra los participantes adquirieron recursos de apoyo, que les permitieron identificar propiedades de esos objetos (e.g., parte de la solución de la segunda actividad). Como resultado de la interacción con el software, los estudiantes descubrieron propiedades de las figuras geométricas y lograron formular conjeturas apoyados en sus observaciones mediante la manipulación de las representaciones figurales de los objetos geométricos. Estos recursos les permitieron establecer *puentes* entre la representación del objeto matemático y sus propiedades; así generaron conceptualizaciones del objeto geométrico en cuestión.

Podemos conjeturar, finalmente, que cuando se emplean representaciones geométricas, su visualización involucra siempre dos operaciones visuales que tienden a asimilarse dentro de un mismo acto por parte del sujeto: (i) distinguir varias formas dentro de una figura, y (ii) identificar estas formas o su configuración representada, usando información previa plenamente reconocida. En nuestro trabajo, resaltamos el hecho de que el uso de la tecnología en la resolución de las actividades propuestas permitió a los estudiantes distinguir propiedades geométricas de los triángulos, en términos de sus áreas, dentro de una figura (triángulo, segunda actividad). Nuestros resultados –en ambiente de papel-y-lápiz– concuerdan con lo expuesto por Duval (2003), quien afirma que las personas no especialistas en visualización, con frecuencia, no toman en cuenta características de la representación, o bien, interpretan erróneamente las representaciones y crean contenidos con significado personal, que no se parecen en lo absoluto a las representaciones matemáticas institucionales (e.g., parte de la solución de la primera actividad). Es cierto que el SGD favorece el aprendizaje de conceptos geométricos, pero no es sólo este ambiente que debemos tomar en cuenta como herramienta de enseñanza, sino que es crucial también el uso de papel-y-lápiz, pues ellos se complementan con la finalidad de lograr un éxito (parcial) en la visualización de objetos matemáticos, además de que favorece en la generación de conjeturas de conceptos abstractos surgidos de tales representaciones. Este trabajo permite apuntar el rumbo de nuevas investigaciones, en las cuales se pretende responder la pregunta de investigación ¿qué tipo de figuras u objetos, analizados a partir de su visualización, promueven la abstracción matemática? Esta y otras preguntas pretenden responderse en el futuro.

Notas finales

[1] Estudiantes, alumnos y participantes son tomados como sinónimos en este artículo.
[2] Esta es la traducción de la discusión original, la cual se efectuó en español.

In this paper we report the results obtained when implementing Activities related to the visualization of geometric representations in paper-and-pencil and technological environments with students[1] grade 11. This is a qualitative research supported by both the representation theory and the visualization of mathematical objects using technological aid. We discuss how the use of technology promotes learning mathematical concepts through the analysis of static and dynamic geometric figures. Our results show that the technological tool contributes to the visualization of geometric representations arisen from working with Activities; however, paper-and-pencil as a working tool becomes necessary to give rise to conjectures about the meaning of abstract concepts linked to these representations.

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

Background and research problem

For some years, mathematical visualization has been researched as a resource to achieve learning mathematical concepts (e.g., Arcavi, 2003; Duval, 2003, 2006; Phillips, Norris, & Macnab, 2010; Presmeg, 2006; Zimmermann & Cunningham, 1991, among others). These authors have developed numerous studies with the objective of exploring different ways of "visualizing" abstract mathematical objects in paper-and-pencil and technological environments, which support and mediate the emergence and development of mathematical concepts. Due to the large number of geometric representations found in different media –paper-and-pencil and technological–, there is the need (in the student) to identify the characteristics and properties they have. In this article we seek to provide an answer to the following question: how does visualization influence the learning of mathematical concepts when the paper-and-pencil and technological environments intervene?

Conceptual framework

The conceptual framework used in this research is supported by the contributions about visualization in mathematics (e.g., Duval, 2003, 2006; Arcavi, 2003; Phillips et al., 2010; Zimmermann & Cunningham, 1991; among others). Accordance with the Dictionary of the Royal Spanish Academy, the word "seeing" can be understood as synonymous with "visualization" and means "to form in the mind a visual image of an abstract concept" (RAE, 2014). Several researchers in mathematics education have contributed to what should be understood by "visualization". Here are three of them: (i) "Visualization is the individual's ability to produce a representation that, in the absence of any visual perception of the objects represented, through which allow observe as if they were really in front of the eyes" (Duval, 2003, p. 48); (ii) "visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings (Arcavi, 2003, p. 217); (iii) visualization is the process of creating mental images either using paper-and-pencil or technology as well as the effective use of such images in mathematical discovery and understanding of the objects under study (Zimmermann & Cunningham, 1991, p. 3). These and other contributions on the meaning of mathematical visualization demonstrate cognitive character of this concept. We will now describe some concepts involved in the visualization of geometric representations both in paper-and-pencil and in technological environments.

Visualization in paper-and-pencil environment

The representation of mathematical objects involves analysis of their institutional meaning. If the object is a geometric figure, its characteristics must be identified; besides, a treatment of the object with the sense of sight must also be done. Nonetheless, regardless of the conceptions, the nature and the existence of mathematical concepts - of those who learn or those who teach-, many mathematicians (e.g., Arcavi, 2003; Duval, 2003, 2006; Phillips et al., 2010; among others) consider that this way of constructing knowledge consists of "seeing", and such visualization must be done by the senses, the imagination and the intelligence (Duval, 2006). Due to the fact that visualization triggers non-conscious automatic processes in the subject which depend on anything that might be kept in his or her memory, visual learning is direct. This allows learners to discriminate and identify in less than a tenth of a second the various elements of sight and relationships (Duval, 2003).

Through the visualization of representations, the individual can access the properties, characteristics and give meaning to the mathematical object analyzed (Phillips et al., 2010). Sometimes, visualization is used to describe visual representations; on other occasions, it is used to

determine how a certain specific representation works in the understanding of or solving of mathematical problems; it might also be used to define the cognitive activity of the subject when using visual representations. The ideas that emerge from visualization in mathematics lead (students) to enrich visual contents whose use is beneficial both in representation tasks and in handling concepts of this discipline. According to Phillips et al. (2010), in visualization one may distinguish: (a) physical objects perceived by the sense of sight (e.g. illustrations, animations, computer-generated displays, etc.); (b) mental objects collected and processed in the mind in the form of mental schemes, mental imagery, constructions and mental representations; (c) cognitive functions expressed in visual perception, manipulation and transformation of visual representations in the mind, making concrete the abstract ways of thinking and imagining facts.

Phillips et al. (2010) point out that these distinctions are relevant to understand the context of the visualizations and to establish effective visualization applications in the classroom. In order for the student to understand and use geometric representations in an adequate manner, he or she must first understand the meaning of straight lines, arrowheads, angle markers, numbers, and variables (Phillips et al., 2010). Since visual representations are not necessarily clear to the students, they must have a considerable knowledge of symbols and representation conventions so that they have meaning. Visualization involves cognitive activities like interpretation and abstraction of that which the mathematical object represents (Duval, 2003; Arcavi, 2003; Hitt, 1995; Zimmermann & Cunningham, 1991, among others). However, the impossibility of direct access to mathematical objects, outside any representation, inevitably causes confusion in students (Duval, 2003).

Visualization in technological environment

Nowadays there are a vast number of computer programs (e.g., GeoGebra, Maple, Mathematica, Cabri-Geometry, Matlab, among others) used in mathematics teaching. This type of tool promotes visualization as a way of teaching mathematical concepts, and becomes a key factor in mathematical education; particularly, the use of any Software of Dynamic Geometric (SDG) makes the inclusion of visualization in teaching geometric concepts an easy thing because any SDG has dynamic images which allow measurement or the addition of auxiliary lines using the tools available in the software; one may even simply explore a geometric figure. When using dynamic representations to solve geometry problems through visualization, handling is facilitated, sketching or construction of the figures used becomes easier and it is possible to modify them in real time if needed (Hitt, 1995). A relevant feature of SDG is its versatility in terms of use which stimulates the interest and participation of the students when solving geometry problems. Different authors (e.g., Arcavi, 2003; Hitt, 1995; Phillips et al., 2010; Zimmermann & Cunningham, 1991, among others) agree that visualization using dynamic models allows students to understand the concepts or the meanings that may be extracted from representations; therefore, it plays an important role in the students' development of analytical thought, something that does not often happen in a paper-and-pencil environment.

Method

The study is qualitative. The participants were 12 Mexican high-school students [grade 11] paired in six teams. The study was divided in two phases: the first one focuses on visualizing geometric figures using a static environment (paper-and-pencil) in order to extract the properties which, according to the student's knowledge, are implicit in the figure; the second one includes the visual analysis of the figure in a dynamic environment (technological). Using SDG (GeoGebra), the student validates the properties of the figures visualized in paper-and-pencil. During the work with the activities in this environment, we seek for the students to conjecture about the general behavior of the figures when any parameter of the figures changes, to find invariants –if there are any, and to describe and explain the mathematical objects visualized.

The first activity was taken from the article *A cognitive analysis of problems of comprehension in learning of mathematics* by Duval (2006, p.117). In this activity the author proposes that, from a sketch of the graph (see *Figure 1*), students find the length of segment $\overline{ED}$, taking as fixed data the length of the segments $\overline{AB} = \overline{DC} = 4$ *cm* and $\overline{BC} = 7$ *cm*. The author reported three types of answers: 9% answer 3 *cm* (mathematical answer), 39.6% answer 3.5 *cm* (direct measure of the segment) while 24.4% provide other answers, including no answer. The students' results had the paper-and-pencil environment as a referent.
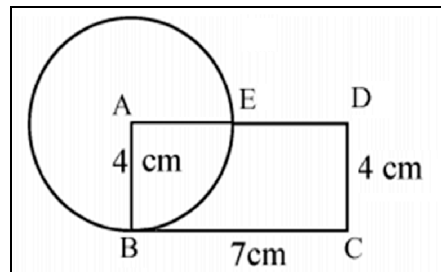
Figure 1. Calculate the length of segment $\overline{ED}$, using paper-and-pencil environment.

The second activity was adapted from the book *Plane and Solid Geometry* (Wentworth & Smith, 1913, p. 33). In our study, the students were presented with a right triangle ABC which according to their characteristics, allow students to identify its properties (implicit and explicit) in order to calculate its area. In the activity, we seek for the students to identify whether there exists a proportional relationship between the area of the triangles *ABC* and *XYZ*, the latter formed from the midpoints of $\triangle ABC$ (see *Figure 2*). To determine if there exists (or not) a relationship between the areas, the students are asked to calculate the areas of the triangles mentioned; if there is any relationship, the student must explain if such relationship may (or may not) be maintained for any triangle regardless its size or shape
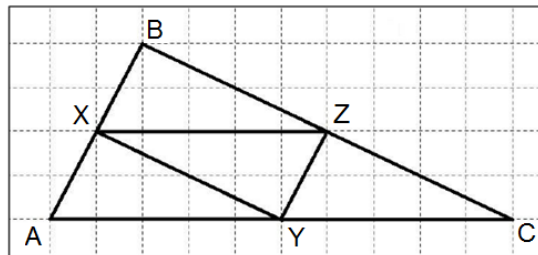
Figure 2. Identifying the characteristics of a triangle, using paper-and-pencil environment.

Data collection included the written record of student responses to the activities, which were video-recorded, in addition to students' responses to questions asked by the researcher during the activities. The activities presented to participants were to be addressed using paper-and-pencil and technology (GeoGebra). At the same activities they were asked to validate the results obtained in both environments. They were asked to validate their results in the same Activities using technological (GeoGebra) and paper-and-pencil environment by confronting them.

Data analysis and result discussion

The analysis of the collected data was based on the students' interpretation and on the number of common events and their solution in both environments, taking into account videos of the discussions (student-student and student-teacher) during the activity and the written record once the problem was solved. In the next section, the results of the implemented activities are analyzed.

First Activity

In the first activity, during the work in paper-and-pencil environment, the following results were obtained: five teams detected that the value of the segment $\overline{ED}$ was 3.5 *cm*, while only one team considered the properties of the circle after they realized that the real value of the segment $\overline{ED}$ was 3 *cm*. According to the results obtained, we may state that the visualization of this geometric figure produced by the students tends to acquire a simpler form (Duval, 2003); this way of "seeing" makes the students perceive the segments $\overline{AE}$ and $\overline{ED}$ as equal. This statement is based on the fact that most of the students identified that point E (intersection of the circle with segment $\overline{AB}$) was located at the midpoint of segment $\overline{AB}$. An example of this way of visualizing is evident in the work of Team 1 (see *Figure 3*) in which it is shown that a difficulty to visualize correctly lies in the manner of articulating the implicit and explicit information given (Arcavi, 2003; Duval, 2003, 2006). From the results obtained, we can state that the way in which the students saw the figure (in a paper-and-pencil environment) generated an impression at first sight that prevented them from reaching the correct visualization of the geometric object shown (Duval, 2003, 2006).

(a)

$\overline{BC} = 7$

$\overline{BC} = \overline{AD}$

$\overline{ED} = \dfrac{1}{2}\overline{BC}$

$\overline{ED} = 3.5$

the segment $\overline{ED}$ is half of the segment $\overline{BC}$ since $\overline{AD}$ and $\overline{BC}$ are equal.

(b)
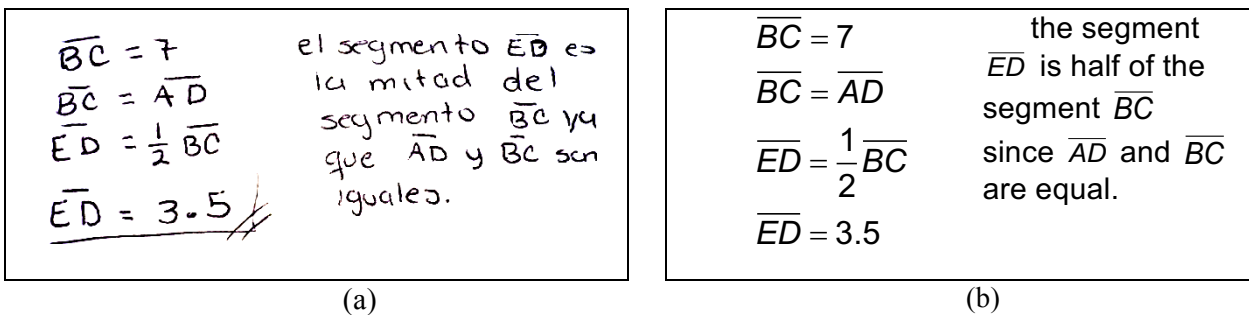
Figure 3. (a) Influence of visual perception when calculating the length of segment $\overline{ED}$, using paper-and-pencil environment. (b) Transcription of the answer provided in (a).

In the second part of this activity (technological environment), the students reproduced *Figure 1* with SDG (GeoGebra) and using its tools calculated the length of segment $\overline{ED}$. Once the problem was solved with SDG, they compared the result obtained with the software to that obtained with paper-and-pencil. When using GeoGebra, all of the teams concluded that the length of segment $\overline{ED}$ was 3 *cm* because point E is not located at the midpoint of segment $\overline{AD}$ (see *Figure 4*).
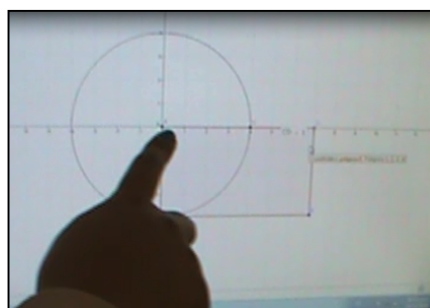
Figure 4. The students realize that point E is not located in the midpoint of segment $\overline{AD}$, using the technological environment.

Next, we discuss an excerpt of what was presented by Team 4[2] (students 4A and 4B) when they contrasted the results obtained in paper-and-pencil with those obtained in the technological environment:

[1] *Teacher:* According to *Figure 1* [*paper-and-pencil environment*], what is the length of segment $\overline{ED}$?

[2] *Student 4A*: Since segment $\overline{BC}$ and segment $\overline{AD}$ are equal. Then, if this [*point E*] is at the center [*midpoint of segment $\overline{AB}$*], it is equal to the half [*3.5 cm*].

[3] *Teacher*: Using the software [*technological environment*], did the result change or remain the same?

[4] *Student 4A*: We realized that the radius of the circumference is 4 *cm* and the radius in all the parts of the circumference is always the same, so it measures 4 *cm* although [*visually in the figure in paper-and-pencil*] it is half of $\overline{AD}$.

[5] *Teacher*: Finally, what would be the value of [*the length of*] $\overline{ED}$?

[6] *Student 4A*: Three, because the distance of $\overline{AE}$ is 4 *cm* and segment $\overline{AD}$ is 7 *cm*.

[7] *Teacher*: In which way did the software help you to visualize the figure?

[8] *Student 4A*: At first sight, it looks as if it were half [*paper-and-pencil environment*] and when we put it in the computer [*GeoGebra*] the units of the figure look well-defined [*they notice the properties of the mathematical object*].

[9] *Teacher*: What can you say about the printed figure [*paper-and-pencil environment*]?

[10] *Student 4A*: There is a figure that is not proportional [*according to its characteristics*].

Second Activity

During the second Activity (in a paper-and-pencil environment), the students were asked to identify whether there is a proportional relation between the area of triangles *ABC* and *XYZ*. Only Team 6 had difficulties calculating the areas; they identified the height of triangle Δ*XYZ* incorrectly (they considered the measure of the segment $\overline{YZ}$ as 2 cm), and concluded that the areas of triangles *ABC* and *XYZ* were different and therefore, they did not find any relationship.

When the rest of the students were asked whether the relation found (if they had found any) could or could not be maintained for any triangle [right or not], they said: (a) Team 4: the relation is not maintained "because the angles are not always the same"; (b) Teams 2, 3, and 5: the relation "could be maintained as long as the midpoint rule is obeyed in each segment or side of the triangle" [*they mean triangle* Δ*ABC*] (Team 5); (c) Team 1: "the relation is maintained because the points that divide the sides of the triangle *ABC* are midpoints"; (d) Team 6: "there is no relation between [the areas of] the triangles"; they do not provide arguments as to why.

In the second part of this activity (technological environment), the students used GeoGebra to look for the relations between the areas of the triangles. All the teams concluded that the areas of the triangles Δ*XYZ*, Δ*AXY*, Δ*YZC* and Δ*XBZ* were equal and that the area of triangle *XYZ* is a fourth of the area of the triangle *ABC*. Afterwards, the students were asked to move points *A*, *B* and *C* (vertices of Δ*ABC*) with the software tools in order to obtain new triangles whose side lengths were different from one another. They were also asked to identify why this relationship can or cannot be maintained for any triangle. Here is an excerpt of what Team 5 (students 5A and 5B) presented:

[11] *Teacher*: When moving points *A*, *B* and *C*, is the relation [*he means that the area of* Δ*XYZ is a fourth of* Δ*ABC*] maintained for any triangle?

[12] *Student 5A*: As long as the midpoint of every side is respected, there will be [Δ*ABC*] four triangles inscribed with the same shape as the largest triangle and each of them will always be a fourth of the total area (*Figure 5*).

[13] *Teacher*: When might this relationship be lost?

[14] *Student 5A*: When the points are no longer midpoints [*he means points X, Y, and Z*] because we realized [*that*] when moving the points [*he means vertices A, B, and C*], the areas [*of the interior triangles*] were still equal and that the sum was equal to the largest [*area of* Δ*ABC*]

because the midpoints made that proportion; but we were only able to see that when we did different cases with the software.

Using the software, all the teams managed to generalize the relation of the areas of the triangles formed by the midpoints of every side of any triangle.
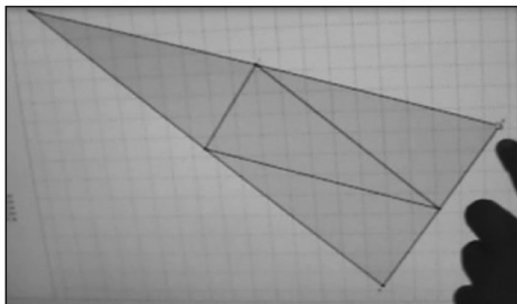
Figura 5. Generalization of the relation of the areas, using technological environment.

Conclusions

Answer to this question (c.f., p 1 of this document.) how does visualization influence the learning of mathematical concepts when the paper-and-pencil and technological environments intervene in it? According to the evidence arising in the implementation of activities, we can say that the interpretation by students of the representation of geometric objects, either in paper-and-pencil or in technological environments, depends on the student's previous knowledge and the context (work on paper-and-pencil or technology, types of activities, discussions: student-student, student-teacher), which are essential and allow students to interpret and make sense of the representations, and thus understand the mathematical object.

Starting from the recognition of the properties linked to the representation of geometric figures, the students were able to propose conjectures that ultimately provided them with the guidelines to define concepts related to visual representation. When working with paper-and-pencil, the geometrical representation offers the student little view [information] about the object because it cannot be manipulated, since it is a static figure. On the other hand, when using the software GeoGebra, the participants gained support resources that allowed them to identify properties of these objects (e.g., part of the solution of the second activity). As a result of interaction with the software, students discovered properties of geometric figures and managed to formulate conjectures supported by observations made while manipulating dynamic representations of geometric objects. These resources helped students to create bridges between the representation of the mathematical object and its properties which lead (the students) to generate conceptualizations of the geometric object in question.

When geometric representations are used, their visualization always involves two visual operations that tend to be assimilated in a single act by the subject: (i) distinguishing several shapes inside a figure, and (ii) identifying those shapes or their represented configuration using previous information widely known. In our paper, we highlight the fact that the use of technology in solving the proposed activities allowed students to distinguish geometric properties of triangles, in terms of their areas, within a figure (triangle, second activity). Our results [paper-and-pencil environment] are in accordance with what Duval (2003) explains when he says that people who are not specialists in visualization simply overlook characteristics of the representation, or they may incorrectly interpret the representation, or make up representations of contents with a personal meaning even if they are not similar at all to the institutional mathematical representations (e.g, part of the solution of the first activity). It is true that the SDG promotes learning geometric concepts, but this is not the only

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environment we should take into account as a teaching tool. The use of paper-and-pencil is a key factor as well since the two environments complement each other to achieve a (partial) success in the visualization of mathematical objects, besides it favors the creation of conjectures on abstract concepts that arise from such representations. This work points to directions for further research, which may seek to answer the research question: what kind of figures or mathematical objects, analyzed through visualization, promote mathematical abstraction? This and other questions are intended to be answered in the future.

Endnotes

[1] Students and participants are taken as synonyms in this article.

[2] This is a translation of the original discussion, which occurred in Spanish.

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DEVELOPING AN UNDERSTANDING OF CHILDREN'S JUSTIFICATIONS FOR THE CIRCLE AREA FORMULA

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In this study we investigated eighth grade students' informal justification for the circle area formula to expand accounts of the measurement knowledge for middle-school age students. Data were collected during three paired interviews of a three-year teaching experiment. Here we describe schemes students exhibited as they operated on measurement tasks at a level we have described as "conceptual area measurer"; the tasks prompted the use of square units to quantify a figure that is not rectilinear. We found students could follow and rehearse a rationale for the validity of the circle area formula with substantive opportunities for movement and figural operations with units, or with decompositions from unit images that coordinated circle and rectangle images.

Keywords: Measurement, Learning Trajectories (or Progressions), Geometry and Geometrical and Spatial Thinking

Area measurement is an important part of elementary and middle school mathematics; unfortunately, many students do not have an adequate understanding of area measurement concepts (Outhred & Mitchelemore, 2000). Many elementary students can remember standard formulas for shapes such as rectangles; however, area measurement is still problematic (Lehrer, 2003). This could be because students are taught the area formula through rote memorization (Simon & Blume, 1994). "Rather than memorize particular formulas for certain shapes, they need to understand why the formulas work" (Strutchens, Martin, & Kenny, 2003). The Common Core State Standards for 7th grade recommends students be able to give an informal justification for the circle area formula (National Governors Association Center for Best Practices, & Council of Chief State School Officers, 2010). In this paper we set out to explore 8th grade students' understanding of the area formula for a circle, including their ability to apply and reason about area formulas for circles.

We expect that asking students to justify the use of the area formula for circles provides an effective context for assessing advances in students' area measurement knowledge. The purpose of this study is to describe and analyze students' thinking as they found the area of circles and developed an informal justification for the circle area formula by coordinating with the area of a triangle, and by tiling with squares. We also hoped to extend a hypothetical learning trajectory (HTL) on area measurement by addressing measures of non-rectilinear shapes.

Research Question
How do eighth grade students develop an informal justification for the circle area formula?

Theoretical Framework

To investigate students' development of understanding for area measurement, we needed a tool to identify varying levels of understanding of area. Thus, we used a hypothetical learning trajectory (HLT) for area measurement developed by Barrett et al. (in press) that describes levels of sophistication. An HLT has three parts: an instructional goal, developmental progressions to characterize mental schemes and actions pertaining to the knowledge goal, and instructional activities to help students progress along the progression (Clements & Sarama, 2014). The instructional goals and activities follow from particular schemes and certain mental actions on objects characteristic of successive levels in the trajectory. An HLT on area measurement has guided the development of our tasks.

The following area HLT levels (Barrett et al., in press) are relevant for the present study because the tasks were designed to scaffold the students to a more sophisticated level. These three levels of thinking address the degree to which children may integrate and coordinate figural images and internal, conceptual images by analyzing parts of figures. By re-organizing a figure into essentially the same collection of components, yet within a different overall shape, children may accomplish the first, least sophisticated of these levels (ARCS). The more sophisticated levels are achieved as children abstract the measures of regions to define area measures as constructions that are products of other linear measures, measures which refer to highly indexed, linear collections of units arranged along a second, orthogonal dimension in arrays. The highest level indicates the most flexible, algebraic grasp of products from linear quantities, taken as inputs to a functional account of the area measurement.

- *Area row and Column Structurer* (ARCS): children at this level can decompose and recompose partial units to create whole units
- *Array Structurer* (AS): children at this level have an abstract understanding of the area formula for rectangles
- *Conceptual Area Measurer* (CAM): children at this level have an abstract and generalizable understanding of the rectangle area formula, they are able to restructure regions to find area, and can provide a justification for the restructuring of the shape.

In the interview sequence reported on here, we prompted students to connect the measure of circles to that of rectangles as a consequence of our reading of the mental actions on objects most likely to be enacted at the Conceptual Area Measurer level of the HLT and because we had found the students expressing related schemes of decomposition and recomposition of area measures.

Methodology

As part of a three-year longitudinal teaching experiment (Steffe & Thompson, 2000) on children's thinking and learning about length, area, and volume, we investigated four children's thinking on area of circles. The students were in eighth grade at a public school in the Midwest. For this report, we used data from three 25- to 30-minute semi-structured interviews with four students (Kari, Lindsey, Joey, and Tanner). We interviewed students in pairs, with Kari and Lindsey as partners and Joey and Tanner as partners. The interviews were videotaped and transcribed and the researchers then analyzed the data. The interviews took place during November and December of 2015.

For the first task, we asked students to find the area of a circle and tell us if they could explain why the circle area formula worked. We used this as an opportunity to see the prior knowledge the students had about area of circles. In the second task, we asked students to compare the area of a square radius to the area of a circle. We created a display with two orthogonal radii serving as adjacent sides of a square, having an area of one radius squared. In this approach, the interviewers

encouraged the students to tile over the circle with cutout squares radii. We saw this activity as an opportunity for students to make a connection to the circle area formula (i.e., $A = \pi r^2$). We hoped students would either: (a) estimate that it would take between three and four square radii to match the area of the circle, or (b) use the formula to assert that approximately 3.14 square radii will always cover the circular region.

Next, students watched a video of a circle and its interior being transformed into a triangle (see Figure 1). We designed this video to help students relate the area of the circle to the area of the triangle. We expected to activate their scheme for decomposing and recomposing space.
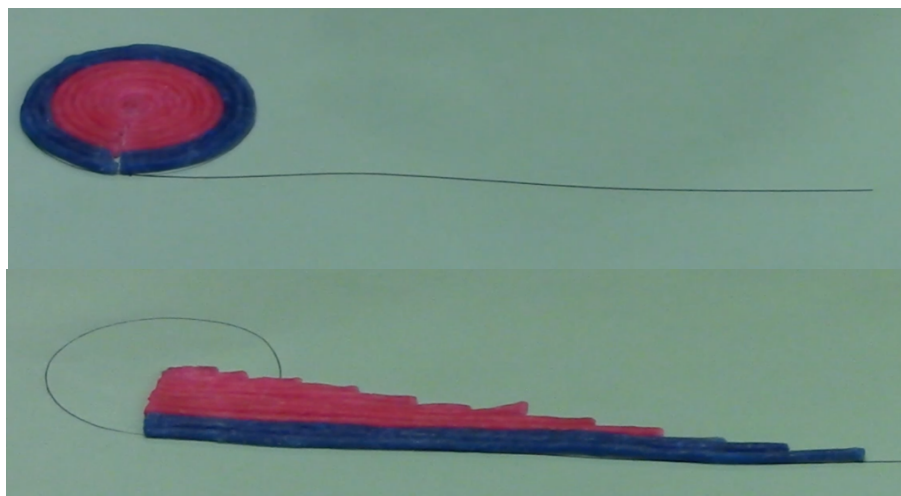
Figure 1. Circle Transformed to a Triangle.

For the third task, the students were asked to reflect on the transformation in the video and to find the area of a circle through relating the area of the circle to the area of a triangle. When they expressed the area of the triangle with an invented expression, we asked them to apply their invented expression to state the area of the circle without relying directly on the standard formula. In the fourth task, we asked students to provide an informal justification of the circle area formula, both by reviewing their prior work and by synthesizing what they had observed in the prior tasks.

Results and Discussion

Next we present descriptions of the students' work and responses for each of the four tasks. Following that, we sketch a generalized account of the reasoning we observed and the knowledge that was within their grasp given this set of tasks, and we comment on the relation of such knowledge to an existing framework characterizing relevant levels of sophistication for students' knowledge about area measurement.

Task 1: Find the area of this circle. If you use a formula, explain how you know that it will give you the correct area or where it comes from.

All four students correctly calculated the area of the circle using the standard area formula $A = \pi r^2$ during the first interview on this topic. Despite knowing the formula and how to compute the area, none of the four students gave an explanation about why the formula makes sense or its relation to that of a rectangle. One student, Lindsey, attempted to develop an explanation of the formula. She split the shape using a diameter and said the formula may have something to do with the height and the base.

Although all of the students successfully found the area of the circle using the standard formula, they did not explain any part of the formula besides telling us the formula and that π was 3.14. This

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could be an indication that students were taught the formula through memorization alone. Students' responses to this task lead us to claim they were not yet operating at a Conceptual Area Measurer (CAM) level of the HLT for area measurement. We make this claim because they used the standard formula for area of a circle but did not demonstrate an abstract understanding of that formula, which is characteristic of the CAM level.

Task 2: Compare the area of the square radius to the area of the circle.

Kari and Lindsey made a guess that it would take four square radii to fill the circle. When they compared the area of the square radii with the area of the circle algebraically, they found the difference, not the ratio, which did not help them interpret how many square radii it takes to fill the circle. Later, using cut out square radii, they determined that it would take about two or three squares. When they were prompted to reflect back on the formula, Lindsey said, "oh π is 3.14 and so it would be 3 or a little bit more that would fit in the circle." Although this may have helped Kari and Lindsey interpret the formula, they still reported that they could not justify why the formula worked.

In the third interview, Kari and Lindsey were asked this question again. They were able to say it takes 3.14 square radii to fill the circle but they may not have taken the square as an object that occupied "radius square units" of the circle area. Instead, Lindsey dragged her finger along two sides of the square and said each showed a length of a radius. Students operating conceptually with area often use a sweeping motion within or across the area being discussed, (Dougherty, 2008). In contrast, Lindsey's gestures may indicate she was treating the sides operationally, as factors that would be used to feed into a calculation for a product of "radius squared". Later, Lindsey labeled another square shape with edge length of "radius" by writing "radius squared" on the interior, indicating an advance beyond the operational approach.

In the first interview Joey and Tanner concluded it would take three square radii to fill the circle. They did not relate the square radii to the standard formula. At the beginning of the third interview they went back to this task. The interviewer asked how many squares it would take to fill the circle? Joey said, "3 point something." The interviewer then asked them if the formula helps them answer that question? Tanner said, "3.14 because the radius square times π is the area of the circle." The interviewer explained to the students this was the beginning to understanding the standard formula for area of a circle but was not yet a justification for the formula.

Based on this interview, we claim Kari and Lindsey were able to interpret the formula as a statement of the number of square radii it would take to fill a circle but struggled to justify why. Joey and Tanner did not articulate a connection between the square radii and π at the end of the first interview, but at the end of the third interview they described the relationship clearly. Apparently this task allowed these students to develop an understanding of the circle area formula as an approximate value, in relation to the number of square radii units needed to cover the circle. Still, it did not help them justify why the formula specifies π square radius units.

Task 3: Finding the area of a circle by transforming it into a triangle

After watching the video of a circle being transformed into a triangle, Kari and Lindsey were given a page that had the circle and triangle shown in the video. They told the interviewer the triangle had the same area as the circle. They labeled the height of the triangle r because they could see the height of the triangle was concurrent with the segment showing a radius of the circle. However, they would not label the base of the triangle circumference, but the were willing to label it C. Lindsey said, "[we can] label it circumference of a circle, but not really because it is not circular." Kari agreed and said, "but if we rolled the circle out it would go to here" as she pointed to the end of the triangle. They agreed they could label it C. We think Lindsey and Kari did not view circumference as a measure but only as a name of part of a circle. They found the area of the triangle by first measuring the length of the base and height of the triangle and then multiplying the base (circumference), height

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(radius), and $\frac{1}{2}$. They checked to see if the triangle had the same area as the circle by using the standard area formula for a circle and found their calculation was about one square centimeter different. With the help of the interviewers they concluded the circle had the same area as the triangle; they found a difference between area measures of 18.75 and 19.63 but they needed to account for measurement error. They concluded they could find the area of the circle using the formula $A = Cr\frac{1}{2}$ (see Figure 2).
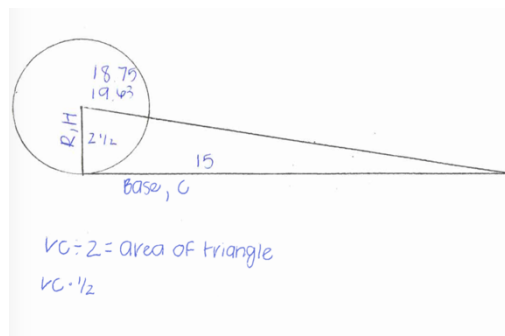
Figure 2. Kari and Lindsey's work for Task 3 on the picture of the circle to triangle transformation page given to them.

In the second interview, Kari and Lindsey, were given another circle and asked to calculate the area without using the standard formula. They drew a triangle that had an altitude the same as the radius of the circle and a base the same as the circumference. They used their invented formula $a = Cr\frac{1}{2}$ to find the area of the circle. After they found the area using their invented formula, they checked their answer with the standard formula (i.e., $a = \pi r^2$) and found the answer to be the same. This suggests a conceptual advance in that they expressed the area of a circle in terms of a related expression involving a triangle that was a reconstituted collection of parts of that circle.

Similarly, after Joey and Tanner watched the video of the circle being transformed into a triangle they found the area of the circle from the video (same page was given as Figure 2), by working with the triangle and using $a = bh\frac{1}{2}$. Joey and Tanner were then given a different circle and asked to find the area without using the standard formula. As we had hoped, their strategy was to try to create a triangle that would have the same area as the given circle. They knew the height of the triangle would be the radius, which they measured correctly. They were not sure how to calculate or measure the circumference of the circle, which they recognized would be the other side length needed for the triangle. With some prompting from the interviewer, they used a wikki-stix to find the distance around the circle. Once they found the measure of the circumference they found the area of the circle by find the area of their created triangle using the triangle area formula. Next, with guidance from the interviewer, Joey and Tanner wrote $\frac{rc}{2} = \frac{bh}{2} = r^2\pi$ on their paper. They then checked their answer by finding the area of the circle using the standard circle area formula. They had a difference in their answers by one square centimeter but concluded the difference was due to a measurement mistake.

We found evidence that students could follow our guidance and logic to interpret a more tangible formula for the area of a circle. Kari and Lindsey seemed to be able to complete the transformation of the circle into a triangle without guidance from the interviewer because they knew how to find the circumference of the circle. Joey and Tanner were able to complete this task with the help of the interviewer giving them a wikki-stix and later the interviewer told them the formula for finding the circumference of a circle.

Task 4: Justifying the circle area formula

   Kari and Lindsey began their informal justification for the circle area formula by drawing a circle and a radius. Lindsey said, "So the circumference is the circle and if you lay that out it would be this, which is the base of the triangle, which we are calling C, cause it is like the circumference of the circle. Then the height is like the radius. And to get the area of the triangle you do base times height which is like C times r … times a half." Kari said that the triangle had same area as the circle. They then set πr^2 equal to $Cr\frac{1}{2}$. They substituted $d\pi$ in for C, followed by $2r$ for d and simplified (see Figure 3).

Figure 3. Kari and Lindsey's work on Task 4.

Figure 4. Tanner and Joey's work on Task 4.

   In the third interview with Joey and Tanner, they were asked to justify the circle area formula. To reply, they revisited previous work and recalled this equation: $\frac{rc}{2} = \frac{bh}{2}$. They substituted $2\pi r$ for the circumference leaving them with $A = \frac{r2\pi r}{2}$. They simplified this equation and were left with the standard circle area formula. This algebra was not easy for them and it took them some time to decide if $r \times r$ was $2r$ or r^2 (see Figure 4).

   All four students took for granted the equivalence of the circle and the triangle and used it in their defense for the informal justification of the formula for area of a circle. At first they all wanted to use the standard formula they knew for area of a circle to justify their invented formula. The students were able to restructure the circle into a triangle and relate the shape to the area of a triangle and provide justification for the transformation and the circle area formula. We note here that the operation used in the video display of representing the circle and the triangle with an identical set of strips, and reorganizing to show the transition between the two figures involves an over-simplification, but we believe it was productive. This scenario disregards the contortion of the inner

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and outer edges of these unitary strips. However, as we approach the limiting case for the width of the strips this distortion becomes negligible. Thus, our imagined conservation of a collection of strips serves as a thought experiment more than a comprehensive argument. Nonetheless, it is conceptually sound, as the knowledge can be expected to mature when a student gains a sophistication level suited to learning the principles of the calculus, later in their academic career. We note too, that others have experimented with similar spatial morphing, which children may see as space conserving (e.g., Lehrer, Jenkins, & Osana, 1998; Kara, 2013).

Conclusion and Implications

Prior to the interviews, the students were not able to explain the standard circle area formula and at the end of the interview they were able to produce an informal justification for the circle area formula. We claim, these tasks supported students' development of a more meaningful understanding of finding the area of a circle. With scaffolding, the four students were preforming at the CAM level because they were able to restructure a circle into a triangle to find an area measure using algebraic expressions and operations. As well, they were able to provide an informal justification of this transformation and defend the circle area formula.

After the first task we found the students were able to compute the area, but they were not able to explain or interpret the formula. Our findings from this task support those of Outhred and Mitchelemore (2000) that elementary and middle school students do not have a sufficient understanding of measurement and those of Lehrer (2003) that students can remember standard formulas but not have a understanding of the formula or area measurement.

To our surprise, these students used the standard formula to justify their invented formula instead of using their invented formula to justify the standard formula. This could be because they had been taught the standard formula and did not learn about the formula as measurement from units and unit iteration for covering such an object. We conjecture that if they were to invent their own formula first, they could use their formula to develop the standard formula, which would drastically alter their conception of the standard formula. They may have conceived of the standard formula as an arbitrary construction that the teacher was merely relaying to them, and moreover, assumed it did not have a practical basis in physical measures with units of area. Students often accept statements like this as valid statements (theorems in action) for practical use without testing or challenging them. Setting measurement activities as empirical tasks is unusual, especially the measure of the circle, given that a formulaic computation is available, requiring only the measure of a radius. By problematizing the formula for measuring area, we found that students in 8th grade were capable of taking a novel unit square, the unit with a side length of one circle radius, and using it to measure both the circle and the triangle for area. By relating a circle to a triangle through a physical transformation we were able to relate the area formula for a circle to the area formula for a triangle. By working with the imagistic and the symbolic representations at the same time we claim students were able to recognize an informal argument to justify the formula for measuring the area of a circle.

Thus, we found that students in Grade 8 are able to recognize the validity and figural veracity of the standard formula for computing the measurement of the area of a circle in terms of its radius. This finding informs and allows us to adapt the row of the area HLT for Conceptual Area Measurer, indicating that students at a conceptual understanding of area measuring in terms of arbitrary square units should also be expected to recognize extensive algebraic manipulation of the area formula to relate it to the formula for finding the area of a triangle and add the four tasks to the HLT. Future research will be completed with these students to see they are able to describe and complete an informal justification for the circle area formula on their own.

Acknowledgments

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HIGH SCHOOL STUDENTS' FORMING 3D OBJECTS USING TECHNOLOGICAL AND NON-TECHNOLOGICAL TOOLS

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We analyzed the ways in which two high school students formed 3D objects from the rotation of 2D figures. The students participated in a task-based interview using paper-and-pencil, manipulatives, and Cabri 3D. The results indicated that they had difficulty using paper-and-pencil to rotate 2D figures to form 3D objects. Their difficulty stemmed from thinking of the rotation in a 2D context. Although the use of manipulatives helped them reason about 3D problems, they still had difficulty representing 3D objects correctly. However, the students were able to relate the rotations applied to the 2D figure to the resultant 3D object using Cabri 3D.

Keywords: Geometry and Geometrical and Spatial Thinking, Technology

Introduction

Forming 3D objects from the rotations of 2D figures (e.g., forming a cone from the rotation of a right triangle about one of its legs) is important for the study of calculus (Baartmans & Sorby, 1996) and for developing students' visualization skills. Nevertheless, there is little research about students' forming 3D objects by rotating 2D figures and how different tools might influence those processes. Although *Common Core State Standards for Mathematics* (National Governors Association Center for Best Practices, Council of Chief State School Officers [NGA & CCSSO], 2010) states that students should be able to "identify the shapes of two-dimensional cross-sections of three-dimensional objects and identify three-dimensional objects generated by rotations of two-dimensional objects" (HSG.GMD.B.4) there is little guidance about how teachers can assist students in learning this important skill and the tools that may be useful. Research is needed to understand how students reason about 3D objects generated by rotations of 2D objects using different tools. The purpose of this research study is to understand how high school students' form 3D objects using two different types of tools: non-technological tools that consist of paper-and-pencil and physical manipulatives, and the technological tool Cabri 3D. Our research question is "How do high school students form 3D objects using different learning tools?"

Theoretical Framework

According to Duval (2006), changing one semiotic system into another system (e.g., statement to equation; figure to statement) using denotations, and coordinating these systems are essential for acquiring mathematical knowledge. Duval refers to *semiotic registers* that allows for representing mathematical objects, making transformations within the same semiotic system (e.g., figure to figure), and creating an equivalent representation in another semiotic system. Two types of transformations of semiotic representations are characterized: *treatment* and *conversion*. Treatments are transformations within the same register (e.g., rotating a triangle 360 degrees). Conversions involve transformations from one register to another one (e.g., finding an algebraic equation for a given graph). Treatments and conversions are given through *discursive* and *non-discursive* representations/apprehensions. Discursive representations are given through speech (articulation of thoughts) using mathematical properties, symbols, definitions, etc. (Duval, 1995).

Students who recognize shapes utilizing *perceptual* apprehension perceive geometric shapes by "figural organization laws, and pictorial clues" (Duval, 1995, p.145). Therefore, students are likely to be misled by perceptual characteristics of geometric shapes. Students who utilize *operative* apprehension make physical or mental manipulations of a geometrical shape such as dividing it into

sub-figures, changing the orientation of it, shrinking-enlarging it, etc. (Duval, 1995). For example, in Figure 1, a student may state that the points are collinear if he or she uses perceptual apprehension. On the other hand, the student who utilizes operative apprehension notices the partial line segments are on different surfaces of the cube and/or changes his or her orientation toward the cube.
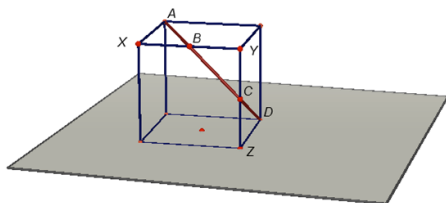
On the cube, point B is located on $\overline{XY}$, point C is located on $\overline{YZ}$. Are points A, B, C, D collinear? Why/why not?

Figure 1. A task related to apprehension of 3D objects.

Gorgorió's (1998) research indicated that students with low spatial abilities had difficulty describing the movements of rotations mentally and providing enough spatial information about 3D objects. Students may not think that they are to spin 2D figures to form 3D objects when they are given tasks in a paper and pencil environment. Especially students who use perceptual apprehension; they may consider this rotation in a 2D context. Under this circumstance, students may have difficulty making a conversion from one semiotic system to another one. There may be certain features of a task that prompt the student for particular actions. For example, the teacher may provide additional cues on the paper-based task such as including a spiral arrow in the question to suggest a rotation about a particular axis. Using manipulatives, students may be better able to see the different locations of the 2D object as it rotates that are not possible to see and difficult to draw in 2D. However, in both cases the learner is still responsible to bring the pieces together, and imagine the final 3D object because "most physical actions on physical manipulatives do not leave a trace sufficiently complete to reconstitute the actions that produced them" (Kaput, 1995, p.167). With the availability of dynamic geometry software (DGS), students can drag points, objects etc., observe the path of motion by activating the trace tool of DGS without being required to remember all of its locations and consider what is formed (Schumann & Green, 1997). With the help of the trace tool, technology allows students to observe and interpret the outcomes of fixed properties under different circumstances (Hollebrands & Dove, 2011).

In the current research study, we examine high school students' formation of 3D objects from the rotation of 2D objects. This study will investigate how students use manipulatives, and technology to form 3D objects.

Methods

The participants were selected among 29 students (18 females, 11 males) from a rural high school enrolled in a second year mathematics class. Students were given a spatial ability test. We selected six students (two above average, two average, two below average) to participate in one 90-minute clinical interview during 2013-2014 spring academic semester. In this report, results from two students, Andrea (female) and Pete (male), will be reported. After consulting their mathematics teacher and considering their test scores, we identified Pete as having low spatial abilities and Andrea as having high spatial abilities. The mathematics teacher of the participants reported that it had been over a year since they looked at 3D geometry, and all they would have done would be finding volume of different shapes. At the beginning of 2013-2014 spring academic semester they studied 2D transformations. None of the participants used Cabri 3D or any other DGS.

The interviewees were given five tasks. The tasks were sequenced beginning with paper and pencil. Afterwards, the participants solved the same tasks using manipulatives and then Cabri 3D. Square-centimeter grid paper was used for presenting the paper and pencil tasks, and students had

access to a ruler and compass. In this report, results from the first two tasks are presented. In the first task, students rotated a rectangle about a line that passed through one of the longer sides. In the second task (Figure 2), students were asked to rotate a rectangle 2 cm away from the axis of rotation. In these tasks, the student needs to match the statement (e.g., intersects the plane perpendicularly, rotate the rectangle 360 degrees about line *KL*) with the figure given in the task (Figure 2) utilizing operative apprehension. In other words, a conversion from statement to figure (or vice versa) is needed. Also, a treatment from figure to figure that involves spinning the rectangle 360 degrees in a continuous motion is required.

> Task 2: Suppose the line in the figure (Figure 2) intersects the plane perpendicularly. In the figure, rectangle *ABCD* is 2 cm away from line *KL*. Draw the three-dimensional object that is formed if you rotate the rectangle 360 degrees about line *KL*.
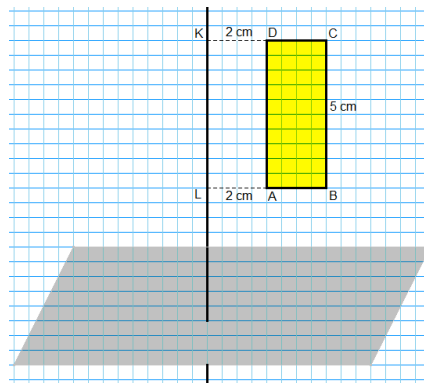
Figure 2. The figure given in Task 2.

A video camera, audio recorder, and a program that recorded the computer screen only when students used Cabri 3D were used to capture the participants' work. Because the participants were not familiar with Cabri 3D, the interviewer spent the first ten minutes teaching them the basic features of the program using some preliminary activities. The interviewees learned how to rotate objects and examine them from different perspectives, mark, label and trace points using Cabri 3D menu tools.

After the interviews, the researchers constructed verbatim transcripts. We identified the interviewees' apprehensions and production of registers within a semiotic system, by focusing on how the participants rotated 2D figures, what types of 3D objects they formed, and if they related the given lengths of 2D figures with the 3D objects (height, radius, etc.). We illustrated the participants' interpretation and constructions within and across the learning tools they used by comparing and contrasting the participants' representations and reasoning with regard to each tool they used.

Results

Results from the Participants' Uses of Paper and Pencil

In the first section of the interviews, the interviewees used paper and pencil to work on the tasks. Andrea and Pete stated that after rotating the rectangle 360 degrees it would arrive at its original location. Andrea summarized her thinking by saying: "basically, you take this figure and you rotate it 360 degrees by the line *AD*. So, that would just be the same figure cause you're rotating it 360 degrees just puts it back where it is." The interviewer asked her to describe how she rotated the rectangle 360 degrees. Andrea denoted the rotation step-by-step by rotating the rectangle 90 degrees each time separately.

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Andrea described the rotation as flipping although her rotation method involved rotating the rectangle about its centroid. After rotating the rectangle, she produced a semiotic representation in 2D. Andrea stated that it would be impossible to get a 3D object. She explained her thinking as follows: "if you start with a 2-dimensional object, I don't really think that you can turn into a 3-dimensional object." In the rest of the tasks, Andrea was consisted with her thinking and refused to produce a semiotic representation in 3D geometry, and said: "no matter what you do to it, it's still going to be the same flat shape it is. It's gonna stay 2-dimensional."

Pete conducted the 360-degree rotation by rotating each corner of the rectangle (Figure 3a) and $\overline{AD}$ (one of the sides of the rectangle) to exemplify how he rotated the rectangle. Similar to Andrea, Pete rotated the line segments 90 degrees each time separately, and stated that the rectangle would be back to its original location. Afterwards, he showed the rotation using an eraser as shown in Figure 3(b), he rotated the rectangle about its centroid. Then, he drew a rectangular prism and said: "most rectangles that one puts in a 3D perspective, it's gonna be a rectangular prism." After Pete had denoted the edge lengths, the researcher pointed at the height of the rectangular prism, and asked how he identified the length of it (Figure 3c). Pete said that he was looking at the object from the top view, and added:

It states that 2 cm is right here (points at one of the short sides of the rectangle), 5 (cm) is here (points at one of the long sides of the rectangle). So, it never tells you the height but it does tell you the 5 cm on this line segment. So, I kinda took that as if possibly it could be saying that how tall it is. It is on a grid, it's not 3D, it could have that (inaudible) this is how tall it is…

(a)   **(b)**   **(c)**
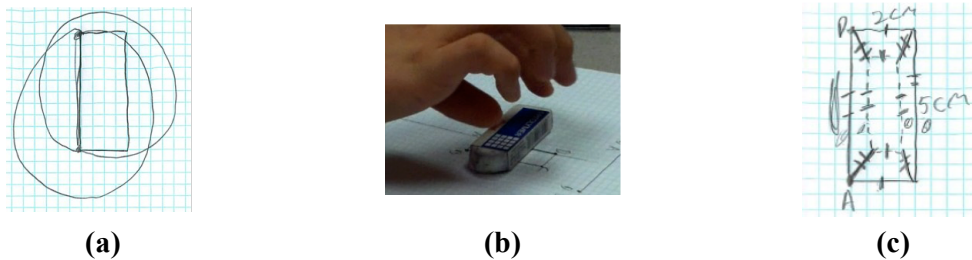
Figure 3. (a) Pete's illustration of the rotation, (b) Pete's illustration of the rotation using an eraser, and (c) Pete's illustration drawing for the first task.

In the second task, Pete rotated the rectangle similar to the first task (90-degree rotation each time separately) and associated the lengths of the rectangle with the edges of the rectangular prism he drew by saying: "if it is telling you to rotate it at, the rectangle 360 degrees about line *KL*, so you'd probably- rotate it towards *KL* that 2 cm could be telling us the height of the object like I said the height on the side." In his semiotic representations, Pete rotated the rectangles 360 degrees through their centroids and put them back in their original positions. He produced a semiotic representation in 3D geometry converting 2D figures into polyhedrons by matching one of the lengths he observed in the tasks.

Results from the Participants' Uses of Manipulatives

In the second session of the interviews, the interviewees solved the same tasks using manipulatives. The use of manipulatives helped Andrea and Pete utilize operative apprehension to reason about the problems since they had rotated the rectangles through the centroid of them in a 2D system using paper and pencil. They made sense of the crucial statements given in the tasks. Namely, the rotation line was perpendicular to the plane and rotating the rectangles about the line implied

spinning objects. They noticed that the statement meant to spin the rectangles. However, Pete said that the same object would be formed although he was aware of the blind spots of the rotation. He demonstrated the rotation using the eraser (Figure 4(a)) and said: "when I am doing it this way, it's rotating, I am rotating it as towards me so it'd look like straight line (when the rectangle is rotated 90 degrees, it looked like a straight line) kind of but a 3-dimensional object."

| (a) | (b) | (c) |
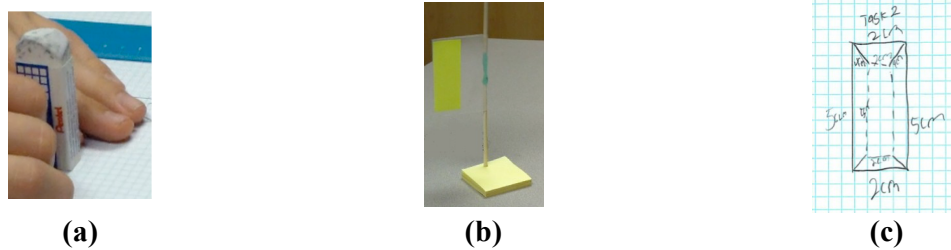
Figure 4. (a) Pete's illustration of spinning using an eraser, (b) Pete's use of manipulatives for the second task, and (c) Pete's modified drawing for the second task.

When Pete was reasoning about the second task using manipulatives (Figure 4(b)), he changed the height of the rectangular prism from 2 cm to 4 cm by adding *KD* and *CD* together by saying: "the 2 cm is here and here but also we added to the 2 cm that it is away. And, if you add them together so that it maybe 4 cm as the height of it from the ground now that I am looking at like this" (Figure 4(c)). Although Pete was doing a right treatment demonstrating how the rotation took place, he had difficulty coordinating the representations in the treatment. He focused on the outcome of the rotation and made little connection between his 3D drawings (e.g., Figure 4(c)) and how the rectangles were rotated.

On the other hand, Andrea held the same belief that one cannot form a 3D object by rotating a plane figure by saying:" it just goes around, stays the same. No matter what you do, if you flip 360 this way and go back to where it was...But no matter what you do if you spin it this way, this way, flip it upside down, it stays the same." She had difficulty thinking of the rotation in a continuous motion. However, using manipulatives, Andrea was able to notice the circular motion of the rotation. She described the previous rotation as flipping, and the new rotation using manipulatives as spinning. While Andrea was solving the third task in which she rotated a semi-circle, she highlighted that it would look like a sphere if one spins it fast enough by saying: "I assume the figure, if you're looking at it when you're spinning fast enough it will look as if it a whole sphere but it still won't be a sphere."

| (a) | (b) | (c) |
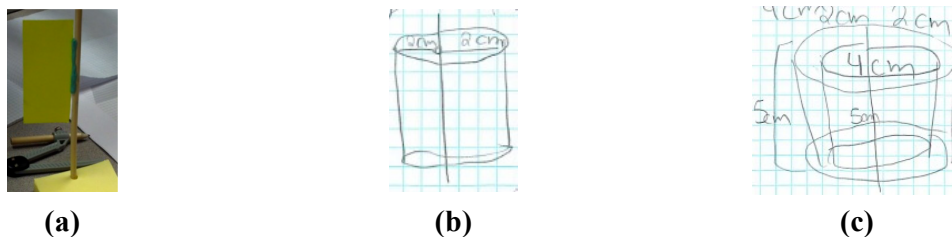
Figure 5. (a) Manipulatives designed for the first task, (b) Andrea's drawing for the first task, and (c) Andrea's drawing for the second task.

Afterwards, the researcher asked what she would form by rotating a rectangle fast enough. Andrea thought it would form a rectangular prism. However, soon after she rotated the rectangle using manipulatives (Figure 5(a)), she noticed that it would form a cylinder by saying:

If you spin it fast enough it would look like all the edges are coming up. Oh, actually, probably look more circular instead of actually rectangular. It would look like a flat, like a cylinder instead of a rectangular prism if you're spinning it so it could be a cylinder instead of a square thing (rectangular prism). Cause if you're spinning it'd look circular on the edges but look flat on the top.

Although Andrea figured out the correctly formed 3D objects after rotations, she mislabeled the radii of circles in her both drawings. Also, she misrepresented the location of line *AD* by drawing it at the left edge of the cylinders instead of in the center of them. The third task in which a semi-circle was rotated 360 degrees helped Andrea revise the radii lengths of cylinders and the location of line *AD*. While identifying the object after she rotated a semi-circle, Andrea highlighted that a sphere would be formed by saying: "yeah, like that makes a sphere. Because half of the line is kind of going down the middle of it. This is a semi-circle on this side, and if you're spinning fast enough, you can kinda see a sphere form out of it." Afterwards, the interviewer showed that she drew the line in the middle when she formed a sphere, while the line was at the edge of the cylinders in the previous tasks. With the prompt of the interviewer, Andrea modified her drawing as shown in Figure 5(b) and Figure 5(c).

Results from the Participants' Uses of Cabri 3D

In the third session of the interviews, the interviewees used Cabri 3D and the *Trajectory* tool. Because Andrea reasoned about the tasks correctly using manipulatives, she did not modify her drawings; instead she verified her drawings and restated her thinking. For example, in the second task, she looked at the object from the top view to indicate the inner cylinder as shown in Figure 6(a) by saying:

Andrea: You can see there is a cylinder in the middle and a cylinder on the outside (Figure 6(a)). So, you see both of the shapes and stuff.
Interviewer: So, why did we get a hole in it?
Andrea: Because the lines are going outwards, they make it, right there, they made it so where it would have a space between this cylinder and this line and the outward cylinder.

| (a) | (b) | (c) |
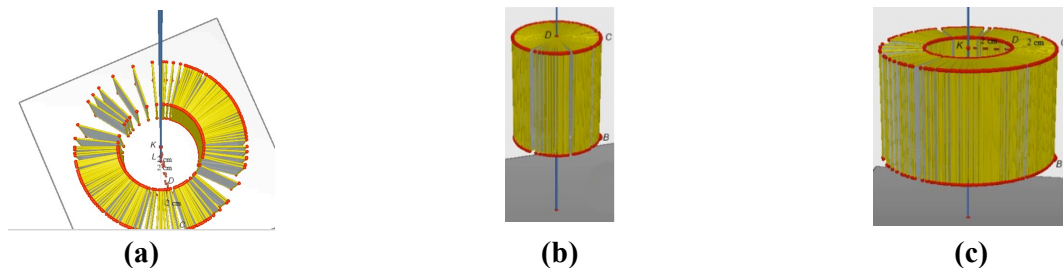
Figure 6. (a) The rectangle in the second task traced out by Andrea, (b) The rectangle in the first task traced out by Pete, and (c) The rectangle in the second task traced out by Pete.

Andrea appreciated Cabri 3D by saying: "this one (Cabri 3D) helped a lot more because you can actually show everything that was going on..." On the other hand, Pete changed his reasoning by observing the objects formed in Cabri 3D. After he rotated the rectangle (Figure 6(b)), he said "I observe that it made a cylinder" and added "because you are dragging around 360-degree motion that it's tracing the whole time you go around. It's tracing the object during it went." The interviewer asked him to compare and contrast what he was thinking before and what he observed. He explained his previous thinking by saying "I was thinking before that it wanted to know the 3-dimensional shape of the object after you rotated it. So, I was thinking that the object that it wanted to know was

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what the shape would be after you rotated it, not the rotation would look like - the shape that rotation made." Using Cabri 3D, he made a connection between how the rectangles were rotated and the resulting 3D objects. Similarly, in the second task, Pete interpreted what he observed (Figure 6(c)) as follows:

> A cylinder but it's hollow in the middle… Because of the fact that it's 2cm away, and instead of like I was thinking, what I was thinking telling you the height of it, of the object. Instead of that, it's telling you that's how far away the rotation going around the object was.

While using Cabri 3D, he observed the rotations from different perspectives to articulate his thinking. The interviewer asked Pete to explain how he considered the tasks using three different learning tools. Pete explained his thinking as follows:

> I was thinking on this one (manipulatives) just like I was on this one (paper and pencil) that it was wanting to know what the 3-dimensional object was, and not what the actual rotation of the object would make if you were to trace it all the way around.

Discussion and Implications

In this research study, we analyzed Andrea and Pete's formation of 3D objects by rotating 2D figures using paper and pencil, manipulatives and Cabri 3D. They were reasoning about the problems primarily affected by perceptual apprehension because they made treatments rotating objects in a 2D semiotic system. They had difficulty understanding how they were asked to rotate the rectangles about the axes of rotations. In other words, they had difficulty making a conversion from the figures to the statements given in the tasks (or vice versa). As a consequence, Andrea was unable to produce a representation in a 3D semiotic system. Pete produced representations matching one of the lengths he observed in the tasks to denote the height of 3D objects. In Pete's drawings, there was little connection between the rotations and the resulting 3D objects. He had difficulty associating the rotation of 2D figures with the resultant 3D objects. Similarly, Gorgorió's (1998) research indicated students with low spatial abilities had difficulty describing the movements of rotations. Pete viewed prisms as a 3D form of 2D figures (e.g., rectangle $\rightarrow$ rectangular prism). These findings suggest a need to conduct research to examine how students can make connections between extruding (e.g. forming a cylinder translating a circle in a linear and continuous motion) and spinning 2D figures (e.g., forming a cylinder spinning a rectangle about one of its sides).

When Andrea and Pete were given manipulatives to solve the tasks, they observed the circular motion of the rotation utilizing operative apprehension. Pete focused on the outcome of the rotations, and provided a similar semiotic representation drawing rectangular prisms. On the other hand, Andrea, at first, held the same belief that one cannot form a 3D object by rotating a 2D figure until she rotated a semi-circle. Then, she identified the correct formed objects although she mislabeled the measures of the cylinders. After using manipulatives, the description of the rotation that Andrea used shifted from flipping to spinning. However, she needed to go beyond to identify the resultant objects because as Kaput (1995) emphasized manipulatives do not always provide sufficient information. She managed to form the objects associating the statements given in the tasks with the semiotic representations she produced. However, Pete who had low spatial abilities could not identify the objects. Schumann and Green (1997) emphasized that imagining a point's continuous motion in mind and conceive its path to look for mathematical relationships is closely related to students' visual imagination capabilities.

When Pete used Cabri 3D, he was aware of the continuous curricular motion of the 2D figures. Based on his observation of the objects formed in Cabri 3D, he attended to the properties of the formed objects. He compared and contrasted the representations in Cabri 3D with his previous drawings. With the feedback of Cabri 3D, Pete considered the rotation as a continuous process

different from his previous methods that focused on outcomes of the rotations. As Hollebrands and Dove (2011) state, technology allows students to observe and interpret the outcomes of fixed properties under different circumstances with the help of the trace tool. On the other hand, Andrea verified her reasoning based on the feedback DGS provided, and supported her previous thinking.

Using manipulatives, the interviewees observed different locations of 2D figures as they helped them to think about the problems in a 3D context. However, the interviewees had difficulty imagining the resulting 3D objects. Manipulatives that give a concrete evidence about forming 3D objects as shown in Figure 7 can be given to students. Future research can examine short- or long-term influences of using different learning tools. Also, we had students use Cabri 3D for comparing and contrasting their previous thinking. New research is needed to indicate how students form 3D objects by the rotations of 2D figures.

Figure 7. Some hands-on materials related to spinning figures.

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YOUNG CHILDREN UNDERSTANDING CONGRUENCE OF TRIANGLES WITHIN A DYNAMIC MULTI-TOUCH GEOMETRY ENVIRONMENT

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This study examined how small groups of second-grade children developed understandings of the concept of congruence while collaboratively exploring and solving problems with dynamic representations of triangles using Sketchpad on the iPad. One case study is presented to illustrate how young learners can infer geometrical relationships between congruent triangles and co-construct mathematical strategies to create congruent triangles using these technologies.

Keywords: Geometry and Geometrical and Spatial Thinking, Technology, Problem Solving

Introduction

Congruence is an important mathematical idea for humans to understand the structure of their environment. Congruence is embedded in young children's everyday experiences that allow them to develop intuitive senses of this geometric relationship. Understanding the concept of congruence provides strong foundations for learning more advanced mathematical processes such as area and volume measurement (Huang & Witz, 2011; Wu, 2005). However, prior research has revealed a variety of students' difficulties in learning congruence at both the elementary and secondary grades (Clements & Sarama, 2014; Wu, 2005). Wu (2005) claims that the teaching of this concept is focused on the static informal definition "congruence is same size and same shape" (p. 5), which does not relate congruence to planar transformations, while the precise mathematical definition of the concept is based on rotations, translations and reflections. Wu notes that middle-school students have difficulties in understanding the precise mathematical definition of congruence and fail to grasp how it underlays other mathematical processes. Clements and Sarama (2014) state that the natural development of congruence also represents critical challenges for young children because they tend to analyze only parts of the shapes (e.g. length of one side) but not the relationships between these parts (e.g. lengths of all the sides) and privilege aspects of the shapes that are salient perceptually (e.g. orientation) rather than aspects that are mathematically relevant (e.g. number of sides). Thus, young children fail when one of the two figures is rotated or flipped or when the figures are unusual for them (e.g. long and thin triangles, scalene triangles, hexagons). The authors suggest that traditional teaching of geometry in early grades is implemented in rigid ways, which means that children are exposed to only prototypical shapes and have little experience with non-examples or variants of shapes. Students' difficulties can endure until adolescence if not well addressed educationally, limiting their access to formal mathematics in higher grades (Clements & Sarama, 2014). Furthermore, although learning congruence is important for the growth of advanced mathematical thinking, its teaching has been traditionally relegated to middle school (Huang & Witz, 2011; Wu, 2005). However, prior research has shown that from birth to 7-8 years of age, children spontaneously develop Euclidean geometry knowledge about two-dimensional shapes including triangles (Shustermann, Lee & Spelke, 2008) as well as intuitive ideas of congruence (Clements & Sarama, 2014). This suggests that second-grade children could engage in informal reasoning about congruence and benefit from the early implementation of the concept as groundings for its future formal learning.

Researchers have stressed that utilizing digital interactive technologies in early childhood education can promote new ways of mathematical thinking in young learners (Clements & Sarama, 2014; Hegedus, 2013; Sinclair & Moss, 2012). The use of dynamic geometry software such as

Geometer's Sketchpad® (Jackwic, 2009, hereon Sketchpad) could support young children's reasoning on properties of two-dimensional shapes and facilitate their access to more complex concepts (Sinclair, deFreitas & Ferrara, 2013; Ng & Sinclair, 2015). The addition of multi-touch devices could foster direct interaction with the mathematical configurations and collaborative behaviors that, in turn, could support the co-construction of shared mathematical meanings (Hegedus, 2013). This study posits that combining *Sketchpad* with the iPad through the application *Sketchpad®Explorer*, could enhance young children's learning experiences of congruence by helping them grasp in a dynamic way what means 'same shape and same size', so that they can link these properties to continuous geometric motions and to a variety of triangles. Moreover, children could work in small groups manipulating the dynamic shapes directly and simultaneously on the iPad, to also benefit from gestural expressivity and social interaction. Such an environment could help students overcome some of the learning challenges stated above. However, research on early learning of congruence is scarce. Furthermore, little is known about how the use of these digital multimodal technologies in small groups could benefit young children's development of foundations on congruence. This study aimed to design and implement a sequence of exploratory and problem-solving activities using *Sketchpad* on the iPad in order to examine the ways in which small groups of young learners reason about and understand congruence ideas while collaboratively working with dynamic representations of triangles. The question was: *How do small groups of second-grade children make sense of the concept of congruence within a collaborative Dynamic Multi-touch Geometry Environment?*

Theoretical Framework

This study is grounded on sociocultural theories of situated learning that see human activity as an integral part of the process of knowing that is mediated by both social interaction and cultural artifacts, such as digital interactive technologies. The theoretical framework of *semiotic mediation* related to the use of dynamic geometry environments and haptic technologies for the development of children's mathematical reasoning (Moreno-Armella, Hegedus, & Kaput, 2008; Hegedus, 2013; Sinclair & Moss, 2012) guided the research. The construct of *semiotic mediation* is central to understand how the use of multimodal technologies can nurture young children's co-construction of understandings about congruence. *Sketchpad* is a computer micro-world that enables users to continuously manipulate and transform, into a drawing-like space, a variety of geometrical objects that are pre-defined mathematically (Sinclair & Moss, 2012). Students can utilize the function tool *dragging* and, after any dynamic transformation, these objects preserve their defining mathematical properties, even if other characteristics vary. These affordances can mediate children's access to a variety of representations of mathematical objects and ways of thinking about the underlying properties (Hegedus, 2013; Sinclair & Moss, 2012). In this study, the dragging tool could mediate children's access to multiple representations of congruent triangles and the discovery of the underlying congruence relationship. Multi-touch horizontal tablets allow for physicality of learning, multiple inputs and co-location of students, facilitating small-group collaboration and haptic representations (Dillenbourg & Evans, 2011; Hegedus, 2013). Mediation of visual dynamic feedback and multi-touch input could foster young children's mathematical inquiry entailing reasoning and discovery, as they are able to conjecture and generalize while interacting with peers and the technology, as well as richer mathematical discourse, gestural expressivity, and understanding of geometric concepts such as congruence.

Methodology

The study entailed the design and implementation of an educational intervention strategy based on collaborative inquiry and problem solving within a dynamic multi-touch geometry environment (hereon DMGE). A sequence of seven activities was implemented in small groups of students for the

early learning of congruence and similarity. Thirteen children (7-8 year olds) from five second-grade classrooms of a middle-SES public elementary school from Massachusetts, U.S., participated in the study. Children included girls and boys from various cultural backgrounds and were organized into five groups –two groups of two students and three of three students. This educational strategy was implemented as part of the afterschool program. A qualitative multiple-case study research approach was the method of inquiry to analyze small-group work on the tasks. This paper focuses on the three first activities of the sequence, designed to promote informal understandings of congruent triangles from a dynamic and multimodal perspective: Two exploratory activities (one task each one) and one problem-solving activity (three tasks). In Activity 1 and Activity 2, children were shown two congruent triangles of contrasting colors, and were asked to drag one of them and describe what happened with the other triangle. In Activity 1, both triangles could be continuously rotated, resized, and translated by dragging one of them, adopting different positions on the screen, but after any dynamic transformation the triangles always remained congruent (Figures 1a). In Activity 2, both triangles could be continuously transformed by dragging one of them, adopting different orientations and positions between them, but they always remained congruent (Figures 1b). In Activity 3, children were shown a referent triangle and a non-congruent triangle over a grid, and were asked to make the non-congruent triangle identical to the referent triangle (Figures 1c). This activity had three tasks with increasing degree of complexity based on the type of triangle (e.g. right, scalene). All the activities showed the area of each triangle at the top, which was called the Size Marker tool.
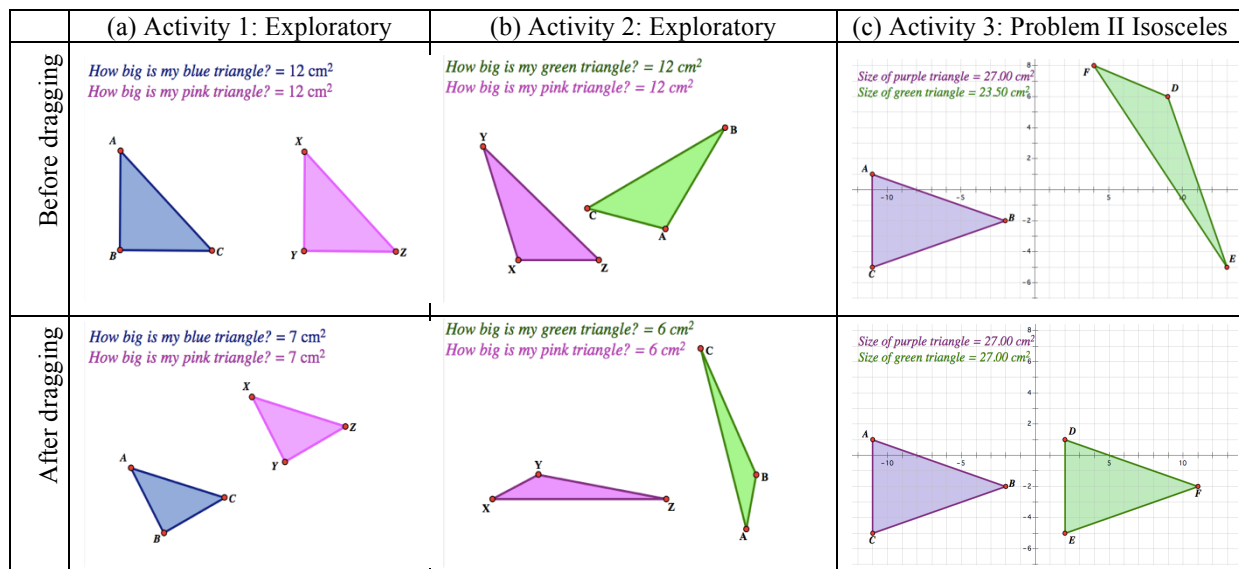
Figure 1. Sequence of exploratory and problem-solving activities for congruence.

The task-based interview with a semi-structured interview protocol was the primary data collection method. Each small group of children had one iPad with the activities developed in the DMGE and was observed and interviewed while solving each activity. The entire sequence of learning took place during four 1-hour sessions, once a week during four consecutive weeks, which were fully videotaped, transcribed and codified for analysis. Discourse analysis of children's interactions within each group was the data analysis method (Wells, 1999). The analytical framework included: (a) Children's ways of thinking about congruence (e.g. one or two relationships between attributes, discovering congruence invariance, measurement, representation of attributes), (b) Collaborative patterns, and (c) Uses of the technology. These aspects were analyzed from children's discourse -utterances, actions, gestures-. Coding consisted of a stepwise iterative process of seeking redundancy, using a first cycle-process coding method and a second cycle-pattern coding method.

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Results

Partial results from this study are presented in three sections regarding three emergent themes. These results are illustrated with excerpts from one case study correspondent to the analysis of Nathan and Kevin's discussions while interacting with each other, the researcher and the DMGE, in Activities 1 and 3. This group was selected because children planned the strategy in Activity 3 before using it, different to other groups. Actions are presented between braces, gestures underlined between brackets, and utterances are presented in normal format (between quotes only within the narrative).

Dynamism Mediated the Discovery of Congruence Relationships between Triangles

The first relevant finding of the study is that *dynamism* mediated young children's discovery of geometric relationships related to congruence of triangles within the DMGE. In Activity 1, I asked Kevin and Nathan: "I would like for you to drag the blue triangle (hereon BT) and tell me what happens with the pink triangle (hereon PT)". Initially, children showed an explorative use of the dragging function, systematically examining different continuous motions of BT such as turning around it, resizing it and, sliding it up and down, and observing the PT's behavior. When Kevin dragged BT up and down several times he began identifying *one relationship between attributes* of both dynamic triangles referred to their same movements as he said "Oh! Now when I move the triangle, if you move it up and down *{drags BT up-and-down}*, that one moves just up and down *{shows PT} [moves right hand back-and-forth]"*. Kevin' statement implied dynamism as he talked about the up-and-down motion of the triangles. Nathan began dragging BT, turning around several times and stretching it until the triangles got increasingly bigger or turning around and shrinking it until the triangles got increasingly smaller, while Kevin observed the screen. I had asked them to explore more, when the following discussion took place.

> **Excerpt 1.** *Case Kevin and Nathan, Activity 1 (BT: Blue Triangle; PT: Pink Triangle).*
>
> 1 *Kevin*: Ok! *{Drags BT stretching and shrinking the triangles two times}* Oh! May be, I think when you move the blue triangle that makes the blue triangle bigger and then also that makes the pink triangle bigger and it also moves?
> 2 *Researcher*: Yeah? What do you think Nathan?
> 3 *Nathan*: Whenever you make the blue triangle bigger *{drags BT stretching the triangles}* or smaller *{drags BT shrinking the triangles}*, they both are always equal, the same size *{drags BT turning around several times}*
> 4 *Researcher*: Yes? Can you show me that?
> 5 *Nathan*: *{Drags BT stretching the triangles, shrinking the triangles, turning around the triangles, stretching the triangles, shrinking the triangles, translating the triangles}*
> 6 *Researcher*: What do you think Kevin about what Nathan says?
> 7 *Kevin*: Um, well like *{observes what Nathan does on the screen}*, they're, yeah, they're always like the same size *{shows the triangles}* and they're, they both have the same lengths of edges *[extends two hands as horizontal parallel lines]*
> 8 *Researcher*: Can you show me that? I want to see
> 9 *Kevin*: Like they both, they both have the same lengths on the sides *{shows one side in PT; then shows the correspondent side in BT; then shows another side in PT and the correspondent side in BT; then shows the last side in PT and the correspondent side in BT}*

The Excerpt 1 shows that both children began inferring *two relationships between attributes* of the dynamic triangles such as same change of size and same type of movement, for instance when Kevin said "Oh! May be I think when you move the blue triangle that makes the blue triangle bigger and then also that makes the pink triangle bigger and it also moves?". They also discovered *two invariant relationships between attributes* of the dynamic triangles such as same change of size and same size,

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for instance when Nathan said "Whenever you make the blue triangle bigger or smaller, they both are always equal, the same size"; or same size and same lengths of the edges as when Kevin said "they're always like the same size and they're, them both have the same lengths of edges". Kevin and Nathan' statements implied dynamism as they talked about the triangles' movements or about their changes of size, which are attributes of dynamic triangles. This dynamism mediated the discovery of the invariant relationship related to "same size", which was evidenced in the use of words such as "whenever", "always" or "also", along with the words "same" and "equal" to make explicit the condition that the triangles are always equal because of the same size, independently of the type of movement. Although they did not say "same shape", Kevin talked about the "same length of edges" as another attribute different from size. Moreover, further on Kevin was able to specify which were the equal edges by showing them by pairs of congruent sides in both triangles. The children found out what was invariance in the activity and were able to formulate these relationships in their own words as a permanent rule of the two triangles. The Excerpt 1 also shows that the use of the technology evolved from exploratory to demonstrative. The children accompanied their statements with actions of dragging and gestures (e.g. pointing out) intentionally directed to demonstrate to each other and me, as the interviewer, what they were thinking. A collaborative behavior was seen when Kevin built on and extended Nathan's idea about "same size" (Line 7).

Gestures Mediated the Co-Planning of Strategies to Create Congruent Triangles

The second relevant finding is that gestural expressivity mediated young children's collective planning of strategies to create congruent triangles within the DMGE. Children used gestures on the iPad to represent a congruent triangle and properties of congruent triangles. The Excerpt 2 is a discussion from Activity 3, Task 2 (Level II: Isosceles) in which Kevin and Nathan planned a strategy to make the green triangle (GT) identical to the purple triangle (PT, see Figure 1c).

Excerpt 2. *Case Kevin and Nathan, Activity 3a (PT: Purple Triangle; GT: Green Triangle)*

1 *Nathan*: (To Kevin) We can move the F down *[points out from GT's point F towards a place down on the grid in front of PT's point A]* and then the D, the E over here *[points out from GT's point E towards a place down on the grid in front of PT's point B]* and then the D over here *[points out from GT's point D towards a place down on the grid in front of PT's point C]*.

2 *Kevin*: We have to equal up as there *{shows that the Size Marker of the referent triangle PT is 27 cm.}* so, that one *{shows PT}* is 27 centimeters *{shows PT' Size marker}*, and that one…

3 *Nathan*: Yeah! 27, so we try to make it 27 *[nods]*

4 *Kevin*: That one is a little thinner so *{shows GT}* (Inaudible)

5 *Researcher*: How do you say Kevin?

6 *Kevin*: The purple triangle *{shows PT' Size Marker}* is like 27 centimeters *[extends 2 fingers on PT's area making a big space between fingers]* and that one is only 23.50 centimeters *[extends 2 fingers on GT's area making a smaller space between fingers]* and this one is a little skinnier *[extends 2 fingers on GT's area making a smaller space between fingers]*

7 *Researcher*: Yeah?

8 *Kevin*: Like it is skinnier *[extends two fingers on a thin area of a GT's angle leaving little space between fingers]* and then is a little fatter at the top *[extends two fingers on a wide area of a GT's angle leaving a wide space between fingers]* and then gets skinnier *[extends two fingers on another thin area of a GT's angle leaving little space between fingers]* and that one is a little fatter *[extends two fingers on a wide area of a PT's angle leaving a wide space between fingers]* than that one *[extends two fingers on a thin area of a PT's angle leaving little space between fingers]*.

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9 *Nathan*: It is skinnier over here *[extends two fingers on a thin area of a PT's angle leaving little space between fingers]* and fatter here, and then gets fatter *[extends two fingers on a wide area of another PT's angle leaving a wide space between fingers]*

10 *Kevin*: That one gets skinnier *[extends two fingers on an thin area of a GT's angle leaving little space between fingers]* and then gets skinnier again *[extends two fingers on another thin area of a GT's angle leaving little space between fingers]*. I just think that it's a little fatter *{shows PT}* than that one *{shows GT}*. We have to make it *{shows GT}* the same lengths *{shows 2 sides of PT}* and everything.

11 *Nathan*: Yeah! *[Nods]*

The Excerpt 2 shows how Nathan and Kevin explicitly discussed a strategy to create a congruent triangle before implementing it. The pre-planned strategy consisted of relocating the points of the GT and aligning them with the points of the PT (referent). Nathan used hand gestures to represent the trajectories of the three GT's points to become a congruent triangle as well as their new location on the grid just in front of the PT, displaying what I call an 'imagined' spatial representation of the new triangle. This representation implies dynamism, as it involves imagined trajectories of the points. It is also embodied, as Nathan used his fingers to show to Kevin the new positions of the points. It also implies informal processes of measuring, specifically, the estimation of distances among the points of the imagined triangle, and the alignment of the points of the two triangles. As a result, the imagined triangle had an approximate shape and size to the PT's shape and size. Kevin and Nathan also talked about another informal measuring component, equaling up the size of the two triangles through the use of tools such as the Size Markers of both triangles (Lines 2 & 3). When Kevin began talking softly about the triangles' attributes, I asked him to repeat what he said and he elaborated on his idea to explain it (Lines 4 & 5). The children began making explicit attributes of the referent and the non-congruent triangles by comparing them informally; for example, they compared their shapes using words such as "thinner", "skinnier" and "fatter". Simultaneously, they utilized hand gestures featured by the use of two fingers on the triangles and intended variation of the space between fingers, to represent the triangles' areas or the areas of their angles and to plan equaling up their sides' lengths (Lines 6, 8, 9 & 10). This process of co-planning the strategy revealed how children were aware of the relationships between attributes of the two triangles such as equal sizes and equal shapes. Uses of the technology were characterized by children's utilization of special tools to solve problems, for instance the grid to imagine the new triangle (Line 1), or the Size Markers to compare sizes (Lines 2, 3 & 6). Children were less explorative than in Activity 1; the use of their fingers was mainly demonstrative so that even without dragging they could represent both trajectories and properties of the triangles as discussed above. The Excerpt 2 also shows the emergence of a collaborative pattern to co-construct strategies. First, children proposed new ideas to each other, discussed actions of their strategy, and explained and justified their ideas demonstrating them to others through gestures (Lines 1, 6, 8, 9 & 10). Second, children built on each other's ideas. For instance, Kevin proposed an idea and Nathan adopted and extended it (Line 3 & 9), or Kevin expanded a Nathan's explanation and made a conclusion (Line10).

Informal Measuring as Mathematical Focus of the Co-Creation of Congruent Triangles

The third relevant finding is that young children used emergent informal ways of measuring as the mathematical focus to assure that two triangles had the same shape and size during the collective creation of congruent triangles. The Excerpt 3 presents a discussion from the process of implementation of Kevin and Nathan' strategy during Activity 3.

Excerpt 3. *Case Kevin and Nathan, Activity 3b (PT: Purple Triangle; GT: Green Triangle)*

1 *Nathan*: *{Re-locates GT's point F on the grid in front of PT's point A, slides GT's points D and E towards left, slides back them towards right and closes the space between them}*

2 *Kevin*: *{Looks at Size Markers}* No! Try to make them both fit *{shows Size Markers; then makes GT taller, opening space between points D and E}* (Triangle is tall but thin)

3 *Nathan*: Ok! *{Makes GT fatter opening space between point F and points E and D; then makes GT taller opening space between D and F}* (…)

4 *Kevin*: (…) We have to align all with that (to Nathan) *{shows the points in PT}*

5 *Nathan*: I'm trying to *{makes GT taller, opening space between points D and E}*

6 *Kevin*: Let's make it bigger *{shows GT}* (GT is on the top-right while PT is on the bottom-left)

7 *Nathan*: I'm doing it bigger *{smoothly slides point E up-and-down on the grid many times}*

8 *Kevin*: Move, move this one *{shows point D}*

9 *Nathan*: You can't move the D. It's a thing!

10 *Kevin*: *{Slides points F and D towards left on the grid}*

11 *Nathan*: 26 *{looks at GT, looks at Size Markers}* (PT or referent is 27 cm.)

12 *Kevin*: *{Slides points F and D back towards right}* I will move D *{Keeps sliding point D}*

13 *Nathan*: 28 *{looks at GT, looks at Size Markers, shows Size Markers}* You cannot move D!

14 *Researcher*: What are you trying to do?

15 *Kevin*: We are trying, we are trying to make it equal, so is like the same *{drags GT }*

16 *Nathan*: One more time *{superposes GT's point D on PT's point A, GT's point E on PT's point C, and GT's point F on PT's point B}*. Then I will move this one over here *{slides GT's point F on the same line of PT's point B locating F in front of B; slides GT's point D on the same line of PT's point A locating D in front of A}*

17 *Kevin*: That one, there it goes! *{Looks at Nathan moving D}* You wanna make it exactly the same?

18 *Nathan*: Yes! *{Slides GT's point E on the same line that PT's point C locating E in front of C. With finger verifies that C and E are aligned measuring distance among them, and adjusts E}*

19 *Kevin*: Yeah! *{Looks at GT and looks at Size Markers, shows Size Markers}* Not, not, move! Ok!

20 *Nathan*: *{Adjusts point F one square on the grid}* Yeah! (GT has the same shape)

21 *Kevin*: 27 centimeters! *{Looks at Size Markers}* Ok! (Size Markers show triangles are 27 cm.)

The Excerpt 3 shows that children implemented and enhanced the strategy they had planned. Their first attempt consisted of relocating the points of PT on the grid one by one. However, they aligned just one of the GT's points with one of the PT's points, using it as the only spatial reference to locate the other points (Line 1). Then they adjusted the sides while checking permanently the Size Markers (Lines 2-13). The triangles were approximate but not exact in shape and size. The first informal way of measuring emerged when Kevin proposed to align all the points of both triangles (Line 4), but children did not reach an agreement (Lines 6, 7, 8 & 9). When I asked them what they were trying to do (Line 14), Kevin had clear their goal as he said "we are trying to make it equal, so is like the same". Nathan also restarted the task suggesting that he was aware that their strategy did not assure that the triangles get the same shape and size (Line 16). Further on, three informal methods of measuring emerged. First, Nathan superposed all the points of GT onto the points of PT, which produced a congruent triangle (Line 16). Second, he dragged each point of GT towards right, one by one, using the x-axis of the grid to align the points of both triangles on the same grid lines (Lines 16 & 18). Third, he adjusted the sides, measuring with his finger the distances between the points of both triangles (Lines 18 & 20). Superposition of two triangles, alignment of the points of two triangles using the x-axis and estimation of distances with fingers between the points aligned are informal ways of measuring that allowed him to correctly equal up the length of all the sides. Kevin

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was aware of the value of aligning, so he helped giving ideas and controlling the Size Markers (Lines 17, 19 & 21). The use of informal measuring and the regular monitoring of the Size Markers helped children construct a congruent triangle. The role of the technology was focused on the constructive use of the dragging tool and other tools provided to solve the task (e.g. Size Markers, grid, x-and-y axes). The collaborative pattern consisted of distributing and coordinating two functions: Nathan superposed, aligned, verified distances amongst and adjusted the points of GT while Kevin monitored the sizes of the two triangles in the Size Markers until they had equal shape and size. They also exchanged relevant information, suggesting ideas and building on each other's ideas.

Conclusions

This paper illustrated an educational strategy envisioned to cross borders among ages/grade levels for allowing young children in early childhood education (k-2) to access to complex concepts, such as congruence, that are traditionally taught at middle grades. The implementation of the strategy involving children's participation in activities designed in a DMGE, showed three relevant findings: (a) mediation of dynamism for discovering invariant geometric relationships between congruent triangles, (b) mediation of gestural expressivity for co-planning strategies to create congruent triangles, and (c) emergence of informal ways of measuring to assure that two triangles have the same shape and size. Children's collaborative patterns included the use of varied modes for conveying ideas (e.g. dragging actions, gestures), building on each other's ideas and distribution of tasks related to the exploration and the solution of the problems. Uses of the dragging tool had three functions: explorative, demonstrative and constructive. Children also used others tools such as the Size Markers and the grid. The implementation of DMGEs has critical implications for the learning of geometric concepts in both schools and informal settings such as afterschool programs.

Acknowledgments

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FROM PROBLEM SOLVING TO LEARNING THEORIES: UNPACKING A THREE-STAGE PROGRESSION OF UNIT AREA ITERATION

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Based on the findings of a larger study on middle school students' problem solving behaviors, we identified three sub-components for understanding the iteration of unit area in this paper: the mechanism of iteration, an identical unit, and the specificity of the identical unit. A three-stage progression on the acquisition of these components is elaborated through the analysis of the students' work.

Keywords: Learning Trajectories (or Progressions), Geometry and Geometrical and Spatial Thinking, Problem Solving

Background

The research community's knowledge on how individuals learn/understand mathematical concepts serves as an important resource when investigating problem solving processes. The findings of mathematical problem solving studies, in return, could serve as a foundational resource for developing learning theories. In our previous study on middle school students' problem solving behaviors, which was designed to reveal students' ways of knowing and thinking by unpacking the relationship among mathematical concepts, cognitive behaviors, and metacognitive behaviors (Zhang, 2010), we proposed a concept development framework for areas based on Vygotsky's (1962) concept formation theory, Berger's (2004) appropriation theory, and Battista's (2012) Cognition-Based Assessment (CBA) levels. The framework served as a platform for designing and analyzing problem solving interviews. One of the findings of the study suggested that the concept development framework of areas could be further refined to characterize sub-components to be more explicit as a guideline for teaching and studying this particular concept. This paper elaborates on a three-stage progression of the iteration of unit area based on the findings.

Theoretical Framework

Vygotsky's concept formation theory and Berger's appropriation theory were used as the structure of the proposed framework, while Battista's CBA levels served as a key reference for the specific stages along with the corresponding cognitive behaviors in the framework.

Vygotsky's theory proposes a framework for an individual's concept development within a social environment, while Berger's theory proposes an interpretation of Vygotsky's theory in the domain of mathematics by adjusting certain stages. Both theories break down any concept development into three phases: heap, complex, and concept. In the heap phase, the learner associates a sign with another because of physical context or circumstance instead of any inherent or mathematical property of the signs. In the complex phase, objects are united in an individual's mind not only by his or her impressions, but also by concrete and factual bonds between them. In the concept phase, the bonds between objects are abstract and logical.

The formation stages for the target concept of study (the concept of area), guided by the two theories described above, are illustrated in Figure 1. The developmental stages were refined based on the pretest responses from 44 middle school students and guided the selection of the interview tasks as well as the analysis of the relationship among mathematical concepts, cognitive behaviors, and metacognitive behaviors emerged from the interview results.
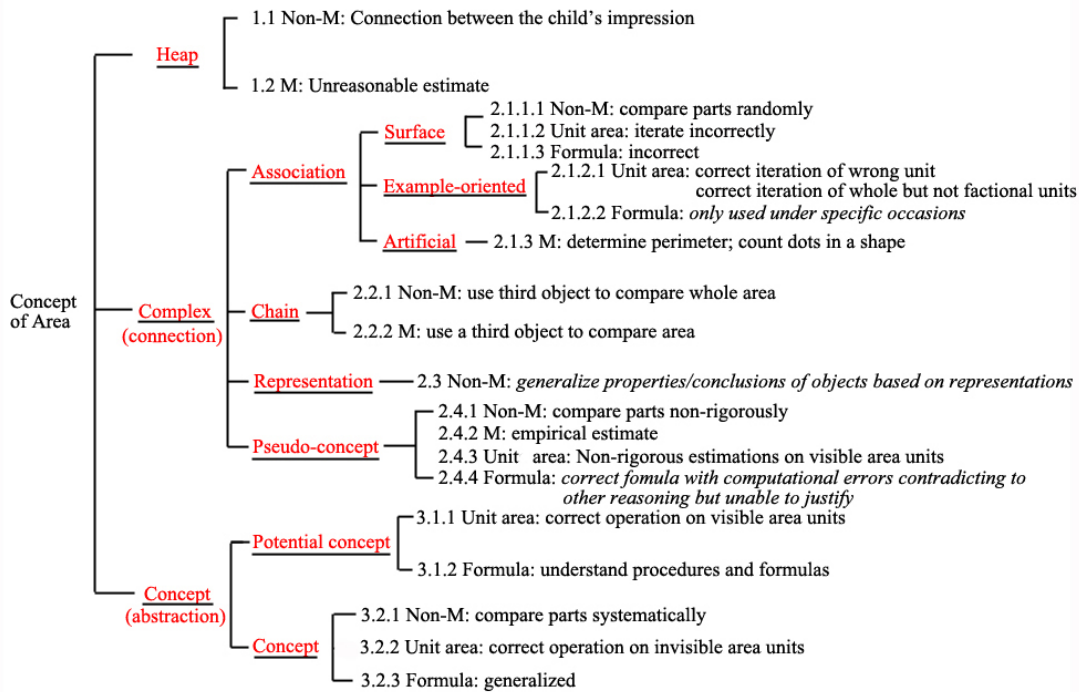
Figure 1. Developmental stages of the concept of area.

In the concept framework, there are three major components under the concept of area: non-measurement reasoning, unit area, and formula; this paper focuses on the unit area component. The specific stages involved in the progression of unit area iteration include: 2.1.1.2. Surface Association Complex – Unit area (iterate incorrectly), 2.1.2.1. Example-oriented Association Complex – Unit area (correct iteration of wrong unit and correct iteration of whole but not fractional units), 3.1.1. Potential concept – Unit area (correct operation on visible area units), and 3.2.2. Concept – Unit area (correct operation on invisible area units).

Methods

Participants

Five individuals from a population of 44 sixth grade students were selected to participate in interviews. A pretest was administered to the 44 students and each individual's developmental status revealed in the responses was categorized as "overall low" (all responses were rated as Heap and non-Pseudo-concept Complex stages), "varied" (responses were rated across Heap to Concept stages), and "overall high" (responses were rated as Pseudo-concept Complex and Concept stages). Among the five participants, one exhibited a low status, one exhibited a high status, and three exhibited varied statuses.
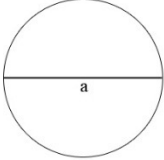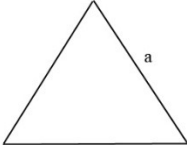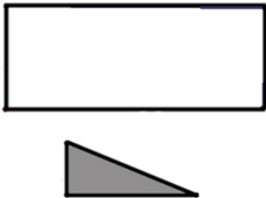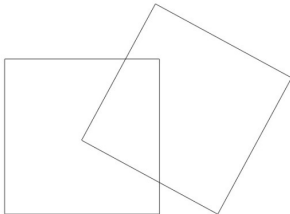
Instrument and Data collection

The five participants were interviewed individually. Each interview consisted of two parts. During the background interview part, the participants' mathematics background information, their beliefs about mathematics, and their views on the value of mathematics for their lives were elicited. During the second part, problem solving interviews, the participants worked on specific mathematical tasks, while interviewer interventions were limited to eliciting clarifications, explanations, or justifications when needed.

Five problems were used during the interviews. All problems were related to the concept of area and allowed the participants to tackle the tasks from different stages of concept development. The problems were designed to potentially cover a wide range of concept stages. The process and rationale of instrument design was reported in a previous paper (Zhang, Manouchehri, & Tague, 2015).

The students' working examples referred in this paper are from three out of the five interview questions, which are illustrated in Table 1.

Table 1: Interview problems

| **Compare areas problem** |
| Which of the regions shown below has the largest area? How would you order them? |

| **Shaded Triangle problem** |
| How many of the shaded triangles shown below are needed to exactly cover the surface of the rectangle? Please explain your answer. |

| **Intersected Area problem** |
| Two squares, each *s* on a side, are placed such that the corner on one square lies on the center of the other. Describe, in terms of *s*, the range of possible areas representing the intersections of the two squares. |

In the Shaded triangle problem, the measurement of the rectangle and the triangle was deliberately removed to test how the participants may solve the problem under this condition. The participants had been expected to determine or question the relationship (2:1 ratio) between the measures of the rectangle and the triangle.

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Data analysis

Data analysis consisted of three aspects: concept stages, cognitive behaviors, and metacognitive behaviors. First, each participant's key cognitive behaviors during each problem solving episode were documented. Second, a summary of observed concept stages and metacognitive behaviors during the episode were catalogued and noted. Finally, a cross analysis of the observed concept stages, metacognitive behaviors, and the relationship between them concluded the analysis phase. This process was followed for each of the five tasks used.

Results

One of the findings indicated that the progression of the iteration of unit area included three components: *the mechanism of iteration, an identical unit,* and *the specificity of the identical unit.* The data suggested a specific order of stages for the acquisition of these components.

Stage 1 – understanding the mechanism of iteration

When an individual understands the *mechanism of iteration* yet is not able to visualize a given shape as the unit, s/he is at 2.1.1.2. Surface Association Complex – Unit area stage and would iterate different units (e.g. triangles of different shapes and sizes).

The findings suggested that an individual could have different visualization abilities for different shapes. For example, when solving the Shaded Triangle problem, a participant who was not able to visualize the given triangle as a unit, iterated 10 triangles with varied shapes and sizes to cover the entire rectangle (Figure 2). Being asked whether she could use a different approach to solve the problem, she rotated the triangle and formed a small rectangle as a unit, then she was able to correctly iterate the small rectangle as a unit to cover the entire rectangle (Figure 3).
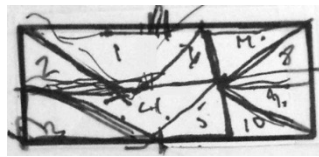
Figure 2. Incorrect visualization of a triangle as a unit.
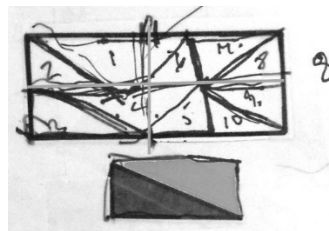
Figure 3. Correct visualization of a rectangle as a unit.

Stage 2 – understanding the mechanism of iteration and an identical unit

When an individual understands the *mechanism of iteration* and the *identical unit*, s/he is at 2.1.2.1. Example-oriented Association Complex – Unit area stage and could correctly iterate an identical (but not the given) unit.

When solving the Shaded Triangle problem, a participant who correctly iterated eight triangles in the rectangle (Figure 4) claimed that one could iterate differently as long as the total number of triangles was eight (as in Figure 5).

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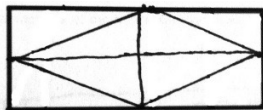
Figure 4. An individual's correct visualization of unit rectangles.

Figure 5. The same individual's Example-oriented Association Complex – Unit area reasoning.

This participant was not aware of the specificity of the identical unit, but he was able to justify the validity of the answer "eight triangles" and abandoned the numerical answer obtained by the correct formula which was 7.57 (the number was incorrect due to inaccurate measures). His Example-oriented Association Complex – Unit area level of understanding was not revealed until the end of the interview. It was triggered by the prompt of "can there be two answers to a problem?" which was intended to elicit his evaluation on the two different answers obtained from visual and formulaic approaches.

Stage 3 – understanding the mechanism of iteration, an identical unit, and specificity of the identical unit

When an individual understands the *mechanism of iteration*, an *identical unit*, and the *specificity of the identical unit*, s/he is at 3.1.1. Potential Concept – Unit area stage or 3.2.2. Concept – Unit area stage.

At the Potential Concept – Unit area stage, an individual is not required to (flexibly) define the area unit prior to the processes of decomposing (e.g. assigning proportional relation between partial squares and whole squares) and reconstructing (e.g. converting partial squares to whole squares) but relying on standard visible area units. For example, when solving the Compare Areas problem, a participant drew unit squares on the circle to find its area (Figure 6) since he forgot the area formula for circles.
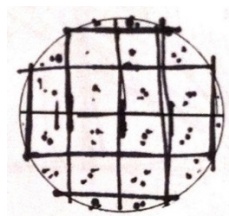
Figure 6. A Potential Concept – Unit area reasoning.

While at the Concept – Unit area stage, an individual takes full control over the area units in terms of their shapes, sizes, orientations, and other properties during the restructuring. For example, when solving the Intersected Area problem, the participant defined the overlapping area (a quadrilateral) as her unit, iterated it three times to cover the entire square (Figure 7), and reached the conclusion that the intersected area is a quarter of the whole square.

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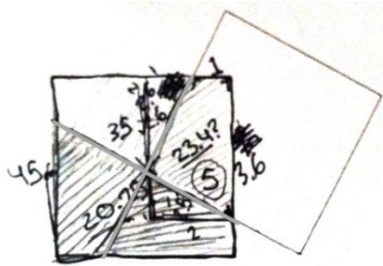
Figure 7. A Concept – Unit area reasoning.

Conclusion

Among the three components under the concept of area (non-measurement reasoning, unit area, and formula), we identified three sub-components for understanding the iteration of unit area: the mechanism of iteration, an identical unit, and the specificity of the identical unit. A specific order of the acquisition of these components was outlined through the analysis of the participants' work as the three stages in the previous section. Each stage corresponds to one or two stages in the concept framework.

Comparing to the four stages in the concept framework (Figure 1) and the progression of area-unit iteration outlined in CBA levels (Table 2), the proposed three-stage progression provides a more explicit conceptual breakdown of the iteration of unit area. In this way, more accurate assessments of students' knowledge can be developed and used, both in research and in the classroom.

Table 2: CBA levels for unit area iteration

Level	Sub-level	Description
M0		Student uses numbers in ways unconnected to appropriate area-unit iteration.
M1		Student incorrectly iterates area-units.
	M1.1	Student iterates single area units incorrectly.
	M1.2	Student decomposes shapes into parts incorrectly.
	M1.3	Student iterates area-units incorrectly, but eliminates double-counting errors.
M2		Student correctly iterates all area units one by one.
	M2.1	Student correctly iterates whole units, but not fractional units.
	M2.2	Student correctly iterates whole units and simple fractional units.
M3		Student correctly operates on composites of visible area-units.

Both concept development theories and learning progressions emphasize the non-hierarchical nature of their stages and levels, i.e., individuals may jump around the stages/levels without

following a specific order (Smith et al., 2006). Whether the three sub-components can be acquired in different orders remains to be examined by future studies.

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TEACHERS' RECOGNITION OF THE DIAGRAMMATIC REGISTER AND ITS RELATIONSHIP WITH THEIR MATHEMATICAL KNOWLEDGE FOR TEACHING

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We examine responses from a national sample of high school mathematics teachers to a questionnaire, which had been developed to study teachers' recognition of a system of hypothesized norms that stipulate that geometry proof problems are to be posed using a diagrammatic register. We report on the psychometric properties of the questionnaire, as well as the relationship between these mathematics teachers' mathematical knowledge for teaching geometry (MKT-G) and their stances on breaching those norms. Although Herbst et al. (2013) hypothesized that the system consisted of five distinguishable sub-norms, the factor structure of the questionnaire suggested that two of those norms might not truly be distinguishable. We also found a positive and significant relationship between teachers' MKT-G and their stances on breaching two of the determined components of the system of norms.

Keywords: Geometry and Geometrical and Spatial Thinking, High School Education, Reasoning and Proof, Measurement

Behavior in social situations involves participants negotiating their way around norms-- ways of behaving in a given social situation that are routine and tacitly expected by those familiar with that situation. Norms are unremarkable when complied with, but elicit comment when breached (Garfinkel, 1963). Our group has been working on bringing these ideas to scale by using visual representations of instructional situations in the context of online questionnaires. This paper reports on a piece of such work, following earlier investigations of the diagrammatic register in geometry proofs (Herbst, Kosko, & Dimmel, 2013).

The present study tests the psychometric properties of Herbst et al.'s (2013) DRN (a.k.a., N3) instrument. We conjectured that teachers with higher MKT-Geometry scores (see Herbst & Kosko, 2014) might be more likely to strategically breach a norm of the diagrammatic register. We examined covariation between the DRN data and data collected with the MKT-G instrument. This work is significant as any efforts to improve mathematical practices in classrooms must come to terms with the norms that undergird ordinary practice (Cobb, Zhao, & Dean, 2009) and because the results suggest that increasing mathematical knowledge for teaching geometry might provide resources for teachers to depart from the norms of ordinary practice.

Theoretical Framework

Our work builds theory to describe the work of teaching. We assume that describing the work of teaching mathematics requires attention to the specificity of the subject matter transactions between students and teacher. To operationalize such specificity we model the transactions between particular items of subject matter knowledge and the tasks in which students can lay claim to them. *Instructional situations* (e.g., doing proofs, solving equations; Herbst, 2006) are modeled by articulating sets of norms that describe what teacher and students are expected to do in those tasks and what knowledge and skills the accomplishment of the task counts toward. The diagrammatic register norm states that proof problems are presented in a diagrammatic register. Herbst et al. (2013) decomposed it into five sub-norms: (SN1) Properties: The statement of the problem does not make explicit properties of betweenness, intersection, separation, collinearity, or concurrency, which are left for the diagram to communicate; (SN2) Diagram: The teacher provides a diagram for students to use while doing the proof; (SN3) Labels: The teacher assigns a proof problem with an

accompanying diagram where all the points needed in the proof are labeled (but not necessarily all points); (SN4) Statement: The proof problem is stated using symbols and labels for elements of a diagram; (SN5) Accuracy: When a teacher provides a diagram accompanying a proof problem, the diagram is accurate.

These norms describe the defaults for setting up problems in the situation of doing proofs, they do not necessarily describe what would be optimal for student learning. Arguably, it would enrich the mathematical experience of students if they could be expected to do themselves what ordinarily would be done by the teacher. Various questions can be asked that contribute to this theory: To what extent do teachers consider adherence to the various aspects of this norm to be appropriate (in comparison with alternatives that might also be compelling)? And: To what extent is mathematical knowledge for teaching geometry related to a teacher's disposition to deviate from these norms?

Methods

Data

As described by Herbst et al. (2013), the DRN instrument (a.k.a., the N3 instrument) contains 30 items that target the 5 sub-constructs of the diagrammatic register norm. Each item asks participants to compare the appropriateness (using a 6-point Likert-like closed response format) of two possible ways of setting up a proof problem: one that we conjecture to be normative and another one that breaches one of the sub-norms (but is otherwise normative). Eight items were designed to measure SN1, five to measure SN2, five to measure SN3, five to measure SN4, and seven to measure SN5. The MKT-G instrument has been described at length by Herbst and Kosko (2014).

The DRN and MKT instruments, as well as a background survey that included a question about participants' years of experience teaching geometry, were administered as part of a larger study to a nationally-distributed sample of high school mathematics teachers, using the Lesson*Sketch* online platform. Participants were randomly sampled from more than 10,000 public secondary schools in the United States, identified using the NCES 2012-2013 School Universe data set. The effective sample of those who completed the background survey, the DRN, and the MKT-G instruments was 300 high school mathematics teachers. The minimum number of years of experience teaching high school geometry that any teacher in the sample had was 1, the maximum was 35, the mean was 6.75, and the standard deviation was 5.96.

Measures

Diagrammatic register norm endorsement. To determine whether the items that are in the DRN instrument measured five distinguishable sub-norms, we conducted an exploratory factor analysis (EFA), after recoding certain items so that high values in the scale of each item indicated departures from the norm. We split the sample into two random sub-samples, then conducted an EFA using the first and a CFA using the second (for more information on this approach, see Duffy, et al., 2012). Once we had determined the factor structure of the items, we created factor scores by taking the mean of each participant's rating of the items that loaded onto each factor. We also created a DRN total score by taking the mean of a participant's ratings of all 30 items. Finally, we calculated alpha scores and average inter-item correlations (IIC) to determine the internal consistency of the set of items in each factor.

Mathematical knowledge for teaching geometry (MKT-G) scores. We used a two-parameter Item Response Theory (IRT) model to create MKT-G scores, after removing four items that the Item Characteristic Curves (ICC) suggested would not discriminate well between individuals. The minimum MKT IRT score was -2.20, the maximum was 2.17, the mean was -0.0000043, and the standard deviation was 0.902.

Analysis

An Ordinary Least Squares (OLS) regression model was then created, in which the DRN total score was regressed on the MKT IRT score and participants' years of experience teaching geometry. After finding that the relationship between the MKT-G score and DRN total score was statistically significant, even when controlling for years of experience teaching geometry, we decided to create four other similar stepwise OLS regression models, each of which used one of the DRN factor scores, rather than DRN total score, as the outcome variable. This was done in order to understand whether participants' stance on breaching the DRN was dependent on which of the sub-norms was breached.

Results

In terms of the EFA, we considered both the criterion of retaining factors with eigenvalues larger than 1 (Kaiser, 1960) and the criterion of retaining components above the point of inflection on a scree plot (Cattell, 1966), as well as factor loadings and fit statistics. Together, these suggested that four factors undergird the DRN instrument. The CFA confirmed that structure to the extent that no standardized item loading was less than 0.3 in the CFA and the fit statistics were reasonable - RMSEA: 0.066, CFI=0.853, TLI=0.841, SRMR=0.091, just short of the typical cut-points of RMSEA<=0.05, CFI>=0.95, TLI>=0.95 and SRMR<=0.6 (Hu and Bentler, 1999). According to that model, the items that were designed to target SN1, SN3, and SN5 loaded onto three factors, in the way that they were expected to. However, the items designed to target SN2 and SN4 loaded onto the same factor. We will hereafter refer to those factors as S1:PRO, S3:LAB, S5:ACC, and S2S4:DNS, respectively.

The means of the DRN total score, SI:PRO, S2S2:DNS, S3: LAB, and S5:ACC scores (described earlier) were 2.86, 3.32, 2.56, 3.38, and 2.42, respectively. The standard deviations of those scores, in the same order, were 0.54, 0.94, 0.89, 0.56, and 0.78. Cronbach's alpha for the DRN total score and each of the factor scores ranged from 0.6909 and 0.8522, and their average inter-item correlations (IIC) ranged from 0.1723 to 0.4344, both of which suggest that the entire set of items as well as the set of items that loaded onto each factor had good internal consistency (Clark & Watson, 1995).

The main take-away from the regression models is that there is a significant, positive association between teachers' comfort with breaches of the DRN and their MKT-G, which seemed to be due to the also significant and positive association between their MKT-G and comfort with breaches of the S2S4:DNS and S3:LAB components of that norm, independent of their years of experience teaching geometry. When the DRN total score was regressed on MKT and years of experience teaching geometry, the MKT regression coefficient and associated standard error were 0.14 and 0.04. When the S2S4:DNS score was used instead they were 0.36 and 0.06. When S3:LAB score was used, they were 0.08 and 0.04.

Discussion

A main finding of this study is the discovery of the somewhat unexpected, but nonetheless comprehensible, factor structure of the DRN instrument. Another is the discovery of relationship between teachers' endorsement of the diagrammatic register norm (as well as two of its subnorms) and their level of MKT. Upon reflection, we imagine that SN2 and SN4 items loaded onto the same factor because all of our proof problems that complied with SN4 (the expectation that the proof problem will be stated using symbols and labels for elements of a diagram) included a diagram and, therefore, complied with SN2 (the expectation that the teacher will provide a diagram for students to use while doing the proof).

In terms of MKT being a significant predictor of S2S4:DNS and S3:LAB, but not S1:PRO or S5:ACC, we would argue that understanding this result also requires careful consideration of each of the 5 subnorms. For example, we would expect that the Accuracy subnorm (SN5) was fairly easy to

recognize, regardless of one's MKT, as there is a sense among teachers that even accurate diagrams can be misleading, and so providing inaccurate diagrams could make something that is to be regarded with suspicion even more problematic. SN1 (properties) deals with subtle issues of positioning. On the other hand, SN2 and SN4 are arguably more directly related to the kind of mathematical knowledge teachers readily have.

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LONGITUDINAL PREDICTIONS OF SIXTH-GRADE GEOMETRY KNOWLEDGE

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In the current study, we examined the contributions of shape and pattern knowledge at four early time points to formal geometry knowledge in sixth grade within a longitudinal sample of over 500 low-income children. Shape knowledge at the beginning of pre-k predicted geometry in sixth grade, over and above general math and cognitive skills. However, at the end of pre-k, kindergarten and first grade, pattern knowledge was a unique predictor of later geometry, and shape knowledge was not. Results highlight the important roles of two important, but understudied components of math knowledge – early shape and pattern skills – and their contributions to the development of formal geometry knowledge.

Keywords: Geometry and Geometrical and Spatial Thinking, Learning Trajectories, Standards

Introduction

Mathematics knowledge begins to develop at a young age, and this early math knowledge matters. General math skill in pre-k and kindergarten predicts math achievement across primary and secondary school (Watts, Duncan, Siegler, & Davis-Kean, 2014). Research is now needed to identify *specific* early math skills that help predict and improve later achievement in specific math areas. Many theories of mathematics focus on the development of numeracy knowledge, that is, skills necessary for understanding numbers and number relations (see Sarama & Clements, 2004). However, math knowledge extends beyond numeracy knowledge. The goal of this research is to focus on two important, but understudied, components of math – early shape and pattern knowledge – and test their contributions to formal geometry achievement in middle school.

Exploring shapes and patterns is a common mathematical activity for young children (Ginsburg, Lin, Ness, & Seo, 2003) and may be an important contributor to general math development. Knowledge of shapes and their properties is considered foundational to later geometric thinking (National Research Council, 2009) and is included in the Common Core State Standards for Mathematics as early as kindergarten (CCSSM; National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA & CCSSO], 2010). Children first learn to classify typical shapes and then to describe the definitional features of both two- and three-dimensional shapes. According to the learning trajectory theory, these shape skills form the building blocks for later geometry achievement (Clements, Wilson, & Sarama, 2004). However, no evidence to date links early shape knowledge to later math outcomes, including knowledge of geometry.

Pattern knowledge includes the ability to identify, extend, and describe predictable sequences in objects or numbers, and it has been recognized by math education researchers as a core skill for mathematical thinking (Papic, Mulligan, & Mitchelmore, 2011; Warren & Cooper, 2007). The first type of pattern young children engage with are repeating patterns, such as the colors red-blue-red-blue-red-blue. Children's knowledge of repeating patterns becomes systematically more sophisticated from pre-k to kindergarten (Rittle-Johnson, Fyfe, Loehr, & Miller, 2015), and several school-based interventions have shown that instruction on repeating patterns supports general math achievement at the end of the school year (e.g., Kidd et al., 2014). However, patterns are not included in the Common Core State Standards at any grade level, and thus receive little attention.

Method

Participants were drawn from a longitudinal study. The sample included 513 low-income children originally recruited at the beginning of their pre-kindergarten year in the U.S. (56% female,

80% black). Children were initially assessed at four early time points: beginning of pre-k (*M* age = 4.4 years), end of pre-k (*M* age = 5.0 years), end of kindergarten (*M* age = 6.1 years), and end of first grade (*M* age = 7.0 years). These children were re-assessed five years later when most students were near the end of sixth grade (*M* age = 12.1 years, 17% had been retained a grade and were in fifth grade). Students were distributed across 51 middle schools.

The outcome measure of interest (administered in sixth grade) was the *Geometry* subtest from the KeyMath 3 Diagnostic Assessment (Connolly, 2007), a standardized math test. The geometry subtest measures a student's spatial reasoning as well as his ability to analyze, describe, compare, and classify two- and three-dimensional shapes.

For early math predictors (administered in pre-k, kindergarten, and first grade), we assessed children's early *shape and pattern knowledge*, which were measured using items from the Research-based Early Math Assessment (Clements, Sarama, & Liu, 2008). Table 1 provides example items. Shape items (*n* = 14 – 23 depending on time point) focused on identifying, creating, and defining shapes. Pattern items (*n* = 4 – 7) focused on copying, extending, or identifying patterns made out of colored shapes or cubes. We also assessed children's general math achievement using the *quantitative concepts* and *applied problems* subtests from the Woodcock Johnson Achievement Battery III (Woodcock, McGrew, & Mather, 2001). Quantitative concepts assesses the knowledge of basic math concepts, symbols, and vocabulary. Applied problems assesses the ability to analyze and solve various math problems.

Table 1: Example items from the early shape and pattern knowledge subscales

Knowledge Subscale	REMA Item #	Item Description
Shape Knowledge	N05	Given a mat with 26 different shapes on it, the child is asked, "Can you point to all the triangles?"
	NG6	"Is this shape a square? How do you know?"
Pattern Knowledge	G04	The child is shown an ABA_AB pattern and asked, "Can you find the missing piece in this pattern?"
	G30	The child is shown an ABBABB shape pattern. "Make the same kind of pattern here, using these blocks."

We also assessed four non-math predictors to control for general cognitive and academic skills. These included a measure of early *reading* skill (The Woodcock Johnson Letter-Word Identification), a measure of *narrative recall* that varied by time point (the Renfrew Bus Story or the Woodcock Johnson Story Recall), and teacher ratings of *work-related skills* (Cooper-Farran Behavioral Rating Scale;), and *self-regulation* (using the Instrumental Competence Scale).

Results

Reflective of the disadvantaged nature of the sample, age- and grade-equivalent scores on the KeyMath geometry assessment indicated that students were approximately two years behind in geometry. Recall, the assessment was administered near the end of sixth grade when students were an average of 12.1 years old. However, the average grade-equivalent score in this sample was 4.8 (*SD* = 2.1) and the average age-equivalent score was 9.5 (*SD* = 2.0).

In the early years, children's shape and pattern knowledge increased from pre-k to first grade. For example, at the beginning of pre-k, children solved an average of 3.3 shape items and 0.6 pattern items correctly. By the end of first grade, children solved an average of 9.4 shape items and 3.7 pattern items correctly. At each of the four early time points, scores on the shape and pattern

subscales were moderately correlated with geometry knowledge in sixth grade (for shape scores, rs = .34 – .45, ps < .001; for pattern scores, rs = .25 – .49, ps < .001).

The primary goal was to test whether shape and pattern knowledge at early time points predicted formal geometry knowledge in sixth grade, after controlling for other general math and non-math skills. We ran multi-level regression models at each early time point with students nested in their sixth-grade schools. The results are presented in Table 2.

Table 2: Longitudinal predictions of geometry knowledge in sixth grade

	Predicting Geometry in Sixth Grade From Four Early Time Points			
	Beginning of Pre-K	**End of Pre-K**	**End of Kindergarten**	**End of First Grade**
Math Predictors				
Shape Knowledge	.16 (.05)**	-.02 (.05)	-.03 (.04)	.06 (.05)
Pattern Knowledge	-.01 (.05)	.25 (.05)***	.14 (.04)***	.11 (.05)*
Quantitative Concepts	.27 (.05)***	.30 (.06)***	.20 (.05)***	.18 (.05)***
Applied Problems	.16 (.05)**	.13 (.05)*	.24 (.04)***	.23 (.05)***
Non-Math Predictors				
Reading	-.08 (.05)	-.05 (.05)	.09 (.04)*	.09 (.05)
Narrative Recall	.04 (.05)	.17 (.04)***	.10 (.03)**	.09 (.04)*
Work-Related Skills	.02 (.08)	.03 (.07)	.05 (.06)	.21 (.06)**
Self-Regulation	.05 (.07)	-.06 (.07)	.11 (.07)	-.03 (.06)
Control Variables	Included	Included	Included	Included

Note. All models include these control variables: age in sixth grade, current grade level, gender, ELL status in pre-k, ethnicity, pre-k school type (public or Head Start), and socio-economic status (respondent's education level and income level). *p < .05. **p < .01. ***p < .001.

First, we showed that past findings on the importance of early math to later achievement generalize to later geometry. Specifically, across all early time points, children's scores on the quantitative concepts and applied problems tests predicted their sixth-grade geometry knowledge (βs = .13 – .30). Second, the contributions of general math knowledge to sixth-grade geometry were often substantially stronger than the contributions of general cognitive/academic skills, including reading, narrative recall, work-related skills, and self-regulation.

Third, the contributions of early shape and pattern knowledge differed. At the beginning of pre-k, shape knowledge was a significant predictor of sixth-grade geometry knowledge (β = .16, SE = .05, p < .05). However, at the three remaining time points (end of pre-k, end of kindergarten, and end of first grade), pattern knowledge was a significant predictor (βs = .11 – .25), and shape knowledge was not. For example, at the end of pre-k, a one standard deviation increase in pattern knowledge was associated with a quarter of a standard deviation increase in sixth-grade geometry knowledge (β = .25, SE = .05, p < .05), over and above controls. As a note, if we excluded patterning from the model, shape was still not a significant predictor at the end of pre-k or kindergarten, but it was a significant predictor in first grade (β = .13, SE = .04, p < .05).

Conclusion

We evaluated the role of two non-numeracy math skills – shape and pattern knowledge – in the development of geometry achievement. We found that shape knowledge was only predictive prior to formal schooling (at the beginning of pre-k). However, early pattern knowledge from the end of pre-k to first grade consistently predicted middle school geometry knowledge over and above

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general math and cognitive skills. These results are consistent with recent research foregrounding the importance of early patterning skills, including intervention work that found a preschool patterning intervention led to greater numeracy knowledge at the end of kindergarten than typical preschool instruction (Papic et al., 2011).

These findings provide some limited support for the learning trajectory theory that suggests early shape knowledge is foundational to later geometry knowledge (Clements et al., 2004). However, they also suggest a more prominent role for patterning. Indeed, the current Common Core State Standards include shape knowledge as a key component as early as kindergarten, yet fail to include pattern knowledge at any grade level. Contrary to these recommendations, the current results suggest that math standards should include repeating pattern knowledge in kindergarten and first grade (i.e., copying, extending, and identifying predictable sequences), and that more research is needed on the importance of shape knowledge.

Acknowledgments

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COMMON CORE GEOMETRY TEXTBOOKS: OPPORTUNITIES FOR REASONING AND PROVING

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Reform efforts in the U.S. have placed an emphasis on reasoning and proving. Yet, previous geometry textbook analyses have found limited opportunities for reasoning and proving and lack of opportunities to engage students in high levels of rigor. This analysis investigated how purported Common Core aligned textbooks address standards about proving geometric theorems. Like previous studies conducted with pre-Common Core textbooks, the findings for opportunities to construct proofs and engage students in high levels of rigor were less than expected. These findings suggest that publishers and practitioners need to ensure that students are given adequate opportunities to construct proofs and gain conceptual understanding.

Keywords: Curriculum Analysis, Reasoning and Proof, High School Education, Standards

Engaging in proving is essential to the practice of mathematicians (Herbst & Brach, 2006; Otten, Gilbertson, Males, Clark, 2014), and mathematics educators have suggested that reasoning and proof should play a role throughout all of mathematics curriculum (Herbst & Brach, 2006; Thompson, Senk, & Johnson, 2012; Sears & Chávez, 2014). Despite these recommendations, proof is predominately found in high school geometry courses and textbooks (Otten et.al., 2014). Teachers observed using conventional geometry textbooks, "adhered to the pedagogical suggestions and implemented proof tasks as presented in the teacher's edition of the textbook" (Sears & Chávez, 2014, p. 774). Yet some teachers have claimed that geometry textbooks may impede the instruction of proofs (Cirillo, 2009). This is important because some research has suggested that textbooks may have more impact in mathematics classrooms than in other content areas (Remillard, 2005). Thus analysis of high school geometry textbooks may provide insight to the extent proof is included within secondary classrooms.

Commenting across a set of studies, Cai and Cirillo (2014) noted that mathematics textbooks are lacking in opportunities for students to engage in reasoning and proving. In an analysis of six U.S. secondary geometry textbooks, less than 10% of the student exercises required students to construct proofs, and most were about "particular mathematical statements rather than general statements" (Otten, et.al., 2014, p.71). In two conventional high school geometry textbooks, Sears and Chávez (2014) found that 128 out of 977 and 79 out of 1066 tasks were proof tasks; of the proof tasks 15% and 13% respectively required "higher level demands (doing mathematics)" (p. 774). These findings are troubling given the expectation for students to learn proof in geometry. The Common Core State Standards for Mathematics (CCSSM) (National Governors Association Center for Best Practices & Council of Chief State School Officers [NGAC & CCSSO], 2010) sustain the emphasis on proof, and require higher cognitive reasoning skills (Porter, McMaken, Hwang, & Yang, 2011). Here the analysis sought to answer the following research questions: 1. To what extent do purported CCSSM-aligned geometry textbooks provide reasoning and proving opportunities as suggested by CCSSM geometry standards about "proving geometric theorems?" 2. To what extent do CCSSM-aligned geometry textbooks provide opportunities for high levels of rigor as required by CCSSM shifts?

Method

Two textbooks were analyzed for this study. Engage New York Geometry (Engage NY) was developed by educators in New York State after the CCSSM were developed. The Pearson Common Core Geometry Edition (Pearson) is a conventional textbook which was adapted to the CCSSM.

These two books were chosen due to limited curriculum analysis on CCSSM textbooks—Engage NY has not been included in the above studies. Additionally, both of these textbooks have the potential to reach a wide audience throughout the United States.

Eleven lessons from Engage NY and 15 lessons from Pearson were coded for topics in CCSSM about proving, reasoning and proving, and level of rigor. These lessons were selected based on the publishers' assignment of the standards "Prove Geometric Theorems": HSG.CO.C.9-11(NGAC & CCSSO, 2010) to particular sections. These particular standards require students to prove theorems about lines, angles, triangles, and parallelograms. Focusing on lessons that address standards about proving may provide insight into how CCSSM textbooks are providing activities that engage students in proof construction. Specifically, items within the exposition and student exercises were analyzed—including items intended to be read or completed by the student within sections (lessons) of a chapter. Chapter introductions, mid-chapter reviews, end of chapter reviews, and assessments were excluded.

Analytical Framework

The analytic framework had three dimensions. The methodology and framework was inspired by the Surveys of Enacted Curriculum (SEC) Methodology in which curricula is measured for CCSSM alignment based on content match, expectations for student performance, and instructional content (Martone & Sireci, 2009).

The first dimension of the framework determined how much of the text directly addressed content in Common Core Standards HSG.CO.C.9-11 by identifying items that may give opportunities to engage in activities related to lines and angles, triangles, and parallelograms.

The second dimension adapted from Otten et.al. (2014) investigated ways in which the standards items may provide opportunities for reasoning and proving, hereby denoted as the hyphenated reasoning-and-proving. Types of statements—general, particular, or general with particular instantiation, were analyzed. The exposition was analyzed for types of justifications—deductive arguments, examples, or left to the students. The student exercises were analyzed for types of reasoning-and-proving activities such as developing a proof, developing non-proof arguments, determining truth values, and conjecturing.

Additionally, in order to account for recommendations of Common Core, the third dimension indicated levels of rigor. The development of this dimension drew from three sources: Educators Evaluating the Quality of Instructional Products (About EQuIP, 2012) rubric, SEC criteria for cognitive demand (SEC, 2015), and Smith and Stein (1998) levels of cognitive demand.

<div align="center">Results</div>

Dimension 1 Analysis: Addressing Topics in the Standards

Individual exercises may have received individual codes at the statements level. Thus most of the data analysis refers to "statements." As expected, the findings indicated that most student exercises addressed topics within the chosen CCSSM—99.5% in Engage NY and 94.5% in Pearson. In both textbooks most of the statements addressed topics in theorems about lines and angles or parallel lines and transversals and had less emphasis on theorems about triangles and parallelograms. Additionally, Pearson had several statements that addressed theorems about triangles and parallelograms that were not specified within the standards. The uneven distribution and focus of topics may indicate a lack of alignment to the standards.

Dimension 2 Analysis: Reasoning-and-Proving Opportunities

Given the focus of this analysis on lessons about proving geometric theorems, opportunities for engaging in reasoning-and-proving seemed relatively low-- 48.47 % and 48.8% of statements in Engage NY and Pearson, respectively, addressed reasoning-and-proving.

Reasoning-and-Proving in the Student Exercises. Four types of reasoning-and-proving activities were most common in both textbooks, construct a proof, develop a rationale, investigate a statement (truth of claim), and make a conjecture. However, the distribution of activities contrasted greatly within both textbooks. For example, the highest percentage, 23.39%, of the activities in Pearson focused on developing a rationale or non-proof arguments. Whereas Engage NY had the highest percentages in making conjectures (including filling in the blanks of conjectures), 21.74%, followed closely with 19.75% of the activities focused on constructing proofs. Table 1 indicates the most frequent statement and activity types in both textbooks.

Table 1: Reasoning-and-Proving in the Exercises--Types of RP Activities

Textbook	Statements	RP Statements	RP Statements (%)	Construct a Proof (%)	Develop a Rationale (%)	Investigate a Statement (%)	Make a Conjecture (%)
Engage NY	253	125	49.41	19.76	5.93	0.40	21.74
Pearson	838	392	46.78	6.09	23.39	7.40	5.73

Opportunities for Constructing Proofs. Despite expectations for these lessons to focus on proof, this analysis had similar findings as Otten et. al. (2014). This investigation found that most of the proofs in the student exercises required students to construct proofs about particular statements. Pearson's construct a proof statements consisted of only 6.09% total; 4.65% of those statements required students to construct proofs about particular statements. Engage NY provided more opportunities for students to construct proofs 19.76%, but 14.62% of them were about particular statements, rather than general statements.

Dimension 3 Analysis: Levels of Rigor

The third dimension identified which exercises and statements fell into four levels of rigor: memorization (first level), procedural skill and fluency (second level), conceptual understanding (third level), and application (fourth level). (See Table 2)

Table 2: Levels of Rigor—Percentage of Statements

Textbook	Memorization		Procedural Skill and Fluency		Conceptual Understanding		Application	
	All	Proof	All	Proof	All	Proof	All	Proof
Engage NY	15.65	0	42.75	10	37.40	86	1.91	6
Pearson	42.27	0	63.90	5.9	18.32	94.10	0.21	0

Most of the statements in both textbooks were determined to be procedural skill and fluency. Many of the exercises required computational procedures, following procedures, or completing problems that were similar to worked examples. The majority of statements in Engage NY received procedural skill and fluency and conceptual understanding codes—42.75% and 37.40% respectively. Pearson's statements were mainly assigned to memorization or procedural skill and fluency—42.27% and 63.90% respectively. Neither textbook emphasized the highest level, application activities—such as conjecturing and constructing proofs of general statements.

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The findings for reasoning-and-proving opportunities indicated that each lesson contained less than five problems that required students to engage in constructing proofs. This may be sufficient to develop deep understanding. However most proof writing problems were determined to provide conceptual understanding versus application level of rigor. Within construct a proof activities, 86% of Engage NY and 94.1% of Pearson's statements were assigned to conceptual understanding. Only 6% of Engage NY construct a proof statements were application level and none of Pearson's construct a proof statements.

Conclusion

The findings in this study indicate that CCSSM textbooks follow the trend found in earlier textbooks of limited reasoning-and-proving opportunities, and may not be providing a level of rigor that will develop deep understanding. Curriculum developers need to consider several elements when writing standards based curriculum. Simply adapting textbooks to new standards does not ensure standard alignment. School leaders and teachers may need to supplement CCSSM textbooks with tasks that give students more reasoning-and-proving opportunities at high levels of rigor. If curriculum developers and textbook companies do not adhere to recommendations of standards, reform efforts may not have their intended impact.

Additionally, the analytical framework developed in this study contributes to consistency within curriculum analysis about reasoning-and-proving by adapting a previously used framework. The additional dimensions for standard alignment and level or rigor provide a holistic approach to analyzing Common Core textbooks—making this a valuable framework that may be applied to additional lessons and Common Core geometry textbooks. That is adapting dimension one would allow for investigations of other standards about proving. Research about enacted curriculum in classrooms using these textbooks could investigate the impact that reform efforts are having within the classroom (Remillard, 2005).

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JUST GO STRAIGHT: REASONING WITHIN SPATIAL FRAMES OF REFERENCE

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In this report, I present differences in two ninth grade students' reasoning within frames of reference in spatial contexts and their coordination of measurements in organizing two- or three-dimensional space.

Keywords: Geometry and Geometrical and Spatial Thinking, Cognition, High School Education

Although coordinate systems are often considered representational tools for reasoning about quantities, little focus is given to students' construction of coordinate systems or their ways of reasoning within frames of reference. Joshua, Musgrave, Hatfield, and Thompson (2015) pointed to the lack of literature conceptualizing frame of reference and discussed the mental actions that are involved in coordinating or combining multiple frames of reference in terms of quantitative reasoning (Thompson, 2011). I distinguish between the construction of coordinate systems to fulfill either of two goals: 1) *spatial organization*, which is to re-present space by establishing a frame of reference and locating points within the space using coordinated measurements; or 2) *quantitative coordination*, which is to coordinate sets of quantities by establishing a frame of reference and obtaining a representational space using geometrical objects. This report presents data from a teaching experiment with ninth grade students to illustrate differences in students' reasoning within frames of reference in their spatial organization. More specifically, I discuss how the students' spatial frames of reference were coordinated to define directionality and used to coordinate measurements in organizing space.

Theoretical Orientation and Constructs

Finding it important to attend to how students construct their mathematical knowledge, I orient my work in modeling students' constructive activities with a focus on schemes and operations (von Glasersfeld, 1995). In discussing the construction of coordinate systems I use Piaget and Inhelder's (1967) distinction between perceptual space and representational space: Perceptual space is constructed in a figurative sense through sensorimotor and perceptual activity on elements of raw material; representational space is constructed through the interiorization of perceptual space entailing a symbolic function in which the individual could regulate spatial behavior in a systematic way. I also adopt Saldanha and Thompson's (1998) notion of a "*tight coupling*" of two quantities which enables one to track "either quantity's value with the immediate, explicit, and persistent realization that, at every moment, the other quantity also has a value" (p. 299); hence, resulting in a multiplicative object of two quantities co-varying.

Methodology

A two-year teaching experiment (Steffe & Thompson, 2000) was conducted at a rural high school in the southeastern US with two pairs of ninth grade students. In this report I discuss two students, Kaylee and Morgan, and their engagement in two tasks from this teaching experiment to present how they each constructed frames of reference and coordinated measurements in locating points or defining the motion of one point to another in two- or three-dimensional spaces. Kaylee and Morgan were selected and paired together based on their initial interviews, which I conducted to specify the levels of units they could coordinate and utilize in reasoning (Lee, 2015, April). The two tasks I discuss were developed to engage students in locating points in two- or three-dimensional space. In the North Pole Task, I asked the students to imagine being in a helicopter hovering over the North

Pole region holding a map showing the one road to the North Pole, the North Pole point (P), and a missing person's location (A) (see Figure 1-(a)). The students' goal was to provide instructions to a rescuer on the ground holding another map so that he could find point A in the region. In the Fish Tank Task, given a cylindrical fish tank, which consisted of fish models submerged in semi-transparent gelatin (see Figure 1-(b)), the students were asked to locate all four fish and to give instructions for Fish 1 to swim to Fish 2. The shapes of the map and tank were selected so that they would not suggest perceptual guidance in students' locating activities.

(a)   (b)
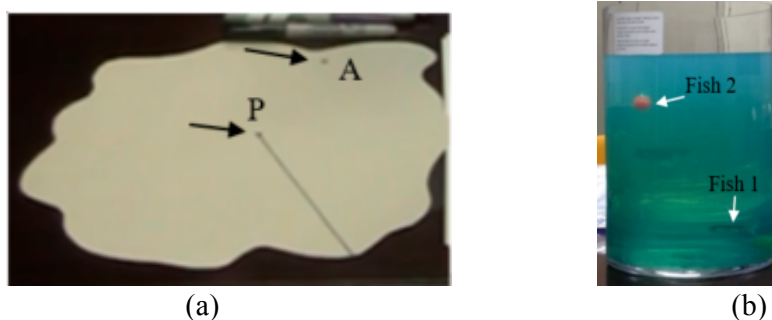
Figure 1. North Pole Task and Fish Tank Task Materials.

Video recordings and student work were collected from the teaching episodes, each lasting for 20–25 minutes. Both on-going and retrospective analyses were conducted (Steffe & Thompson, 2000) based on inferences from observable activities—visual illustrations, verbal descriptions, and physical gestures—the students carried out.

Analysis and Findings

North Pole Task: Locating a point in two-dimensional space

In the North Pole Task, the students demonstrated a difference in their coordination of perspectives and construction of frames of reference in two-dimensional space. Kaylee located point A in reference to point P by defining the direction of the rescuer's movement from P to A in two different ways. First, Kaylee established a frame of reference consisting of an initial ray anchored onto the rescuer's line of sight, a vertex at point P, and a terminal ray through point A. Second, Kaylee established a frame of reference consisting of horizontal and vertical lines with the intersection of the lines anchored at point P. Utilizing these frames of reference, Kaylee coordinated measurements of angle measure and distance to locate point A and to describe the rescuers motion to find the missing person. Kaylee consistently coordinated an environment-centered frame of reference taking her above-the-ground perspective and a viewer-centered frame of reference taking an imaginary on-the-ground rescuer's perspective (Carlson-Radvansky & Irwin, 1993) in developing instructions for the rescuer. Kaylee utilized decentering, unitizing, and coordinating perspectives which allowed her to anticipate results of movements of the rescuer even when the physical movements of the rescuer were not in her perceptual field.

On the other hand, Morgan tended to take a more temporal approach in that she wanted to give in-the-moment instructions to the rescuer or the missing person from the helicopter. Also, in developing instructions for the rescuer, her language was based on her perspective and did not account for the rescuer's line of sight. For example, after Kaylee shared her strategy in using a protractor to locate point A in reference to point P, I asked the students what they would do if they were only given rulers. Morgan connected points P and A with her ruler suggesting that she intended to tell the rescuer to go straight from P to A for a certain distance. Such tendencies highlighted Morgan's focus on attending to an environment-centered frame of reference (Carlson-Radvansky &

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Irwin, 1993) taking her above-the-ground perspective. As a result, her notion of going straight was not coordinated with the perspective of the rescuer. Moreover, her instructions would have required Morgan to make adjustments based on perceptual imagery of the rescuer's movement.

Fish Tank Task: Locating a point in three-dimensional space

    In contrast to the North Pole Task, Morgan demonstrated a coordination of different perspectives. As shown in Figure 2-(a), Morgan first identified the fish in vertical layers of the tank, taking the side view, and then superimposed a grid on the top of the tank, taking the top view of the tank. In thinking about Fish 1 swimming to Fish 2, both students agreed on coordinating these two perspectives. However, they came to a disagreement in how to coordinate them to describe the movement of Fish 1 to Fish 2. Morgan explained that "he [Fish 1] will go up two and then he'll have to go over however many this is [Figure 2-(a)]." Morgan elaborated that "[y]ou only need two straight lines" because "they're already in line. Then he could *just go straight to him*, instead of moving two times." In short, similar to how she wanted to connect point P and A with the ruler in the North Pole Task, Morgan wanted to tell Fish 1 to "go straight" to Fish 2 once they were on the same layer. On the other hand, using her diagram in Figure 2-(b), Kaylee explained that after going up two layers, "let's say they're on the top now. So, he has to go this way and then this way, that's three units, three different measurements."

(a) Morgan's diagram   (b) Kaylee's diagram
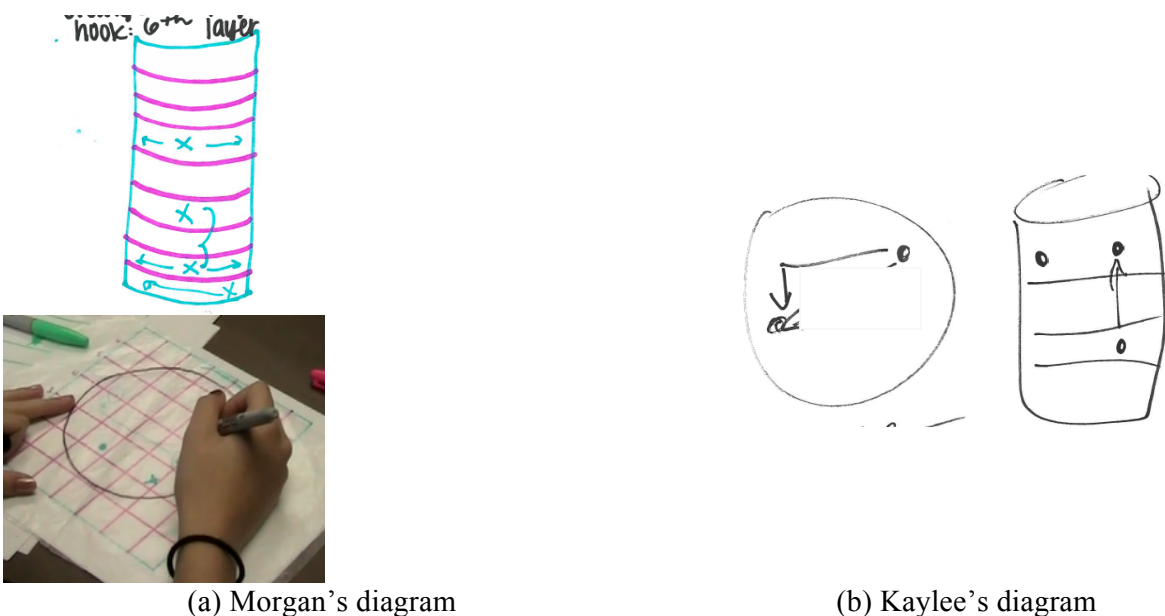
Figure 2. Students' diagrams explaining the movement of Fish 1 to Fish 2 in the fish tank.

    Although both students coordinated the two perspectives (side view and top view of the tank), Kaylee's coordination of the two perspectives was simultaneous whereas Morgan's coordination was sequential. Through disembedding, Kaylee held her frame of reference of the first perspective as a unit structure, translated and inserted her second frame of reference into the first frame of reference. Still aware of each frame of reference as unitized structures, Kaylee was able to track the location of a given point along one spatial dimension with the realization that the point had a specific location along the other two spatial dimensions. Therefore, I claim that the *position* of each point for Kaylee was a multiplicative object (Saldanha & Thompson, 1998) and that her coordinated system of measurements was a multiplicative structure. This allowed Kaylee to define directionality in three-dimensional space by decomposing the spatial movement of Fish 1 to Fish 2 into three spatial

dimensional movements (to which she referred as length, width, and height). On the other hand, Morgan first considered the vertical movement along the layers in a two-dimensional frame of reference anchored to the side view of tank; then, she considered a second movement along a second two-dimensional frame of reference anchored to the top view of tank. As such, Morgan's coordination of the two different perspectives was sequential and thus the position of each point along each spatial dimension was not tightly coupled (Saldanha & Thompson, 1998). Morgan's instructions of telling Fish 1 to "go straight" to Fish 2 would end in different results based on the perspective that Fish 1 is taking in the moment. Therefore, I claim that Morgan reasoned compatibly with Kaylee to the extent that she could coordinate measurements along three spatial dimensions in activity.

Discussion and Implications

A coordination of different perspectives and a simultaneous coordination of frames of reference were found to be crucial in the students' organization of two- or three-dimensional space. Mental operations such as decentering, unitizing, disembedding, translating, and inserting were utilized in this process. Considering that high school students are expected to use coordinate systems for reasoning throughout school mathematics both in algebraic and geometric domains, lack of ways of reasoning within frames of references could become a border for students developing powerful mathematical concepts. The findings of this study can contribute to the understanding of students' ways of reasoning within frames of references and their coordination of measurements in organizing space. Additional research is needed to investigate connections between students' ways of reasoning for spatial organization and quantitative coordination.

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USO DE RECURSOS EN LA RESOLUCIÓN DE PROBLEMAS: ESTUDIO CON PROFESORES EN PRE-SERVICIO

USE OF RESOURCES IN PROBLEM SOLVING: STUDY WITH PRE-SERVICE TEACHERS

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Este artículo reporta cómo la tecnología (Geogebra), el diseño de actividades y el trabajo en parejas –considerado como recurso (Gueudet & Trouche, 2009) – potencian la reflexión de profesores en pre-servicio cuando resuelven problemas de contenido geométrico y algebraico. El estudio está apoyado en dos marcos conceptuales: reflexión-en-acción (Schön, 1983, 1987) y la Aproximación Documental de lo Didáctico (Gueudet & Trouche, 2009, 2010, 2012). El estudio es de tipo cualitativo. Nuestros resultados muestran que el uso de estos recursos, potencia la reflexión de los profesores en pre-servicio durante la resolución de los problemas.

Palabras clave: Conocimiento Matemático para la Enseñanza, Resolución de Problemas, Tecnología, Maestros en Formación

Introducción

Una de las líneas de investigación en educación matemática ha estado enfocada, desde hace algunos años, en el impacto que tiene la falta de conocimiento matemático de los profesores en pre-servicio, en su práctica futura (Ball, Thames, & Phelps, 2008). Esa falta de conocimiento conlleva procesos limitados de enseñanza cuando ellos se integran en sus prácticas escolares. Atendiendo a esta problemática, diversos investigadores (e.g., MacPhail, Tannehill, & Karp, 2013; Tatto & Senk, 2011; entre otros) señalan que debe dársele mayor atención a la formación matemática de los futuros profesores, debido a que su conocimiento matemático es el recurso esencial que utilizará en su práctica futura. Al respecto, Schön (1983) y Freudenthal (1981) mencionan que una manera de contribuir en la reconstrucción del conocimiento –de quienes aprenden– es a través de la reflexión que los propios sujetos hagan sobre los recursos (e.g., Adler, 2000; Gueudet & Trouche, 2009; entre otros) que usan y las acciones que efectúan durante una determinada actividad. En este sentido, la tecnología y el trabajo en grupo deben entenderse como recursos mediadores de la actividad y de la reflexión para quien resuelve problemas (Gueudet & Trouche, 2012; Drijvers, 2013; Gerárd, 2012, entre otros). En este artículo pretendemos responder la siguiente pregunta ¿cómo el diseño de actividades en ambientes de lápiz-y-papel y tecnológico e implementadas en parejas potencia la reflexión de los profesores en pre-servicio cuando resuelven problemas?

Marco Conceptual

Nuestra investigación está apoyada en dos enfoques teóricos: *la reflexión-en-acción* (Schön, 1983, 1987) y la *Aproximación Documental de lo Didáctico* (Gueudet & Trouche, 2009, 2010, 2012). En el enfoque documental los recursos pueden ser de dos tipos: físicos y no físicos. Los recursos tienen origen social y son construidos con el fin de resolver tareas. Durante este proceso documental, el usuario de los recursos construye sus propios *esquemas de utilización* sobre dichos recursos, mientras interacciona con él. La reflexión durante la interacción de los recursos juega un papel importante en la construcción de esquemas. Los esquemas son la parte no observable; es decir, es el conocimiento implícito del sujeto, los cuales guían su acción (Schön, 1983; Vergnaud, 1990).

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Metodología

En la investigación participaron seis futuros profesores (entre 22 y 26 años de edad) de educación secundaria [grados 7, 8 y 9]. Para recolectar los datos se implementaron tres actividades en parejas. Cada una de ellas consiste en resolver un problema de contenido geométrico y algebraico en dos etapas. Primero, se les pidió resolver el problema con papel-y-lápiz y una calculadora *CASIO fx-82MS*; segundo, resolver el mismo problema, pero ahora apoyados de la exploración del problema con Geogebra. La implementación de las actividades fue dirigida por uno de los autores de este artículo, mediante entrevistas semi-estructuradas. Las sesiones de trabajo para cada actividad tuvieron una duración, aproximada, de una hora y media. Éstas fueron video-grabadas.

Análisis de Datos y Discusión de Resultados

Debido a limitaciones de espacio, en este artículo sólo reportamos los resultados de la actividad I, implementada a una pareja de estudiantes. En adelante, los integrantes de esta pareja son nombrados como: E1 y E2 y para referirnos al Investigador se usa IN. Las entrevistas fueron transcritas y evaluadas para cada actividad, tomando en cuenta el marco conceptual.

Actividad I. Papel-y-lápiz

Al principio de esta actividad se les pide a los estudiantes que resuelvan el problema 1 (Figura 1), sólo utilizando papel-y-lápiz y una calculadora [proporcionada por el investigador] para fines prácticos de cálculos matemáticos.

Problema 1

Ambiente de papel-y-lápiz: El cuadrado y sus triángulos

En el cuadrado de vértices *ABCD* de lado 1 *cm* (como se muestra en la Figura 1), la diagonal *BD* se divide en cuatro partes iguales. ¿Cuál es el área del triángulo *DFC*? Ahora divide la diagonal *BD* en cinco partes iguales y luego en 6 partes iguales, ¿cuál es el área del triángulo *DFC* que se forma con la quinta y sexta parte de la diagonal *BD*?

Figura 1. Problema 1, papel-y-lápiz, (tomado y modificado de Mason y Stacey, 1989).

Episodio I

L1 *E1*: Creo que hay cinco formas de calcular el área de un triángulo. La más común es base por altura sobre dos, pero no tenemos la altura. ¡No recuerdo la del semi-perímetro! […]. Ahora, si lo hacemos por ángulos […].
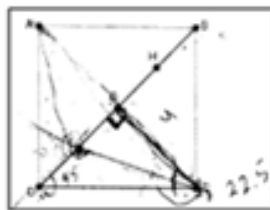
Figura 2. El cuadrado y sus triángulos.

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L2 *E1*: ¡Ah! ¡Aunque podría ser! A ver, sí, el concepto de bisectriz […]. Es que no recuerdo ¿es el de mediatriz o el de bisectriz?

L3 *E2*: Porque de nada sirve que se prolongue [*prolonga el segmento CF con el lápiz*] ¿o sí?

L4 *E1*: Es que si prolongamos [*remarca la línea que trazó E2*]. [*Guarda silencio*] ¡Ah¡ Vamos a hacer lo siguiente […]. Entonces se supone que éste [*señala el triángulo DFC*] sería dos sobre cuatro que esto es igual a […] ¿No son cuatro?

L5 *E2*: No, sería igual a un cuarto […] Porque sería 1 por 1.

En este episodio se observa el uso de recursos previos para comenzar con el planteamiento del problema, la interacción entre estudiantes misma que provoca reflexión entre ellos.

Actividad I. Exploración en Geogebra

Las siguiente parte de la actividad consiste en resolver el mismo problema, pero ahora haciendo uso de la exploración en Geogebra.

Figura 3. Actividad I, exploración en Geogebra.

A continuación, se muestra el diálogo de los estudiantes durante la resolución del problema usando Geogebra.

Episodio IT

L6 *IN*: ¿Es similar lo que encontraron en papel-y-lápiz con lo que observan en Geogebra?

L7 *E1*: Sí, es básicamente lo mismo, sólo que lo que hicimos con papel-y-lápiz lo basamos todo en torno al cuadrado y ahorita si lo basamos sólo al triángulo sería exactamente lo mismo, el área.

L8 *E1*: […] N es igual al número de divisiones [*lo escribe*], el área del triángulo DFC es igual a uno entre N por un medio [*lo escribe*] […].

L9 *IN*: Con el cursor presiona dentro de la pantalla donde dice segmento uno.

L10 *E1*: ¡Sí! ¡Es lo que había dicho! Que todos los triángulos comparten la misma altura.

L11 *IN*: Ahora, presiona el segmento dos con el cursor.

L12 *E1*: ¡Ah! ¡Ya sé que es esto! Se supone que esto está en proporción de esto [*señala los segmentos DG y BE*] y esto está […] DC está en proporción de EC y FE está en proporción de BC y FC está en proporción de BD […] ¡sí ahí está, es el teorema de los catetos! [*Se refiere al Teorema de Pitágoras*.]

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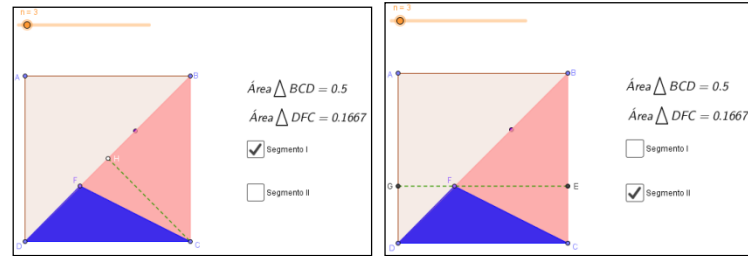
Figura 4. Segmento I y II.

L13 *IN*: ¿Ven alguna relación entre el segmento GD y el triángulo DFC?

L14 *E1*: FD es la hipotenusa del triángulo DGF y al mismo tiempo este DGF debería de estar en proporción [interrumpe E2].

L15 *E2*: […] GD es la altura del triángulo rectángulo DFC. Entonces, para sacar el área es base por altura. Sabemos que la base es uno y la altura es un cuarto de uno, punto veinte y cinco […] y así se puede sacar el área […].

La pregunta del investigador en este extracto de la entrevista se observa que influye para que los estudiantes ´puedan ver otra manera de resolver el problema.

Conclusiones

Los datos surgidos de la actividad I y su posterior análisis nos permiten dar respuesta parcial a la pregunta de investigación. Durante la implementación de la actividad detectamos intenciones secundarias, de los estudiantes, para resolver el problema. Estas intenciones conllevan hacer uso de recursos previos [conocimientos], no sólo para el reconocimiento del problema, sino para el planteamiento de ideas que los motivaron a efectuar acciones que podrían conducirlos a la solución o a encontrar nuevos recursos. Durante la resolución del problema, observamos estados de sorpresa sobre el uso de recursos; los cuales, desde el punto de vista de Schön (1983) conducen a procesos reflexivos. Dichos procesos llevaron a generar nuevas intenciones y cambios de acción instrumentada. La participación del compañero de trabajo muestra, en algunos episodios de la entrevista, cómo conduce a la reflexión, la cual se ve reflejada en los cambios de acciones instrumentadas de su pareja La exploración del problema en Geogebra, permitió comparar las soluciones efectuadas en papel-y-lápiz; pero también generó el surgimiento de sorpresas y reflexiones que los condujeron a efectuar nuevos planteamientos y acciones instrumentadas sobre el uso de recursos.

This article reports how technology (Geogebra), activity design and work in pairs –considered as a resource (Gueudet & Trouche, 2009) – enhance reflection in pre-service teachers when solving geometry and algebra problems. The study is supported by two conceptual frameworks: reflection-in-action (Schön, 1983, 1987) and the Documentary Approach of Didactics (Gueudet & Trouche, 2009, 2010, 2012). This is a qualitative study and its results show that the use of these resources enhances the reflection in pre-service teachers during problem solving.

Introduction

For some years, one of the research lines in mathematics education has been focused on the impact that the lack of mathematics knowledge among pre-service teachers has on their future practice (Ball, Thames, & Phelps, 2008). This lack of knowledge leads to limited teaching processes

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once the teachers start their school practice. Addressing this issue, several researchers (e.g., MacPhail, Tannehill & Karp, 2013; Tatto & Senk, 2011, among others) point out that mathematics training for future teachers should be given a greater deal of attention, since mathematics knowledge is the fundamental resource they will use in their future practice. In this regard, Schön (1983) and Freudenthal (1981) state that one way of contributing to the reconstruction of knowledge—of those who learn—is by means of the subjects' reflection on the resources they use (e.g., Adler, 2000; Gueudet & Trouche, 2009, among others) and the actions they take during a certain activity. Therefore, technology and group work must be understood as mediating resources of the activity and the reflection for those who solve problems (Gueudet & Trouche, 2012; Drijvers, 2013; Gerárd, 2012, among others). In this article we seek to answer the following question: How does activity design in paper-and-pencil and technology environments implemented in pairs enhances reflection in the pre-service teacher when solving a problem?

Conceptual Framework

Our research is supported by two theoretical approaches: *reflection-in-action* (Schön, 1983, 1987) and the *Documentary Approach of Didactics* (Gueudet & Trouche, 2009, 2010, 2012). In the documentary approach, there can be two types of resources: physical and non-physical. Resources have a social origin and are constructed in order to solve tasks. During this process, the user builds his / her own schemes of *utilization* of the resources, while interacting with them. Reflection during the interaction with resources plays a fundamental role in scheme construction. Schemes are non-observable part; that is, they are the implicit knowledge of the subject, which guides your actions nonetheless (Schön, 1983, Vergnaud, 1990).

Methodology

The participants of the research were six future teachers (ages between 22 and 26) of secondary school [grades 7, 8, and 9]. In order to collect the data, three activities in pairs were implemented. Each of the activities consisted in solving a geometry and algebra problem in two stages. First, the participants were asked to solve the problem using paper-and-pencil and a CASIO fx-82MS calculator. Then, they were asked to solve the same problem supported by the exploration of the problem using Geogebra. The implementation of the activities was conducted by one of the authors of the article through semi-structured interviews. The work sessions for each activity had an approximate duration of one hour and 30 minutes. The sessions were video-recorded.

Data analysis and result discussion

Due to space limitations, in this article we only report the results obtained from activity I, given to a pair of students to whom we will refer as S1 and S2 while the researcher will be RE. The interviews were transcribed and evaluated for each activity, taking into account the conceptual framework.

Activity I. Paper-and-pencil

At the beginning of this activity, the students are asked to solve problem 1 (Figure 1), using only paper-and-pencil and a calculator [provided by the researcher] to be used for mathematics calculations.

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Figure 1. Problem 1, paper-and-pencil (taken and modified from Mason and Stacey, 1989).

Episode I

L1 *S1*: I think there are five ways of calculating the area of a triangle. The most common one is multiplying base by height, and then divide by 2, but we don't have the height. I don't remember the one of the semi-perimeter! […] Now, if we do it by angles […].
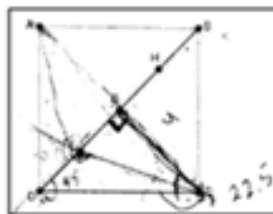
Figure 2. The square and triangles.

L2 *S1*: Oh! It might be, though! Let's see, yes; the concept of bisector […] I just don't remember, is it perpendicular bisector or angle bisector

L3 *S2*: Because extending it is useless [extends the segment CF using the pencil] isn't it?

L4 *S1*: It's just that if we extend [retraces the line originally traced by S2]. [Remains silent] Oh, let's do this […]. It is assumed that this [points at triangle DFC] would be two divided by four which is equals to […] isn't that four?

L5 *S2*: No, it would be equals to a quarter […] because it would be one times one.

In this episode, we see the use of previous resources to start with the problem statement. The interaction between students proved to be a fundamental resource to promote reflection among them.

Activity I. Exploration in Geogebra

The next part of the activity involves solving the same problem, only this time using exploration in GeoGebra. Below, we show the dialog between students while they solved the problem using GeoGebra.

Figure 3. Activity I, exploration in Geogebra.

Episode IT

L6 *RE*: Is what you found in paper-and-pencil similar to what you see in Geogebra?

L7 *S1*: Yes, it is basically the same; it's just that we based everything we did using paper-and-pencil on the square and now, if we base it only on the triangle, it would be exactly the same, the area.

L8 *S1*: […] N is equals to the number of divisions [writes it down], the area of the triangle DFC is equals to one by N times one half [writes it down] […].

L9 *RE*: Using the pointer, click where it says 'segment one' on the screen.

L10 *S1*: Yes! That's what I'd said! That all the triangles share the same height.

L11 *RE*: Now, click on 'segment two' with the pointer.

L12 *S1*: Oh! I know what this is! It is assumed that this in proportion to this [points at segments DG and BE] and this is […] DC is in proportion of EC and FE is in proportion of BD […] Yes, it's there! It is the theorem of the legs of the triangle! [S1 means the Pythagorean Theorem].
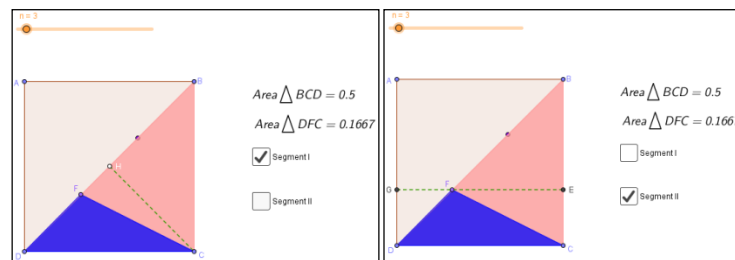
Figure 4. Segments I and II.

L13 *RE*: Do you see any relation between the segment GD and the triangle DFC?

L14 *S1*: FD is the hypotenuse of the triangle DGF and, at the same time, this DGF should be in proportion [E2 interrupts].

L15 *S2*: […] GD is the height of the right-angled triangle DFC. Then, it is base times height to get the area. We know that the base is one and the height is a quarter of one, point twenty-five […], so we can get the area […].

The question asked by the researcher in this excerpt of the interview helps the students to see a different way of solving the problem.

Conclusions

The data obtained from activity I and their subsequent analysis allow us to provide a partial answer to the research question. During the implementation of the activity we detected that the

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students held secondary intentions to solve the problem. These intentions led them to use previous resources [knowledge], not only to recognize the problem but also to lay out ideas which motivated the students to take actions. Such actions might lead them to solve the problem or to find new resources. During problem solving, we observed states of surprise regarding the use of resources. According to Schön (1983), these states lead to reflective processes; the processes generated new intentions and changes of instrumented action. In some parts of the interview, the participation of the peer promotes reflection, visible in the changes of instrumented actions made by the partner. The exploration of the problem in Geogebra allowed the participants to compare the solutions found in paper-and-pencil and generated surprise and reflection. These led the subjects to lay out new approaches and instrumented actions on the use of resources.

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PRE-SERVICE TEACHERS' WAYS OF THINKING ABOUT AREA AND FORMULAS

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In this study, we examine students' understanding of area across three groups – elementary PSTs, secondary PSTs, and non-PST mathematics majors/minors. Analysis of students' solutions to the given tasks did not reveal major differences across these groups; however, further investigation is needed to determine the robustness of these findings.

Keywords: Measurement, Geometry and Geometrical and Spatial Thinking, Teacher Education-Preservice

Introduction

The concept of area and associated formulas are found throughout the K-12 mathematics curriculum. In order to teach these topics meaningfully, teachers must be able to (1) relate formulas to the properties of shapes, (2) use discrete and continuous ways of thinking about area and its relation to linear measurements, (3) understand principles such as area conservation, and (4) recognize equivalent expressions of formulas and how one formula can be derived from another. This list is not intended to be exhaustive, but rather to highlight key aspects of content knowledge essential for teaching area. In this paper, we report on a preliminary study conducted with elementary pre-service teachers (PSTs) and mathematics majors/minors (including secondary PSTs) intended to elicit students' ways of thinking about area and area formulas.

Often, studies of PSTs focus upon only the elementary or secondary level. By administering a task to both groups, we are able to discern productive ways of thinking that cut across these populations, and question some of the assumptions about deficits in mathematical knowledge too often assigned to non-math majors (e.g., elementary-focused PSTs). Further, we administered tasks to mathematics majors/minors including both PSTs and non-PSTs, which allows us to compare the thinking of these two groups.

Related Literature

Beginning in the 1980s, the van Hiele levels have been widely used as a framework within which to classify stages of development of geometric thinking (van Hiele, 1986). In the intervening years, researchers have extended and reformulated the framework. For the purpose of our study, we use the work Clements and Battista (2001) as a construct with which to analyze tasks. These researchers propose that van Hiele levels develop at different rates and in tandem, and that reasoning based on what is visible (visual reasoning) gradually gives way to analysis of properties of classes of shapes (descriptive/analytic reasoning), and then to abstract reasoning based on relations across classes of shapes (abstract-relational reasoning).

Meaningful use of formulas for area depends upon understanding relationships among length measurements, and the coordination of this understanding with the "building up" of an area. However, "many students' measurement reasoning is superficial, with poorly understood procedures or formulas substituting for deep understanding" (Battista, 2007, p. 893). Further, "given the pervasiveness of measure in geometric and graphical contexts, poor understanding of measure might be a major cause of learning problems for numerous advanced mathematical concepts" (p. 902). With respect to PSTs, researchers have documented the challenges students experience in coordinating length measurement, multiplication, and area (e.g., Simon & Blume, 1994). Researchers have also found that the many calculus students have difficulty identifying the correct units for solutions to

area and volume tasks, and that these students lack knowledge of arrays and dimensionality (Dorko & Speer, 2015).

An additional component of geometric thinking necessary for understanding area formulas is conservation of area. In a study of high school and university students, researchers found that a key intuitive notion affecting students' responses to conservation of area tasks was the belief that area equivalence implied congruence (Kospentaris, Spyrou, & Lappas, 2011). These researchers also found that students tended to assume properties and relationships based on the appearance of figures in drawings, which led to erroneous conclusions.

Overall, research suggests that coordinating measurement, area, and area formulas is cognitively complex and students may reach the post-secondary level with an incomplete and superficial understanding of this knowledge domain. Relatively few studies have been conducted at the post-secondary level, and none, that we were able to find, focused on development and justification of non-standard formulas. Thus, our preliminary study of university students' understanding of area concepts and formulas focuses on the following questions:

1. What ways of thinking do PSTs and non-PSTs employ to determine area? To generate and make sense of area formulas?
2. What differences, if any, exist among the 3 populations, elementary PSTs, secondary PSTs, and non-PST mathematics majors?

Study Population and Design

The study population consisted of 39 students enrolled in a large, public university in the U.S. Southwest. Twenty-four of these students were elementary education majors enrolled in one of two required mathematics content courses. The remaining 15 students were all enrolled in a junior level geometry course, and included nine secondary PST mathematics majors, and six non-PST mathematics majors or minors. We note here that we were not able to collect data for all 39 students for any of the tasks administered. This was due in part do to the fact that we administered some tasks at the beginning of the semester when students were still adding and dropping courses, and partly due to general issues of absenteeism.

We selected mathematics tasks in an effort to elicit thinking about area concepts (e.g., measurement, conservation of area) and area formulas, and tasks were administered during the classes described above. Tasks were given across multiple class periods and the time provided for these tasks varied. In this paper, we focus on three tasks out of this set (see Figure 1).
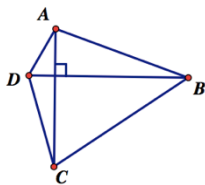
Task A: Why is the formula for the area of a triangle $\frac{1}{2}bh$? (b is the base, h is the height)

Task B: Quadrilateral ABCD is drawn at right. The length of AC is 8cm and the length of DB is 12cm. AC and DB are perpendicular.
 	
 Circle the answer you believe is correct below and explain your choice:
    I do not have enough information to find the area of ABCD.
    I have enough information to find the area of ABCD.

Task C: In Triangle Land, instead of graphing on square graph paper, everything is graphed on equilateral triangle graph paper (in which all the angles are 60 degrees). One triangle on the graph is one unit of area. [*We gave students triangular graph paper.*] Find the formula for the area of a parallelogram in terms of triangular units.

Figure 1. Area concept tasks.

All students worked individually on the tasks. One of the tasks (Task A) was administered to both elementary PSTs and mathematics majors/minors (PSTs and non-PSTs). The second and third tasks (Tasks B and C) were administered only to the mathematics majors/minors.

Analysis of student written work on these tasks consisted of looking for patterns in the data and using the patterns to create and describe initial categories. The data set was coded using the categories, and we created new categories as appropriate. Our methodology utilized aspects of Grounded Theory (Glaser & Strauss, 1967), but we departed from the theory in that our analysis was informed by our reading of related literature.

Findings

Triangle Area Formula

Task A was administered to 19 elementary PSTs, 9 secondary PSTs, and 3 non-PST mathematics majors/minors. Of these 31 students, two students (both secondary PSTs) provided what we considered to be a complete justification. Each of them demonstrated that it was possible to "build" a quadrilateral around *any* triangle, and that the quadrilateral would have twice the area of the triangle. Most students (13 elem. PSTs, 6 sec. PSTs, and 3 non-PSTs) related the area of the triangle to the area of a rectangle and/or parallelogram, but provided an incomplete justification. Some of these students' arguments assumed that all triangles could be formed by dividing a square or rectangle in half. For example: "Because the area of a square is b x h and a triangle is half of a square, so dividing the b x h of a triangle would make sense" (elem. PST). We posit that these students' self-created drawings interacted with their analysis. For example, a student may have begun by drawing a square, and then reasoned visually from the drawing, or a student might have thought "b x h is the area of a square [or rectangle] and then I take ½ of it" and then created a drawing to illustrate this thought. In either case, students' thinking may fall across all three levels of Clements and Battista's (2001) levels – visual reasoning, descriptive/analytic, and abstract-relational. Of the remaining seven students, two (both elem. PSTs) showed evidence of relating the formula to the perimeter of the triangle (e.g., "the area is ½ bh because you only take one side of each . . . instead of multiplying both bases and both heights so you would only need ½"), which is type of thinking well-documented in the literature (Battista, 2007).

Quadrilateral Area Problem

Task B was administered to 5 secondary mathematics major PSTs and 5 non-PST mathematics majors/minors, and we found both similarities and differences in students' approaches to this problem across these two groups. One student in each group believed that they did not have enough information to determine the area because they did not know where AC intersected DB. Both students' explanations indicated that they focused on decomposing the given figure into triangles, finding the areas of the triangles, and then recomposing the shapes. Of the remaining four PSTs, three surrounded the figure with a rectangle, argued that each of the four right triangles in the original figure would be half of its corresponding rectangle, and concluded that the area of ABCD must therefore be one-half that of the surrounding rectangle. The other PST used variables to represent the lengths (i.e., dividing the 12cm length into x and 12-x), decomposed ABCD into triangles, substituted the variables into the formula for the areas of the triangles, and concluded that the area of ABCD was 48cm^2. This method was also employed by one non-PST. Two non-PSTs determined the area by surrounding the figure with a rectangle, but neither student provided an explanation beyond drawing a rectangle around the figure and writing the equation representing the calculation (e.g., multiplying the lengths AB and DC and dividing by 2), The other non-PST deformed ABCD into a rhombus by sliding one diagonal and then the other, found the area, and then argued that both areas would be the same. Overall, 8 of 10 students demonstrated understanding of

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conservation of area in terms of decomposing and composing, with some students showing a preference for the use of variables and equations, and other students showing a preference for surrounding and subtracting.

Non-Standard Area Formula

Task C was administered to 8 sec. PSTs and 4 non-PSTs. Three PSTs and 1 non-PST wrote that the formula would be A = 2bh (or equivalent), where b is the base length in triangular units and h is the height measured at a 60° angle with the base. One of these students (a PST) wrote that the sides of the parallelogram must have a 60° or 120° angle between them, but crossed this statement out, likely indicating that this caveat had been rejected. Three PSTs and 2 non-PSTs included drawings indicating that they may believe that the formula A = 2bh would only apply to 60-degree angle parallelograms. If true, this would be similar to believing that measuring the height perpendicular to the base applies only to rectangles. The remaining three students (2 PSTs and 1 non-PST) all found the correct formula for 60° angle parallelograms; however, the 2 PSTs reverted to a perpendicular measure of height for other parallelograms, and the non-PST multiplied two of the parallel sides for non-60° angle parallelograms. Based on our data from this problem, it appears that in the face of non-standard units, students tended to exhibit visual reasoning (Clements & Battista, 2001).

Discussion

The findings from this preliminary study are in no way intended to be conclusive. Data from these problems will be used to refine tasks, select additional tasks, and plan task-based interviews. What we can say is that for these problems, and with these sets of students, we did not find evidence of *major* differences in mathematical thinking across the three student groups, although the confusion between area and perimeter exhibited by two elementary PSTs bears further investigation. Most elementary PSTs provided solutions to the Triangle Area Formula task that were similar to those provided by secondary PSTs and non-PSTs. With respect to the other two tasks, secondary PSTs and non-PSTs were very similar except in one regard – when prompted for an explanation in Task 2, PSTs were much more likely provide a written argument supporting their conclusions. This could be due to any number of reasons (including an anomaly due to the small sample size) but we look forward to further investigating this phenomenon.

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AREA UNITS WITHOUT BORDERS: ALTERNATIVES TO TILING FOR DETERMINING AREA CHANGE IN DYNAMIC FIGURES

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We report on thirteen clinical interviews with middle school students regarding their conceptions of area growth. The tasks involved coordinating length and area for dynamic geometric figures. Students appealed to tiling conceptions as well as a "distributed" conception of area in which the amount is spread evenly throughout the figure based on a unit with no pre-determined boundary or shape. The distributed conception supplements the tiling conception and supports students in reasoning about area units and quantitative change.

Keywords: Geometry and Geometric and Spatial Thinking, Measurement

A frequent outcome of typical instruction in area is that students become proficient with standard area calculations (e.g., the length times width formula for rectangles) but do not demonstrate well-structured conceptions of area units and space (Izsák, 2005; Lehrer, 2003; Simon & Blume, 1994). Student understanding improves during interventions that emphasize area units as two-dimensional entities, focus on coordinating length units with area units, support students' multiplicative grouping of tiles (e.g., as row-column arrays), encourage partitioning and rearranging geometric shapes, and develop pre-multiplicative and multiplicative reasoning (Clements, Battista, Sarama, & Swaminathan, 1997; Kobiela, Lehrer, & VandeWater, 2010; Lehrer, 2003; Simon & Blume, 1994; Zacharos, 2006). Even so, developing area reasoning is an extended and complex process.

This study examines student reasoning about area *change* in dynamically growing geometric figures. We address the following research question: "What area conceptions and strategies are available to middle schoolers as they reason about linear change between length and area in dynamic figures?"

Literature Review

Lehrer (2003) articulated a number of general properties of measurement that can be applied specifically to area. First, the defined unit must relate to the empirical attribute of area. A physical object might be used as a unit, such as an index card. The unit can relate to area indirectly—for instance, a can of spray paint could serve as the unit to measure the area of a collection of walls, or two perpendicular length units could imply a square unit of area. Once the unit is defined, area is measured by iterating the unit to tile the object, with every unit identical to every other. Standard area units are square tiles by convention. Additional properties of measurement, such as proportionality and additivity, enable shapes to be measured in flexible ways, such as converting between measurements based on a known multiplicative relation between different units.
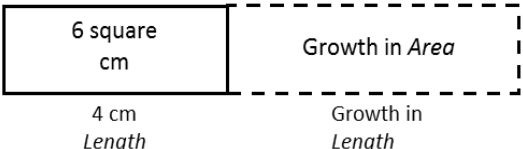
Research has shown that students can experience difficulty mentally structuring tiled regions as row-column arrays, especially in the absence of perceptual cues, and they do not connect arrays to the multiplicative structure of the area formula (Battista et al., 1998; Simon & Blume, 1994). There are recent efforts to rethink the relationship between area, multiplication, and geometric transformations. One effort is to imagine area being produced through "sweeping" one length through another (Kobiela et al., 2010; Smith, 2013). A related effort is to understand area multiplicatively as a composition of two lengths rather than as the additive combination of multiple row-units into an array (Simon & Blume, 1994; Smith, 2013). A third effort is to consider area through transformational reasoning with dynamic figures, attending to change and covariation (Johnson, 2013; see also Simon, 1996).

Method

This study was part of a larger project investigating students' generalizing activity. We conducted thirteen semi-structured interviews with seven boys and six girls in middle school (grades 6-8). Each interview lasted about an hour. We presented the participants with the task sheets shown in Figure 1. The interviews were videotaped and transcribed and we use pseudonyms for all participants. We coded and analyzed the transcripts for variations in student thinking related to area conceptions and strategies.

Task 1—Rectangle
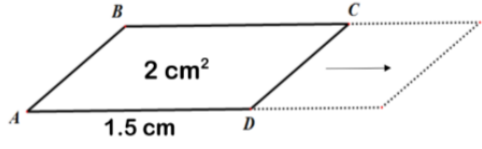
The rectangle below grows along the dotted path as shown in the figure.

6 square cm

Growth in *Area*

4 cm
Length

Growth in *Length*

When the length grows by _____, the area grows by _____.

Task 2—Parallelogram

The figure **ABCD** below is a *parallelogram*. The parallelogram is growing along the dotted lines as shown by the arrow.

2 cm²

1.5 cm

If I increase the length of segment **AD** by _____, then the area grows by _____.

Figure 1. Interview task sheets.

Results

The students employed seven unique strategies. Table 1 lists each strategy, its definition, and a generic example based on Task 1. The counting strategy was infrequent given the form of the tasks and the students' ages. The area formula was used frequently, but not by all students, and not for every question for any one student. Students used the additive strategies, which are inappropriate for these tasks, alongside early multiplicative strategies. Students varied in whether they used one or many strategies as well as in their sophistication and ability to justify their choices.

As students justified their strategies, they sometimes appealed to a tiling or distributed conception of area. First we discuss the tiling conception. Anya was a sixth grader who gave a correct justification of the area formula on the rectangle:

Interviewer: How do you know that the area is the length times the width?
Anya: Because there would be one and a half I guess like kind of rows that way (gestures horizontally) and 4 rows that way (gestures vertically) and that creates sort of like squares or rectangles within the rectangle and there's 4 rows of 1.5 so that would be like 4 times 1.5 so that's how we figure out how many, or how the area would be.

Anya was also able to justify her formula on the parallelogram task by transforming it into a rectangle. Other students could not justify this extension.

Some students circumvented the need for explicit unit tiles by viewing area as an amount that was distributed evenly throughout the shape. In the distributed view, 6 cm^2 can be understood as an expression of the relative magnitude of a region (smaller than 7 cm^2, twice as large as 3 cm^2, etc.) according to a common scale with an implicit unit. The unit has no pre-determined shape; it is a unit without borders.

Table 1: Observed Strategies for Determining Area

Strategy (*n of 13 students who used it at least once*)		Definition	Generic example on Task 1 (a growing rectangle with area 6 cm^2 and length 4 *cm*)
Counting (1)		Fill a region with unit tiles and count the tiles.	Draw an array to produce 6 tiles. If length grows to 8, extend the array to fill the area and count 12 tiles individually.
Formula (8)		Apply a known area formula, e.g. $A = l \cdot w$.	Solve $6 = 4 \cdot w$ for $w = 1.5$. If a new length is 8, solve $A = 8 \cdot 1.5 = 12$.
Additive coordination: Area and length are coordinated through additive comparisons.	*Between measures* (2)	Compare the difference between area and length and preserve that difference after change.	The area of 6 is two more than its length of 4, so a length of 8 would imply an area of 8+2, or 10.
	Within measures (4)	Identify additively the change in one quantity and replicate the change in the second quantity.	If the length increases by 3 from 4 to 7, then the area must also increase by 3 and so 6+3 is 9.
Multiplicative coordination: Area and length are coordinated through multiplicative comparisons.	*Formal proportion* (2)	See the task as proportional and solve formally, e.g. using the cross multiply algorithm.	If the length becomes 8, then the algorithm might be filled in as 4/6=8/x and then solved for x.
	Between measures (3)	Compare the length and area values to one another multiplicatively and preserve that ratio after change.	The area of 6 is 3/2 the length of 4, so a length of 8 means the area is 3/2 of 8, or 12.
	Within measures (12)	Identify multiplicatively the change in one quantity and replicate the change in the second quantity.	If length doubles from 4 to 8, then the area must also double from 6 to 12.

A few students were asked of the given rectangle's area, "Where's the six?" Rather than drawing 6 discrete tiles or units, they indicated the whole region by gesturing, circling, or shading it. When pressed to explicitly show the amount, Robin and Olivia partitioned the region into two rows of three, decoupling the row-column array structure from the perimeter's units of length (Figure 2). Robin also partitioned an adjacent region of growth, 5 cm^2 in size, into 5 columns that did not mesh with his prior array.
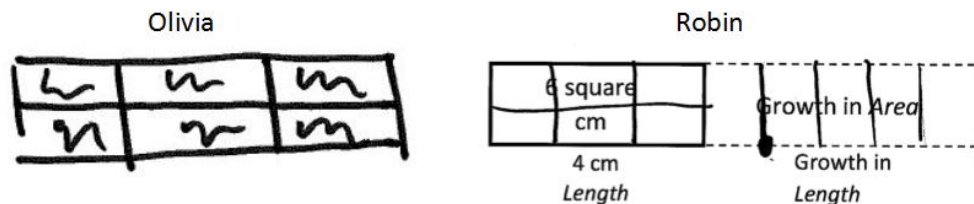
Figure 2. Olivia's and Robin's partitions.

The distributed conception appeared to underlie some students' use of the multiplicative strategy within-measures. For example, Willow was a 6th grade girl who began the rectangle task using exclusively additive reasoning. Then she suddenly made an unprompted shift in strategy that seemed to appeal to a dynamic image of the rectangle's growing path. Willow justified how the quantities of length and area accumulate together in a coordinated fashion without having to explicitly articulate the structure of the unit of measure (Figure 3):

> Unless it will start at 0 because if you start at 0…if you start it from 0 to find out the actual growth then say this is like the first they grew, and this kind of, so this grew by 4 first (gesture A)

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

and then this grew by 6 (gesture B) so this could grow by 4 again (gesture C) and this could grow by 6 again (gesture D).
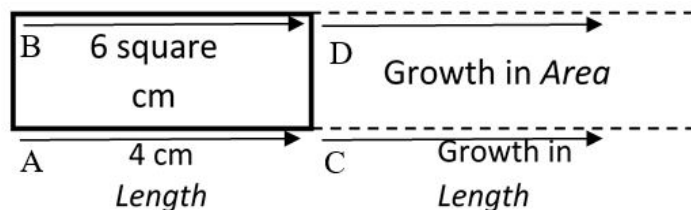
Figure 3. Willow's finger gestures.

There were a couple of advantages to this strategy. The first was that students who reasoned about area growth in this way could engage in meaningful multiplicative reasoning, including forming a composed length-area ratio pair, iterating and partitioning that ratio, finding the unit ratio, and generalizing from the unit ratio to the arbitrary case. The second advantage was that the same reasoning used for the rectangle extended smoothly to the parallelogram with no need to rearrange its parts or recall a formula.

Conclusion

The distributed conception of area is a promising supplement to the tiling conception for making sense of area change. In our study, several students were able to invent strategies to quantify area change and coordinate it with changes in length by drawing on a distributed conception of area and strong multiplicative reasoning.

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RATIONAL RULERS: A TEACHING EXPERIMENT ON FRACTION MAGNITUDE

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Keywords: Rational Number, Measurement, Design Experiments

Utilizing the Measurement Construct of Fractions

A curriculum unit was created and tested with the intent of having fourth grade students understand magnitude in a new way. Students folded and labeled rulers as part of their job working for the Rational Ruler Company. Students were given the total length of the ruler such as 4 units or 2 units as illustrated in figure 1. Students' mathematical discourse focused on arguing and justifying what to label the parts. A common argument among the class was the difference between a part/whole perspective and a measurement perspective of ruler labeling. If students perceived each section of the ruler as part of the whole ruler, they would label each section of an 8 section ruler 1/8. This study found that often children did not pay attention to the indication of 1 unit and incorrectly labeled the ruler (see figure 1). However, over time, the students improved their ability to treat each unit within the ruler as a whole, when partitioned into 4ths, it represented a ¼ of one unit (see figure 2). One finding of this study is that creating the rulers in this fashion provided scaffolding for the transition into decomposing fractions into smaller fractions, fraction addition, and fraction multiplication by a whole number. Additionally, it was confirmed that working with the measurement construct of fractions avoided common fraction misconceptions such as seeing a fraction as two whole numbers which can lead to incorrect assessment of fraction magnitude.
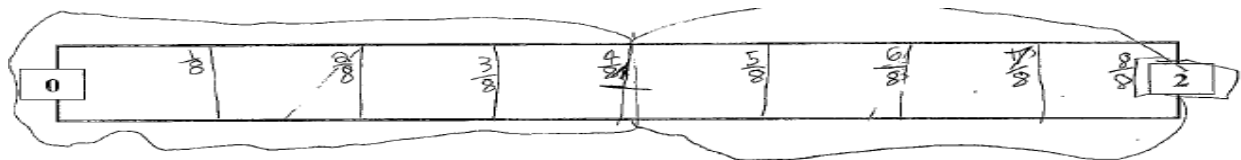
Figure 1. Labeling of ruler disregarding wholes.
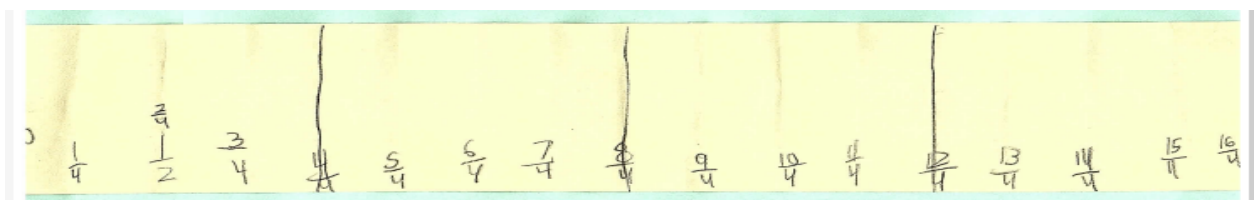
Figure 2. Labeling according to units-measurement perspective.

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CURRICULUM AS A NAVIGATOR: CROSSING BORDERS TO HIGHER LEVELS OF GEOMETRIC REASONING

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The van Hiele model for geometric reasoning has been a useful model to describe students' concept development in geometry and provides a framework for curriculum and instruction. This model, conceived as sequential and hierarchical, suggests that school curriculum and instruction match students' level of geometric reasoning. However, prior studies have indicated that the level of instruction in a traditional geometry course often does not match the level of geometric reasoning with which students begin the course (Senk, 1989; Shaughnessy & Burger, 1985). In this paper, we share a study of how a transformation-based curriculum, designed to address the van Hiele levels, contributes to students' reasoning about a similarity task.

Using the van Hiele model as a framework, this case study explores the geometric thinking of 12 ninth grade geometry students, who were from the same high school course and represented a range of developmental levels. Released items from the 1996 twelfth grade National Assessment of Educational Progress (NAEP) related to similarity and congruence were used as the basis for the protocol to assess students' thinking. Students took the written assessment and a smaller representative sample of students were then asked to explain their responses in an interview. The geometry course in which the students were enrolled used a transformations-based curriculum. We describe the curriculum as transformations-based because the curriculum begins by developing student understanding of geometric transformations in a 2-dimensional plane. All other geometric concepts are then developed from transformations. For example, the concept of similarity is developed from students' work with non-congruence motions, such as dilation.

Findings suggest that when pressed to explain their thinking geometric tasks, students shifted to explanations at lower van Hiele levels. That is, while their initial justification could be characterized at van Hiele Levels 1 or 2, when pressed to extend their explanations, students seemed to draw on reasoning characteristic of van Hiele Levels 0 or 1, respectively. We believe that by using discourse from the lower levels in their responses, students had developed what can be described as a meta-discourse (Wang, 2016), where their thinking reflected what was previously practiced and therefore encompassed in their evolved discourse. We believe that when students are provided with rich content experiences at lower levels, they develop reasoning capabilities. Without experiences at the lower levels, students may be without cognitive tools to articulate, justify, or provide alternate methods. We believe this validates the importance of geometry curricula that provide sufficient experiences at lower van Hiele levels, as such experiences enable the development of higher-level geometric reasoning.

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DESIGNING CONFIGURE: A MICROWORLD FOR LEARNING QUALITATIVE GEOMETRY

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Keywords: Technology, Design Experiments, Geometry and Geometrical Spatial Thinking

Piaget's investigation of the child's representational thinking about the nature of space provided evidence that young children possess early and intuitive topological ideas (Piaget & Inhelder, 1956). As I set out to conduct a teaching experiment (Steffe & Thompson, 2000) in order to construct models (Thompson, 1982) of these ideas with two children, "Amanda" (age 6) and "Eva" (age 7), I realized I needed a tool that could make these ideas visible. I explored a variety of existing dynamic geometry environments, but none proved suitable for the exploration of topological ideas. So in seeking to answer the following research question, I initiated a design-based approach (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) to develop one: *How can a software environment be designed and developed in ways that support fundamental topological representations and transformations such that learners' reasoning about topological ideas are made visible and are able to further develop?*

The poster I am proposing illustrates the iterative design and development process by which a dynamic geometry environment called *Configure* (Greenstein & Remmler, 2009) came to be. *Configure* affords users opportunities for a form of reasoning that resonates with the "rubber sheet" conception of topology. However, so as to remain open to findings of children's conceptions without giving in to a felt need to classify them as topological or not, I gave the name "qualitative geometry" to those forms of reasoning.

In my analysis of the experiments with Amanda and Eva (Greenstein, 2014), I determined that the qualitative equivalence schemes they constructed had come about as a result of interactive mathematical activity mediated by *Configure*. The structural character of their property-based distinctions was evident in names like "cherries," "worms," and "blocks," which they had assigned to equivalence classes they had constructed. Importantly, these properties were determined in relation to imagined, mental operations of possible transformations and not by the shapes themselves. Accordingly, I concluded that it is indeed the case that *Configure* had been developed in ways that support fundamental topological representations and transformations such that learners' reasoning about topological ideas were made visible and were seen to further develop. Three mutually informing considerations were integral in its development: 1) the content of topology, 2) Piaget's model of conceptual development (1970), and 3) the software's usability by children who would and did use it.

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TOWARD THE SAME OPENNESS: RESULTS FROM A TEACHING EXPERIMENT IN ANGLE MEASURE

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While researchers have examined students' angle concepts at the elementary, middle, and undergraduate levels (Lehrer, Jenkins, & Osana, 1998; Mitchelmore & White, 2000; Moore, 2013), little research has explored high school students' understandings of angle concepts. I present data and findings from an eight-month teaching experiment (Steffe & Thompson, 2000) which investigated high school students' understandings of angle and angle measure. I focus on one ninth-grade participant, Hillary, and her activities during the first three teaching sessions of the study. During the first teaching session, Hillary manipulated two hinged pairs of chopsticks—one long pair and one short pair—that served as physical models for angles. While Hillary was able to set the short pair of chopsticks to be both more open and less open than the long pair of chopsticks, Hillary explained that it was not possible to set the short pair to have the same openness as the long pair because they were not the same size. From Hillary's inability to set the pairs to the same openness, I inferred that openness and length were not independent attributes for Hillary (cf. Lehrer, Jenkins, & Osana, 1998).

For the second and third teaching sessions, I designed activities with the intention of engendering changes in Hillary's conception of openness. In these sessions, Hillary engaged in activities that involved placing a laser beam at the vertex of the hinged chopsticks and sweeping the beam from one side to the other. In addition to enacting this sweeping action physically, Hillary enacted the sweeping action in a virtual environment with a dynamic geometry software. In the virtual environment, the sweeping beam left a trace as it swept from one side to the other. By the end of the third teaching session, Hillary set the short pair of chopsticks to have the same openness as the long pair of chopsticks. I argue that three factors were critical for engendering Hillary's progress during these teaching sessions: (1) Hillary associated the sweeping action with openness and could carry out this action independently of length; (2) Hillary could visually compare traces of the sweeping action in the virtual environment for two different pairs of chopsticks; and (3) the traces of the sweeping action in the virtual environment engendered Hillary to develop a superimposition scheme for determining whether or not two pairs of chopsticks had the same openness, regardless of the lengths of the chopsticks.

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HIGH SCHOOL STUDENTS' DRAWINGS OF ANGLES FROM HAPTIC PERCEPTIONS

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In this session, I present data and findings from one task in a teaching experiment (Steffe & Thompson, 2000) focusing on high school students' understandings of angle and angle measure. While researchers have examined students' angle concepts at the elementary, middle, and undergraduate levels (Lehrer, Jenkins, & Osana, 1998; Mitchelmore & White, 2000; Moore, 2013), research focusing on high school students' understandings of angle concepts is scarce. Additionally, investigations of students' angle concepts are critical as researchers have shown that incoherent or limited conceptions of angle measure constitute a cognitive border for the study of trigonometry (Akkoc, 2008; Moore, 2013).

This session focuses on a task in which four, sighted, ninth-grade students were asked to draw objects that they felt with their hands. The objects were hidden from each student's visual field at all times by virtue of a screen. The purpose of the task was to examine how students organized haptic perceptions to form an element in representational space (Piaget & Inhelder, 1967). Each student handled four objects, which were angle models cut from plastic: one model was right; two were acute; one model was obtuse. In the teaching session, the teacher-researcher referred to the models as "objects" and never used the term "angle." Students drew one object before moving onto the next and were not permitted to return to previous objects or drawings. To analyze students' drawings, I superimposed each angular model on the corresponding drawing and measured angular discrepancies. The students' right-angle drawings were essentially indistinguishable from the right-angle models. In the acute cases, I found modest deviations across drawings and models. I observed stark differences between models and drawings across all four students in the obtuse case. For the obtuse models, all students underestimated the openness of the models in their drawings, with two of the four students producing acute drawings for the obtuse models. In this poster, I discuss students' words and actions as they engaged in the task in order to support hypotheses for why the obtuse case resulted in the largest discrepancies.

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UNITS COORDINATING AND SPATIAL REASONING IN THREE DIMENSIONS

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Keywords: Geometry and Geometrical and Spatial Thinking, Cognition, High School Education

Students' units coordinating activities have been linked with their understandings of various mathematical concepts and reasoning, such as fractional knowledge (Steffe & Olive, 2010), multiplicative relationships (Hackenberg, 2010), combinatorial reasoning (Tillema, 2012), and proportional reasoning (Steffe, Liss, & Lee, 2014). The levels of units that students are able to coordinate and hold together mentally in one structure are associated with different mental operations that a student can utilize in reasoning; as such, students' levels of units coordination can be viewed as an operational border in mathematical learning across multiple domains. This poster shares partial data and results from a teaching experiment (Steffe & Thompson, 2000) with two ninth-graders, Kaylee and Morgan, to explicate their spatial reasoning in three dimensions in relation to the levels of units and the mental operations they used when reasoning.

In this poster, I present an analysis of Kaylee and Morgan's engagement in two tasks from this teaching experiment. In the first task, students were asked to locate four fish or describe the motion of one fish to another in a cubic fish tank. In the second task, the students were asked to reason with cubic blocks of various sizes. For example, they were asked to find the total number of unit cubes in each cubic block and the number of unit cubes that would be painted if the exterior of the block was painted. In the first task, Kaylee utilized mental operations which produced three levels of units to construct frames of reference and coordinated the frames of reference to locate the fish in the three-dimensional space. In the second task, Kaylee utilized her frames of reference to partition the cubic blocks, producing multiple three levels of units structures. Her coordination of frames of reference allowed her to mentally decompose, re-present, and anticipate (Piaget & Inhelder, 1967) the interior of the cubic blocks in the absence of their perceptual availability. In the poster, I compare Morgan's activities with Kaylee's in the tasks and propose that the mental operations that produce three levels of units are necessary for powerful spatial reasoning in three dimensions. Additionally, I consider implications for secondary students and propose future research directions.

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SPATIAL COORDINATION AS A PRECURSOR FOR QUANTITATIVE COORDINATION: USING ONE POINT TO REPRESENT TWO POINTS

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Saldanha and Thompson (1998) explained thinking covariationally as "holding in mind a sustained image of two quantities' values (magnitudes) simultaneously" (p. 1). Often, Cartesian coordinate systems are used as static representations of the mental covariation outlined by Saldanha and Thompson. We distinguish between two uses of coordinate systems: 1) *spatial coordination*—to re-present space by establishing frames of reference to locate points within the space; and 2) *quantitative coordination*—to coordinate sets of established quantities in a representational space. In this poster, we argue that spatial coordinations are necessary (i.e, border) if Cartesian coordinate systems are to be productive tools for students' quantitative coordination and subsequent covariational reasoning.

We conducted a two-year teaching experiment (Steffe & Thompson, 2000) with two ninth-grade students, Kaylee and Morgan, to investigate their constructions of coordinate systems (Lee, 2015, April). The first author served as the primary teacher-researcher and the second author served as the witness for the teaching experiment. In this poster, we present one task, the Ant Farm Task, in which Kaylee and Morgan were asked to describe the location of two points moving along two different line segments using a single point. Students worked within a dynamic geometry environment. In the environment, the screen was imagined to be the floor of a room on which two ant farms—long, thin rectangles that could be rotated and repositioned within the dynamic environment—were resting. Each ant farm contained an ant—a point that moved haphazardly along the longest segment connecting midpoints of opposite sides of each rectangle. Students were asked to find a way to locate both ants using a single point. Students were presented with the spatial situation absent of any quantitative values (e.g., an ant's distance from the end of the tube). Although we expected this task to be trivial for both students due to their experiences using the Cartesian plane, we found that the spatial coordination was not immediate for either student. In this poster, we present an analysis of Kaylee's and Morgan's strategies in the Ant Farm Task. Additionally, we consider implications for secondary students. In particular, we hypothesize that students must develop spatial coordinations, as in the Ant Farm Task, prior to engaging substantively in quantitative coordination and covariational reasoning involving two-dimensional Cartesian coordinate systems.

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OPPORTUNITIES TO INTERACT WITH QUADRILATERALS AND THEIR DEFINITIONS IN ELEMENTARY EDUCATION TEACHER TEXTBOOKS

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Keywords: Teacher Education-Preservice, Curriculum Analysis, Geometry and Geometrical and Spatial Thinking

Rationale

Despite the importance of developing geometry skills at an early age, many students do not receive high cognitive demand geometry tasks during elementary or middle school (Battista, 2007). If we expect future elementary school teachers to cross the border from their own perhaps underdeveloped past experiences to becoming effective geometry teachers it is especially important to ensure that pre-service teacher preparation programs provide opportunities to learn about and fully understand the geometric content covered in elementary school. This study examines the opportunities to interact with quadrilaterals and their geometric definitions found in nine mathematics content books intended for elementary education majors.

Methods

A collection of nine popular text books were selected via a series of searches on textbook sales websites. Sample problems were collected from the end of chapters which introduced definitions of quadrilaterals and examined according to a coding scheme based on Bloom's taxonomy (Sosniak, 1994), with consideration for relations to the van Hiele levels (Burger & Shaughnessy, 1986) and the Common Core State Standards (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). The codes included; recognition, describe, classify, relationships/hierarchy, and create. Problems were also coded to identify those related to teaching contexts rather than exclusively content knowledge.

Results and Implications

Results revealed a reasonable variety of problems across all the text books examined, but that most individual text books contained limited variation. These results imply the need for instructors of pre-service teachers to examine their selected text and supplement their instruction with additional problem types as needed. The results also identify appropriate textbooks to reference when locating examples of each problem type. Similar analyses could be done for other subject areas as the elementary content standards continue to evolve over time.

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TEACHERS' USE OF DYNAMIC GEOMETRY ENVIRONMENT IN PROVING THE INSCRIBED ANGLE THEOREM

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Purpose and Background

Despite the emergence of new types of digital technologies and the importance of proofs and proving in teaching and learning of mathematics (Tall et al., 2012), only little research is focused on teachers' proving process in Dynamic Geometry Environment (DGE) (de Villiers, 2004). The current study uses Geometer's Sketchpad® (Jackiw, 1991, 2009) to examine: (a) what conjectures do two high school teachers formulate using DGE to explore the Inscribed Angle Theorem (IAT)? And (b) what are the ways that these teachers prove their conjectures? One way to describe the IAT is that the measure of an inscribed angle is half of that of the intercepted arc.

Framework & Methods

Mathematical proving process consists of: (a) formulation of a conjecture and (b) construction of mathematical proof (Sinclair & Robutti, 2013). Becky and Jill (pseudonyms) were certified high school geometry teachers. Data were collected from a 90-minute videotaped clinical interview. All recordings were transcribed and then analyzed using two cycles coding. First cycle includes descriptive coding. In second cycle, I used pattern coding to identify patterns. Task 1 was designed to examine the relationship between the inscribed angle C, the arc AB, and the central angle O. Task 2 was designed to explore the relationships of two inscribed angles that intercept the same arc or alternate arcs (when dragging point D on minor arc AB). The participants could drag points on the circle and measure different properties.

Results

Becky and Jill formulated five conjectures related to similar ideas but focused on different components. For instance, both Becky and Jill formulated the IAT conjecture but Becky discussed the relationship between the inscribed and the central angles and Jill described the relationship between the inscribed angle and the arc that it intercepts. The different formulations impact their proving process. Additionally, three mathematical proofs constructed by Jill and one proof constructed by Becky used only DGE and were verbal. This is important because that the narratives in these proofs were mainly based on theory. Becky and Jill's formulation of the IAT conjecture and relying on previous result (assuming that the IAT has been proven true) seemed to play an important role in their attempt to prove other conjectures.

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YOUNG CHILDREN USING ANALOGICAL REASONING TO UNDERSTAND SIMILARITY IN A DYNAMIC MULTI-TOUCH GEOMETRY ENVIRONMENT

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Geometrical similarity is a critical concept for the development of advanced mathematical thinking; however, it is a difficult idea for young children to grasp (Lehrer, Strom, & Confrey, 2002). This study investigated second-grade children's ways of reasoning while working in small groups on similarity activities created in a dynamic multi-touch geometry environment (hereon DMGE). The DMGE combines *Sketchpad*, offering visual dynamic feedback, and the iPad offering multi-touch experiences. The study is part of a larger research involving the design and implementation of an educational strategy to nurture in early childhood education (k-2) intuitive senses of congruence and similarity within the collaborative DMGE. The study was implemented with five small groups in the context of the afterschool program of a public elementary school of Massachusetts, U.S., and included thirteen second-grade students (7-8 year olds). The strategy entailed a sequence of seven exploratory and problem-solving activities for congruence and similarity. The poster focuses on the fourth activity, designed to explore two dynamic similar triangles. A qualitative multiple-case study based on task-based interviews and artifacts was the method of inquiry. Discourse analysis was the data analysis method for the small group work. Theories of *semiotic mediation* related to the use of digital interactive technologies guided the whole research (Moreno-Armella, Hegedus, & Kaput, 2008; Sinclair & Moss, 2012).

One main finding was that the young children used analogical reasoning to make sense of the similarity task. Dynamism of the multiple representations of the similar triangles and gestures mediated this form of reasoning. One case study illustrates how children's analogies emerged as a genuine way to 'mathematize' both the activity and their world. The students intuitively used prior everyday experiences to understand the new situation and find out invariant mathematical relationships, even if they did not explain them formally. They inferred that both similar triangles always have the same shape and same type of movement but different sizes, and identified 'objects' in their world that have, in a certain way, the same mathematical structure. The role of the DMGE was critical, as children utilized the dragging tool with their fingers to transform the triangles and the continuous motion of these multiple representations helped them infer the geometrical relationships. Moreover, children in the small group built on each other's ideas to understand these relationships. Implications for early childhood education are discussed.

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DEVELOPING STUDENTS' THINKING OF DYNAMIC MEASUREMENT

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Keywords: Measurement, Learning Trajectories (or Progressions), Design Experiments, Technology

Although much research has been conducted on the teaching and learning of geometric measurement, what is available shows students' difficulty in understanding area and volume (Clements and Sarama 2014). Moving from an additive reasoning of adding square tiles to a multiplicative reasoning of combining two dimensions in an area formula is extremely difficult, especially if students only understand multiplication as repeated addition. As Piaget et al. (1960) states, "the child thinks of the area as a space bounded by a line, that is why he cannot understand how lines produce areas" (p. 350). This project aims to explore measurement in a *dynamic* way in order to resolve those difficulties and assist students in developing a conceptual understanding of area. An intuitive way for students to visualize a meaning for area in a dynamic way is to view it as a 'sweep' of a line segment of length *a* over a distance of *b* to produce a rectangle of area *ab* (Confrey et al. 2012). For instance, one can imagine taking a paint roller of length *a* and rolling it a distance of *b* to produce a rectangle of area *ab* (Figure 1).
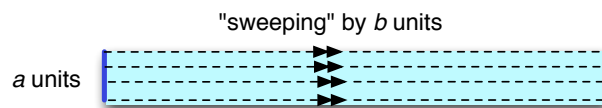
Figure 1. Area as 'sweeping" (reproduction from Confrey et al. 2012).

Our conceptualization of the area formula involves students considering both length (e.g. of the roller) and width (e.g. the distance of rolling) simultaneously as attributes that define the area (e.g. the space generated). In this poster we present a) the type of dynamic tasks and tools that we used for developing students' DYME reasoning, b) the forms of DYME reasoning developed as a result of students' systemic engagement in these dynamic tasks, and c) how DYME experiences assisted students' development of the area formula.

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BUILDING A CONCEPTUAL UNDERSTANDING OF FRACTIONS THROUGH A SIX-WEEK INTERVENTION IN EARLY ELEMENTARY

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Students struggle to change from additive to multiplicative thinking (Moss, 2005) that can be attributed to curricular emphasis on procedures rather than development of important fractional understanding (Bezuk & Cramer, 1989; Behr & Post, 1992; Cwikla, 2014). This study was developed as a partnership between a university and local elementary school for content support and to enhance student achievement and understanding of fractions. Research questions included 1) What do students understand about fractional concepts at kindergarten and third grade? and 2) Are these understandings impacted by a research-based sequence of lessons taught over a six-week period at each grade level? The study seeks to foster conversations about fraction curriculum in light of mathematics standards and examine the need for cross grade curricula discussions to remove borders and barriers in early elementary grades.

Methodology

The intervention was a sequence of six grade-level appropriate lessons that focused on developing fraction concepts using an exploratory approach with a variety of models. Each 30-45 minute lesson designed using research literature, was taught by pre-service teachers once a week for 6 weeks in a small group setting. The results presented in this proposal are for 116 Kindergartens and 115 3rd graders that were part of a larger study conducted from 2007 to 2013 with students in grades K-3 in an urban elementary school in Central Texas, USA with majority African American and Hispanic population and 83% low socioeconomic status.

Pre-and post-assessments were developed by the research team utilizing a variety of sources such as: *Iowa Test of Basic Skills* (ITBS), *California Achievement Test* (CAT), *Texas Assessment of Knowledge and Skills* (TAKS) tests and Lamon's (2005) work on student fractional understanding. Data analysis using a one-tailed t-test was conducted to determine the impact of implementing a sequence of research-based lessons on key fractional concepts to students at K and 3rd grades.

Results

In Kindergarten all means increased from pre to post with 16 of the 27 questions statistically significant ($p < .05$). In 3rd grade while the means increased on 19 of the 24 questions only 8 showed statistically significant increase ($p < .05$). Findings and implications will be shared.

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Chapter 5

Inservice Teacher Education/Professional Development

Research Reports

Brief Research Reports

Posters

MATHEMATICS TEACHERS' PERSPECTIVES ON FACTORS AFFECTING THE IMPLEMENTATION OF HIGH COGNITIVE DEMAND TASKS

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While there are instructional practices researchers claim teachers should be engaging in, (e.g., use of technology, implementing high cognitive demand tasks) often times teachers are either not engaging in these practices or not successful with implementation (Henningsen & Stein, 1997). I conducted a research study from the perspective of three middle school mathematics engaged in professional development around the implementation of high cognitive demand tasks (Smith & Stein, 2011) and whether or not they could identify when the demand of task was lowered and maintained and what factors contributed to either instance. This report details the teachers' perspectives and hopes to contribute to the body of knowledge around those providing professional development to teachers and how to use teachers' perspectives to shape the professional development to support teachers' use of instructional practices.

Keywords: Teacher Education –Inservice/Professional Development

Smith, Bill, and Hughes (2008) stated, "mathematical tasks that give students the opportunity to use reasoning skills while thinking are the most difficult for teachers to implement well" (p. 132). Teachers should be able to select tasks appropriately and implement those tasks while maintaining demand in order to support students' mathematical thinking (Henningsen & Stein, 1997). These assertions are based from research on teachers' implementation of high cognitive demand tasks that occurred from the researchers point of view (e.g., Henningsen & Stein, 1997; Stein, Grover, & Henningsen, 1996). My goal was to understand from the perspective of middle school mathematics teachers what they cited as impacting the demand when implementing high cognitive demand tasks. In doing so, I was hoping to break down the barrier between teacher and researcher by talking with the teacher and understanding what they claimed made the implementation of a high cognitive demand task difficult and they cited as supporting their efforts to maintain demand. Prior work has solely been from the point of view from the researcher and I wanted to see if teachers could identify the factors that either led to the decline of demand or helped maintain the demand of a task and do more than just provide the account of the lesson from my point of view. If teachers could identify those instances, I maintain that someone working with the teacher would then be able to help a teacher build upon their strengths in maintaining demand and support the teacher around what was lowering the demand of a task. By offering the teachers' perspectives, those working with teachers can anticipate possible roadblocks and provide the necessary support needed when implementing any new instructional practice being learned. The research questions that guided my study included: What are teachers' perspectives of their classroom practices as they implement high cognitive demand tasks? What factors do teachers identify as affecting the implementation of high cognitive demand tasks?

High Cognitive Demand Tasks

Cognitive demand refers to the amount of effort a student needs to expend to think about a problem, and mathematical tasks are categorized in two levels, low and high cognitive demand. Smith and Stein (2011) outlined and characterized four different sublevels of the demands of tasks: memorization tasks, procedures without connections tasks, procedures with connections tasks, and doing mathematics tasks. Memorization and procedures without connections are low cognitive demand tasks while procedures with connections and doing mathematics are high cognitive demand

tasks. This study focuses on teachers implementing high cognitive demand tasks. Research on high cognitive demand tasks to date has focused on whether teachers could identify high cognitive demand tasks and if teachers maintained the level of demand during implementation as determined from the researchers point of view (Henningsen & Stein, 1997; Stein et al., 1996). Stein and colleagues (1996) wanted to observe and portray the nature of the mathematical task as implemented by teachers and found the teachers had success selecting and setting up high cognitive demand tasks but were not as successful in maintaining the intended level of cognitive demand throughout the lesson. Stein and colleagues characterized factors related to the maintenance or decline of cognitive demand during task implementation, but these observations were entirely from the perspective of the researchers. In a follow-up study Henningsen and Stein (1997) looked more closely at the classroom factors associated with the implementation of high cognitive demand tasks, specifically tasks classified as *doing mathematics*. The researchers looked at how tasks at that level affected student engagement and found many factors associated with both the maintenance and decline of demand. Although the researchers described and characterized factors associated with the implementation of high cognitive demand tasks, they used archival data and therefore were unable to include the teachers' perspectives on what happened during task implementation. In these studies, the researchers were unable to converse with teachers to gain their perspectives on whether they realized they lowered demand and if so, their reasons for lowering demand.

Teachers' Perspectives

Research on teachers' perspectives seeks to give voice to teachers. Some research studies on gaining teachers' perspectives involved the researchers understanding topics that have been documented as being beneficial to student learning (eg. technology in the classroom, teaching thematic units) but lack the teachers' perspectives of the reality of implementing such structures in the classroom (Handal & Bobis, 2004; Wachira & Keengwe, 2010). Both Wachira and Keengwe (2010) and Handal and Bobis (2004) wanted to identify barriers that kept teachers from implementing different aspects of instructional practices that research has shown to positively influence students' learning of mathematics. Wachira and Keengwe wanted to gain urban teachers' perspectives on integrating technology in their mathematics classrooms to see if urban teachers' perspectives aligned with research on the benefits of using technology in the mathematics classroom, giving their purpose as follows:

> While the use of technology has been found to be an effective means to produce growth in students' understanding of mathematics content, research findings indicate that few teachers integrate technology into their teaching to enhance student learning. This study sought to explore urban teachers' perspectives on barriers that hinder technology integration in their mathematics classroom. (p. 18)

Handal and Bobis (2004) wanted to capture teachers' perspectives on teaching mathematics around a central theme instead of content and understand what barriers were preventing teachers from teaching thematic units, even though a thematic approach can motivate students and deepen their conceptual understanding of mathematical topics. Barriers teachers cited as preventing them from implementing new instructional practices included no access to technology, unreliability of technology, lack of technology support, lack of time, lack of knowledge, lack of confidence, lack of curricular coherence, and a mismatch from the content being taught and the content on the end of year state test (Handal & Bobis, 2004; Wachira & Keengwe, 2010). Proposed actions to help teachers overcome perceived barriers included strengthening teacher support and building learning communities of teachers.

Theoretical Framework

The theoretical framework guiding my study was the task implementation framework, as shown in Figure 1, developed by Stein and colleagues (1996).
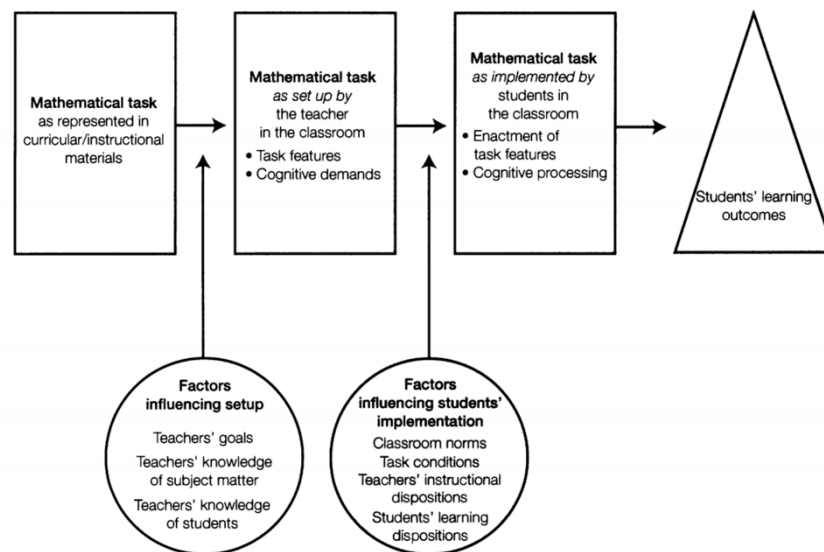
Figure 1. Task Implementation Framework (Stein et al, 1996, p. 459).

I chose to situate my study within this framework because it highlights factors that affect the implementation of high cognitive demand tasks. The task implementation framework "proposes a set of differentiated task-related variables as leading toward student learning and proposes sets of factors that may influence how the task variables relate to one another" (Stein et al, 1996, p. 458). The purpose was to identify factors that changed the task as it was implemented. While a task may start out at one level, during implementation the level may change due to the factors listed in the framework. The level of the task can change between successive phases of implementation: between the task as represented and the task as set up by the teacher or between the task as set up by the teacher and the task as implemented by students. I used the task implementation framework to identify factors affecting implementation when the participants in my study enacted high cognitive demand tasks.

Settings and Participants

I worked with the seventh grade mathematics team at a middle school in the Southeastern United States with approximately 700 students. The participants reported that 75% of the students were eligible to receive free and reduced lunch with 4% of the population being English language learners. The population of the school consisted of 60% African-American, 25% White, 7% Latina/o, 4% Asian, and 4% multi-racial. The three teachers in my study, Mr. Cone, Mrs. O'Neill, and Mr. Fielder all had less than five years experience. Mr. Cone in his 5th year of teaching and taught both mathematics and social studies. His bachelor's degree was in history and decided to get a master's in middle school education where he added a concentration in mathematics. Mrs. O'Neill was in her third year of teaching after being a stay at home mother for 25 years and only taught mathematics. Mrs. O'Neill went through a state certification program where she needed to pass the state certifying exam and be employed by a school. She was hired by the school and earned her certification in a year taking classes online and after school. Mr. Fielder taught mathematics and was in his first year of teaching after completing his bachelor's in middle school education with concentrations in social studies and mathematics. Mr. Fielder was in his second year and he was a student teacher in Mr. Cone's classroom the year before.

Data Collection and Analysis

I collected data through classroom observations and interviews. I provided professional development before school started on high cognitive demand tasks and then initially observed each teacher during one of their class periods for the first two and one-half weeks of school. I then observed each teacher multiple times throughout the semester. I planned with the teachers for the specific implementation of two high cognitive demand tasks, the Figure S Task and the Border Problem (see Figure 2) and made sure to observe and interview the teacher after each of those lessons.

Figure 2. Figure S Task, (adapted from Smith, Hillen, & Catania, 2007, p. 41) and Border Problem (adapted from Boaler & Humphreys, 2005).

During classrooms observations I took field notes and audio recorded and transcribed the interviews. I interviewed each teacher three times with a semi-structured (Patton, 2002) interview that lasted between 40 and 60 minutes. The interviews included questions on conceptions of high cognitive demand tasks, if they thought they maintained or lowered the demand of tasks, when they thought instances of maintaining or lowering demand occurred, and factors that affected the implementation of high cognitive demand tasks.

The goal for data analysis was to gain teachers' perspectives of factors that affected the implementation of high cognitive demand task. The data I used included observations and individual interviews. I went through each teacher's interview line-by-line and found instances where the teacher made a comment about a factor that affected the use or implementation of high cognitive demand tasks. I created narratives separating the data into instances where the teacher cited factors that led to the decline of demand and instances where the teacher cited factors that maintained the demand of the task. Once I had each teacher's narrative, I went through and coded using the factors from the task implementation framework. I coded each teacher's narrative individually first and then looked for common themes across all cases, making note of where the teachers had common factors.

Because I was comparing my perspectives to the teachers, I included a member check in my data analysis that involved checking back with participants to determine whether the analysis accurately represented their experiences. I wanted, to give the participants the opportunity to examine my work and offer advice if they saw a different interpretation (Lincoln & Guba, 1986). I provided each teacher with a copy of my analysis of his/her implementation of tasks in response to the first research question and my analysis of all the teachers in response to the second research question to get their input. Specifically, I wanted to find out if they agreed with the factors I identified as influencing the implementation of high cognitive demand tasks in their classrooms and whether I had omitted any important factors. I emailed each teacher the summary of my analysis telling each to read over the

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analysis and then get back with me on a time so I could meet with each teacher separately. Mr. Fielder responded quickly to the email saying he agreed with my analysis and it was not necessary to meet. I met with both Mr. Cone and Mrs. O'Neill on separate occasions and after discussion the analysis was kept intact.

Teacher Identified Factors Influencing Implementation

Teachers' Instructional Dispositions

Each teacher's instructional disposition was a factor influencing implementation. Stein and colleague's (1996) defined teachers' instructional dispositions as the "features of [teachers] pedagogical … behaviors that tend to influence how they approach classroom events" (p. 461). Examples of teachers' instructional dispositions include the extent to which a teacher is willing to let a student struggle with a difficult problem and the of assistance that teachers typically provide students during that struggle (Stein et al, 1996). All of the teachers said they often guided students to the right answer. They gave different reasons as to why they led students, but each had the tendency not to allow students to productively struggle with the task. Mr. Cone was not apt to let students struggle with the problem but was working on being able to question students instead of just giving them the answer. Mr. Fielder lowered the demand during the Border Problem when the students struggled with finding the border of any size square. He had not intended to lead, but when students could not come up with an expression for any size square, Mr. Fielder showed them how to take each of the numerical expressions and turn it into an algebraic expression rather than asking questions or offering hints to help them find the expression themselves. Mr. Fielder recognized this behavior saying:

> The only struggles I had were in those classes where I had to give it away. It was frustrating because I know that theoretically we are supposed to allow students to develop these ideas themselves, just kind of point them in the right direction. To have to tell someone about this idea and them not get it, that is frustrating because they can't get it on their own and then you try to tell them about it and then they get more confused so then it's almost like you have no idea, and that's the frustrating part.

Mrs. O'Neill also acknowledged that she struggled with leading students too much. During the implementation of the Figure S task, she said she explicitly helped students who were stuck because she was afraid the students would shut down without her help. She said she led students too much during the Border Problem when they were getting an incorrect answer. She initially told a group of students they were getting the wrong answer because they were double counting the corners, but when she realized that she was being too directive, she adjusted her questioning to help the other groups of students arrive at that conclusion without her doing it for them. She said she initially was guiding her students through tasks because she wanted to make sure she had enough time to accomplish her goals. Thus, instead of letting students struggle, she gave them the information needed to solve the problem.

Teachers' instructional dispositions positively influenced students' implementation when the teachers held back and allowed students to productively struggle with the task. Examples of teachers holding back included using questioning techniques or referring students to work with each other instead of relying on the teacher for ideas. Both Mr. Cone and Mr. Fielder pointed to questioning students and not giving away answers as reason for maintaining the demand of tasks. The teachers recognized both instances of maintaining demand due to questioning but also lowering demand due to giving away the information too soon. Mr. Fielder said he had success with the Border Problem because he maintained demand with his on-level students by pulling back and not leading students to the answer. He said he realized the students were getting it on their own, and he could lead less and

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watch the students come up with the ideas of generalizing any size square on their own. Mr. Fielder said he was becoming okay with allowing students to struggle with the mathematics and if they could not finish the task in one day, he would allow them to take two days to struggle with the task. By engaging with the task for two days, they may flail but not give up entirely. Mr. Cone said that he was getting better at using questions, and his goal was to focus students' attention on the mathematical concepts with questions rather than statements. He said he had to make a conscious effort not to give away pieces of information during implementation and he was getting better at letting students "carry more of the cognitive burden" because he tried to answer the students' questions less frequently and encouraged the students to engage with their peers more.

Task Conditions

The task condition related to class time affected the implementation of the task because often there was not enough time in class for students to grapple with the task, thus lowering demand. Mr. Cone attributed class time as a reason for not maintaining the demand saying he did not think the students discussed how to generalize for either problem. Mr. Cone described the implementation of the Figure S task by saying; "I am leaning towards failure because we didn't have a lot of time in class to do it. Students worked on it, but they weren't able to share their thinking. I didn't have a chance to continue that discussion." Mr. Cone claimed he altered tasks because he did not have the time for students to grapple with the mathematics so he would often make the connections for his students instead of letting them try to figure it out. Mrs. O'Neill identified lack of class time as a reason for lowering demand, saying she often gave the answer away because that would save her time in class to be able to get to each part of the lesson.

The task conditions helped the teachers maintain demand related to students abilities of building on prior knowledge. The teachers commented on the task conditions as a reason for the success of the lessons. Mr. Fielder said the Figure S task was successful because the students did not need a lot of prior knowledge to engage with that task. He had the same sentiment with the Border Problem and explained that the task was both easy to set up and implement because the students could engage with the task. He noted that the Figure S task built on itself in a way that students did not need much support from the teacher. He also said the Border Problem had an easy set up in that all he needed to do was give simple instructions, and the students were able to access the task. Mr. Fielder posed the tasks as challenges or problems for the students to solve because he said lessons were successful when he found tasks that he could set up as challenges and have students engage in without realizing they were doing mathematics. Mr. Cone attributed the success of the Border Problem to it having a low entry floor while also allowing students the option of being creative. Mrs. O'Neill spoke of the nature of the Border Problem and noted that all levels of students could access the task; saying, "The kids that were more advanced, they were looking for different, more complex strategies for solving it, so it was a puzzle for your weakest learner as opposed to the one that is more advanced." As an extension, she had her accelerated group create their own patterns to give other students to come up with the generalization for the pattern.

Students' Learning Dispositions

The teachers explained that high student engagement in the class as well as the ability to have a discussion around the task contributed to maintaining the cognitive demand during successful implementations. Mrs. O'Neill said she was successful with task implementation when she was able to have a discussion around the mathematics with the students sharing multiple solution methods because she enjoyed hearing multiple ways the students had solved the task which was different from the beginning of the year when the students just wanted to know how to get the answer. Mr. Cone said success with tasks implemented in his classroom was due to high student engagement. He said that through the implementation of tasks he had become more reflective about his practice of

interacting with students, thinking about how he can clarify, extend, or prompt students' thinking based on what the student says, which he had not thought of before. Mrs. O'Neill claimed students' learning dispositions lowered the demand of the task saying students relied on her or other adults in the room for the answer and would not productively struggle with the task. She wanted students to work by themselves or ask a neighbor if, but often they wanted her to give them the answer. She said that it just took time for the students to get to know each other to feel comfortable talking with their neighbor.

Discussion

The teachers recognized factors that affected the demand of tasks during implementation. Each of the teachers attributed giving away information to lowering the demand and not giving information away to maintaining the demand. According to the task implementation framework this relates to the teachers' instructional dispositions and their inclination as to when to allow students to struggle with the task. The teachers often did not allow students to grapple with the problem. Instead, they intervened to help students find the answer. The teachers realized they were lowering the demand and provided justification for leading students, claiming that if they did not give the information or help the student, the student would have disengaged with the problem out of frustration. From the perspective of the teachers, lowering the demand was acceptable if it kept a student engaged. The teachers attributed task conditions, students' learning dispositions, and teachers' instructional dispositions as relating to maintaining the demand of tasks. The teachers explained that the tasks having multiple solution methods and multiple entry points allowed students to engage successfully with the task. The teachers noted that students at all levels engaged with the high cognitive demand tasks, which was not always common in their classes. The teachers also attributed their instructional dispositions to helping maintain the demand of the task when they intentionally did not provide students with answers or lead them to solution methods.

The teachers were often able to identify the factors that helped to maintain or lead to the decline of demand. This was encouraging because in the future when they implement high cognitive demand tasks hopefully when there is an instance where they start to lower demand, such as giving a leading answer, they may be able to recognize and correct themselves to ask a question or not provide the student with the answer but still give support. The teachers being able to pinpoint when the demand was lowered can help those providing professional development to then provide support around those areas. Because the teachers all realized they lowered demand when giving students the answers when they struggled, I should have taken that information and then provided more targeted support on how to scaffold in such a way that supports student thinking, but does not explicitly give the student the answer.

Overall, there were many factors identified by Stein et al. (1996) that the teachers identified as contributing factors affecting the demand and illuminate the complexity around implementing high cognitive demand tasks. The teachers' ability to identify these factors illustrates teachers can identify when they maintain and lower demand. Each teacher was able to identify what led to the decline of demand and if I had a chance to continue working with the teachers, I would strive to help each with areas that needed improvement such as helping not give information away to students, or helping to plan a task that fit into the allotted class time. I argue those who work in a professional development capacity with teachers need to break down possible barriers between researcher and participant by listening to teachers' perspectives. Researchers can ascertain what interventions will benefit teachers and will fit within the confines of teachers' many responsibilities and abilities. Listening to teachers can help individualize professional development plans and provide targeted support for each teacher. While I initially implemented one professional development session with all three teachers, when analyzing the data from interviews and observations, it became apparent that each teacher needed different interventions and support to help him/her implement high cognitive demand tasks as related

to what each claimed lowered or maintained demand. Researchers can use teachers' perspectives to provide the support necessary to help teachers become successful when implementing new instructional practices in their classroom.

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REPRESENTING VIDEO CLIPS OF STUDENTS' THINKING IN A MATHEMATICS CLASSROOM AS ANIMATIONS

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Video clips and animations have been used to encourage discussions of teaching in professional development. While both representations are useful for teacher noticing of students' thinking, animations may be more viable for sharing with a wider audience. We describe a process for representing a video clip of students' thinking as an animation which preserves the focus on students' actions by using the cK¢ framework and video clip selection criteria. We found two types of clips, explaining clips and working clips. We address challenges with explaining clips where students did not provide enough verbal evidence of their thinking, and challenges with working clips where students worked on different conceptions at the same time. This study is relevant for designers of professional development who wish to increase the uses of a classroom video clip and broaden the set of resources available for promoting teacher learning.

Keywords: Design Experiments, Teacher Education-Inservice/Professional Development, Technology

Introduction

Teacher professional development has used both discussions of videos and animations to help teachers notice students' thinking (e.g. Chieu, Herbst, & Weiss, 2011; van Es & Sherin, 2010). When teachers study videos from their own classrooms with other teachers, they can develop a better understanding of their own practices (Borko et al., 2008). However, using video clips recorded in the classroom of a practicing teacher can create challenges because it is important to maintain a stance of critical inquiry while also supporting teachers (Lord, 1994). In addition, compliance with protocols that require protecting the identity of study participants can limit the availability of video to a wider audience, especially when scaling up a professional development intervention, While animations have been used in the past as a way to design stories of classroom instruction (Herbst, Nachlieli, & Chazan, 2011), they also have potential to allow professional developers to show examples of actual students' thinking using a different representation than a video clip. Representing a video clip as an animation could allow professional developers to show the representation to a wider audience and provoke teacher noticing of student thinking differently. In this paper, we describe a process we used to represent video clips of students' thinking as animations while attempting to preserve the nature of the original video clip, and how we addressed challenges with the differences in the representations.

Using Video and Animations in Professional Development

Video Clips

Video clips of actual mathematics classroom instruction are a popular tool in teacher professional development (e.g. Borko et al., 2008; Coles, 2013; Sherin & van Es, 2005). One use of video clips is a video club (Sherin & Han, 2004), where teachers gather to watch a video clip of students' work in one of their own classrooms and identify and discuss the students' thinking. In a typical video club, the focus is on using evidence found in the video clip to identify what the student was thinking (van Es & Sherin, 2010). Video clubs have been shown to encourage teacher growth in focusing on student thinking over teacher moves (Sherin & Han, 2004), interpreting rather than evaluating students' actions (Sherin & van Es, 2005), and using evidence to back claims about students' thinking

(Sherin & van Es, 2005). The video clip is typically around five minutes in length (Sherin & van Es, 2005), and can be of whole class discussion (e.g. Sherin & van Es, 2005), group work (e.g. Borko et al., 2008), or students' work at the board (van Es, 2009). It is important that the video must highlight something mathematically interesting in the class (van Es, 2009).

Animations

Animations have also been used to encourage teachers' discussions in professional development. One example of the use of animations is the ThEMaT project (Herbst & Chazan, 2003), which uses animations to provoke teachers' discussions of what is typical in mathematics instruction by showing interactions that deviate from the norm. Outside of mathematics education, animations have also been used to promote character education (Bailey, Tettegah, & Bradley, 2006) and teach classroom management to pre-service teachers (Smith, McLaughlin, & Brown, 2012). While a goal of the use of animations is to encourage teacher learning (Chieu, Herbst, & Weiss, 2011; Nachlieli, 2011; Smith, McLaughlin, & Brown, 2012), a contrast with video clips is that animations can be designed to showcase specific aspects of teaching (Herbst & Chazan, 2006). Researchers have described additional benefits of animations as the ability to abstract a case so that teachers could identify with it as their own classroom and students by removing distinctive physical features from the classroom and animated characters (Chazan & Herbst, 2012; Herbst & Chazan, 2006). Overall, animations allow for the intentional design of specific cases of instruction in a way that can be presented to teachers in a generalized setting.

Comparing Videos and Animations

Researchers have found that teachers are able to discuss and analyze classroom interactions across multiple representations of teaching (Herbst & Chazan, 2006; Smith, McLaughlin, & Brown, 2006). When pre-service teachers were shown either a 3-D computer animation or a live action video of a classroom management scenario, there was no difference found in teachers' analysis (Smith, McLaughlin, & Brown, 2006). Herbst and Chazan (2006) found that, though teachers did remark on differences in the temporality of events when shown stories of teaching as animations, comic books, and slide shows, they still were able to discuss the stories as if they were real episodes from a classroom. In addition, while they found differences in the types of statements made, Herbst and Kosko (2014) found that animations and video were equally useful for eliciting teacher evaluations. Research comparing teacher reactions to video clips and animations suggests that animations are a valid representation of students' thinking when using video clips is not viable.

Research Questions

Because teachers have the ability to notice similarly when analyzing video clips and animations (Herbst, Aaron, & Erickson, 2013), and animations make it feasible to share video clips with a wider audience as previously described, we designed a process for representing video clips of students' thinking as animations and addressing challenges that we encountered during that process. Specifically, we wish to address the following questions:

1. How can a video clip be made into an animated vignette while preserving the nature of the original representation?
2. What challenges arise when creating an animated vignette from a video clip of students' thinking?

Because the animations we created were slides with voice-recorded dialogue, we could not create an exact replica of the original video clip with animated characters. However, we wanted the experience of watching the animations to be as close as possible to the experience of watching the original video clips. The first question addresses how we created a process that would allow us to

represent video clips as animations with slides and dialogue while maintaining the same criteria of the video clip that caused us to identify it as worthwhile. However, we found challenges in the process both with how to represent entire clips and with individual aspects of the clips. The second question discusses what challenges we encountered and how we addressed those challenges.

Theoretical Frameworks

cK¢ Framework

We used the cK¢ framework (Balacheff, 2013) to understand how students were thinking during the video clips. The cK¢ framework examines students thinking in terms of conceptions, which are composed of a problem which a student attempts to solve with a series of operations, using a set of semiotic resources, and verifying with a control structure (Balacheff, 2013). The cK¢ framework was useful for the analysis of the video clips because identifying the students' conceptions was a key purpose of viewing and discussing the original video clips. By analyzing the clips with a focus on each student's conceptions, we could ensure we represented the video clip as an animation in a way that preserved each of the original conceptions.

Criteria for Criticizing Video Clips

After we identified the students' conceptions using the cK¢ framework, we used the criteria of *window*, *depth*, and *clarity* defined by Sherin, Linsenmeier, and van Es (2009) to analyze how to represent the video clips as animations. The first criteria, *window*, refers to how well the clip affords the viewer an opportunity to determine what the student is thinking, while *depth* refers to how substantive the student's ideas are, and *clarity* refers to how well one can determine the student's thinking from the video clip (Sherin, Linsenmeier, & van Es, 2009). Because the depth of the students' ideas was dependent on the actual student operations, and not how they were represented, we did not consider the depth of the student idea when deciding how to represent the video as an animation. However, understanding the window into students' thinking for each conception and the clarity of the conception provided a framework for preserving the nature of each video clip when representing it as an animation.

Methods

Data for the project comes from video recorded in the classrooms of five Geometry teachers in high-needs schools that participated in a two-year professional development study group funded by the National Science Foundation, focusing on noticing and using students' prior knowledge. Video clips used in this paper were recorded only during the first year of the professional development, and were used for video club discussions during the professional development study group sessions. The animations to be created were a series of still frames of students working at a small group, with scripted audio. Because the original video clips were determined to be worthwhile examples of student thinking by the framework of window, depth, and clarity described by Sherin, Linsenmeier, and van Es (2009), and we desire to compare the video and animated representations of the classroom interaction, our goal was to maintain the nature of the original video clip as much as possible.

In order to design the animations, we first developed a template, using the students' conceptions framework by identifying the operations that students performed and semiotic resources used (Balacheff, 2013) in the video. We viewed the video in order from start to finish, coding each conception with corresponding operations and semiotic resources. To address the need to create both audio and visual slides for the animations, we described in the template what visual and verbal evidence demonstrated the students' operations for each conception. We then used the operations,

semiotic resources, and evidence to decide what should be included in the animated version of the video clip. We will discuss this process further to answer the first research question.

Results

Designing a Process to Preserve the Nature of the Original Video Clip

When using the template to recommend how to represent the video clips as animations, the central question we asked for each student operation was how to represent that operation to retain the window into students' thinking while not affecting the clarity. In order to do so, we examined the verbal and visual evidence shown by the student for each operation. Then, we asked if the window into students' thinking changed without the visual evidence. If it did not, we did not need to make a new slide for that operation, and relied on the script to represent the student's idea. If it did, we decided what needed to be added to the animation. Our guidelines were to preserve the script if possible, and write a recommendation for a slide to show the visual evidence. Figure 1 shows a flow chart of our process.
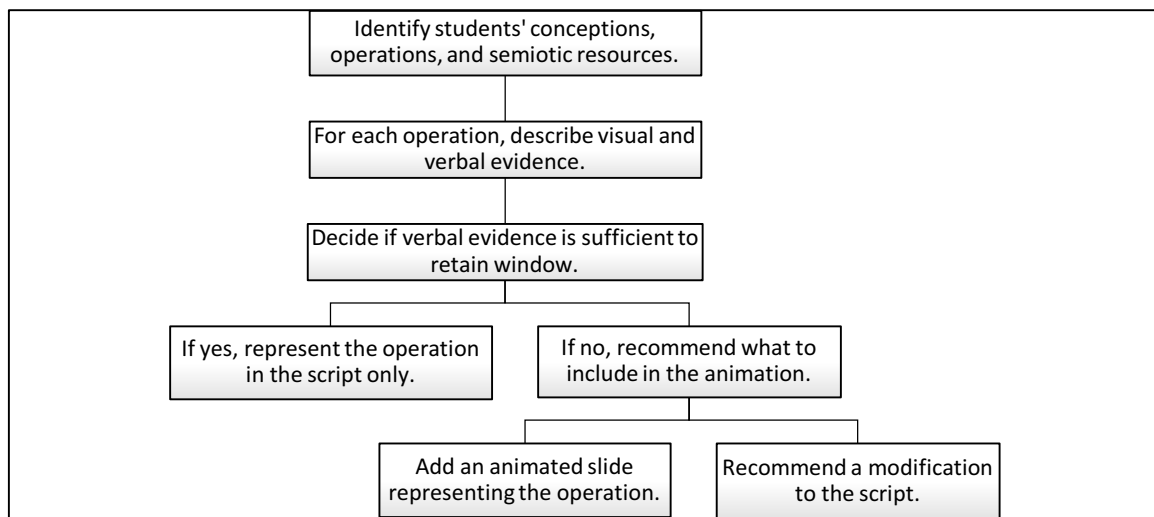
Figure 1. Flow chart of the video to animation process.

As an example of the video clip to animation process, Table 1 shows the template for a video clip of students working on a problem about dilation in the context of one-point perspective. The students are given a diagram with a vanishing point, two houses drawn to be the same size in one-point perspective on the right, and two trees drawn to be different sizes in one-point perspective on the left. In this specific video clip, the students are examining the figure in order to determine which tree is larger.

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Table 1: Example of the video to animation template

Student	Description	What is visible at the beginning of the conception	Time Code Video	Operations	Semiotic Resources	Verbal evidence	Visual evidence	Is visual evidence needed?	Necessary evidence to retain window	Recommended modification
Stephanie	The front house is missing a line.	Students' worksheets with answers for question 1	0:40	Describe that she doesn't like the missing line.	Worksheet, pencil as pointing device	She says she doesn't like that there's a line missing	Pointing (missing line)	Yes	Where is the missing line	Slide: Show the paper with Stephanie pointing to the missing line.
Cassie	Use appearance to decide the larger tree	Same as before	0:58	Use appearance to determine the front tree is larger.	Worksheet, finger as pointing device	"But this tree does seem bigger than this one."	Pointing (trees)	Yes	Which tree looks bigger	Script: Replace "this tree" with "the front tree"
Stephanie	Use where the perspective line crosses the tree to decide which is larger.	Same as before	1:00	Examine where the perspective line crosses each tree.	Worksheet, finger as pointing device	"But if you look at it where this line crosses and it crosses right there it gets smaller."	Pointing (intersection of perspective lines and trees)	Yes	What intersections she is examining	Slide: Stephanie pointing to the perspective line crossing the larger tree.
				Determine the trees are the same size because of how the line crosses.		"I am saying yes because they are on the same vanishing point thing."	None	No	None	
				Write her answer on her paper.		None	None	No	None	
Stephanie	Determine what to call the lines.	Same as before	1:30	Ask what to call something on her paper.	Worksheet, pencil as pointing device	"What would you call this? Because that is the vanishing point."	Points to the vanishing point	Yes	What is she determining the name of	Slide: Stephanie pointing to the vanishing point.
				Explain that the point already there is the vanishing point.		"Because that is the vanishing point."	None	No	None	
				Decide to call it "perspective lines".		"Yea let's just call it that."	None	No	None	
Donell	The trees are the same size, and look different because of how far you are from them.	Same as before, with more written explanation in the answers to the questions.	1:50	Decide the trees are the same size because you are closer to the front one.	Worksheet	"I said yes because the closer you are to it the bigger things look. "	None	No	None	
				The second tree and house look smaller because they're further away.		"But the only reason why the second tree and house look smaller is because they are further away."	None	No	None	

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First, we used what was visible at the beginning of the clip to determine what the initial slide in the animation would be. At the beginning of the clip, Stephanie is addressing the figure of the houses, one of which has a line missing that she remarks about. Her visual evidence to support this conclusion is pointing at the line. While she verbally claimed that a line was missing, the visual evidence of which line is necessary to retain the window into Stephanie's thinking. This required a new slide showing Stephanie pointing to this line. Then, when Cassie uses appearance to determine the larger tree, she claims, "But this tree does seem bigger than this one." Again, the window into her thinking decreases if we do not know which tree she is referring to. However, because there was a slide added directly before and after this comment, we chose to modify the script to retain the window, replacing "this tree" with "the front tree". In the next operation, Stephanie again points to indicate which line she is referring to, so we recommended a slide showing her pointing to the lines she was referring to. Stephanie does not show visual evidence in the next two operations, so no modifications were necessary to retain the window of those operations. However, in the next operation, she asks what to call the vanishing point by referring to it as "this" and pointing, so we added a slide of her pointing at the vanishing point. In the next two operations, she explicitly describes her thinking, and the script is sufficient to provide a window into her thinking. The final conception of the video, from Donell, is a case where the script does not provide a clear window into his thinking when he says, "I said yes because the closer you are to it the bigger things look." However, in the video, Donell does not show any visual evidence of what "it" is, so adding a modification to the slides or script would provide a greater window into his thinking than was shown in the original clip. The same is true with his final comment. As a result, no slides were added to show Donell's thinking.

Challenges in the Process of Representing Videos as Animations

Explaining clips vs. working clips. During the process of representing video clips as animations, we encountered two main types of challenges, challenges with the clip as a whole and challenges with individual operations. In terms of the clip as a whole, we found two main categories of clip, explaining clips and working clips. Explaining clips were those where the students were explaining their thinking to each other, such as the example above, whereas working clips were those where the students were working on new ideas. Some clips included both explaining and working.

Challenges with explaining clips. In general, explaining clips went smoothly using the process we previously described. These clips showed one student describing one conception with one set of visual evidence at a time, and students typically provided verbal evidence to describe their thinking. Most of the modifications we made involved adding slides or making small changes to the script in order to retain the window when students used vague language like "it" or "this one". The main challenges with explaining clips were difficulties representing certain individual operations in the animation.

In an explaining clip, it was difficult to represent when a student showed their work to another student, rather than specifically describing the actions they performed. In these cases, we chose to create a slide showing the student's work. We chose not to add a verbal description to the script because a verbal description would provide a higher window into students' thinking than the exclusively visual evidence provided in the original video clip. A second challenge was when students referred to multiple quantities in the video clip, but were either not specific about the quantities or specified by pointing. In this case, we had two options; to either add a visual slide showing the student pointing, or change the script to add a verbal description. We considered which modification would least affect the window and clarity of the original clip. For example, in Stephanie's first conception above, she is referring to a specific line on the paper, but there was no easily identifiable name for the line and it was the first new slide in the animation, so we chose to create a slide showing her pointing at the line. On the other hand, in Cassie's first conception, she is

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referring to one of two trees and adding the word "front" to clarify which tree she referred to only modified one word of the script.

Challenges with working clips. Working clips were more challenging to represent as animations because of the nature of the entire clip, rather than challenges with individual operations. Working clips typically showed students independently working on different conceptions simultaneously, students spoke less, and each student showed different visual evidence for their own conception. One adaptation we made to the process for a working clip was to analyze the clip one student at a time, rather than analyzing the conceptions sequentially. This allowed us to remain consistent with the template by examining one conception at a time. After analyzing each individual student, we integrated slides showing the progression of all three students' conceptions back together as close to the original clip as possible.

One challenge in working clips was when a student did mathematical work without explicitly describing it. In some cases, the same clip showed students explaining their operations later. In these cases, we chose to include the explanation of the operations in the animation rather than the student working silently. This allowed the students' thinking to be shown in the video without repeating the same conception, while also retaining the clarity from the original video clip. If the student did not describe the work later, we showed visual slides representing the work without modifying the script. This remained consistent with the original video, where viewers needed to watch the student work on the problem without verbal evidence of what was being done.

Conclusion

Videos and animations have both been used to elicit teacher noticing of students' thinking (Chieu, Herbst, & Weiss, 2011; van Es & Sherin, 2010). Because researchers have found that teachers are able to notice similarly across different representations of students' thinking (Herbst & Kosko, 2014), and due to the ability of animations to reach a wider audience and represent a more general classroom (Chazan & Herbst, 2012), we designed a process to represent video clips we showed in a video club as animations. We found two types of video clips: explaining clips and working clips. While we were typically able to represent an explaining clip as an animation by following the process we designed, we addressed challenges with individual student operations that did not provide enough evidence to retain the window or clarity of students' thinking by making small changes to the script or adding slides to show visual evidence. Working clips were more challenging to represent as animations due to the lack of verbal evidence given by students while working. However, we were able to adapt the process of representing video clips as animations by examining each student individually, making recommendations for what visual evidence would be necessary to retain the window into students' thinking, and integrating the students' conceptions together as slides. Overall, using the dimensions of window and clarity allowed us to both consistently choose whether and when to adapt the script or create visual slides while also ensuring we avoided providing too much evidence in the animation and reducing the ability for teachers to notice student thinking themselves. Further work could compare teachers' reactions to video clips and animations of the same interaction. We believe this work is useful to researchers and designers of professional development as a process for creating animations from video clips in order to increase the number of uses of video clips, create animations that are as realistic to the original video clip as possible, and widen the audience for sharing examples of students' thinking.

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NEGOTIATING MEANING: A CASE OF TEACHERS DISCUSSING MATHEMATICAL ABSTRACTION IN THE BLOGOSPHERE

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Many mathematics teachers engage in the practice of blogging. Although they are separated geographically, they are able to discuss teaching-related issues. In an effort to better understand the nature of these discussions, this paper presents an analysis of one particular episode of such a discussion. Wenger's theoretical framework of communities of practice informs the analysis by providing a tool to explain the negotiation of meaning in the episode. Results indicate that the blogging medium supports continuity of discussions and can allow for the negotiation of meaning, but that a more nuanced treatment of the construct is necessary.

Keywords: Teacher Education-Inservice/Professional Development, Technology, Learning Theory, Informal Education

Introduction

> I've been throwing around the term "pyramid of abstraction" recently, and there was some great pushback on Twitter this evening. This post is my attempt to clarify what I mean, and why I think it is a useful perspective to building students' knowledge. (Kane July 31, 2015)

Mathematics teacher Dylan Kane wrote the passage quoted above in a blog post after a series of Twitter interactions with mathematics teacher bloggers Dan Meyer and Michael Pershan. The series of interactions were prompted by Dylan's original blog post in which he explained ideas about a metaphor for mathematical abstraction as pertaining to his teaching practice. The discussion that resulted from this formulation presents an instance of a negotiation of meaning around a metaphor related to the teaching and learning of mathematics. Most importantly, it provides an example of how mathematics teachers are engaging in the practice of blogging, which is the focus of this paper. It should be noted, however, that this paper is not concerned with teacher knowledge (Ball, Thames, & Phelps, 2008; Shulman, 1986), but rather with the social interaction among mathematics teachers that the blogging practice affords.

Mathematics Teacher Blogging

Mathematics teachers around the world are choosing out of their own will to create, organize, and manage their own personal blog pages. On these publicly visible virtual pages, they make relatively regular posts, which are most often related to their work as mathematics teachers. These posts can include written and/or media enhanced recounts from their teaching experiences, links to interesting resources, and responses to posts that other bloggers have made.

Since blog pages are individually managed and no universal blogging platform exists, finding like-minded bloggers may at times be difficult. To mitigate this issue, many bloggers extend their practice through Twitter, a universal micro-blogging platform that allows users to create searchable profiles. There are currently 431 profile listings on the Math Twitter Blogosphere directory, most of whom have both a Twitter handle and a blog page listed on their profile. Although micro-blogging on Twitter is slightly different in nature than blogging due to the 140 character limit on each post, which forces users to communicate ideas more concisely, it is still considered a form of blogging (Ebner, 2013). Blog pages allow users to explore ideas deeply, but Twitter makes it easier for like-minded bloggers to connect. Mathematics teacher bloggers often use both mediums to express ideas, linking between the two when necessary. For instance, some bloggers include snips of Twitter conversations

in their blog posts, and other bloggers link to their blog posts within their Twitter posts. In this way, teachers are communicating with peers across the world.

This sort of collegial interaction is unusual for teachers because teaching is generally an isolated profession (Flinders, 1988; Lortie, 1975). Mathematics teachers' typical contact with other mathematics teachers is limited to that of department meetings and teacher lunchrooms, where teachers usually plan lessons and grade papers individually (Arbaugh, 2003). Sparse one-time professional development initiatives are not generally found to be effective in stimulating teacher collaboration (Ball, 2002), and they don't often extend into more regular professional development opportunities because they require time, funding, and facilitation.

However, there are hundreds of mathematics teacher bloggers who seem to be overcoming these constraints. It is evident that they spend hours writing publicly about their daily practice, posting resources, and sharing their dilemmas with no compensation and no mandate. This unprompted, unfunded, and unevaluated teacher activity is a rich phenomenon of interest that deserves attention. Ironically, this phenomenon is largely unstudied. Empirical investigations related to blogs in mathematics education are limited to studies on the utility of blogging as a pedagogical tool within either a mathematics course (Nehme, 2011) or a mathematics education course (Silverman & Clay, 2010; Stein, 2009). These studies do not account for the autonomous and self-driven nature of the blogosphere and there is no clear work in mathematics education exploring the activities of these teachers.

Research Question

As such, this study is guided by the overall research question of what the mathematics teacher blogosphere affords for teachers who engage in blogging in relation to their practice. To this end, in this paper I pursue an investigation of one episode that exemplifies the type of interaction that is possible within the mathematics teacher blogosphere.

Theoretical Framework

Since blogging is an individual practice that is made public (Efimova, 2009), a mid-level theory that accounts for situated participation is desirable. C*ommunities of practice* (Wenger, 1998) is one such theory: it is a social theory of learning where learning is considered as increasing *participation* in the pursuit of valued enterprises that are meaningful in a particular social context. *Practice* is at the heart of Wenger's (1998) *communities of practice*, and a key aspect of *practice* is the ability to motivate the social production of *meaning*. The continuous production of *meaning* is termed as the *negotiation of meaning*, and is further defined by the duality between *participation* and *reification*. For Wenger (1998), *participation* is "a process of taking part [as well as] the relations with others that reflect this process" (p. 55), and *reification* is "the process of giving form to our experience by producing objects that congeal this experience into 'thingness'" (p. 58). Both *participation* and *reification* shape the participant and the community in which they participate in an ongoing manner.

According to Wenger (1998), *participation* and *reification* imply each other, require and enable each other, and interact with each other. Wenger (1998) notes that the benefit to viewing the *negotiation of meaning* as a dual process is that it allows one to question how the production of meaning is distributed. He notes that there is "a unity in their duality, [because] to understand one, it is necessary to understand the other, [and] to enable one, it is necessary to enable the other" (Wenger, 1998, p. 62). Various combinations of the two will produce different experiences of *meaning*, and together they can create dynamism and richness in meaning if a particular balance is struck.

Wenger (1998) notes that "when too much reliance is placed on one at the expense of the other, the continuity of meaning is likely to become problematic in practice" (p. 65). If *participation* dominates, and "most of what matters is left unreified, then there may not be enough material to anchor the specificities of coordination and to uncover diverging assumptions" (p. 65). However, if

reification dominates, and "everything is reified, but with little opportunity for shared experience and interactive negotiation, then there may not be enough overlap in participation to recover a coordinated, relevant, or generative meaning" (p. 65). In essence, *participation* allows for re-negotiation of *meaning*, and *reification* creates the conditions for new *meanings*. In this interwoven relationship, the two aspects work together to drive the process of *negotiation of meaning*.

The interplay between *participation* and *reification* is different in each unique social situation, contributing to a different experience of *meaning* for participants of that *practice*. What is of interest in this paper is how mathematics teacher bloggers experience the social production of *meaning* within the blogging *practice*.

Method

In order to be able to view mathematics teacher blogger conversations on Twitter, I have spent over a year collecting subscriptions. Every user sees different posts based on who they subscribe to, and it is important to note that my pervasively subjective position influences what I can notice in this ultra-personalized and dense virtual environment.

Initially, my position was predominantly that of a 'lurker' in that I had not made significant contributions to the blogosphere. This position changed slightly after my attendance to the MTBoS conference 'Twitter Math Camp 2015' (TMC15). Physically meeting many of these bloggers connected me to them more than before. As my subscription list grew, I was also able to view more of their conversations. It was during this time after my return from TMC15 that I encountered a particular conversation between Dylan, Michael, and Dan that I flagged as interesting based on my theoretical framework and for its power to illustrate a possible mode of interaction in the blogosphere.

After the conversation took place, I used *storify.com*[1] to identify all tweets related to the conversation from the feeds of each of the participants (Dylan, Michael, and Dan), and rearranged them chronologically. I copied the written content of each post, the name of the participant, and the time stamp, and pasted it into an offline spreadsheet document. I also included the written content of any blog post that was linked to in the Twitter posts within this document. This reconstructed conversation made of Twitter and blog posts comprises the data set for this paper.

This data set was then coded according to Wenger's (1998) *negotiation of meaning* so that conclusions could be drawn about participant experiences of the social production of *meaning*, as embedded within the *practice* of teachers in the blogosphere. As part of the analysis, moments of *participation* and *reification* were coded and labelled at each instance. In general, *participation* was considered to be any action that a blogger took as part of the blogging *practice*, and *reification* was considered to be any trace that was left from a *participation*. The interplay between these aspects was then considered in relation to the data, and conclusions were drawn about the nature of the *negotiation of meaning* as exemplified in this case of mathematics teacher blogging.

In what follows, a reduced version of the reconstructed conversation is presented in the results section, and is then reviewed in terms of Wenger's (1998) *negotiation of meaning* construct in the analysis and discussion section.

Results

On June 25, 2015, Dylan writes a blog post about his ideas regarding how teachers can help students deal with solving difficult mathematical problems by helping them become better equipped to transfer prior knowledge to new contexts in mathematics. He refers to the popular 'ladder of abstraction' metaphor, and claims that it is incomplete.

> I often hear references to the "ladder of abstraction" — the idea that students' understanding begins with the concrete, and climbs a metaphorical ladder as it becomes more and more abstract. I think this is a useful metaphor, but is also incomplete. (Kane June 25, 2015)

Dylan then suggests an alternative to the metaphor that would make it more useful in classroom practice and more reflective of how he has experienced students learning mathematics.

> I think the metaphor of a ladder of abstraction would be better replaced by a pyramid of abstraction . . . I worry that the ladder of abstraction metaphor leads me to believe that, once a student understands one concrete example of a function, they are ready for a more abstract example. While some students may be, I want to focus on building a broad base first, and then moving up the pyramid after we have spent time analyzing the connections between the examples and the underlying structure. (Kane June 25, 2015)

A month later, on July 17th, Dylan writes a reflection about his experiences at PCMI, a three-week summer mathematics institute. In this post, he discusses the importance of letting students engage in productive struggle within mathematical problem solving, without oversimplifying. Just after making this point, he uses a hyperlink (italicized below) to refer back to the June 25th blog post in which he had introduced the idea of a 'pyramid of abstraction.'

> This gets at something I wrote about recently that I called the *pyramid of abstraction* – that students build abstract ideas from looking at connections between a wide variety of examples, rather than simply jumping from concrete to abstract. (Kane July 17, 2015)

Just as for his June 25th post, Dylan publishes a link to his July 17th post on Twitter (Figure 1).

Figure 1: Dylan's Twitter post linking to his blog post.

Shortly after publishing the link to his July 17th blog post on Twitter, Michael Pershan responds in agreement with Dylan's reference to the 'pyramid of abstraction' metaphor (Figure 2).
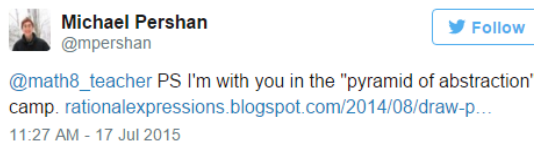
Figure 2: Michael's response to Dylan's post.

Fifteen days later on July 31st, Dan Meyer challenges Dylan and Michael on Twitter by asking about the meaning of the apex of the 'pyramid of abstraction' (Figure 3).
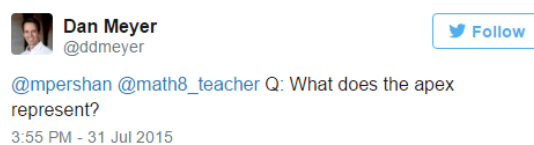
Figure 3: Dan's question to Dylan and Michael.

This is followed by a series of interactions on Twitter between Dan, Michael, and Dylan that occurs over the course of a few hours. This series of interactions is presented in transcript form below. Some comments have been removed for brevity.

> *3:55 Dan:* Q: What does the apex represent?

4:13 Michael: Like, the uber-apex? Or the apex for a skill family?

4:23 Dylan: an abstract principle that can be transferred among multiple contexts-think of all math as nested pyramids of abstraction

4:24 Michael: And at the very tippy tippy top something like "do math," I guess.

4:28 Dan: I'm asking about the significance of the pyramid's tippy-top in this newfangled metaphor.

4:29 Dan: I don't think Dylan's explanation works. I can always abstract the "abstract principle" he puts at the top.

4:30 Dan: Abstraction has no end. Seems to me you guys are in a jam.

4:31 Dylan: maybe abstract isn't the best word. Heart of this for me is knowledge that will transfer to a new context

4:35 Dan: Pyramid of something-other-than-abstraction then? Curious what it is you're trying to describe.

4:38 Dylan: I'm defining abstract as knowledge that transfers. I want to stick with that, but it's worth defining more carefully

4:39 Michael: I'm staying away from such a tough problem as trying to define "abstract"!

4:40 Michael: When I am interested in pyramids of abstraction, it's an attempt to describe mathematical thinking.

4:40 Michael: There are strategies that we use that often represent bundles of strategies, and so on.

4:55 Dylan: my premise is that students need many representations to abstract from, hence a pyramid.

5:07 Dan: But abstraction requires multiple instances /by definition/.

5:08 Dan: Just saying this pyramid thing complicates an already complicated concept.

A few hours later, at 9:28PM on July 31st, Dylan publishes a blog post in response to this conversation, starting with the post initially quoted in the introduction of this paper, and proceeding to explain how he has defined 'abstraction.'

> I'm defining abstraction very specifically. A student abstracts a concept, or builds abstract knowledge, if they can apply that knowledge in multiple contexts. (Kane July 31, 2015)

He then discusses why he thinks a 'pyramid of abstraction' is useful in teaching mathematics and why this metaphor matters to him as a teacher.

> There are a set of teacher actions we can take to facilitate *transfer* — that moment when a student applies a concept they understand to a context they haven't seen before. That's what I'm chasing. (Kane July 31, 2015)

In the last section of his blog post, he lists issues he still has with the metaphor, acknowledging that it is possible the metaphor obfuscates the concept he wants it to represent, and asking the reader if it makes sense to them. He also concedes that there may be no end to abstraction, but that he personally doesn't believe this is true.

> If points in the coordinate plane are an abstract concept, and linear relationships are another abstract concept, and functions are another abstract concept, do we end up in an infinite pyramid of abstraction? I don't think so. I think each of those ideas can then be a building block for broader mathematical concepts. (Kane July 31, 2015)

Finally, he admits he is integrating knowledge transfer and abstraction into one idea, and implicates that this may be a result of his own understanding of 'abstraction.'

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Do I Actually Understand Abstraction? I'm redefining abstraction a bit, and I'm also lumping knowledge transfer as one giant idea, which it might not be. (Kane July 31, 2015)

Analysis and Discussion

Initially, Dylan *participates* in thinking about his past formulations about problem solving and writes these ideas in the blog post that he publishes on June 25th. Within this process of *participation,* Dylan engages in a process of *reification* when formulating his 'pyramid of abstraction' metaphor. He also produces a public trace of this process, which can be seen as the product of his *reification*. This trace is made even more public than a blog post because Dylan also links to it on Twitter.

On July 27th, Dylan *participates* again in this same manner by posting a blog post and linking to it on Twitter, but this time with a focus on recapturing his experiences from a professional development encounter. In this process of *participation*, he uses a hyperlink to reference a *reification* he made on June 25th regarding his formulation of the 'pyramid of abstraction' metaphor. In this way, the June 25th *reification* has prompted further *participation* around this topic on July 27th, resulting in a new *reification*, and a *re-negotiation* of meaning. The concept now has a rich history, and is traceable through the use of hyperlinks.

When Michael publicly agrees on Twitter with Dylan's metaphor, a *participation* and *reification* in itself, it makes Dylan's 'pyramid of abstraction' *reification* even more public, and catches the attention of Dan fifteen days later, who then challenges the metaphor. The power of Dan's *reification* (3:55) is that it is pervasively public. Dan currently has 46,324 followers, Michael has 4,761, and Dylan has 1190. The nature of Twitter is that if one member tags another in a post, only those who follow both members see the post in their feed. Since Dan, Michael, and Dylan share followers as mathematics teacher bloggers, the number of people who see such a post is large.

This now very visible *reification* prompts a series of *participations* and *reifications* from all three of these members as well as from any 'lurkers' who may be reading. In particular, Dan's question about the apex of the pyramid (3:55) is a *reification* that prompts *participation* from both Dylan and Michael, who *reify* their interpretations of the 'pyramid of abstraction' metaphor. Dylan *reifies* his focus on knowledge transfer as the implication of the pyramid metaphor (4:23), and Michael *reifies* his vision of the pyramid metaphor as a skill family (4:13) that ultimately comprises mathematics (4:24). In this way, the *meaning* of the metaphor is being *negotiated*.

Dan subsequently *participates* by posting that he does not think Dylan's explanation works (4:29) and that there can be no 'top' to abstraction (4:30). This *reification* stimulates a *re-negotiation* of the term 'abstraction.' Dylan *participates* further by *reifying* his focus on knowledge transfer (4:38) while Michael *participates* by *reifying* that the focus should not be on defining abstraction, but rather on how students bundle strategies in mathematics (4:39-4:40). This shifts the *negotiation of meaning* to a focus on the applicability of this metaphor to mathematical learning, and Dylan *reifies* that his intent with the metaphor is to explain that students need multiple representations of a concept in order to be able to abstract mathematical meaning (4:55). Dylan then restates his impressions of the conversation and refines his concept of the 'pyramid of abstraction' metaphor in the blog post he publicizes a few hours later. This post may be seen as a reflective form of *participation* in which Dylan *reifies* a heightened level of awareness regarding the 'pyramid of abstraction' metaphor. Perhaps the most powerful *reification* Dylan makes in this blog post is that 'abstraction' is not ubiquitously defined. This ultimately reflects Dylan's experience of *meaning* in this context.

Wenger (1998) states that "having a tool to perform an activity changes the nature of that activity" (p. 60). It is clear that the blogging tool has done just that for these mathematics teachers because unlike in a face to face discussion, the practice of blogging results in *participations* that directly produce permanent, public, and traceable *reifications*, making them prime contenders for prompting *participation* and holding participants accountable for their statements. As such, the

continuity of *meaning* within this medium clearly does not suffer as Wenger (1998) warns can happen if too much emphasis is placed on either *participation* or *reification*.

Further, the asynchronicity of the medium allows participants to take time between responses, which may imply various degrees of *participation* in the practice, and in turn, various degrees of *reification*. Unfortunately, Wenger's (1998) construct does not provide a mechanism for such differentiation. He defines each component broadly, including a wide variety of interactions, most of which would occur in a face to face setting. However, in the case of blogging, *participations* include not only writing posts, but also restating, questioning, and devising examples, while *reifications* include not only publicized posts, but also ideas that have now attained a state of 'thingness' in the community, which here is most prominently the 'pyramid of abstraction' metaphor.

Conclusions

I have illustrated that the blogging medium changes the nature of mathematics teacher discussions because its design allows for *participation* and *reification* to be closely intertwined in a way that their co-evolving and co-implicated relationship drives the *negotiation of meaning* among participants, ensuring continuity of *negotiation*. Such continuity is an important affordance of the mathematics teacher blogosphere experience, and it can be attributed to three important features: asynchronicity, permanency, and publicity. Asynchronicity allows users to *participate* and *reify* to different degrees depending on how long they take to respond, permanency allows *reifications* to prompt further *participation* even days or months later, and publicity allows *reifications* of *participations* to be visible by many 'lurkers,' whose presence makes *participants* accountable for their *reifications*.

I have also revealed that a more nuanced treatment of *participation* and *reification* is needed for mathematics teacher blogging in particular. Some *participations* are quick and spontaneous tweets, while others are prolonged and accompanied by reflective activity. Some *reifications* are mere traces, while others are concepts such as a 'pyramid of abstraction.' Heightened levels of awareness resulting from a prolonged *negotiation of meaning* can also be considered *reifications*. There is currently no terminology within Wenger's (1998) construct to refer to such higher order *participations* and *reifications*.

Finally, the construct of *negotiation of meaning* also has the power to expose *what* is being negotiated. In this case, teachers found it valuable to invest time into engaging in *negotiating the meaning* of 'abstraction' and the metaphor of a 'pyramid of abstraction' as opposed to a 'ladder of abstraction.' Looking for more of these instances may help identify teaching-related issues of interest to mathematics teachers. Most evidently, the development of ideas can be traced within the blogging medium. Each idea has a history and a future. The history is traceable, and the future is quickly unfolding.

Endnotes

[1] *Storify.com* is an online service that allows users to curate content from various social media sources, and arrange it in any order, with narration if desired. It also makes it possible to automatically arrange content chronologically.

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FROM THE UNIVERSITY TO THE CLASSROOM: PROSPECTIVE ELEMENTARY MATHEMATICS SPECIALISTS' PEDAGOGICAL SHIFTS

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This project focuses on the development of prospective Elementary Mathematics Specialists (EMSs) in a K-5 Mathematics Endorsement Program. Program courses emphasized elementary mathematics content and pedagogy while providing opportunities for participants to evidence their learning through classroom teaching practice, all in an attempt to facilitate pedagogical shifts toward more standards-based instruction and dialogic discourse. Data presented include scored observation rubrics of classroom practices, professional portfolio reflections on implementation of new teaching practices, and semi-structured, individual interviews.

Keywords: Teacher Knowledge, Teacher Education-Inservice/Professional Development, Problem Solving

Purpose

Many in the field of mathematics education are proponents of Elementary Mathematics Specialists (EMSs) and the critical role they have in elementary schools of supporting effective mathematics instruction. The recent unified position of several prominent mathematics education organizations, including the Association of Mathematics Teacher Educators (AMTE, 2013), asserts every elementary school in the U.S. should have access to an EMS and that advanced specialist certification should be offered via intensive preparation programs. One salient challenge associated with EMSs' preparation is their development of new ways of teaching in elementary classrooms.

Adopted by most states in the U.S., the Standards for Mathematical Practice in the *Common Core State Standards for Mathematics* (CCSS-M; National Governors Association Center for Best Practices [NGACBP] & Council of Chief State School Officers [CCSSO], 2010) and the National Council of Teachers of Mathematics' *Principles to Actions* (NCTM, 2014) both strongly depict *standards-based learning environments* (SBLE) that foster dialogic discourse and conceptual understandings of mathematics. In such learning environments, students make conjectures about their mathematical ideas and explain their thinking and reasoning while teachers value students' multiple perspectives and carefully craft mathematical discussions by using their ideas to bring the classroom to shared mathematical understandings. This pedagogical approach requires thoughtful planning and questioning on the part of the teacher and is a key shift away from traditional classroom instruction. An important goal of EMS preparation is the development of these pedagogical competencies.

Our university is located in one of the limited number of states in the U.S. offering EMS certification (i.e., a K-5 Mathematics Endorsement [K-5 ME]). Key goals for participants in the K-5 ME program are pedagogical shifts toward alignment with a SBLE, development of the *specialized content knowledge* (SCK) necessary for teaching mathematics, and changes in mathematical beliefs. Research in general on EMS preparation is limited, and our previous research on this particular program shows teachers had significant: increases in SCK, changes in pedagogical beliefs toward a cognitive orientation, and increases in mathematics teaching efficacy (Swars, Smith, Smith, Carothers, & Myers, 2016). This study in particular extends the existing inquiry on the program by focusing more so on the translation of learning in the K-5 ME courses into instructional practices. It was guided by the following research questions: What pedagogical shifts do prospective EMSs experience as they complete a K-5 ME program? How do prospective EMSs describe their pedagogical shifts during a K-5 ME program?

Literature Review

EMSs are generally considered to be teachers, teacher leaders, or coaches with the expertise to support effective elementary mathematics instruction and student learning in the classroom, school, or other levels (AMTE, 2013). Over time, the roles of EMSs have mostly been considered in two ways, including EMSs who work with students and EMSs who work with teachers (AMTE, 2013; Reys & Fennell, 2003). In general, the specific roles and responsibilities of EMSs vary according to the contextual needs and plans of schools and school systems, with an increasing number of schools utilizing EMSs in some manner (Gerretson, Bosnick, & Schofield, 2008). The extant studies on EMSs have focused on improving instructional practices, designing coaching programs, and improving student achievement, with overall results showing positive impacts of EMSs on teacher development and student learning (Campbell & Malkus, 2011; McGee, Polly, & Wang, 2013).

A body of research shows that classroom pedagogy has more influence on improving student learning than the use of particular curriculum materials (Brown, Pitvorec, Ditto, & Kelso, 2009; Remillard, 2005; Tarr et al., 2008). The *Principles and Standards for School Mathematics* (NCTM, 2000) and the CCSS-M recommend the intersection of mathematical content and process standards requiring a pedagogical approach different from the traditional direct instruction in computational skills still found in many U.S. classrooms. Many of these suggestions are grounded in constructivist compatible instruction, where teachers: engage students in real-life contexts; provide students with original, non-routine problems; and develop a classroom community grounded in dialogic discourse intended to develop students' individual and shared understandings of mathematical concepts and practices in ways that nurture their abilities to problem solve, reason, and communicate mathematically (Charalambous & Hill, 2012; Cobb & Jackson, 2011; Tarr et al., 2008). Teachers are creating SBLEs and ensuring their instructional tasks hold high levels of cognitive demand (Porter, 2002; Stein, Smith, Henningsen, & Silver, 2009). According to the Tarr et al. (2008) study, improved student learning and achievement was connected to the extent of enactment of such a SBLE.

Methodology

This inquiry used a descriptive, holistic case study design. The case was the described experience of implementation by a group of elementary teachers in an EMS preparation program. We focused on their efforts to enact instructional practices in their classrooms drawn from their learning in the program. Collected data were both qualitative and quantitative in nature.

Participants and Setting

Participants were 13 elementary teachers enrolled in a graduate level K-5 ME program at a large, urban university in the southeastern U.S. The teachers worked at one urban, elementary school recently converted to a charter school. They held varying teaching positions at the school, including those of classroom teacher, small group resource teacher, and instructional coach, but all worked with the primary grades.

A foremost goal of the program was the development of a deep and broad understanding of elementary mathematical content, including the *specialized content knowledge* (SCK) for teaching elementary mathematics (i.e., the "mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students" [Ball, Thames, & Phelps, 2008, p. 399]). The program also focused on high-leverage teaching capabilities in the elementary classroom, including: (a) selection and implementation of mathematical tasks with high levels of cognitive demand, (b) use of multiple mathematical representations, (c) use of mathematical tools, (d) promotion of mathematical dialogic discourse, explanation and justification, problem solving, and connections and applications typical of a SBLE, and (e) use of children's thinking and understandings to guide instruction. Learning during course sessions occurred via: (a) active inquiry and analysis of the mathematics in the elementary curriculum, specifically the CCSS-M, (b) study of children's thinking and learning by way of video

clips and teaching cases, (c) examination of examples of classroom practice through video clips and teaching cases, and (d) scrutiny of the research base related to mathematics teaching and learning in the elementary grades.

The program was two semesters in duration and included four 3-semester-hour mathematics content courses for elementary teachers that integrate pedagogy, plus one 3-semester-hour practicum course providing an authentic residency. The four courses were: Number & Operations, Algebra, Data Analysis & Probability, and Geometry & Measurement. Each course was offered for 7 weeks, meeting 1 evening per week for 5.5 hours at the elementary school. The Number & Operations course had a significant focus on Cognitively Guided Instruction (CGI). Key program assignments included: clinical-style interviews of children's mathematical understandings; selection, adaptation, or generation and analyses of worthwhile mathematical tasks; an in-depth data design, collection, and analysis project; and critical examination and presentation of extant research on elementary mathematics education via synthesis papers. The participants also completed a 3-semester hour practicum course during the second semester of the program that provides an authentic residency enacting the synthesis of content knowledge and problem-based pedagogy emphasized in the program. Practicum assignments included the creation of a portfolio demonstrating expertise in teaching elementary mathematics, analyzing impact on diverse student learning, and technology integration. Successful completion of the practicum and all four of the content/pedagogy courses led to recommendation for the K-5 ME.

Data Collection and Analysis

Quantitative data were collected using scored observations of classroom teaching practices with what we have called the Standards-based Learning Environment Observation Protocol (SBLEOP), which documents the degree to which the teacher facilitates and the students experience a SBLE (Tarr et al., 2008). This observation protocol was adapted from an observation tool in the Wisconsin Longitudinal Study (Romberg & Shafer, 2003) and slightly modified for the K-5 ME program. It consists of a rubric assessing the extent to which specific mathematics classroom learning events are apparent during an observed lesson, using a scale of 1 to 3, with a higher score indicating more alignment with a SBLE. Five of the classroom events were included as data in this study.

Qualitative data include professional portfolios with teacher reflections, as well as six individual interviews. The professional portfolios with teacher reflections and two classroom observations, conducted by the university supervisor, were completed as part of the authentic residency course in the second semester of the program, and provided the opportunity to document: the degree to which classroom instruction included worthwhile mathematical tasks and evidence of a SBLE; analysis of diverse student work and achievement using formative assessments and remediation; technology integration; and personal reflections on teaching practices. This documentation experience served as a cumulative reflection for the participants and a summative evaluation for the university supervisor. Portfolios were required to have at least ten enacted lesson plans with at least one lesson from each of the four mathematical domains of the program's courses. Given the participants' consistent focus on the Number and Operations domain for the classroom observations, as well as the majority of the additional six chosen lessons included in the portfolios, teacher reflections were drawn from the Number and Operations section of the professional portfolio for analysis. The semi-structured, individual interviews were conducted with six randomly selected participants within two weeks of completion of the program, exploring their experiences with implementation and documentation (i.e., the portfolio, classroom observations, reflections, etc.) during the program.

The qualitative and quantitative data were intended to reciprocally illuminate and extend the findings, particularly in the drawing of conclusions. The interview and teacher reflection data were analyzed using line-by-line open coding that generated numerous meaning units (i.e., embedded coherent and distinct meanings), which were then documented in a coding manual. These meaning

units were then compared across cases and as consensus was reached between the researchers, coded meaning units were collapsed and renamed until final shared themes were determined. Descriptive statistics were used for analysis of the items on the SBLEOP.

Results

Quantitative Findings

The SBLEOP from the authentic residency course in the final semester of the program provided categorical data for five classroom events from two classroom observations. We used the same approach as Tarr et al. (2008) to convert these data to numerical data. We summed the individual scores (1, 2, 3) from the two observations, then determined whether each event for each participant should be rated as high (5-6), medium (3-4), or low (2). These categorical codes were then assigned numerical values of 2 (high), 1 (medium), or 0 (low). These numerical values were then summed across the five classroom events to find a composite score for each participant ranging from 0-10, which indicate teachers' enactment of a SBLE. Table 1 shows the percentages of participants who were rated as low, medium, or high by classroom event and the percentages of participants in each range of composite scores. The composite scores show that 85% of these teachers enacted a SBLE at a high level (7-10), and the remaining 15% enacted a SBLE at a medium level with a composite score of 6 out of 10.

Table 1: Percentage of Participants by Score for SBLEOP and Composite Score

Classroom Event	Scores with Numerical Codes of 0, 1, 2		
	Low (0)	*Med (1)*	*High (2)*
CE1 Making Conjectures	31%	62%	8%
CE2 Conceptual Understanding	0%	8%	92%
CE5 Explaining Strategies	0%	15%	85%
CE6 Multiple Perspectives	0%	0%	100%
CE7 Using Student Statements	0%	54%	46%
	Sum of Event Scores (0-10)		
	Low (0-2)	*Med (3-6)*	*High (7-10)*
Composite Scores	0%	15%	85%

Qualitative Findings

The analysis of the interview and reflection data revealed several commonalities across participants' experiences with classroom implementation. These themes tell a story of movement through shifting beliefs and practice across the program. Participants began in a place of *skepticism*, troubling this new way of teaching and learning mathematics, with resistance that came from doubt and uncertainty. It was not until participants *tried it on*, by putting pieces into practice and experimenting in the classroom, that they began to see things differently. This move to the classroom was often credited to the implementation assignments and documentation requirements (i.e., classroom observations and creation of the portfolio) from their authentic residency course, which served as *shifters,* or impetuses for change in their practices and beliefs about teaching and learning mathematics. Ultimately, though, we found that participants *need more support* in their classroom endeavors. The changes they were making were challenging and significant in scale, and the feedback and support needed to maintain these new practices were acknowledged.

Skepticism. When these participants started the endorsement courses, they were coming from a place of skepticism. This community of teachers was immersed into a very rigorous program of study, and the initial reaction was resistance. Participants were holding on to well-entrenched habits and customary practices, which provided a level of comfort and safety. These long standing teaching

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practices were troubled in the endorsement courses. Their first course, Number & Operations, introduced them to CGI, including the problem types, solution strategies, and the power of a problem-based pedagogical approach when implemented with rich dialogic discourse in the classroom. Participants watched videos in the course of CGI implementation and voiced skepticism because they thought: (a) it took too much time to develop such a pedagogical routine, (b) their own students were not capable like those from the videos, and (c) the expectations for classrooms were unrealistic.

Participants initially believed that incorporating CGI would be too difficult because it would take a long time to develop an aligned pedagogical routine with students. This assumption came from a place of skepticism about the capabilities of students' thinking and learning in their own classrooms. When reflecting on watching the videos in class, one interviewee said, "I was still wondering, ok, how long had they been exposed? It just was really scary especially um with me teaching the kids who are struggling in math… will they be able, you know, to do this?" However, that same interviewee came to think about her students in a new and promising way: "I'm thinking, like, these kids aren't, you know, as LOW as people think…"

The biggest place of skepticism was in this disbelief about what children can do mathematically and how they can learn: "I was a non-believer at first… I'm like, word problems are the hardest things that kids have to do in math. They struggle with that the most, and to just see it is unbelievable." This skepticism about their own students quickly diminished as they began to witness firsthand their learning in the classroom, as one participant asserted, "I think for me the biggest a-ha was my mindset about what I thought kids could do… so for me, it's just planning it out and knowing that kids can." Participants started to see their students "can do more than we give them credit for", and "they can surprise you with what they know".

Trying it on. In the authentic residency course experience, participants were required to complete two passing classroom observations of their mathematics instruction. The observation rubric was highly conducive for a CGI lesson, so nearly everyone chose to try that on. In this trying on, this implementation of CGI, participants felt anxious about their execution, wondering if they were "doing it right". They also felt uncomfortable in not knowing if it would work. In trying it on, though, and engaging in it for themselves, they grew in confidence about the effects of CGI and also their impact as mathematics teachers. They were valuing the mathematics, the thinking involved, the conceptual understanding and problem solving skills, the multiplicity of strategies, and the explanations their students were using. Said one participant, "Kids that can't even read word problems necessarily work them out and have the strategies and the conceptual understanding… it's just that powerful."

The mathematics that participants noticed was another focus in their stories of implementation. They spoke of "witnessing a shift in their [students'] mathematical thinking," and their students "feeling successful in their ability to solve problems" and "going from procedural to conceptual understanding". The discourse that participants facilitated in their classrooms also led to important shifts in their thinking about how children learn mathematics. Finding (for themselves and from their students) that there are multiple ways to solve problems, many ways to think about the same mathematics, and numerous perspectives on the same story problem. Most importantly, participants began to value discourse, this opportunity for the class to hear and learn from each other, as a way for students to explain and justify their thinking. One participant asserted, "I think [CGI] has been the best thing I have learned in math as a teacher thus far, hands down. Nothing compares to it… I think that's the most important thing I have learned in math thus far." Additionally, they found it built a stronger classroom community.

Having 13 prospective EMSs as participants in the same school, working on the same coursework and implementing the same practices, built a strong community of teachers as learners as well. This support system provided a push to ask questions, to try new things, and to persevere. This

parallel, between what CGI did for their mathematics classroom community and what a collective implementation did for the school community, is noteworthy.

Shifters. In connecting their learning of theory, methods, and content from the courses to enactment in the classroom setting, there are many factors that contributed to shifts in instructional practices and beliefs. These factors, or "shifters", include: receiving and using critical feedback, program coursework (e.g., clinical interviews and worthwhile mathematical task collections), implementation of and reflection on classroom observations, documentation in the professional portfolio, listening to children's mathematical thinking, trying on new and different instructional practices, and deepening their own mathematical understanding, to name a few. In particular, interviewees spoke specifically about the value of the implementation (classroom observations) and the documentation (portfolio), in their shifts: "I think I would've continued to do it the way I was doing it and probably would've kept getting frustrated with not getting the results that I was looking for. So I think that really helped me, having [the university supervisor] come in here."

Facilitating this buy-in and meaningfully shifting the prospective EMSs' instructional practices and beliefs was not easy. These salient experiences, these "shifters", generated a change that pushes beyond imagination into reality: "There's no way that I would have ever bought in to this had I not had the practicum experience." By taking their course learning and having a space as a community to implement and document over time as they tried on these new teaching practices in their own elementary classrooms, participants began to notice their beliefs shifting, their practices shifting, and their attitudes toward mathematics shifting. Said one participant, "The mantra that I have lived by every year, I want to reinvent myself, and I will say that every year. I say it and then [in] the summer I'm like, did I reinvent myself? I don't think I did. But I think if I reflect THIS time I can say I reinvented myself."

Need more support. Across the data, there was an emphatic call for further support and feedback. Each participant made note of their appreciation for the feedback they received; however, two evaluative classroom visits provided limited time and support from the university supervisor. One participant in particular recommended some sort of follow-up as they tried on new practices: "just come back and check on us". Another participant, one of several, mentioned the need for non-evaluative spaces for feedback and support. For their two observations, participants would choose lessons that they felt confident implementing; but, in their struggles and changing instructional practices and beliefs, they often had questions and a need for support apart from their evaluation: "It would be great for us to kind of still have mentors, somebody to come and check in on us."

Another consistent desire voiced by the teachers was sustainability. Participants recognized that they learned a lot in one year, and they voiced an aspiration to make sure they continue to implement these new teaching strategies. Some ideas for this sustainability were to implement ongoing professional development, to have more classes over the summer, and to have more opportunities to bring university supervisors or mentors into their classrooms.

Discussion

EMSs provide crucial and needed mathematics expertise in elementary schools. However, there is limited research on how to best prepare EMSs and in particular support the difficult process of pedagogical shifts in classrooms. This case study explored the experiences of one group of prospective EMSs as they sought to connect their learning of theory, teaching methods, and content knowledge in program courses with their classroom instructional practices.

The quantitative and qualitative data revealed the prospective EMSs were facilitating instruction mostly aligned with a SBLE. The SBLEOP showed the largest gain across the two observed lessons was the classroom event of enacting lessons that foster development of conceptual understanding. Prospective EMSs were encouraging and valuing students' multiple strategies and perspectives and creating learning opportunities grounded in dialogic discourse as they valued students' mathematical

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statements and used them to build discussion or develop shared understanding, and students were explaining their mental reasoning and problem-solving strategies. However, they struggled to provide frequent opportunities for their students to make conjectures (i.e., informal generalizations) about mathematical concepts and processes in the context of problem-based learning. From our own experiences, overcoming this difficulty requires more time for classroom discourse, deeper understandings of the mathematical connections to and within the content by both teachers and students, and higher expectations that young children can make important connections and generalizations.

These teaching practices evidenced on the SBLEOP did not come easily for the participants. Skepticism and disbelief marked their perceptions of their own students' mathematical capabilities. When viewing videos of children's mathematical reasoning and explanation during the course sessions, the teachers were highly dubious that such could occur in their own classroom realities, due to students' inabilities and time constraints of a pedagogical routine grounded in problems. However, experimentation with what they were learning in the courses, particularly a problem-based pedagogy aligned with CGI, made them believers. Trying it on and seeing the benefits for and capabilities of their own students were instrumental in the prospective EMSs changing their practices and beliefs about teaching and learning mathematics.

The qualitative data revealed the key supports for trying on new instructional practices, including the authentic residency course assignments. Shifters, like supervisor feedback, clinical interviews and worthwhile mathematical tasks assignments, classroom observations, and the creation of the professional portfolio, pushed participants to make these pedagogical changes. Many participants began the program from a place of skepticism, but after implementation (trying on new teaching practices) and documentation (portfolio assignments and reflections), changed their practices and beliefs. Participants identified the importance and profound impact of implementing a SBLE in their classrooms, notably the use of CGI as a guide for instruction. However, the prospective EMSs voiced the need for more support. They wanted help and feedback beyond the two enacted lessons in the program and also after program completion. Participants wanted opportunities for non-evaluative classroom observations with reflection and feedback.

In sum, it is vital that the efficacy of EMS preparation programs, such as the one in this study, be carefully studied in order to determine their impact. Based on our findings, the participants in this program connected their learning in the courses with their classroom teaching practices. Even more importantly, the participants found immense value in that connection and desire continued support in maintaining these new practices. It seems the implementation and documentation expectations during the authentic residency course supported this very difficult translation, facilitating the ultimate goal of the program of pedagogical shifts.

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EARLY CAREER ELEMENTARY MATHEMATICS TEACHERS' NOTICING RELATED TO LANGUAGE AND LANGUAGE LEARNERS

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There has been limited attention to early career teachers' (ECTs) understandings and practices related to language in teaching and learning mathematics. In this qualitative case study, we drew upon frameworks for teacher noticing to study the language practices of six early career elementary and middle school mathematics teachers. We describe multiple themes that cut across teachers' noticing related to language and language learners, and discuss one theme (i.e., Perspectives on multiple languages) in more detail, including evidence of specific forms of noticing. Implications for teacher education and professional development are discussed.

Keywords: Elementary School Education, Teacher Education-Inservice, Instructional Activities and Practices, Equity and Diversity

Various calls have been issued to better prepare teachers to address the mathematics learning needs of a growing multilingual student population (Diversity in Mathematics Education Center for Learning and Teaching [DiME]; 2007; Grossman, Schoenfeld, & Lee, 2005; National Council of Teachers of Mathematics [NCTM], 2014). However, despite a sociopolitical turn in teacher education (Gutiérrez, 2013), serious attention to equity has been slow to gain hold. Mathematics teacher educators have argued that a key component of equitable mathematics instruction is drawing on the diverse experiences and understandings that children bring to the classroom, including children's mathematical thinking and children's linguistic and cultural funds of knowledge, or what we refer to as children's multiple mathematical knowledge bases (Turner et al., 2012). Prior research has documented preservice elementary teachers' learning related to children's multiple mathematical knowledge bases in the context of mathematics methods courses (Turner et al., 2012). Yet few studies have investigated how preservice teachers take up these practices in early career teaching. In particular, there has been limited attention to early career teachers' (ECTs) understandings and practices related to *language and language learners* in teaching and learning mathematics (Janzen, 2008). Given the increasing linguistic diversity among public school students (National Center for Education Statistics, 2014) and the fact that as many as 88% of teachers work with English learners (Karabenick & Noda, 2004), increased attention to teachers' language practices is critical.

Noticing includes attending to classroom actions and interactions, as well as reflecting, reasoning, and responding (Hand, 2012; Jacobs, Lamb, & Philipp, 2010; Mason, 2011; van Es, 2011). In this study, we drew on Jacobs et al.'s (2010) and Jacobs, Lamb, Philipp, and Schappelle (2011) three components of noticing (attending, interpreting, and deciding to respond), to investigate early career teachers' understandings and practices related to language and language learners. While Jacobs et al. (2010, 2011) focused on teachers' noticing of children's mathematical thinking, in this

study, we extended the noticing framework to explore teachers' noticing related to language. Specifically, we investigated the following research question: How do ECTs notice language in planning, enacting, and reflecting on mathematics lessons?

Literature Review and Analytical Framework

Mathematics Teachers' Practices with Language

Several emerging areas of research relate to teachers' practices and understandings about language and language learners in mathematics. For example, McLeman, Fernandes and McNulty (2012) studied preservice teachers' beliefs, and found that opportunities to learn about English learners supported non-deficit oriented views. In addition, broader sociopolitical forces have been found to shape teachers' understandings of language in mathematics teaching. Barwell (2014) documented consequences of institutional mandates to use English as the sole language of instruction. Barwell found that in mathematics classrooms taught by monolingual English speaking teachers, linguistically diverse students' use of language was painstakingly monitored for grammatical accuracy. Researchers have also investigated the impact of professional development (PD) on mathematics teachers' understandings and practices with language. For example, Ross (2014) found that PD specifically focused on working with English learners in mathematics classrooms correlated with teachers' increased self-efficacy. Additionally, Takeuchi and Esmonde (2011) found that participation in an inquiry-based PD program positively changed mathematics teachers' discourse about linguistically diverse students and families. Teacher participants initially described language as a barrier for English learners when learning mathematics. During the project, teachers' discourse began to shift and teachers ultimately focused on the importance of making their students' linguistic diversity more visible in their classrooms (Takeuchi & Esmond, 2011). Similarly, Chval, Pinnow, and Thomas' (2014) found that focused PD supported a third grade teacher in understanding that language could be used to build students' mathematical knowledge and, conversely, mathematical instruction could support language development. This led to the teacher providing more explicit language instruction during her mathematics lessons for all of her students.

In summary, prior studies have focused on the role of teacher beliefs and PD experiences in shaping teachers' practices related to language in mathematics. In this study we draw on frameworks of noticing to make sense of how teachers notice language in planning, enacting, and reflecting on mathematics lessons. Noticing frameworks are particularly well-suited for understanding how teachers make sense of complex situations in classrooms (Sherin, Jacobs, & Phillips, 2011), such as teaching and learning mathematics with students of diverse linguistic backgrounds. Moreover, given that what teachers notice and how teachers interpret what they notice impacts what teachers do in the classroom (van Es & Sherin, 2008), a focus on teacher noticing is warranted.

Teacher Noticing in Mathematics

In framing our study, we found that Jacobs and colleagues' definition of *professional noticing of children's mathematics thinking* provided a useful foundation from which to build (Jacobs et al., 2010; Jacobs et al., 2011). Noticing consists of a set of three interrelated skills. First, *attending* to children's strategies includes focusing on "noteworthy aspects of complex situations" and discerning patterns in children's mathematical strategies and understandings (Jacobs et al., 2010, p. 172). Second, *interpreting* children's mathematical understandings involves teachers reasoning about children's strategies and how they construct a picture of children's understanding based on details of a child's work and research on children's mathematical thinking. Finally, *deciding how to respond* on the basis of children's understandings reflects the decisions teachers make for instruction and whether these decisions draw on specifics of children's thinking as well as research on children's learning and development (Jacobs et al., 2010, 2011).

Noticing skills develop over time, and much of the existing research has focused on teachers' development of noticing through teacher education or PD programs (McDuffie et al., 2014a, 2014b; Star & Strickland, 2008; van Es, 2011). For this study we take a different approach in three ways. First, instead of looking for changes in teachers' noticing over time, our aim was to understand what and how teachers notice during their first years of teaching to investigate noticing skills as teachers begin their career. In other words, our intent was to map the noticing terrain for early career teachers (ECTs) so that we might better understand what is possible for those new to teaching and also glean what areas will need support for teachers' professional development across their career. Second, instead of focusing on teachers' noticing of children's mathematical thinking, we shift the object of noticing to *language* in mathematics teaching and learning. Although we view language as tightly linked to children's mathematical thinking, our intent was to bring language to the foreground of study. Third, unlike Jacobs et al.'s (2010, 2011) focus on interviewing teachers to examine how they notice children's mathematical thinking, we extended our data collection to include classroom observations, and correspondingly, we studied both teachers' decisions for responding and their actions resulting from in-the-moment decisions during lesson enactments.

Methods

We used a qualitative case study design (Creswell, 2013; Stake, 1995), to study the practices of six early career elementary and middle school mathematics teachers. The ECTs were part of a larger study (Aguirre et al., 2012; Turner et al., 2012; McDuffie et al., 2014a, 2014b) that followed participants across math methods courses, student teaching, and into their first or second year of teaching. The ECTs attended one of two universities located in different regions of the U.S.

Participants

Table 1 outlines the background and teaching context of each ECT participant.

Table 1: Participating Teachers

Teacher	Grade	Teachers' Linguistic and Ethnic Background	Students' linguistic background	Linguistic Context of Instruction
Evelyn	Y1 & Y2: 7th	English and some Spanish; Mexican American	Some bilingual (Spanish L1); All English proficient[2]	English; non ELD[3] classroom
Estelle	Y1 & Y2: 2nd	English; European American	Many students bilingual (Spanish, L1); All English proficient	English; non-ELD classroom
Padma	Y1: 4th Y2: 3rd	English; Indian American	All students bilingual (Spanish, L1); Range of English proficiency	ELD classroom
Kara	Y1: 5th	English; some Spanish; European American	All but one bilingual; All English proficient	English; Spanish encouraged
Natalie	Y1: K	English; some Spanish; European American	Many bilingual (Spanish or dual L1); All English proficient	English; Spanish encouraged
Elena	Y1: 1st	Bilingual (Spanish/ English); Mexican American	All bilingual (Spanish L1)	Bilingual 90/10 Spanish/English

[1]All teacher and district names are pseudonyms.
[2]English proficient as determined by district language assessment and placement policies.
[3]ELD refers to English Language Development classrooms for English learners.

Data Sources

Data sources included classroom observations of mathematics lessons, coupled with pre and post observation interviews. Observations were clustered so that we observed a sequence of mathematics lessons on two or three consecutive days. We observed 8-12 mathematics lessons per year in each ECTs' classroom. We recorded detailed field notes for each lesson, and collected lesson artifacts including student work samples. We conducted pre-observation interviews prior to each set of observed lessons, and post observation debriefs for the first and final lesson observed. These interviews probed ECTs' perspectives and reasoning, and provided opportunities for ECTs to recount, interpret and respond to key moments from lessons. We conducted interviews at the beginning, middle and end of the year to capture reflections about teaching and learning across the year (not just at the level of the lesson), and information about their teaching contexts (e.g., leadership, PD, curriculum, assessment, policies.) Interviews lasted approximately one hour, and were recorded and transcribed for analysis.

Data Analysis and Analytical Framework

Through multiple and iterative cycles of analysis, we conducted within-case analysis and cross-case analysis for these cases of teaching (Creswell, 2013; Stake, 1995). As part of the larger project, we conducted preliminary analysis with first-cycle coding to summarize segments of data and identify themes relative to our research foci (Miles, Huberman, & Saldaña, 2014). This initial phase resulted in a code book including the following codes that were relevant to the study reported here: context and background; language; connections to students; connections to family/community; equitable participation. When creating the code book, we developed decision rules for the coding process and descriptions for each code. For example, we defined a *stanza* (Miles, Huberman, & Saldaña, 2013) of text as including both the question and the participant's response, as well as additional text needed for context.

During the second phase of data analysis we used the code book to code all transcripts in the qualitative data analysis software HyperResearch (Researchware, 2011). During this phase we sorted the data by topic and continued generating themes. For example, themes related to language such as: acquiring vocabulary; multiple meanings of words; mathematical discourse; multiple languages. To achieve *interpretive convergence* (Miles, Huberman, & Saldaña, 2013) and ensure consistency in coding data so that all data on a topic were identified with appropriate codes, two researchers independently coded approximately one third of the transcripts, and then met to discuss and resolve any discrepancies.

For a third phase of analysis, we focused on our six ECT cases. For each participant, we generated a *narrative compilation* (Creswell, 2013) of practices related to language. These compilations included representative and compelling examples, along with non-examples, from all data sources to test emerging themes (confirming, refuting, or investigating further). These three phases of analysis, along with research and theory in the field, led us to teachers' noticing skills regarding language in their practice.

In the fourth phase of analysis, we adapted Jacobs et al.'s (2010, 2011) definitions for each of the three components of noticing (*attending*, *interpreting*, and *deciding to respond*) to include language as an object of noticing. We expanded *deciding to respond* to include decisions evidence in lesson enactments. We identified noticing patterns for each participant based on coding data over these two dimensions (language and noticing) and created an analytic memo for each ECT's language-related practices. Finally, we looked across ECTs for larger patterns to build our cross case analysis.

Findings

We found that ECTs demonstrated all three forms of noticing language as they planned, enacted, and reflected on mathematics lessons. More specifically, we identified themes that cut across various

teachers' noticing related to language. Given space constraints, we only briefly describe each theme below, discuss the final theme in more detail, and evidence how specific forms of noticing were evidenced.

Sense-making of Mathematical Terms by Eliciting Students' Ideas

One strategy that ECTs used to promote student sense-making of mathematical vocabulary was asking students to discuss and generate their own definitions for key mathematical terms. ECTs noticed that when they repeatedly elicited students' ideas about key vocabulary, students moved beyond memorized definitions to deeper understandings. A related pattern was that ECTs tended to introduce vocabulary as a way to help students describe and name their experiences with mathematics concepts.

Connections Among Language, Concepts, and Everyday Contexts

ECTs often began mathematics lessons with connections among language, mathematical concepts, and everyday contexts. ECTs explained that they aimed to engage students' interests and leverage students' experiential knowledge to support students in making sense of both the mathematical concepts and the associated terminology.

Multiple Opportunities for Students to Hear, Say and Use Key Mathematical Vocabulary

ECTs attended closely to students' needs to hear and say new terms repeatedly throughout a lesson. ECTs highlighted key vocabulary through voice inflection, encouraged the use of terms in classroom talk between students and with the teacher, and provided multiple prompts for students to use new vocabulary (e.g., choral response, talk to a partner and say the term, sentence frames).

Emphasis on Precise Use of Mathematical Vocabulary

ECTs often held students accountable for precise use of language, and stopped to question students when mathematics terms were missing or not used correctly. ECTs displayed mathematical vocabulary in the classroom, and reminded students to be precise with the use of these words during small and whole group discussions, and in written descriptions of solutions.

Expectations for Justifying Reasoning and Explaining Thinking

Most ECTs evidenced a consistent emphasis on mathematical discussion, and expected students to justify solutions and explain their strategies to others. ECTs viewed mathematical discussions as a key component of students' mathematical learning, and as a context for students to practice using and making sense of mathematical vocabulary.

Perspectives on Multiple Languages

Most ECTs evidenced noticing that reflected a resource orientation towards students' home languages. That is, ECTs interpreted inclusion of home languages, and students' spontaneous use of multiple language during lessons as supportive of student learning. Notably, ECTs held this perspective despite the fact that three of six ECTs were teaching in contexts that mandated English as the language of instruction. Beyond this commonality, ECTs evidenced more variation in their perspectives and practices towards the use of multiple languages than was evident in the other themes. Three of the ECTs attended closely to challenges with mathematical language that some English learners in their classrooms faced. For example, when Evelyn noticed students using Spanish as they worked on mathematics, even though she was not able to fully understand their conversation, she positioned Spanish as a resource to support student learning and encouraged students to continue talking and thinking with Spanish. In one instance, a student started talking through one of the lesson tasks in Spanish, and then looked at Evelyn and remarked "I'm sorry, I can't do that." Evelyn responded by encouraging the student to continue (Year 1, Post observation interview).

While ECTs in general supported connections to students' home languages in mathematics instruction, their reasons for doing so varied. For example, Elena maintained that teaching mathematics in the home language of students allowed students to focus on the concepts and deepen their understanding. This stance was consistent with the bilingual education model at her school, and Elena taught mathematics instruction almost exclusively in Spanish. Elena's primary goal was to ensure student understanding, and she used (and encouraged students to use) both Spanish and English to support learning. For example, when mathematics worksheets were only available in English, Elena translated the directions and problem text into Spanish, so that students had access to both languages to support their sense-making (Post Observation Interview). For Estelle, occasional connections to students' home language were aimed at increasing student interest and engagement, and "helping students feel proud" of their bilingualism. She explained that when she asked students to contribute Spanish translations of key words in her lessons, "they had this big smile on their face like, 'Wow! I have something that Ms. Estelle doesn't have!'" (Y1, Middle of the year interview). As noted above, Evelyn also used connections to students' first language to support mathematical understanding, but responses were also aimed at honoring students' identities. Evelyn was aware of the struggles and lack of support her mother experienced as a bilingual student learning mathematics in an all-English instructional environment, and was determined to offer a different experience for her students (Year 2, End of year interview).

Two of the six ECTs evidenced noticing related to multiple languages that in some instances reflected a mixed (deficit/resource) orientation toward language other than English. The deficit-based ideas included perspectives such as: a lack of English caused student confusion, a lack of English proficiency in parents served as a barrier to students, and when students use only English in math class this evidences progress/understanding. For example, on one hand Padma expressed strong support for students' bilingualism and consistently praised families for supporting students Spanish language development. But on the other hand, she did not view attending to multiple languages as part of her role as a mathematics teacher. She positioned English as the (only) language of school mathematics and did not invite the use of Spanish as a resource to support students' learning. She explained:

> The problem is, I don't know enough [Spanish]. The only one [term in Spanish] I remember was during my student teaching, when we were doing polygons, so like, septagon, seventh grade in Spanish is *septimo*, so that's what I used last year [when I taught the names of polygons] (Year 2, post observation interview).

Padma framed her own lack of proficiency in Spanish and her school language policy as shaping her noticing and responses related to multiple languages. Specifically, Padma explained that the school's philosophy regarding speaking any language other than English was that "it can't happen," because "we're here to teach the students English" (Year 2, post observation interview).

Finally, in some cases a mismatch existed among what teachers attended to related to multiple languages, what teachers planned for instruction, and what they actually enacted. Three of the ECTs claimed to welcome multiple languages in instruction, and described plans for using multiple languages, but the teachers' descriptions did not always match what we observed. For example, Kara explained that she attempted to use cognates whenever possible to support student understanding. She noted:

> Well I try and use as many cognates as I can. I don't know a lot, but I try and sometimes I make them up and I shouldn't because they're not what they're supposed to [be]. And then so I make sure I run them by my bilingual teachers first (Year 1, beginning of the year interview).

She explained she "welcomes" students to speak Spanish during math ("my kids are always welcome to use Spanish when they're learning math. I never limit them to English, ever"). However,

connections to languages other than English were not evident during the lessons observed, either by Kara or her students. It is possible that Kara connected to multiple languages in lessons that were not observed, but it is also likely that Kara's noticing of multiple languages was limited to attending in her plans and reflections, and that she did not evidence responding to multiple languages in her lesson enactments. This mismatch between some ECTs' *vision* for noticing language and their *enacted practices* related to language may have reflected tensions between school contexts that on one hand encouraged the use of Spanish in instruction, and on the other hand emphasized students' acquiring academic language in English.

Significance

Patterns in ECTs' practices indicate that beginning teachers *can* engage in complex work of noticing and drawing on language in teaching and learning mathematics. We also identified key supports and challenges that have implications for teacher education. Extended experiences in classrooms with multi-lingual students might support learning ECTs in attending to the role of language in teaching and learning mathematics and in learning about diverse students and families. Similarly, on-going prompts to consider ways to draw on language in instruction might sharpen teachers' noticing of students' resources. These findings deepen our understanding of how to support equitable instructional practices that meet the learning needs of a culturally and linguistically diverse student population.

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IDENTIFYING BARRIERS TO TEACHER GROWTH IN IMPLEMENTING PROBLEM SOLVING BY REFLECTING ON LESSON STUDY

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As teacher educators, we are challenged to design professional development materials that build a perspective on problem solving as an avenue to achievement of content objectives. This qualitative study examines the impact of the intentional design of school-based Lesson Study as a follow-up to a content institute. Data analysis identified four barriers to the critical transformation of practicing teacher mindset about implementing high cognitive demand tasks: time, expectation of student mastery, perception of student readiness, and proficiency in orchestrating productive discussions. Identifying ways in which teacher educators can close gaps in teachers' problem solving perspectives is essential to changing the perceptions of problem solving from an activity to a content-driven learning experience.

Keywords: Teacher Education-Inservice/Professional Development, Problem Solving, Teacher Beliefs, Teacher Knowledge

After an eight-week content institute, mathematics teachers in grades 3-8 from 37 schools collaborated to implement strategies for productive mathematical classroom discourse in the form of Lesson Study (LS). Many of the teachers discussed how pivotal the professional development (PD) sessions were in envisioning problem solving in their classrooms. However, their beliefs did not translate to practice. As university facilitators, we studied the evolving relationship between participants' beliefs and actions as they transferred knowledge from the content institute to LS planning, implementation and reflection.

Theoretical Framework

Classroom implementation of rich mathematical tasks is challenging for many teachers, as problem solving standards are at times foreign and inconsistent with previous beliefs. Engagement in rich mathematics in PD can help teachers to view problem solving as an exploration of connected ideas instead of as an isolated activity (Schifter & Riddle, 2004). It is important to address teachers' pedagogical beliefs in concert with content-focused PD to foster a growth mindset (Boaler, 2011; Dweck, 2006) toward mathematics teaching and learning. While engagement with content can support teachers in shifting their instructional practices, innovative ideas are not always accepted nor sustained (Foss & Kleinsasser, 2001). LS provides an opportunity for continued PD (Garet, Porter, Desimone, Birman, & Yoon, 2001) through interaction, collaboration and collective professional "noticings" (Jacobs, Lamb, & Phillip, 2010; Mason, 2011). Therefore, we designed a tailored LS protocol to provide multiple iterations of a specific lesson for this collective and organic noticing to take place (Murata, 2011) and to improve the potential for these innovative strategies to be maintained in practice.

Research Context

University facilitators, including the authors, modeled mathematical inquiry in the development of computational fluency and proportional reasoning as they highlighted strategies for a problem

solving mindset in the classroom. An adapted protocol (Smith, Bill, & Hughes, 2008) provided a framework for developing lessons of high cognitive demand, and *Orchestrating Discussions* (Smith, Hughes, Engle, & Stein, 2009) defined five practices to support teachers in using student responses to explore content objectives. After the content institute, facilitators supported vertical teams as they designed and implemented an iterative LS protocol that included three phases: 1) One teacher piloted the lesson and shared the results; 2) A second teacher enacted a formal LS in which all members observed the classroom lesson; and 3) The remaining members individually enacted the lesson with contextual modifications based on the LS reflection and their students' learning needs.

In the following sections, we present the analysis of two LS groups focused on proportional reasoning. The first group, The Patriots, consisted of three general educators and one Spanish immersion teacher. They enacted "The Fencing Problem" adapted from the problem "Surrounded and Covered" (Noyce Foundation, 2013) in which students calculated perimeter with the goal of maximizing area. The second group, Mason's Pride, consisted of eight teachers, five general educators, two special educators, and one mathematics specialist. They enacted "The Pizza Problem" (Erickson, 2015) in which students each received a part of the problem, and collaboration was essential to identify the required proportions.

Method

Purposeful sampling was utilized as the nature of the teachers' interactions in theses groups warranted further exploration (Patton, 2002). Data for this research included coursework from the content institute, formal lesson plans, video-recorded and transcribed conversations from LS debriefs, participant written reflections, and field notes. As data was collected we shared insights and reflections with one another in weekly meetings and captured our thoughts in researcher memos. We identified relevant information through the use of open coding (Merriam, 2009). We collaborated to categorize codes after we examined documents individually. Upon completion of the analyses, categories were grouped and labeled with a theme. The reduction of these categories led to the emergence of a select few themes. Incorporating multiple sources into the research design was a purposeful decision that provided validation for our findings and helped to achieve construct validity (Maxwell, 2005).

Findings

Teachers did not demonstrate the anticipated growth in viewing problem solving as a content-driven learning experience and expressed views of problem solving as an isolated activity. For the two LS groups we examined, data suggested that there were four barriers to the critical transformation of teacher mindset.

Time

The greatest lesson learned from both groups piloting the lesson was the amount of classroom time required to implement the task. After the first lesson iteration, both groups highlighted a need to extend the allotted time to allow students more opportunity to productively struggle before providing the additional guidance. As a result, they extended the time for problem solving during the formal LS. However, as teachers implemented the third iteration of the lesson in their own classrooms, many expressed a lack of confidence in the students' ability to productively struggle with the problem and an uncertainty in how to manage the timing of the lesson. Teachers expected the lesson to take only one math block and were frustrated when the lesson was not finished in the allotted time.

Expectation of Student Mastery

Teachers in both groups valued the idea of students persevering on tasks, yet they were challenged by the student frustration they perceived. The tendency to guide and control the lesson

was difficult for some teachers to resist as many of the teachers felt driven by the needs of their students to arrive at a correct response. One host teacher expected mastery of the content objectives within the one lesson and was disillusioned when some students never arrived at the answer. "The children in my classroom started to learn the process of how to have rich math discussions with their peers, but fell short of mastering it" (Erin, Mason's Pride, individual lesson reflection). The expectation of student mastery during initial experience was mirrored when group members implemented the lessons in their own classrooms.

Perception of Student Readiness

Based on their perceptions of student readiness, teachers implemented their lessons in ways that decreased the cognitive demand of the task. Many teachers did not believe their students capable of accessing the problem and over-accommodated to ensure student success. Kate, a math interventionist and member of The Patriots, did not offer the problem to her students the problem as she believed it was too challenging for the large percentage (90%) of English Language Learners in her classroom. Instead, she scaffolded the problem to provide step-by-step guidance. Melissa, a special education teacher, engaged her students in similar problem solving, but at a "more basic level to give them the experience they need."

Orchestrating Productive Discussions

Each LS group defined objectives for their task based on mathematical content and process standards and purposefully planned how they would approach the group discussion at the end of the lesson. The lesson template prompted teachers to preplan the higher-level questions that would support a productive discussion. However, while all participants were able to implement aspects of their purposefully planned discussion, the challenge that many faced was orchestrating a productive discussion that supported exploration and achievement of the lesson content goals.

Many teachers were unable to connect student problem solving with standards-based objectives due to novelty of the experience. They shared that not knowing what to expect caused them to take more time than expected evaluating students during the problem solving process. As a result they felt rushed sequencing the discussion to support student learning. Both the novelty of the experience and lack of time heightened the disconnect between individual perception of problem solving as an activity and the institute goal of implementing problem solving as a content-driven learning experience. "I am struggling to see the practicality of the lesson...it is too much effort that goes into one lesson that makes it very difficult for teachers to use on a daily basis" (Carlos, The Patriots, individual lesson reflection).

Implications

Even though participants attended our institute and had the opportunity to engage in deep mathematical thinking, many teachers still viewed problem solving as an activity separate from developing mathematical proficiency. Teachers are challenged by leading student-centered discussions and require additional support in developing their ability to use problem solving to achieve a mathematical content objective and to impact student learning.

Identifying ways in which teacher educators can not only facilitate highlighting content connections during problem solving but also relate problems back to the mathematical objectives will foster productive teachable moments. Analysis of LS provides important evidence of teacher thinking and informs our development of future PD opportunities. Lessons learned will assist mathematics teacher educators in breaking down the barriers in teacher beliefs about problem solving that relegate opportunities for engagement with rich mathematical tasks to extra time in the curriculum pacing.

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TRANSFORMACIONES COLECTIVAS DEL ESPACIO FÍSICO-INTELECTUAL EN AULAS DE MATEMÁTICAS

COLLECTIVE TRANSFORMATIONS OF THE PHYSICAL-INTELLECTUAL SPACE IN MATHEMATICS CLASSROOMS

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La investigación enfocada en mejorar la práctica docente de las matemáticas ha creado y perpetuado un discurso de experto/novato. Estas dicotomías crean una división—por consiguiente una frontera—al generar conocimiento que crece entre investigadores relativamente más que entre maestras y maestros. Posterior a una breve crítica de estas metodologías, proponemos una metodología que reconoce la complejidad de los diferentes contextos que le dan vida a la enseñanza de las matemáticas. Compartimos ejemplos de un proyecto de investigación que muestran las transformaciones colectivas de la enseñanza mediante la multivocalidad de maestras/os, estudiantes, padres e investigadores.

Palabras clave: Métodologías de Investigación, Capacitación Docente / Desarrollo Profesional

Los estudios enfocados a transformar la enseñanza de las matemáticas tienen un lado que genera ideas y otro que las inhibe. Tomemos como ejemplos las investigaciones que enfatizan "*best practices*" (Ball, 2000), o la profesionalización de la enseñanza basada en interpretaciones del marco teórico de Goodwin (1994), o la reciente acumulación de estudios sobre "*teacher noticing*" (Sherin, Jacobs, & Philipp, 2011). El lado generador de ideas mueve a los investigadores a refinar ideas y constructos. El lado que las inhibe—no tan evidente como el lado generador—surge al observar críticamente las metodologías empleadas en estos estudios, las cuales separan el proceso de enseñanza de los contextos complejos en los cuales existe. Estas metodologías asumen que la práctica docente se puede descomponer con propósitos de análisis (Jacobs & Empson, 2015), y asignan un grado mayor de responsabilidad a un solo agente: el/la maestro/a. El resultado es una serie de propuestas que promueven jerarquías e incluso trayectorias de lo que el/la maestro/a debe saber, practicar, y notar. Estamos conscientes que todo enfoque implica un desenfoque, lo que nos lleva a refleccionar en la seriedad de este desenfoque.

El análisis de una realidad que ha sido fracturada perpetúa el discurso expero/novato de acuerdo al cual el/la maestro/a posee o carece de ciertos conocimientos. Lo que nos preocupa acerca de estas dicotomías es que surgen de una visión tácita que supone que la enseñanza es un fenómeno estable que se puede analizar y categorizar de tal manera. Estas dicotomías a la vez crean una división—por consiguiente una frontera—misma que no ha sido suficientemente interrogada. Nuestra interrogación es la siguiente: Aunque estos enfoques de investigación sofistifican el conocimiento del investigador, dicho conocimiento aún no logra traducirse a una transformación de la práctica docente, al impedir que los/las maestros/as logren percibirse en la realidad fracturada que ocupa nuestro estudio. Cuando esto sucede, la posibilidad de transformar la práctica docente se debilita. Nuestro enfoque es diferente, ya que busca la transformación colectiva, la que emerge y se observa a través de procesos participatorios colectivos que incluyen a estudiantes, maestras/os, padres de familia, e investigadores.

Perspectiva teórica

Para explorar la complejidad inherente a la transformación de la práctica docente es indispensable no reducirla con propósitos de análisis. Para ello nos adherimos a las ideas de Davis y Simmt (2003) acerca de cómo estudiar la trascendencia de un colectivo:

Complexity is not just another category of phenomena, but an acknowledgement that some phenomena are not deterministic and cannot be understood strictly through means of analysis (i.e., literally, by taking apart or cutting up). A different attitude is required for their study, one that makes it possible to attend to their ever-shifting characters and that enables researchers to regard such systems, all at once, as coherent unities, as collections of coherent unities, and (likely) as agents within grander unities (p. 140).

Para estudiar las transformaciones en la enseñanza de maestras con estudiantes que provienen de comunidades vulnerables, nos enfocamos en la multivocalidad de maestras, estudiantes, padres, e investigadores de educación matemática. Aunque presentamos estas voces en espacios separados, la resonancia de estas voces es la que crea un colectivo coherente con una gran agentividad y una diversidad interna que es auto regulada por los mismos participantes (Davis & Sumara, 2006). Alineados al principio de la investigación cualitativa de *mostrar* en lugar de *dictar* (Tracy, 2010), proporcionamos ejemplos de transformaciones ocurridas en un proyecto de investigación en diferentes aulas de matemáticas a través del tiempo. En estos ejemplos vemos a los participantes en medio de transformaciones, experimentando tropiezos, retos, y logros situados en una colectividad compleja. Nuestro propósito es ilustrar los senderos de enseñanza que las maestras recorren y que, lejos de ser aislados, se entrelazan y comunican con las realidades igualmente complejas de los estudiantes, padres, e investigadores. A lo largo de estos senderos, las maestras no experimentan un conocimiento individualizado del tipo que se propone en muchos estudios, sino que ellas con los demás participantes hacen camino al andar.

Transformaciones colectivas desde la perspectiva del investigador

Durante el primer año de un proyecto de investigación enfocado en la reciprocidad como proceso generador de transformaciones colectivas, empecé a notar cómo al llegar a observar la clase de matemáticas, por lo general había un estudiante excluído de la instrucción (*time out*) por mal comportamiento. Estos estudiantes tendían a ser los mismos semana tras semana. Decidí acercarme a ellos para indagar los motivos de su exclusión. Sus faltas eran mínimas: inatención, hablar fuera de turno, ponerse se pie sin permiso, interrumpir a un compañero, no compartir materiales. Uno de estos estudiantes era Ray, un niño afroamericano, el más alto de su clase de segundo grado. Ray era inquieto, curioso, y le gustaba ir a observar lo que otros grupos hacían. Su curiosidad era tal que muy fácilmente se distraía con todo lo que él notaba en el aula. Esto lo metía en problemas y como castigo casi todas las semanas era excluído de la instrucción. Al acercarme a Ray en su aislamiento físico de los demás, ambos comenzamos a discutir los mismos problemas de matemáticas que su maestra discutía con el resto del grupo. Al usar preguntas abiertas encaminadas al significado de los problemas, empecé a notar que las ideas de Ray diferían de las que los dos escuchábamos de sus compañeros/as—a distancia, desde nuestra esquina en el aula. Por ejemplo, el sentido numérico de Ray era evidente. Un día su maestra mostró a los estudiantes un grupo de objetos y les pidió que contaran cuántos objetos había. Una niña contó 16 en lugar de 15. Al preguntarle a Ray si era correcto, me explicó que en realidad eran 15 porque "*there is 10 and 5 more makes 15, because 15 is 5 more than 10.*" Cada semana seguí escuchando de Ray ideas sobre importantes relaciones numéricas. Un día decidí pedirle a su maestra traer de regreso a Ray durante la discusión final de grupo. Habiendo hallado un espacio físico e intelectual para Ray en la discusión de grupo, Ray compartió con sus compañeros sus ideas matemáticas, subrayando cómo él notaba importantes relaciones numéricas en los problemas. Al día siguiente del regreso de Ray a su participación en grupo, Ray me tomó del brazo y, con un sentido de urgencia, insistió en que me sentara cerca de él. Tal parecía como si yo me hubiera convertido en su pasaporte de participación, en su presentador, en alguien que él necesitaba para lograr entrar a un espacio que se le había negado. Poco a poco, y sobre

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todo después de mi acercamiento a Ray. el cual hice evidente ante su maestra, dejé de encontrar a Ray en su acostumbrado sitio de *time out*.

Transformaciones colectivas desde la perspectiva de los estudiantes

A veces solo se necesita una sola oportunidad para que los estudiantes nos ayuden a transformar la instrucción matemática. Este fue el caso de un grupo de estudiantes latinos/as de tercer grado durante una unidad de estimación. Después de una primera semana de instrucción enfocada en la técnica de redondeo, la maestra no lograba entender por qué los estudiantes parecían no entender la técnica: "*I don't know why they're not getting something that seems so easy, you know?*" Ambos decidimos entrevistar a los estudiantes para entender sus perspectivas, para que nos dijeran lo que ellos notaban en los problemas de estimación. Lo que a simple vista parecía una aplicación errática o inconsistente del método de redondeo tenía, de hecho una lógica compleja. Por ejemplo, los niños nos informaron que algunos ejemplos incluían números tan simples que no era necesario estimar. También hallaban un conflicto linguístico: al leer las palabras *estimar* y *total*, esto activaba dos procesos distintos: total significaba para ellos un cálculo exacto, mientras que la palabra estimar algunos la entendían y otros simplemente no sabían su significado, lo cual los hacía a veces estimar y otras veces calcular. Estas perspectivas llevaron a la maestra a revisar los problemas de manera que reflejaran la perspectiva de los estudiantes. Empezamos a usar palabras como más o menos, casi, y aproximadamente, en lugar de estimar. Revisamos los números incluídos en estos problemas, así como los contextos para hacerlos más relevantes y con un cierto propósito. La transformación fue tal, que la maestra comenzó a enseñar la estimación no como una técnica que se debe memorizar, sino como una herramienta útil a través de todo el currículo de matemáticas.

Transformaciones colectivas desde la perspectiva de los padres de familia

Como preámbulo a una unidad de medición de tercer grado en un aula bilingüe, la maestra decidió invitar a un grupo de padres y madres de familia para que platicaran con sus estudiantes acerca del uso de la medición en sus trabajos. Al llegar, tomaron asiento en la parte de atrás del aula de clase. La maestra y el investigador los invitamos a pasar al frente a manera de un panel. Estos adultos hablaron y demostraron el papel de la medición en trabajos tan variados como la limpieza (cuánto líquido limpiador se deben mezclar con cuánta agua), el arreglo de mobiliario en un cuarto (incluyendo los muebles del aula de clases), las compras del supermecado (pesar dos kilos de frijol), y el corte de metales mediante rayo laser (para la construcción de muebles de metal). Esta plática despertó la curiosidad de los estudiantes, llenando el aula de preguntas que los padres contestaron. La visita de los padres generó transformaciones multiples del espacio físico-intelectual en esta aula de matemáticas. Por un lado, los estudiantes desarrollaron un reconocimiento de las conecciones entre las matemáticas escolares y las de la vida diaria. Así mismo, en entrevistas subsecuentes expresaron respeto hacia los conocimientos de sus propios padres y la relevancia de este conocimiento en el contexto escolar. La maestra por su parte rediseñó los problemas de medición con ejemplos que los padres habían mencionado durante su vista. Y aún después de varios años, esta maestra sigue enseñando medición con un enfoque que coloca al estudiante como centro desde el cual toda noción de medición deriva su significado. Por ejemplo, el largo de un brazo, el tamaño de una mano, o el marco de una puerta no son mediciones al azar sino que existe una interconneccíon que los estudiantes exploran, alteran, reinventan, analizan, y refleccionan a fondo antes de formalizar el tema.

Transformaciones colectivas desde la perspectiva de las maestras

La coplaneación de unidades que las maestras eligen por ser difíciles de enseñar y aprender ha servido para que estas maestras creen transformaciones a nivel colectivo. Por ejemplo, una maestra y yo planeamos una unidad de medición del tiempo. La planeación ocurrió a dos niveles: primero al

nivel de las ideas más trascendentales, y día a día estas ideas las complementábamos con lo que apredíamos de los estudiantes. Durante la unidad de estudio, los estudiantes lograron reinventar importantes conecciones entre mediciones lineales y mediciones del tiempo, lo cuál les permitió aprender el concepto con comprensión. Al año siguiente, una maestra de segundo grado del mismo proyecto, amiga y colega de la maestra de cuarto grado, solicitó nuestra colaboración porque había escuchado todo lo que los estudiantes habían aprendido. En este caso, la maestra y yo usamos ideas similares pero para segundo grado. Este tipo de transformaciones se asemeja a las descritas por Davis y Sumara (1997) quienes describen cómo "It was not long before we became aware that the complex weave of classroom activity had spilled into other communities through what appeared to be a mix of deliberate communication, casual conversation, and unconscious imitation" (p. 115). Estas transformaciones no son efectuadas solo por las maestras, sino que pertenecen al colectivo de maestras, alumnos, e investigador. Por ejemplo, al final de una de estas unidades de estudio, una maestra escribió la siguiente reflección: "*When the child takes a new or different direction than we expect, we have the potential for new common resources*."

Nuestras consideraciones colectivas

Los ejemplos presentados sugieren que las transformaciones observadas se dan como parte de esfuerzos colectivos. Esta colectividad—particularmente su diversidad interna, su auto regulación, y el efecto cascada del conocimiento emergente que desarrolla—nos obliga a no romper la naturaleza compleja de estas transformaciones. Parecido al proceso de una cámara fotográfica, nuestro interés es ampliar el enfoque utilizado para así analizar la formación y el desarrollo de una transformación. Creemos que una de las características más importantes observada en los ejemplos anteriores es la auto sustentabilidad del cambio (Franke, Carpenter, Levi, & Fennema, 2001). Así mismo, el cambio no recae en un solo individuo, sino que se construye a partir de la diversidad interna de grupos. Para comunicar nuestra diversidad interna como investigadores, ofrecemos nuestras perspectivas de cómo entendemos las transformaciones que hemos ilustrado. Higinio: El cambio y su sustentabilidad en las aulas de matemáticas va mucho más allá de las acciones individuales de las maestras. Me interesa estudiar cómo un aula es en realidad un sistema complejo que aprende, que se auto regula, y que está en continua generación de conocimiento. Luz: Con frecuencia me pregunto cómo podríamos construir un puente entre las investigaciones y la complejidad de lo que ocurre en un aula de clase. Esta complejidad require más de una voz y muy a menudo, son las maestras las que tienen mucho que aprender de los estudiantes, padres y comunidades. Carlos: Las transformaciones colectivas de la práctica docente presentan una esperanza y un desafío. La esperanza está en que, como facilitador/a, la responsabilidad de una transformación de este tipo no es solo mía. El desafío parte de mi rol como docente o investigador/a, al trabajar con otras personas para propiciar—o no—un proceso generativo complejo que permita que el pensamiento, planeación, y acción colectiva sucedan.

El lado generador de ideas del que hablábamos al principio nos ha permitido desarrolar un conocimiento muy importante que no deseamos ignorar. Más bien, lo que queremos es descubrir formas de situar este conocimiento dentro de la complejidad del proceso de enseñanza. Nuestras experiencias en colaboraciones con comunidades diversas nos llevan a cuestionar el discurso de experto/novato porque quienes efectúan estas transformaciones son grupos complejos. Lo aprendido de estas experiencias nos anima a proponer que es posible re-enfocar nuestras investigaciones de manera que nos ayuden a entender la transformación de la enseñanza matemática como un proceso complejo que requiere asumir una responsabilidad de grupo.

Research focused on improving the teaching of matemathics has created and perpetuated an expert/novice discourse. These dichotomies create a divide—therefore a borderline—by generating knowledge that grows relatively more among researchers than teachers. Following a brief

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methodological critique, our methodology recognizes the complexity of the contexts in which mathematics teaching lives. We share examples from a research project to show collective transformations of teaching through multivocality of teachers, students, parents, and researcher.

Keywords: Research Methods, Teacher Education-Inservice/Professional Development

Most studies that focus on transforming the teaching of mathematics have an aspect that generates ideas and another aspect that inhibits them. Let us take as examples the research that emphasizes "best practices" (Ball, 2000), or the professionalization of teaching based on interpretations of Goodwin's theoretical framework (1994), or the more recent studies on teacher noticing (Sherin, Jacobs, & Philipp, 2011; Jacobs, Lamb, & Philipp, 2010). The generative aspect of this research moves researchers to refine ideas and constructs. The inhibiting aspect—perhaps not as evident as the generative aspect—emerges from a critical consideration of the methodologies that these studies employ and that separate teaching from the complex contexts in which it exists. These methodologies assume that the teaching practice can be decomposed for analysis (Jacobs & Empson, 2015), and assign a great deal of responsibility to one single agent: the teacher. The result is a series of claims that promote hierarchies and even trajectories of what teacher know, believe, practice and notice. We are aware that every focus implies a defocus. Our intention is to invite participants into a reflection regarding the seriousness of this defocus.

The analysis of a fractured reality perpetuates an expert/novice discourse according to which teachers possess or lack certain knowledge. These dichotomies emerge from the assumtion that teaching is a stable phenomenon that can be analyzed and categorized accordingly. These dichotomies create a divide—therefore a borderline—which has not been interrogated enough. Our interrogation is this: Although these research approaches add sophistication to our researcher knowledge, such knowledge resists translation into transformative teaching practice, as teachers may not be able to see themselves in the fractured reality we study. When this happens, the possibility for transforming the teaching practice weakens. Our approach is different, since it is focused on the collective transfromation, the kind that emerges and can be observed through collective participatory processes that include students, teachers, parents, and researchers.

Theoretical Perspective

To recognize the complexity inherent to the process of transforming the teaching practice requires not reducing it for analytical purposes. To achieve this we adhere to Davis and Simmt's (2003) ideas about how to study complex phenomena:

> Complexity is not just another category of phenomena, but an acknowledgement that some phenomena are not deterministic and cannot be understood strictly through means of analysis (i.e., literally, by taking apart or cutting up). A different attitude is required for their study, one that makes it possible to attend to their ever-shifting characters and that enables researchers to regard such systems, all at once, as coherent unities, as collections of coherent unities, and (likely) as agents within grander unities (p. 140).

To study the teaching transformations in classrooms with students from non-dominant communities, we focus on the multivocality of teachers, students, parents, and math education researchers. Although we present these voices in separate spaces, their resonace is what creates a coherent collective with internal diversity that is self-regulated (Davis & Sumara, 2006). Aligned with the principle of qualitative inquiry of *showing* rather than *telling* (Tracy, 2010), we share moments of transformations observed in one research project in different classrooms across time. We showcase participants in these transformations as they experience stumbles, challenges, and achievements situated in a complex collectivity. Our purpose is to illustrate the teaching pathways

that teachers walk through that, far from being isolated, intertwine and enter into dialogues with the equally complex realities of students, parents, and investigators. Along these pathways, the teachers do not experience an individualized knowledge of the kind proposed in many studies, but rather participants *hacen camino al andar* [form paths by walking].

Collective Transformations from the Reseacher's Perspective

During the first year of a research project focused on reciprocity as a process for collective transformations, I began to notice that upon arriving to do my classroom observations, often there were students excluded from instruction (*time out*) due to bad bahavior. These students tended to be the same week after week. I decided to join them instead of the whole group to find out the reasons for their exclusion. Their mishaps were minor: inattention, speaking out of turn, standing up without permission, interrupting a peer, not sharing materials. One of these students, Ray, was an Afroamerican child, the tallest of his second grade class. Ray was hyperactive and curious. He liked to observe what others were doing. His curiosity was such that he was easily distracted with everything that he was noticing in the classroom. This often got him in trouble and as a punishment he was excluded from instruction almost every week. When approaching Ray in his physical isolation, he and I began to discuss the same math problems that his teacher was discussing with the rest of the group. By using open-ended questions focused on the meaning of problems, I began to notice that Ray's ideas differed from those that he and I were listening from his peers—at a distance, from our corner in the classroom. For example, Ray's number sense was evident. One day his teacher showed the class a set of objects, asking them to count them. A girl counted 16 instead of 15. When I asked Ray if that was correct, he explained that it was actually 15 because "*there is 10 and 5 more makes 15, because 15 is 5 more than 10.*" Every week I continued hearing ideas from Ray about important numerical relationships. One day I decided to ask his teacher to bring Ray back into the final whole group discussion. Having found a physical and intellectual space for Ray in the group discussion, Ray shared with his classmates his mathematical ideas, emphasizing how he was noticing important numerical relationships in the problems. The day after Ray returned to his group participation, Ray took me by the arm and, with a sense of urgency, insisted that I sit next to him. It was as if I had become his passport to participation, his presenter, someone that he needed to be able to enter a space that had been denied to him. Little by little, and especially after my working closely with Ray, which I made evident to his teacher, I stopped finding Ray in his habitual time out corner.

Collective Transformations from the Students' Perspectives

Sometimes all it takes is one single opportunity for students to help us transform mathematics instruction. This was the case in a group of third grade Latino/a students during an estimation unit. After the first week of instruction focused on the rouding technique, the teacher could not understand why students seemed not to understand the technique: "*I don't know why they're not getting something that seems so easy, you know?*" The teacher and I decided to interview the students to understand their perspectives, so they could tell us what they were noticing in the estimation problems. What at first sight looked like an erratic or inconsistent application of the estimation method had, in fact, strong sense making. For instance, students informed us that the numbers in some examples were so easy that estimation seemed unnessesary. They also found a linguistic conflict: in reading the words *estimar* and *total*, two distinct processes were activated: total meant for them an exact calculation, whereas the word estimate some understand but others simply did not know its meaning, which led them to sometimes estimate and other times calculate. This student noticing led the teacher to revise the problems to reflect the students' perspectives. We began using words such as *más o menos*, *casi*, and *aproximadamente*, instead of estimate. We revised the numbers in the problems, as well as the contexts to make them more relevant and purposeful. The

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transformation was such that the teacher began teaching estimation not as a technique to be memorized, but as a useful tool across the mathematics curriculum.

Collective Transformations from the Parents' Perspectives

As a preamble to a third grade measurement unit in a bilingual classroom, the teacher decided to invite a group of parents to talk with students about the use of measurement in their jobs. The parents initially sat in the back of the room. The teacher and researcher invited them to move to the front as a panel. These adults demonstrated the role of measurement in jobs as varied as cleaning (how much cleaning solution to mix with water), furniture arrangement (including the classroom furniture), grocery shopping (weighing two kilograms of beans), and laser cutting metalwork (for metal furniture). This conversation awoke students' curiosity, filling the room with questions that parents answered. Their visit generated multiple transformations of the physical-intellectual space in the classroom. On the one hand, students recognized connections between school math and everyday math. On the other hand, in subsequent interviews they expressed respect for their parents' knowledge and the relevance of this knowledge in the school context. The teacher, in turn, redesigned the measurement problems using the examples the parents had mentioned. Even after several years of this transformation, the teacher continues teaching measurement with a focus that positions students at the center from where every measurement notion derives its meaning. For example, the length of an arm, the size of a hand, or the height of a door frame are not random measures but there exists a connection that students explore, alter, reinvent, analyze, and deeply reflect upon before formalizing the topic.

Collective Transformations from the Teachers' Perspectives

The coplanning of units that teachers find difficult to teach and learn has helped teachers to create collective transformations. For example, a foruth-grade teacher and I co-planned a unit on measuring time. The planning occurred at two levels: first, at the big ideas level, and then these big ideas were shaped by the students' own ideas on a day-by-day basis. During this unit, students were able to reinvent important connections between linear and time measurement, which allowed them to learn the concept with meaning. The following year, a second-grade teacher in the same project, who was a friend and colleague of the fourth-grade teacher, asked for our help because she had heard about how much the fourth-grade students had learned. This teacher and I used similar ideas but adapted for second grade. These kinds of transformations resemble those described by Davis and Sumara (1997): "It was not long before we became aware that the complex weave of classroom activity had spilled into other communities through what appeared to be a mix of deliberate communication, casual conversation, and unconscious imitation" (p. 115). These transformations are created not by the teachers alone, but they belong to the collective of teachers, students, parents, and researcher. For example, at the end of one of these units of study, one teacher wrote the following reflection: "*When the child takes a new or different direction than we expect, we have the potential for new common resources*."

Our Collective Considerations

The above examples suggest that transformations occurred as collective efforts. This collectivity—particularly its internal diversity, self-regulation, and the cascade effect of the emerging knowledge—urges us not to break the complex nature of these transformations. Similar to the process of a photographic camera, we are interested in amplifying its focus so we can analyze the formation and development of a transformation. We believe that an important characteristic observed in the examples provided is the self-sustainability of change (Franke, Carpenter, Levi, & Fennema, 2001). Similarly, change is not in the hands of one individual, but it is constructed from and within the group's internal diversity. To communicate our internal diversity as researchers, we now offer

our perspectives of how we understand the illustrated transformations. Higinio: Change and its sustainability in math classrooms extend beyond teachers' individual actions. I am interested in studying how a classroom is in fact a complex system that learns, that self-regulates, and that it continuously generates new knowledge. Luz: I often wonder how to build a bridge between research findings and the complexity of what happens in a classroom. This complexity requires more than one voice and, quite often, the teachers are the ones who have much to learn from students, parents, and communities. Carlos: Collective transformations present a hope and a challenge. The hope is that the responsibility for a transformation of this kind is not exclusively mine. The challenge is how as a collaborator working with others, I can occasion a generative process by which complex thinking, planning and acting can occur.

The generative aspect of the research reviewed has allowed us develop knowledge that we do not wish to ignore. Rather, we want to find ways to situate this knowledge within the complexity of the teaching process. Our experiences collaborating with diverse communities lead us to question the expert/novice discourse because those who learn how to create transformations are complex groups. What we have learned from and with these groups encourage us to think that it is possible and desirable to refocus our research so it can help us understand the transformative teaching practices as a complex process that requires assuming a group responsibility.

Acknowledgement

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EXAMINING THE RELATIONSHIP BETWEEN CERTIFICATION PATH AND TEACHING SELF-EFFICACY

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In this paper we explore the relationship between certification path and teaching self-efficacy for teachers of K-12 mathematics. We focused on teaching self-efficacy for a content area that spans the K-12 curriculum: fractions, ratios, and proportions. Initial findings indicated that alternate certification is significantly correlated to teaching self-efficacy. However, in multiple regression analysis of certification after including other variables that predicted teaching self-efficacy, the relationship between certification path and teaching self-efficacy was no longer statistically significant. Findings suggest that the apparent relationship between alternate certification and teaching self-efficacy is likely due to other factors (age, highest grade taught, MKT) that differentiate alternately certified and traditional teachers.

Keywords: Teacher Beliefs, Teacher Education-Preservice, Teacher Knowledge

In this paper, we question a border within mathematics teacher education: the path to certification. In order to understand the relationship between certification path and teacher beliefs, we compare teachers who were certified after a traditional 4-year undergraduate degree in education with those who took an alternate path. Paths to teacher certification are part of an ongoing policy debate which continues to rage in the wake of the Every Student Succeeds Act becoming law (Sawchuk, 2015). Alternate certification is often promoted as a way to bring people with high levels of knowledge into classrooms. In this study, we focused on a different consequential teacher characteristic that might describe those who pursue different paths: teaching self-efficacy: a belief in one's ability to help students learn.

Studies have shown that higher teaching self-efficacy has an impact on student learning (Chang, 2015; Fox, 2014). Thus self-efficacy for teaching is important for mathematics teachers. Because K-12 math teachers teach different topics, we focused on a central idea: teaching self-efficacy for fractions, ratios, and proportions. The purpose of this study was to answer the following research question: *Do teachers who achieve certification through an alternate path have higher teaching self-efficacy for teaching fractions, ratios, and proportions?*

Theoretical Framework

Teaching self-efficacy (TSE) beliefs are a teacher's own judgments about her capability to teach and her confidence that her instruction will affect student learning (Bandura, 1986; Pajares, 1992). The construct of TSE has been used extensively for several decades and several measures of TSE exist (see review, Tschannen-Moran & Hoy, 2001). Under Bandura's (1986) social-cognitive theory, TSE beliefs determine teachers' "persistence when things do not go smoothly and their resilience in the face of setbacks" (Tschannen-Moran & Hoy, 2001, p. 784), and thus is clearly related to the productive disposition for teaching. Self-efficacy to teach may vary with the content taught. Bandura (1986) wrote that self-efficacy as such was too broad to be useful for research without narrowing one's attention to self-efficacy beliefs that are relevant to the specific situation or activity being researched.

What factors contribute to teachers' self-efficacy? One factor may be the path by which they become certified. Another may be a teachers' academic achievement prior to college. As people age, they tend to be more self-assured, so it is possible that a teacher's age contributes to their self-efficacy In education, there is a general perception that higher grade levels teach more difficult math,

so it's possible that having taught at a higher grade level (regardless of their present grade) may make teachers more confident in their teaching of particular content. Finally, mathematical knowledge for teaching (MKT) may also be related to teaching self-efficacy.

In the last decade, large-scale studies using more sharply focused instruments have found evidence of the expected relationships between teacher knowledge and student achievement. These new instruments share a focus on the content knowledge that teachers' arguably use in practice. Ball, Thames, and Phelps (2008) proposed a framework for content knowledge for teaching "subject-matter-specific professional knowledge," (p. 389). Mathematical knowledge for teaching (MKT) includes pure content knowledge as well as specialized content knowledge (SCK), which is a kind of mathematical knowledge that teachers but few other adults possess.

Methods

Data for this study was collected from a variety of sources. Participants' certification path, age, and highest grade taught was provided from official records for teachers completing the Texas Teacher Training Survey (TTTS). This survey was conducted by the National Research Council in association with the Texas Education Authority and collected data on a representative sample of Texas teachers certified between 2006 and 2010. University selectivity using ratings for undergraduate institution selectivity (Barron, 2001) and was based on self-reports. TSE and MKT data came from a follow up survey of TTTS participants (Jacobson, 2013).

The instrument for TSE was adapted from measures for prospective science teachers (Enochs & Riggs, 1990; Roberts & Henson, 2000). The items measuring TSE were modified to address the domain of multiplicative reasoning by replacing the word "science" with the phrase "topics involving fractions, ratios, and proportions." For example, the question, "I usually do a poor job teaching science" became, "I usually do a poor job teaching topics involving fractions, ratio, and proportion." Overall TSE scores were obtained using a rating scale IRT model.

The MKT instrument was composed of 25 items selected from the Measures of Effective Teaching project (Bill & Melinda Gates Foundation, 2010) that focused on fractions, ratios, and proportions. The items explicitly addressed two kinds of MKT: understanding and evaluating students' mathematical thinking (18 items) and selecting and using tasks and representations (7 items). The selected items could be cross-classified by three topics that make up the domain of multiplicative reasoning: fraction multiplication and division (8 items), fraction and ratio comparison (10 items), and proportional reasoning (7 items). Both TSE and MKT instruments had high internal consistency (Cronbach's α > .9). All items had point-biserial correlations greater than or equal to .2, and all item parameters were acceptable (Crocker & Algina, 1986).

Results

In order to answer our research question, we began with descriptive statistics. We then ran correlations to understand the relationships between pairs of variables. After that, we ran a multiple regression model to predict teaching self-efficacy from the other variables.

An analysis of correlations (see Table 1) shows that for all but one explanatory variable (university selectivity), there was a statistically significant linear relationship between it and the outcome variable. This indicates that these variables may indeed be related to teachers' self-efficacy for teaching fractions, ratios, and proportions. However, there were also significant correlations between several of the explanatory variables.

Our findings seem to indicate that teachers who undergo alternate paths to certification have higher teaching self-efficacy for teaching fractions, decimals, and percentages than those who receive their certificate after completing a traditional undergraduate education program. However, because of significant correlations between our explanatory variables, it is possible that teachers with higher teaching self-efficacy also have other characteristics in common which may be responsible for this

correlation. Thus, we decided to run a multiple regression of certification on the other variables to further understand these relationships.

Table 1: Correlations among teacher variables [A]

	1	2	3	4	5
1. Teaching self-efficacy (TSE)					
2. Traditional Certification Path	-.181*				
3. University selectivity	.049	-.136 *			
4. Math Knowledge for Teaching (MKT)	.284 *	-.106	.252 *		
5. Age	.202 *	-.230 *	.038	.031	
6. Highest grade taught	.310 *	-.217 *	.089	.327 *	.042

[A] N is between 362 (MKT & Age) and 443 (TSE & Age); * $p < .01$

A standard regression model was tested in which teachers' self-efficacy for teaching fractions, ratios, and proportions was predicted from certification path, university selectivity, MKT, age, and highest grade taught. Overall, the model was significant, $F(5, 303) = 11.67$, $p < .001$, and accounted for 14.8% of the variance in efficacy scores, $R2 = .148$. Standard error of regression coefficients are reported in Table 2.

Table 2: Predictions of self-efficacy for teaching fractions, ratios, and proportions

	Self-efficacy for teaching fractions, ratios, and proportions		
	B	SE	Significance level
Constant	-.794	.278	.005
Traditional Certification	-.189	.107	.079
MKT	.259	.069	<.001
University selectivity	-.008	.037	.838
Highest grade taught	.049	.016	.002
Age	.015	.005	.001

Coefficients for three of the explanatory variables – MKT, highest grade taught, and age - were significant ($p < .01$). As the MKT score increases, the predicted self-efficacy also increases. In other words, the more a teacher knows about math knowledge for teaching, the higher self-efficacy they have. As the highest grade taught increases, so does the predicted self-efficacy. Finally, as a teacher's age increases, so does the predicted self-efficacy. After controlling for other variables in the model, the coefficient for certification path was not statistically significant ($p = .079$), and neither was the coefficient for university selectivity ($p = .838$).

Conclusions

With this study, we sought to determine the relationship between certification path and teachers' self-efficacy for teaching fractions, ratios, and proportions after taking into account other information about teachers including the selectivity of undergraduate college/university, MKT, age, and highest grade taught. The regression model indicates that yes, these explanatory variables do account for some of the variance (15% of it) in teachers' self-efficacy for teaching fractions, ratios, and proportions. One limitation of this study may be that the R^2 is not large, different variables than the ones we selected may be required to understand teaching self-efficacy. However, the significance of the model indicates there is definitely a relationship between the predictors we chose and the teaching self-efficacy outcome variable.

We were most interested in certification path because few studies have examined the relationship between certification path and teaching self-efficacy, an established predictor of student

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achievement. The correlation we found between certification path and teaching self-efficacy was very interesting in this regard, and we hypothesized that there would be a statistically significant regression coefficient for this variable when predicting teaching self-efficacy. However, there was not: when other factors including age, MKT, and highest grade taught were included in the model, certification path was still negative but was no longer statistically significant ($p = .08$). These results seem to indicate that neither certification path – alternate or traditional - predicts higher teacher self-efficacy than the other. Instead it is other differences between teacher characteristics that account for the apparent relationship between teaching self-efficacy and certification path. We conclude that teaching self-efficacy may not be a meaningful difference between teachers who follow different paths to certification.

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USING MATHEMATICAL ARGUMENTATION TO ACHIEVE EQUITY IN DISCOURSE

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In this paper, we show how one teacher used software, curriculum and new teaching moves to advance student argumentation and promote equity in discourse. The professional development (PD) she engaged in, Bridging PD, focuses on assisting teachers in diverse urban school communities to infuse argumentation into their practices so that all students actively engage in mathematical argumentation. Bridging PD integrates technology-based curriculum and dynamically-linked representations in the PD and then teachers use these in their classrooms.

Keywords: Teacher Education-Inservice/Professional Development; Equity and Diversity

Literature Review

As demonstrated in recent studies (Knudsen, Shechtman, & Kim, 2012; Choppin, 2014; Cirillo et al., 2014), strengthening the alignment between mathematical communication and argumentation is consistent with current mathematics curriculum policy documents (National Council of Teachers of Mathematics, 2000; the National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). The Common Core State Standards for Mathematics (CCSSM) Standards for Mathematical Practice 3 includes constructing and critiquing mathematical arguments (Cirillo et al., 2014). Several researchers (Herbel-Eisenmann, Choppin, & Pimm, 2012) have also observed that there is a need to increase access for students of color and the poor to "participation in dominant mathematically-based discourse practices, such as argumentation" (p. ix). Further, fostering student involvement in classroom discussions enhances the view that students hold of themselves as mathematics learners and doers (Walshaw & Anthony, 2008, p. 536).

A number of researchers (Hicks, 1998; Walshaw & Anthony, 2008) have also reported that for many teachers, it is a major challenge to include classroom discourse as an integral part of their teaching. In their comprehensive and critical review of the literature on mathematical discourse, Walshaw and Anthony (2008) identified what teachers can do to promote discourse that allows all students to achieve. Among these practices are revoicing to achieve clarity, using students own ideas as a springboard for developing new knowledge, supporting students in making their own conjectures and pressing for justification. The authors observed that when talking about mathematics became acceptable in the classroom, mathematical discussion, explanation and defense of ideas became defining features of a quality mathematical experience (Walshaw & Anthony, 2008).

Bridging PD

A multifaceted professional development (PD) program, Bridging PD prepares teachers to address the challenge of bringing argumentation to diverse urban classrooms. One important part of Bridging PD is providing teachers with curriculum that supports argumentation. Through teaching games, they develop teaching moves to use in this context as well as in their own lessons. The technology allowed the participants to generate different cases, and to test and justify their conjectures. In the 7th-grade unit focused on in this paper, students learn about linear functions within the context of motion, so the student software presents a motion represented as an equation, as a graph and in a simulation of vehicles traveling (Figure 1). In the first lesson of the unit, the

onscreen prompts ask students to create a story to correspond to a graph, check their story with the simulation, create new graphs and stories, and make conjectures about the graphs.
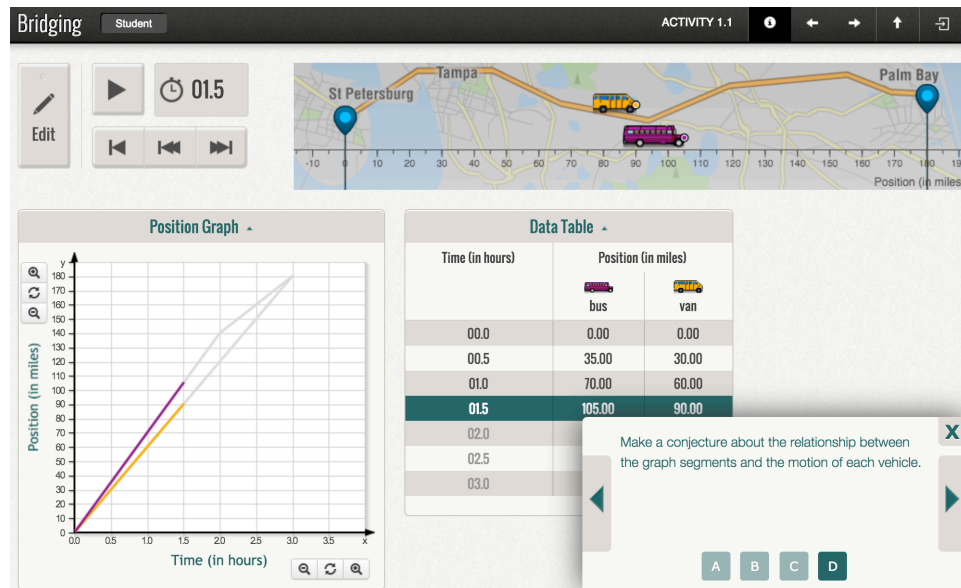

Figure 1. Dynamically linked representations with onscreen prompts.

Methods

Our study has a quasi-experimental design complemented by in-depth case studies for a subset of teachers. One of the case study teachers is Ms. Norris, a 7th-grade teacher with less than three years experience in the classroom. She works in an urban school with about 70% Black, 10% White and 15% Hispanic students. Roughly 50% of the students receive free or reduced-price lunch. The following analysis is based on two observations of her classroom in spring 2015 and two post-PD observations in fall 2015. Notes were created during the observation and cross checked with the videos, using an observation protocol developed to capture teaching practices and classroom norms that support mathematical argumentation. Then, argumentation episodes were parsed out for further analysis for teaching moves and students' responses.

Analysis

In pre-PD observations, Ms. Norris's practice was identified as having "little to no argumentation." She demonstrated how to solve a problem step-by-step to the whole class, students worked on problems individually under her supervision, and selected students shared their work with the whole class.

In November, Ms. Norris's classroom was observed when she was using the 7th grade unit. In the first lesson, students were assigned to make a conjecture about the relationship between motions and graphs. As she circulated among small groups of students working with the software, she visited a pair of students who were working together with an animation, which showed a van and a bus moving at different speeds, and its corresponding graph.

The following excerpt illustrates the teacher's moves for supporting conjecturing:

Ms. Norris: Do you almost have a conjecture?
Kyanna: A conjecture?
Ms. Norris: Yeah, down here okay. So skip this editing again. See if you can jump to how is the graph and the animation related? Like what makes you say about like which bus, like how can you make the bus win, how can you make the van win based on the graph, okay?

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Kyanna: So, we can say if the bus moves further [Bree interrupts]
Bree: No, if you make the van closer to the finish line, the uh
Kyanna: Then the van will win.
Bree: yeah, but if [Ms. Norris interrupts]
Ms. Norris: What do you mean?
Kyanna: If you make the van go the finish line [Bree interrupts]
Bree: No, if you make, if you place the van closer, ahead, ahead of the bus [Ms. Norris interrupts]
Ms. Norris: The start?
Bree: Yeah.
Ms. Norris: It has to win?
Bree: Yeah
Ms. Norris: Try that and see if that works.

The teacher initiated the conversation by prompting for a conjecture about the relationship between the graph and the motion. She then followed up with questions prompting students to clarify their conjecture and to generate a case to confirm it. Generating cases is a structure used explicitly in the Bridging curriculum to support students in finding patterns by exploring various cases, come up with a conjecture from the patterns and use later as evidence to support their conjecture. Clarifying, a move developed in the PD, is important in conjecturing because it leads students to use more precise language and to refine their ideas so that they establish shared understanding about the conjecture and can critique the conjecture with reasons that are not rooted in simple confusion (Thompson & Schultz-Ferrell, 2008). This interaction was brief, but it led students to come up with an extreme case that excited them. Later, Ms. Norris came back to the pair as they called on her.

Ms. Norris: It just shrank. That's okay. Oh, Wow! [Bree and Kyanna giggle] Cool. What that, what does that graph look like? That's a, so making kind of crazy situations like that are actually really helpful for making conjectures because if you notice, what's crazy about this situation? [Ms. Norris points to the representation of the motion] It's like where you were just talking to me about like "Wow." Why did you say that?
Bree & Kyanna: Because it went so fast.
Ms. Norris: Which one went so fast? What's true about its picture? its graph?
Kyanna: [indiscernible]
Ms. Norris: How, that's, you can just use that to make a conjecture. You said, "wow it's going so fast." What's- [Bree points to y-axis on graph on computer]
Kyanna: [Indiscernible]
Bree: Oh, oh, look in the graph. It's 80 and 0. on the [Bree starts writing in her notebook]
Teacher: There you go. [Teacher leaves the pair]

The pair continued their conversation. One of them played the animation again.

In this conversation, the teacher addressed the role of extreme cases in making a conjecture. First, she shared the excitement with the students about their finding of an interesting case, where a van moved really fast while a bus moved slowly. She noted the unusualness of the case students found by labeling it as "crazy." Also, she probed further for the mathematical aspects of their findings by building on what students said—using teaching moves to seek for, clarify and use student ideas as a springboard for introducing new knowledge are effective for promoting student discourse (Walshaw & Anthony, 2008). She pressed students to say more than what they simply noticed about motions and to think about "what is true" about the relationship between the motion and the graph, which later became their conjecture: "If the van/bus is close to the origin, the line closer to the y-axis will be faster." Ms. Norris further clarified with them that they meant the y-axis instead of the origin.

Conclusion

These selections show that this teacher, working in a diverse urban school classroom, is learning to support her students in argumentation, specifically in generating cases that lead to conjectures, with the support of online curriculum. She used students' engagement with what they saw on the computer screen to lead them to more sophisticated conjectures. Starting with students' own ideas, she helped students clarify their thinking and supported them in talking to each other, formulating conjectures, and asking and responding to questions—all practices aligned with the literature on enhancing math discourse. She was, indeed, providing the kind of access to high-level disciplinary practices that have been lacking in many such classrooms. This finding is also consistent with other results from prior Bridging PD studies (Knudsen, Shechtman, & Kim, 2012).

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EPISTEMOLOGIES AND COALITIONS *SIN FRONTERAS* IN MATHEMATICS EDUCATION

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Acknowledging our position as outsiders and researchers in the work with specific communities of students of color and their teachers, we examine our epistemological paradigms through the lens of Latina/o Critical Theory. Analysis of our testimonios revealed that our experiences have influenced our perspectives as mathematics educators and researchers. Findings describe our necessity to "straddle the borderlands" of different epistemologies to responsively and more explicitly address issues of equity, specifically race, in our coalitions with school communities.

Keywords: Equity and Diversity, Teacher Education-Inservice/Professional Development

As researchers and mathematics educators concerned with issues of equity in mathematics education, we believe that forming coalitions—across racial, gender, cultural, and geographical borders—may support the combined work of "allies" when aiming at engaging in actions and goals *sin fronteras*. An ally is a person who helps, supports, or acts in solidarity with another in a particular effort. Through ally work, we aspire to disrupt systems of privilege and oppression in mathematics education. Drawing on ally perspectives, we accept that our knowledge, goals, and values as math educators are influenced and bounded by mainstream mathematics education perspectives. Consequently, we intentionally contrast our stances with borderline epistemologies of a minoritized group, specifically a Latina/o Critical (LatCrit) framework to reflect on our stances in these possible coalitions. The LatCrit perspective aims "to move researchers and educators into spaces of moral and critical practice" (Delgado Bernal, 2002, p. 393).

Theoretical Perspective

We adopted the LatCrit perspective to critically examine our emerging understanding of what it means to be an ally. The five tenets of this framework (Delgado Bernal, 2002) include:

1. *Centrality of race and racism and their intersectionality with other forms of subordination.* Historically, racism has been associated with other forms of oppression such as class, gender, language, and immigration status and among others. LatCrit maintains that at this intersection some answers to theoretical, conceptual, and methodological questions might be found.
2. *Challenge dominant ideologies*. LatCrit values culturally specific ways of teaching and learning as ways of knowing that extend beyond the formal. This stance defies traditional educational deficit frameworks that privilege dominant groups' ideas such as objectivity, meritocracy, color-blindness, race neutrality, and equal opportunity.
3. *Commitment to social justice*. LatCrit epistemologies seek structural and personal political and social change by dismantling structural subordination that empower minoritized groups.
4. *Centrality of experiential knowledge*. LatCrit views experiential knowledge of people of color as a strength. Methods such as storytelling, family history, biographies, *testimonios*, and oral histories, etc. are utilized as processes to learn about these students' unique experiences.

5. *Call for transdisciplinary perspectives*. LatCrit challenges unidisciplinary in research and education and instead encourages the merging of various research methods to examine race and racism in education in historical and contemporary contexts.

Methods and Data Sources

This study involved the participation of fourteen project team members from eight U.S. public universities. Members included nine faculty members (six white females, one white male, one African American male, and one Latino male), four doctoral students (one white female, two Asian females, and one African American male), and one postdoctoral researcher (African American male). Participants engaged in life history and background interviews to document the incoming epistemologies informing this work. Two rounds of interviews were conducted either via phone conference, online, and/or face to face by two white faculty members of the team. The semi-formal interviews included open-ended questions that explored the participants' stance, perspectives, knowledge, experiences, and personal definitions regarding issues of privilege and oppression as well as features of equitable mathematics teaching and learning including *access* and *agency* (Gutiérrez, 2012), and *allies*. All interviews were audiotaped and transcribed. We viewed the interview data as testimonios related to awareness on issues that affect minoritized students. *Testimonio* is a verbal description of a journey that reveals the injustices suffered by marginalized people with the goal of promoting healing, empowerment, and advocacy for similar current and future populations (Pérez Huber, 2010).

To address the research question – *In what ways do the project team members' epistemologies align (or not) with the basic tenets of LatCrit?* – a content analysis of the interviews was developed by two faculty members (Latino and white female) and doctoral students (African American male and Asian female). Content analysis is a "technique for making inferences by objectively and systematically identifying specified characteristics of messages" (Holsti, 1969, p. 14). Guided by LatCrit epistemology and critical race grounded theory, themes emerged from data through critical lens that explicitly addressed issues of equity (Pérez Huber, 2010). Three stages of data analysis were implemented: preliminary, collaborative, and final. In the preliminary phase, themes in the data were identified through initial coding strategies (Charmaz, 2006). The themes were shared and discussed with the team until reaching consensus. This process served as an opportunity to member-check as well as to further reflect on our epistemological stances and provide richer understanding of our work (Kruger, 1988). Results are described through four contrasting and yet complementary examples that help present the major themes from the analysis.

Results

Initial themes were collapsed into three explanatory themes: (a) the recognition of one's power and privilege when working for social justice at the social and personal levels in mathematics education, (b) the acknowledgement that the assessment of our relationships across roles (such as students, mathematics teachers, and mathematics teacher educators) is crucial when considering the development of purposeful coalitions and actions as allies that support agency and access across these roles, and (c) a need for more explicit focus on race, language, and immigration as forms of subordination to promote equitable changes for diverse learners.

Experiential knowledge framing stances about power and privilege

Similarly to other analyses using LatCrit, the members' testimonios overwhelmingly identified experiential knowledge as a source of their perspectives and understandings of issues of equity. For example, Liz described some of her research interests linked to her school experience: "My interest in voice and authority has to do with my own silencing as a math student." Liz's silencing informed her need to further explore a key aspect in mathematics instruction, the relationship between

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students' voices and teachers' authority construed in classroom interactions. Likewise, Frida made use of her experiences as a teacher to support the work of teachers. She shared, "My experiences in the classroom, enlisting the support of parents, that's all about teachers in the service of the children because parents can support the work of teachers reaching into the community and so on." As a teacher, Frida realized the powerful changes that students experience through the combination of understanding the interactions between mathematics and their communities. Both of these previous stances represent an approach towards equitably addressing issues of student diversity.

Moreover, members' *testimonios* revealed that participants have learned that, with teachers and students, equitable work starts with equitable stances, which comprise the acknowledgment of one's own privilege. For example, Liz stated, "In terms of trying to be an ally, you almost have to learn to see the world in a different way and step outside of your own privilege to understand and be genuine in those relationships." Similarly, Tracy shared, "Every time you see injustice, you have to interrupt it. You should be the one. If you have the privilege of power, say something." According to Liz's and Tracy's *testimonios*, awareness of one's privilege is a necessary condition for supporting equity. Simultaneously, this awareness confers a responsibility of knowing when to "use it" and when to "step outside of it" in order to support equitable community work and productive alliances, something that leads to the next theme.

Assessment of relationships and roles yielding tensions in coalitions for change

Social change is a major goal addressed by the tenet of *commitment to social justice*. The *testimonios* highlighted the relevance of coalitions to achieve this goal. Mathematics educators may play a role as allies in this process, but *testimonios* presented an evolving definition of this role. Most participants agreed with Liz "being an ally is something you have to live." Thus, ally work requires purposeful action. Liz used a simile of allies being like 'brokers.'

> People who are brokers, who can sort of exist between teachers and […] (they) exist sort of between spaces can do and say things that people who live in those spaces can't do and say. For example, I can go to the principal and say things and do things on behalf of the teachers without them asking me to and create opportunities for them, without repercussions for me.

Liz's account of a broker represents the role of an ally for teachers because s/he can work at a different level than the teacher community. Thus, people with powerful positions have more opportunity to exert their agency to implement changes, to be allies. As Frida described, however, an ally alone cannot promote the intended change. "I worked sometimes at schools with teachers who really didn't want to work with math professional developers. They weren't interested in the district or school's goals. […] If teachers set their own goals, then they're more likely to work with someone else."

The power of alliances resides in the development of coalitions that are mutually developed and embraced by all parties involved. Contrastingly, other *testimonios* raised concerns about the feasibility of coalitions. For example, Tracy questioned, "Who wants allies and who doesn't? Do you want me, as a white person, as an ally? Are there some people of color who have written that white people can't be allies for people of color? Like, stop calling yourself an ally!" A tension in the role of an ally occurs at the border of a patronizing or a colonizing approach. LatCrit supports critical coalitions among Latinas/os and nonLatinas/os to dismantle structures of oppression (Valdes, 1999), raising the question, not of "if," but "how" and "on what" coalitions can develop.

Focus on the intersectionality of identity to promote equitable changes for diverse learners

Analysis of the *testimonios* yielded recurrent concerns about class, gender, and ability as forms of subordination. Katy, for example, argued:

> My family really influenced the way that I think about the world. I remember most clearly my parents talking to me about gender equality. […(and)] less explicit talk about race. My dad married a black woman so obviously that would have been a huge influence on the way that I thought about race. Still I don't remember explicit discussions as much as on gender.

While identifying experiential knowledge as an epistemological source, Katy described how her context did not include race as a central issue to consider. Consequently, topics such as race, language, and immigration were less prevalent in the *testimonios*. LatCrit, however, identifies race as intersecting with other issues, but yet it places race at the center of analyzing forms of subordination. This recurring theme is a critical call to our team to epistemologically expand towards and more explicitly consider language, immigration, and especially race as forms of subordination to be examined and dismantled in mathematics education (e.g., Foote & Bartell, 2012; Martin, 2009; Moschkovich, 2010; Nasir, Hand, & Taylor, 2008).

Scholarly Significance

The juxtaposition of our epistemological stances identified our *frontera,* our challenge of taking a more explicit stance on issues of race in mathematics education, and not only on other intersecting issues, latter issues which are more evident to the majority of our team. Paying more attention to issues of race in mathematics education would be fruitful not only in relation to inform our stance as allies, but also—as LatCrit sustains—it is at this location where answers to structural subordinating forces might be found. As researchers and mathematics educators, we understand our need to "*straddle the borderlands*" (Anzaldúa, 1987) by embracing the issue that our lived experiences do not match those of diverse students. Thus, our aimed ally-stance in our coalitions with diverse teacher and student communities demands from us to learn to listen, notice and address explicitly, *sin fronteras,* issues of race in mathematics education and research.

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THE SMARTNESS DILEMMA: A CHALLENGE TO TEACHING MATHEMATICS FOR EQUITY

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Abstract: Research on teachers frequently illustrates the persistence of beliefs about students and mathematics that reproduce inequity. The prevalence of these beliefs suggests that they not only belong to individual teachers but also are part and parcel of culturally dominant discourses concerning the nature of mathematical ability. In this paper, I show how these discourses interfere with equity-oriented reforms, focusing on a teacher who was deeply engaged in redefining what counts as "smart in math," dismantling the borders between "smart" and "not smart." In spite of her efforts, the teacher continued to see students' deficits and, more significantly, she constantly doubted the legitimacy of the "smartnesses" she worked hard to see. I call this phenomenon the Smartness Dilemma. Understanding how it functions is important for understanding how to support teachers to disrupt hierarchies of perceived ability.

Keywords: Teacher Beliefs, Equity and Diversity

Introduction

The dominant "ideology of intelligence" in the United States holds that intellectual ability is innate, fixed, and quantifiable on a linear scale, so that some people plainly appear to have more than others (Oakes, Wells, Jones, & Datnow, 1997). Although there is growing recognition that intelligence is not fixed but developed through learning and effort, boundaries between intelligent and unintelligent students remain deeply inscribed in schools, as evidenced by the persistence of classifications such as "gifted," "average," and "remedial."

Amanda Pepper (a pseudonym), the high school mathematics teacher at the center of this paper, consistently strove to erode these boundaries and support all of her students to see themselves as intelligent and capable—an especially ambitious goal because most of her students had histories of struggle in school, particularly with mathematics. Despite their protests, she routinely insisted that they rely on each other to solve complex problems, refusing to step in herself to explain how to reach a solution or to adjudicate a disagreement about content. In doing so, she enacted confidence in her students and gave them meaningful opportunities to learn that they could succeed without her. Redefining mathematical intelligence even more explicitly, Amanda labeled her students and a wide variety of their skills "smart," challenging schools' traditional emphasis on accurate and efficient computation. Near the end of class one August day, for example, she asked students to write about how they were "smart today in math," noting that she had seen them recognize patterns; make predictions; use multiple representations (e.g., tables, graphs, and equations); use technology to help solve problems; persist and continue to try when stuck; ask questions; and communicate their thinking clearly.

Yet even in the midst of naming a multitude of ways that her students were smart, Amanda sometimes found herself stuck in "a deficit mindset." Research spanning several decades has analyzed such mindsets in terms of individual teachers' beliefs about students (e.g., Garmon, 2004; Gomez, 1993; Jackson, Gibbons, & Dunlap, in press). Here, I question boundaries between good and bad teachers as owners of good or bad beliefs. I focus instead on social ideologies of intelligence. I find that dominant ideologies of intelligence create a quandary that I call the Smartness Dilemma: although teachers may strive to see each of their students as smart, the lens of hierarchical ideologies

of intelligence remains before them, bringing borders between "smart" and "not smart" into focus and making many strengths invisible.

Theoretical and Empirical Background

Seeing is a social practice. What teachers notice and how they make sense of it comes not only from their individual knowledge and beliefs but also from social and historical ways of seeing and making sense (Vygotsky, 1986; Wenger, 1998). Hierarchical ideologies of intelligence are deeply engrained in American education, pushing teachers (and others) to see more intelligence in some and less in others. Apparent differences are reified with increasing concreteness. For example, standardized testing produces categories like Advanced, Proficient, Basic, Below Basic, and Far Below Basic (as the proficiency bands are called in California). While these labels are meant to designate only performances, in practice, they "acquire" children (McDermott, 1996), so that it becomes sensible for educators to talk about "my FBBs [far below basics]" and "the bubble kids" (those on the cusp of proficiency).

"Metaphors of hierarchy" are especially salient in mathematics, a discipline that is often assumed to proceed along a "linear path" from the basic to the sophisticated (Parks, 2010). As Parks describes, a linear hierarchy is inscribed throughout mathematics education discourse, shaping everything from casual teacher talk to mathematics textbooks to policy documents. Its ubiquity creates a context in which it is natural to locate students on the path as either on track, ahead, or behind—and unnatural to see intelligence as multidimensional or all students as smart.

Of course, no matter how dominant an ideology is, meanings are never cemented but always in negotiation (Wenger, 1998). Still, as I show here, asserting alternatives to dominant ideologies of intelligence is not merely an issue of changing individual teachers' beliefs.

Methods

This paper draws on data from a study that involved 18 mathematics teachers in two diverse urban high schools. I focus here on Amanda Pepper. Amanda is a White woman who was in her third year teaching at the time of the study. She was an active participant in a district-sponsored professional development (PD) program based on a pedagogical approach called Complex Instruction (CI; see Nasir, Cabana, Shreve, Woodbury, & Louie, 2014). CI asserts that *all* students are not only capable of learning but also already "smart." The PD was designed to support teachers to broaden their views of what counts as mathematically smart and to accordingly transform their instruction. Amanda attended workshops and received sporadic CI coaching. She was held up as a leader and a model in the local CI community.

I conducted observations in CI PD sessions, in Amanda's classroom (n = 8), and in routine meetings of the mathematics teachers at Amanda's school (n = 8) for one school year. I also interviewed Amanda informally throughout the year and once in a semi-structured format at the end. I collected these kinds of data for a total of six teachers in the study but elected to focus on Amanda for the present analysis because she was uniquely clear in articulating a dilemma that is not commonly addressed in the literature on teaching mathematics for equity.

My analysis focused on Amanda's own descriptions of her goals and of the obstacles she encountered as she worked toward those goals. I also drew on transcripts from classroom observations and teacher meetings in cases where they were directly related. I used open coding (Emerson, Fretz, & Shaw, 1995) to develop the themes described below.

Findings

Amanda expressed two broad kinds of challenges: ideological and structural. The structural difficulties she described are likely familiar to many who have worked in under-resourced schools.

They included lack of time, lack of access to expertise, and lack of support for professional learning, as well as high mobility amongst students, teachers, and school leaders.

More central to this paper are the ideological challenges—manifestations of the Smartness Dilemma—that Amanda faced. Specifically, she described continuing to see her students' deficits and doubting the legitimacy of the "smartnesses" that she worked hard to see.

Seeing deficits instead of strengths

At times, Amanda caught herself "in a deficit mindset, where I'm like, [my students] can't do all of these things instead of [focusing on] what they can do." She noted that they rarely came to class "prepared and having studied" and that they weren't "willing to work hard." Yessica was "so lost" because, as a newcomer from Mexico, she spoke very little English and was also frequently absent. Keshia would "probably never be able to manipulate an equation."

These ways of thinking and talking about students draw on dominant ideologies that not only highlight certain types of knowledge and motivation but also link deficits in these areas to particular groups, such as students of color, students from low-income households, English learners, and students with learning disabilities. These ideologies found their way into Amanda's thinking, even when she intended to look for her students' strengths. They presented her with one facet of the Smartness Dilemma: whether or not to trust her own perceptions, and if not, what to do with these apparent "facts" about her students' capabilities.

Doubting the legitimacy of nontraditional "smartnesses"

Although she sometimes dwelled on students' deficits, Amanda was skilled at noticing and publicly acknowledging ways of being smart that are not typically recognized in American schools or society (a capacity that she had successfully been supported to cultivate, unlike many teachers). Even when she focused on strengths, however, Amanda was plagued by doubt—a second facet of the Smartness Dilemma. She described own education in "a very White, one-way school," saying that there was "this traditional view of what smart looks like … I think I still have it in my head." This view made her question whether "I'm out of reality, by wanting to believe that [my students] are so smart in different ways."

The "traditional" voices in Amanda's head were amplified by a number of her colleagues, who vocally reasserted traditional views of mathematics and mathematical intelligence. For example, in a department discussion of how to teach factoring in their Advanced Algebra classes, Amanda advocated for a multi-modal approach using area models as well as algebraic manipulation. Rob, who taught the department's most advanced courses, insisted on a purely symbolic procedure that "shows algebraically what's happening to the factors." Larry interjected that he would ideally like to teach students both methods, but given time constraints, he preferred Rob's way because "it's just like, this method is what math looks like." These teachers' reluctance to embrace CI and their confidence in traditional views came up repeatedly in my conversations with Amanda. They shook her trust in her students, in her own content knowledge, and her own intelligence, leading her to ask questions like: "Do I need to be thinking more critically about it? Am I really, are [my students] really smart? Or am I just trying to compensate?" Even when she felt empowered to push back and ask Rob how his students felt about themselves and mathematics, Amanda described wondering if she was being "soft" and experiencing "this awful twist in my head again of just, constant questions and doubt."

Discussion and Conclusion

In much the same way that Amanda worked to challenge boundaries between "smart" and "not smart" students, her case challenges boundaries between "good" and "bad" teachers. She was successful in noticing a wide variety of "smartnesses," and many in her district viewed her as a

highly skilled and deeply committed CI teacher. Yet dominant ideologies of intelligence were ever present for her, highlighting students' deficits and making her question the validity of the different strengths she saw. This shows that even teachers who are unusually committed to expansive ideas about what counts as mathematically smart are deeply affected by dominant ideologies. Future research should explore the range of these effects for different teachers, taking ideology as an analytic focus. Such a focus can challenge the deficit views of teachers that beliefs-focused research often implies. It can also help the field to better understand the diverse kinds of support that teachers may need to navigate the Smartness Dilemma and other challenges associated with resisting the narrow and hierarchical ideologies that currently dominate American society. We may find that resources like community support and ongoing validation, for example, are just as if not more important than resources for developing technical and other kinds of expertise.

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UNDERSTANDING TEACHERS' PARTICIPATION IN AN EMERGING ONLINE COMMUNITY OF PRACTICE

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Research has shown the importance of teachers' participation in professional communities for supporting instructional change. However, recent studies indicate the potential of online communities for supporting engagement in productive and generative instructional practices that are transferable to teachers' classrooms. This study aimed to better understand the nature of 28 mathematics teachers' participation in an emerging online community. Results indicate a relationship between the extent of teachers' persistence in project activities and their beliefs about their relationship with the community. This study has implications both for the design of professional development programs and the ongoing support provided during such programs for increasing the likelihood of teachers' persistent involvement.

Keywords: Teacher Education-Inservice/Professional Development

Introduction

Research-based instructional practices such as student-centered instruction, the use of cognitively demanding tasks and orchestrating productive mathematical discussions have been shown to be conducive for improved student achievement (Jackson, Garrison, Wilson, Gibbons, & Shahan, 2013). Such instructional practices, however, are difficult for teachers to enact, especially when local institutions are not supportive of such practices (Gamoran et al., 2003; Kazemi & Franke, 2004). In our work, we seek to capitalize on the structure and support of professional communities to facilitate teachers' instructional change (Vescio, Ross, & Adams, 2008). Specifically, we focus on the potential of online communities that lie outside specific classrooms, schools or districts to support teachers development and implementation of high quality instructional practices — even those that may run counter to the norms and practices that are accepted within teachers' local settings. More generally, our work focuses on supporting the emergence of online communities for teachers that provide a context for meaningful and generative professional development (PD) and support emergence of research-based instructional practices. This paper reports on our current efforts to better understand the nature of teachers' participation in this online community and the potential impacts of their participation.

Theoretical framework and related research

To frame our work of cultivating online teacher communities, we draw on Wenger's (1998) notion of communities of practice. Participating in a community of practice involves sharing a history of learning with others. Involved in this history of negotiating meaning and common experiences, participants in a community develop shared practices, common beliefs and a repertoire of tools (Wenger, 1998). Such involvement, however, results in discontinuities between practices of "those who have been participating and those who have not" (Wenger, 1998, p. 103). Thus, boundaries form and communities' practices diverge.

Boundary encounters can be designed to support the crossing of boundaries. Encounters can include one community visiting another and immersing themselves within the host community's

practice. This exposes the visiting community to the host's practices and provides contexts for the intersection, negotiation and coordination of practices. Prolonged engagement in negotiating meaning and shared experiences at the boundary can result in the emergence of boundary practices and, consequently, the emergence of a new community of practice (Wenger, 1998).

Our work in the Emerging Communities for Mathematical Practices and Assessment project (EnCoMPASS) drew from this theoretical perspective and aimed to purposefully design a boundary encounter between a group of mathematics teachers and an existing online community of practice, the Math Forum, consisting of experienced mathematics teachers and teacher educators whose practices and beliefs are aligned with research-based instructional practices (Shumar, 2009). EnCoMPASS began with a face-to-face experience but continued (and currently continues) with online PD activities that focused on analyzing, organizing, and annotating student work and developing feedback for students in a technologically mediated environment (Shumar, 2009). Activities were designed to support teachers' engagement in the existing practices of the Math Forum, an online community that recently celebrated its 20th year supporting online mathematics education. These activities, which were designed to function at the boundary of the EnCoMPASS teachers and the Math Forum, are conjectured to facilitate the emergence of boundary practices and consequently the emergence of a professional community for teachers that is both "their own" and, at the same time, in alignment with the research-based instructional practices that guide the Math Forum's work.

Methods

To better understand teachers' participation and the emergence of community, we documented teachers' engagement in these boundary encounters quantitatively and conducted more in depth qualitative analysis of teachers' use of discourse in project activities. The question we explore in this brief research report is: What is the relationship between participants' participation in the community (investment over time) and their beliefs about the community and their relationship to and with it?

Project data included 28 participants' interactions during two one-week, face-to-face workshops (summers of the first and second years of the project) as well as 7 shorter face-to-face workshops and 20 online classes/workshops from the first three years of the project. Quantitative analysis was conducted using basic social network analytic methods, including counts of participation and engagement in project activities. Qualitative analysis was conducted using a modified grounded theory approach (Corbin & Strauss, 1998), where segments of discourse identified as significant were subjected to theoretical interpretation, resulting in a set of codes that represented similar moments. We report on the connections between participation and one set of codes from this analysis.

Results

This study uncovered a relationship between teachers' participation in project activities and their beliefs about their relationship with the community. First, teachers' overall participation in project activities could be sorted roughly into three categories – those with high, medium and low levels of participation (Table 1). The validity of these categories was further elaborated using more detailed social network methods and reported elsewhere (Matranga & Koku, 2015).

Table 1: Levels of Investment

Level	Participants	Number of activities involved
High	14	6-13
Medium	12	3-5
Low	2	0-2

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

Qualitative analysis of participants' utterances and interactions revealed a distinction between teachers in the three levels of participation and, in particular, differences in the ways that they talked about their relationship and future engagement with the project.

Personal Connection and Community-based Language

Teachers characterized as having a personal connection with the community used community-based language to describe their future engagement. As part of these descriptions teachers showed affect towards the group, specifically referenced their involvement in a "community," or used pronouns such as "we," "us" and "our" to discuss their involvement. The following excerpts are examples of such community-based language:

- "EnCoMPASS has been near and dear to my heart … So, I'm so grateful and thankful for being part of EnCoMPASS and meeting this wonderful group of people. I'm most thankful for the interactions I've had with you, and hopefully we will continue to work together" *(FA, W1)*.
- "And continuing the communication with others, because, you know, I'm learning so much, and I really hope that, as I've said, I'm able to take this back home and really continue working with it and continue being part of this community" *(LA, W2)*.

Egocentric, Outsider Language

Teachers characterized as egocentric discussed their future work by using outsider language. Teachers characterized in this way focused on what "I" want to do, or the ways in which the project can be used as a resource for their own future work. Examples include:

- "Novel approaches are interesting to me, when I look at student work I want to see something that I didn't think of yet, that is what is interesting to me" *(NK, W1)*
- "I am very intrigued on how to incorporate [problem solving] into my instruction. I've never done that, unlike several of you … I have very little exposure to it, so I am interested in pursuing it further" *(LH, W2)*.

Mixed Language

The third way in which teachers talked about their future work related to the project mixed both outsider and community-based language. Teachers characterized in this way tended to discuss what "I" wanted to do but also appeared to be in transition to using community-based language as they referenced specific individuals that they would like to work with in the future.

- "One thing I definitely want to do is be more consistent with my use of [problem solving] in my classroom. I will be lucky enough to have 60 extra classes this year. Instead of 180 we have 240 classes, so I know I can carve a lot of time out, at least 30 extra days of problem-solving in there, so that's really good… I hope to also use [the software] a little bit- Jerry L and Maggie S and I talked about this- to give a task to students, have them submit responses, and we can comment on each other's feedback and support each other … and I'm sure we'll do [Google] Hangouts together as well" *(JA, W2)*

Discussion

Research indicates the importance of community for supporting teachers' instructional change. The goal of the EnCoMPASS project is to facilitate boundary encounters that support the emergence of a self-sustaining and generative online community of mathematics educators. The research reported in this paper aimed to better understand the nature of a group of mathematics teachers' experiences in such encounters. This study suggests that there is a relationship between participants'

persistent involvement in PD activities and the extent to which they used community-based language and illustrated a personal connection to the community.

This finding has two implications. First, this emerging typology can frame additional analysis of participants within each level of investment to better understand why they continued to engage in the project and whether particular activities led to their use of these different types of discourse. One conjecture we have is that specific "community-building" activities or unscheduled work time allotted during face-to-face workshops could be factors that influence the development of relationships and a resulting personal connection to the community.

Second, this typology develops a continuum that relates characteristics of teachers' discourse to their likelihood to continue to engage in community-based PD. Professional developers could leverage this continuum to support the emergence of teacher communities. For example, they could support teachers who are identified as egocentric to foster their valuing of the importance of a support network for implementing high quality instructional practices. In addition, as different discourses emerge, they could purposefully group participants to be more involved with those that use community-based language and appear to value communal relationships.

In our ongoing work, we are collecting and analyzing additional data that can further verify and refine the results reported in this paper. This process includes a chronological analysis of project activities to understand the changing and evolving nature of teachers' beliefs about their relationship with the community and how this is related to engagement with particular project activities/participants. Moreover, we are investigating the content of teachers' interactions to relate teachers' experienced norms and practices to their investment and communal relationships.

In addition, the results reported in this paper, coupled with social network analysis reported elsewhere (Matranga & Koku, 2015) will allow us to make predictions regarding leaders and other individuals that can influence the persistence of the community.

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CROSSING THE UNIVERSITY BORDER: SUPPORTING ELEMENTARY MATHEMATICS SPECIALISTS SHIFTING PEDAGOGY DURING AN AUTHENTIC RESIDENCY COURSE

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This innovative project supports the development of Elementary Mathematics Specialists (EMSs) during their final authentic residency course required for a K-5 Mathematics Endorsement. This coursework provides the opportunity for participants to evidence their learning from their mathematics content courses through classroom teaching practice, with the ultimate goal of pedagogical shifts toward more standards-based instruction and dialogic discourse. This project provides and examines additional coaching sessions focused on classroom discourse, peer collaboration, and mentor feedback. Data collection includes scored observation rubrics of classroom practice, Portfolio Key Assessments, belief surveys, content knowledge assessments, supervisor and participant reflections of coaching sessions, and individual interviews.

Keywords: Teacher Knowledge, Teacher Beliefs, Classroom Discourse, Elementary School Education

Purpose

As the mathematics standards in the U.S. for grades K-12 increase in rigor, so does the need for highly qualified teachers. Teachers require a deep and broad mathematical knowledge to draw from during instruction (Hill, 2010), especially to create *standards-based learning environments* (SBLEs) that encourage dialogic discourse and foster conceptual understandings of mathematics (Tarr et al., 2008). Developing learning environments that prompt students to make conjectures about mathematical ideas and explain their thinking requires thoughtful planning and questioning on the part of the teacher. Teachers must value multiple student perspectives and carefully build whole class discussions using students' statements in an effort to help the class reach shared mathematical understandings. These skills are not common in today's elementary classrooms, even with the widespread adoption in the U.S. of the *Common Core State Standards for Mathematics* (CCSS-M, NGACBP & CCSSO, 2010) that include mathematical practices depicting such a pedagogical emphasis.

With the new mathematical content and practice standards set forth by the CCSS-M, along with the accompanying high-stakes standardized assessments, teachers feel added pressure to change their instructional practices to align with the increased expectations and emphases for student learning. This has contributed to the focused discussion on the important role of elementary mathematics specialists (EMSs), who are mostly considered to be teachers, teacher leaders, and/or coaches with the established expertise to support effective elementary mathematics instruction along with student learning in the classroom, school, or other levels (Association of Mathematics Teacher Educators, 2013). Our university offers an EMS preparation program (i.e., a K-5 Mathematics Endorsement [K-5 ME] program) that contains 15 semester hours of courses, including 12 mathematics content course hours and a 3-hour authentic residency course. Our K-5 ME program aims to develop EMSs that can effectively orchestrate classroom discourse, create and employ a SBLE, meet and exceed the expectations of the National Council of Teachers of Mathematics' *Principles to Actions* (2014), all while supporting fellow teachers. The purpose of this project is to provide increased support during the authentic residency course experience with the aim of supporting EMSs' pedagogical shifts in classrooms and to study the outcomes of these additional supports.

Related Perspectives

Elementary teachers' beliefs about mathematics teaching and learning are often shaped during education programs, but enacting these shifting beliefs can prove difficult for teachers as they rely on past experiences and comfortable practices (Charalambous, Panaoura, & Philippou, 2009; Connor, Edenfield, Gleason, & Ersoz, 2011). "Teacher preparation coursework and professional development offerings must address both mathematical content and pedagogy in ways that advance teachers' subject matter understanding and their understanding of students' emerging conceptions of mathematics while also fostering effective instructional skills and practices" (Campbell et al., 2014, p. 453). When considering EMS preparation programs, practicum experiences serve as the opportunity to implement new pedagogy and demonstrate specialized content knowledge. However, when too little attention is paid to this implementation during and after program completion, teachers often fall back on traditional instructional practices (Fennema et al., 1996; Philipp, 2007). Polly and Hannafin (2011) found that teachers' espoused beliefs did not always match their enacted classroom practices, but that scaffolding and ongoing support can aid this connection.

An additional means of supporting the connection between changing beliefs and enacted practices is through the use of mentorship. For example, Halai (1998) worked as a mentor with teachers as they learned new ways of teaching mathematics and found that a trusting relationship with a mentor supported growth and pedagogical shifts. These findings lend credence to the importance of the practicum phase of an EMS preparation program, with attention focusing on teachers' implementation and pedagogical shifts.

K-5 ME Program Experiences

Our K-5 ME program strives to shape and create EMSs as mathematics teacher leaders. This innovative program produces specialists in the field of elementary mathematics, filling leadership roles in their schools and modeling the impact of advanced specialist certification. The K-5 ME program aims to develop EMSs who can effectively orchestrate classroom discourse, establish and utilize a SBLE, implement the expectations of the CCSS-M, and support their peers in doing so. The participants in this project have completed the first half of the K-5 ME mathematics content/pedagogy course requirements and are working to complete the final half while enrolled in a practicum/authentic residency course. This authentic residency provides them the opportunity to implement what they have learned and document their enacted classroom practices, serving as their final evaluative piece toward obtaining the K-5 ME.

The learning goals for the authentic residency course include that prospective EMSs should: (1) demonstrate the knowledge, skills, and dispositions gained during program coursework via effective classroom practices; (2) apply information gained from feedback to their classroom practices; and (3) apply reflective thinking to their classroom instructional practices. Course assignments include planning, implementing, and reflecting upon at least ten mathematics lessons modeling the content knowledge and pedagogy learned throughout the program's four mathematics courses. Two of these lessons are observed by a university supervisor, using a rubric specifically designed for observing elementary mathematics instruction and the extent to which it is aligned with a SBLE and providing evaluative feedback on the observed teaching practices. Using the Portfolio Key Assessment, an assignment adapted from the state proposed K-5 Mathematics Endorsement Program Portfolio Guidelines, participants submit a portfolio evidencing their content knowledge and pedagogical skills acquired in the four mathematics courses and authentic residency course.

In previous semesters, the authentic residency course provided minimal feedback to the participants due to the evaluative nature of the course assignments, limited time spent with the university supervisor and colleagues, and lack of collaborative mentorship opportunities. Previous research on the program has shown participants need more critical and constructive feedback, more guidance as they implement new teaching practices, and more coaching from their university

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supervisor as well as peers, teacher leaders, and mentors in their schools. This project is an effort to provide that support to endorsement candidates during their authentic residency experience. Ultimately, this project shifts the role of the supervisor, aiming to provide support as a coach and mentor to the participants rather than an evaluator. This allows for more direct and practical feedback, collaboration between participants and their mentor, and a model for future effective teacher leaders.

As part of this innovation, participants engage in monthly coaching sessions aimed at providing support and guidance for K-5 ME candidates as they complete their authentic residency. These coaching sessions maintain a heavy focus on residency course assignments, implementation of new teaching practices, reflection on these implementation experiences, and collaboration with fellow teachers within a safe and supportive environment. Coaching session topics focus on facilitating classroom discourse as a lens to provoke conversation and questions. Further, participants engage in a book club with their mentor. The book, *5 Practices for Orchestrating Productive Mathematics Discussions* (Smith & Stein, 2011), highlights the focus of classroom discourse and provides an extension of participants' content knowledge and pedagogical skills learned through the program's mathematics courses. Sections include: the 5 practices for facilitating effective inquiry-oriented classrooms, anticipating what strategies students will use in solving a problem, monitoring their work as they approach the problem, selecting students whose strategies will enhance discussion and meet the lesson objectives, sequencing those students' presentations to maximize their potential to increase students' learning, and connecting the strategies and ideas in a way that helps students deepen their understanding of the mathematics. Drawing from participants' learned mathematics and venturing into the art of orchestrating discourse can bridge the gap between understanding the content and implementing effective instruction. As many K-5 ME participants have expressed in the past, asking questions and orchestrating discourse are often the most difficult skills to master. By reading together and crafting those skills, participants can build confidence as a teacher leader. Furthermore, in order to extend this practice of collaboration and coaching, participants can video-record their teaching practice. This offers an added opportunity for participants to critically reflect on their own practice while the mentor is able to speak directly to the lesson and provide specific feedback. Celebrating successes together and providing constructive feedback to those that need help are essential practices of teacher leaders.

Data Collection and Preliminary Analysis

Data continues to be collected via observation rubrics of classroom practice, Portfolio Key Assessments, pre- and post-surveys for pedagogical and teaching efficacy beliefs, content knowledge assessments, supervisor notes and participant reflections of coaching sessions and feedback provided, and participants' evaluations of the innovation through individual interviews following their authentic residency course.

Data collection and analysis is ongoing, and preliminary evaluation of this new authentic residency course support already reveals an increase in participants' confidence in putting their learning into action, shifting pedagogical practices, practicing discourse in the mathematics classroom, refining questioning strategies, and completing their required course assignments. Participants' classroom observations show more thorough and thoughtful planning, and portfolios are much more detailed and robust. The relationships that the university mentor has been able to create with participants allow for more collaboration and engagement in new teaching practices. These coaching sessions are opening up the experiences of the authentic residency course into more of an opportunity than a requirement, feeling more supportive than evaluative. It is expected that continued data collection followed by a thorough data analysis will show evidence of participants' strengthened abilities to implement a SBLE, carefully build classroom mathematics discussions, and find value in multiple solution methods and perspectives, allowing these prospective EMSs to become more

effective at orchestrating discourse and implementing the CCSS-M. Further, preliminary data analysis shows the beginnings of pedagogical shifts for these prospective EMSs. Therefore, it is expected that a final analysis of the complete data collection will exhibit participants' pedagogical shifts in their teaching practice, grounded in their changing beliefs about how children learn mathematics, how they teach mathematics, and how to build a classroom environment conducive for both. As this project is designed to better support those shifts in pedagogy, we anticipate it will create a longer-lasting, more effective change for these prospective EMSs.

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THE LANGUAGE OF PROFESSIONAL DEVELOPMENT LEADERS

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Leaders in professional development (PD) initiatives (such as facilitators, principals, coaches) hold a great deal of power in their language, carrying the ability to manage meaning and frame experiences. Rather than working from interview data, this report addresses a gap in leadership research by examining the words used by leaders in their on-the-job interactions. We present an initial framework for capturing leadership language qualities at the macro level (framing the PD's purpose) and micro level (rhetorical crafting in terms of metaphor usage, pronoun choice, and other language selections). Our data come from a larger project evaluating the efficacy of a large-scale sustained PD. Through developing a lens for analyzing leadership, we hope to build a tool to eventually connect leadership with other related PD measures including teacher buy-in, fidelity of implementation, and ultimately outcomes in schools.

Leaders in professional development (PD) initiatives (such as facilitators, principals, and coaches) hold a great deal of power in their language. Their words carry the ability to inspire participants to buy into initiatives, establish themselves as credible leaders, and build positive learning communities. Communication is more than the transmission of information, as language acts to bring meaning to ideas and frame experiences.

In this report, we share a preliminary analysis of leader interactions with careful attention to rhetoric and framing choices of leaders within schools and the PD program. Our data comes from a quasi-experimental study evaluating the efficacy of a mathematics PD program in a midsized, urban school district. We use detailed field notes and video-taped PD sessions to compare language across various leaders. Despite a well coordinated PD, we found consistent differences in framing and rhetoric across leaders at various sites.

On Framing and Language

Fairhurst and Sarr (1996) describe "reality [as] a social construct, and language is its primary vehicle" (p. 19). They go on to explain that leaders' discourse can serve to build frames to explain purposes of innovation, gain interest, to inspire, and to promote a sense of community. Individuals' experiences are shaped by the discursive choices of those around them. We use the lens of *framing* and *rhetorical crafting* to analyze the language of leaders. We use these constructs in a way consistent with Conger (1991) where framing is the defining of major concepts and purpose, and rhetorical crafting is at a finer-grained level. Conger defines framing as "the process of defining the purpose of an organization in a meaningful way" (Conger, 1991, p. 32). We generalized this construct to capture framing of major ideas including, but not limited to, the purpose of our PD. Conger discusses rhetorical crafting as using symbolic language, focusing on emotional power in his writing, to package a message. He goes on to use the analogy of a gift's wrapping paper being "as impactful as the gift itself" (p. 32). For our analysis, we adapt this notion to analyze language choices across leadership interactions.

Leaders in PD

Leadership is essential for positive change in schools (Leithwood, Harris, & Hopkins, 2008). The role of several types of leaders has been explored within PD including principals (e.g. Youngs & King, 2002), PD facilitators (e.g. Linder, 2011), and teacher leaders (e.g. Darling-Hammond, Bullmaster, & Cobb, 1995). Leadership literature varies from describing types of effective leaders (often relying heavily on interview data), to leadership roles, and leadership actions. We aim to build on leadership work by addressing *leadership interactions* directly. That is, our primary source of data are videos and notes from PD sessions where various leaders interact with participants.

Methods

Context of the Study

We are currently conducting a large quasi-experimental study evaluating the efficacy of a studio model PD in a midsized urban school district. We have all grades 3-5 teachers at 25 elementary schools participating in either (a) 3-day summer sessions only or (b) 3-day summer sessions and five 2-day cycles of PD (studio model) throughout the year. The PD focuses on creating mathematically productive classrooms through Best Practices in teaching that promote students developing mathematical habits such as justifying and generalizing (Foreman, 2013). For the schools participating in the studio model, the two-day cycle is split into a leadership coaching day and studio day. During the studio day, one teacher at each school (the studio teacher) opens his or her classroom for a commonly planned and subsequently observed lesson. All teachers at the school work together to plan, refine, and debrief the lessons. This day is preceded by a day of leadership coaching with the principal and math coach, as well as planning with the studio teacher. The PD facilitators work with the principal and coach at each school to (a) help the principal understand the goals of the PD, (b) plan the principal introduction for the next day during which the principal frames the PD, (c) observe in the grades 3-5 math classrooms and connect these observations to teacher implementation of the PD, and (d) plan on how to increase buy-in and sustain the PD between cycles.

Data Collection and Analyzing Leadership

We collected data on two case study schools, School 1 (year 1, 2, and 3 data; 603 students in 2012-2013 with 83.5% receiving free/reduced lunch, 53.3% of 5th graders meeting standards in math) and School 2 (year 1 data only; 358 students in 2012-2013 with 38.5% receiving free/reduced lunch, 75% of 5th graders meeting standards in math). For each PD session, both days were video-recorded and at least one member of the research team took detailed field notes. The field notes were first processed by identifying instances of *leadership interactions.* We used leadership interactions to capture any interaction between participants where (a) one of the participants was in a leadership role; and (b) the communication was substantive. We then analyzed the leadership interactions across three midyear sessions each of which had a different PD facilitator and principal. Initially, we open-coded the leadership interactions to look for trends across discourse. After this initial exploration, we developed categories of rhetorical crafting and identified instances of framing related to the PD. We then returned to the video to assure our categories accurately reflected the conversations.

Preliminary Results

Through our initial analysis, we found that leaders varied in how they framed important aspects of our work and in their discourse choices in a variety of ways such as pronoun choice and usage of metaphors.

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Framing

We analyzed leadership interactions based on the framing of the purpose and nature of the PD, the roles and expectations around teachers, and the nature of mathematical classrooms. Consider the following contrasting principal framing of the PD work from their opening statements to teachers.

Principal S: Some of those, what that looks like is short answers to a question, it could be restating facts or statements, showing procedures, and we got to be getting out of that and instead challenging our students more, bumping up the rigor. A big piece of our work this year is aligning our actions and being really purposeful about this is what we want to see: we want students to be making sense, we want students to be justifying, we want students to be generalizing, making connections to the work, making representations of the work.

Principal A: The group is also flexible, so that when it comes to our homework assignments and that type of thing, I think we can make them more genuinely confirming for the work we're going to be building. Yesterday when we were walking around, I saw a couple, well more than a couple of great things and I want to encourage you guys to keep trying to do these things. They're new and learning to do anything new is the hardest part. I think we're over a big hump in terms of effort and the work in terms of conferences. The hard part's done, we just have to focus in on the gift of the work.

Principal S frames the PD work in terms that are a.) consistent with the focus of the PD such as having students justify and generalize, and b.) as purposeful for benefiting students. In contrast, Principal A frames the PD work in terms that are a.) not specific to any of the PD's focus, and b.) pleasing an external source, "the group", through completion of "homework". The choice of the word "homework" alludes to the PD work being prescribed and perhaps undesirably necessary.

Rhetoric

We also found a number of differences in leaders' rhetorical crafting. We present two example differences: pronoun choice and imagery. Table 1 includes additional categories of rhetoric themes.

Table 1: Sample Themes in Leadership Rhetoric

Sample Rhetorical Crafting Category	Description
Inclusiveness of language	Pronoun choice such as *I* and *you/they* vs. *we*
Orientation towards school/participants	Strengths-based or deficit-based language
Level of specificity	Specific examples or broad statements
Level of personalization	Personalized or generic messages
Use of imagery	Figures of speech (i.e. metaphor) in language

Within our first theme, inclusiveness of language, we present excerpts from two leaders with contrasting crafting. The first tended to favor "I" and "you" statements such as, "The survey is a gift you give yourself. I know how busy you are and how fast you are running." In contrast, the second leader used "we" statements such as, "We're going to work hard to see what we can do so students are engaging with these." This may situate leaders as either part of the group of teachers or external to them.

Leaders also use imagery to manage meaning. In one episode the leader begins the day by saying, "We're going to put on roller skates this morning." This conjures up an image of the leader and

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attention, compared the struggle to "herding cats". This image brings a sense of chaos to the situation where the leader is separate from the participants, trying to manage them.

Conclusion and Discussion

Our examination of leaders' use of framing and rhetorical crafting revealed patterns and themes in their language choice, which could reveal how they establish themselves as leaders, create buy-in amongst their teachers, and develop a positive learning community. This is true both in global framing of ideas and in subtle language choices. For instance, Fiol, Harris, and House (1999) found that charismatic leaders more frequently used inclusive referents such as "we" rather than "I" and "you". Similarly, the use of metaphor has been associated with leadership rhetorical selections where images can either help bring positive meaning to ideas or potentially confuse or skew a message (Fairhurst & Sarr, 1996).

Through analyzing the language of PD leaders, we are beginning to unravel some potential causes for differences in buy-in and enactment of this initiative (Thanheiser, Melhuish, Shaughnessy, & Foreman, 2015). A leader's language choices can serve as a motivating factor, but could also serve to exclude or alienate participants. Our initial framework provides a tool for analyzing rhetoric and future analysis will test the generalizability of the work. Furthermore, we look to connect leadership language with other constructs such as fidelity of implementation and outcome changes such as teaching quality and student achievement.

Acknowledgments

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EXPERIENCIAS NARRATIVAS DE PROFESORES DE EDUCACIÓN BÁSICA CON LA ANSIEDAD MATEMÁTICA

IN-SERVICE TEACHERS NARRATIVE EXPERIENCES OF MATHEMATICS ANXIETY

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El propósito de este estudio fue aportar a la comprensión acerca de la ansiedad al enseñar matemática de profesoras de Educación Básica a través de conversaciones guiadas. Estas conversaciones giraron en torno a las experiencias de los participantes con la matemática como estudiantes y como profesores. La narrativa generada en esta experiencia, provee un discurso poderoso que da cuenta del grado en que la ansiedad matemática que sienten los profesores moldea sus actitudes y sus posibilidades de éxito al impartir la asignatura. El foco de análisis para este estudio fue la comunidad de profesores de Educación Básica, específicamente profesoras que han sufrido ansiedad en el proceso de aprender y enseñar matemática.

Palabras clave: Capacitación Docente / Desarrollo Profesional, Creencias de los Maestros

Antecedentes y Objetivos

La ansiedad matemática en profesoras de Educación Básica es un tópico que ha capturado el interés de investigadores en educación matemática. Las experiencias de ansiedad vividas por profesoras de Educación Básica en sus propios procesos de aprendizaje, puede impactar en cómo ellas se aproximan a la enseñanza de la matemática y en cómo imparten la asignatura en sus salas de clases (Ball, 1988; Bursal & Paznokas, 2006; Stoehr, 2015). Estudios previos relacionados con la ansiedad matemática padecida por profesoras de Educación Básica investigaron si profesoras que la experimentaban podían ser exitosas en la enseñanza de esta disciplina (Beilock, Gunderson, Ramirez, & Levine, 2010; Bursal & Paznokas, 2006). También ha habido preocupación respecto a si profesoras con ansiedad matemática pueden traspasarla a sus estudiantes (Beilock et al., 2010; Sloan, 2010). Actualmente hay acuerdo respecto a que una profesora de matemática segura y competente es vital y necesaria en la clase de matemática (Beilock et al., 2010; Oswald, 2008).

Recientemente académicos han estudiado la ansiedad y seguridad relacionada con la matemática desde un enfoque narrativo, ellos sugieren que la conversación colaborativa entre mujeres es una herramienta muy significativa por la cual las mujeres que participan en ellas pueden expresar ideas y sentimientos, propios o de otros, desde su punto de vista (Stoehr, 2013). Reunir a mujeres en torno a conversaciones guiadas de temas relevantes de estudio, permite que en dicha experiencia afloren tópicos y conclusiones que no se podrían lograr de otra manera (Belenky, Clinchy, Goldenberger, & Tarule, 1986). Involucrar a profesores en conversaciones basadas en la metodología narrativa *Conversations that Matter* (Stoehr, 2013) es una manera de guiar a profesores en un trabajo complejo, intelectual y moral, de pensarse como un profesor (Griffin, 1988).

El objetivo de este estudio es reportar los resultados de un estudio narrativo enfocado en entender la ansiedad matemática vivenciada por profesoras chilenas en servicio a través de un instrumento narrativo basado en *Conversations that Matters* (Stoehr, 2013). Estas conversaciones guiadas giran en torno a las experiencias relacionadas con la matemática de profesores en servicio de Educación Básica, tanto como estudiantes como profesores.

Perspectiva/Marco de trabajo

Perspectiva 1: Mujeres y Ansiedad Matemática

La ansiedad matemática es más que solo no tener afinidad o gusto por la matemática (Vinson, 2001). "Ansiedad matemática se refiere a un estado de respuestas poco saludables que ocurre en algunos estudiantes cuando se enfrentan a problemas matemáticos, manifestándose a través de sentir pánico, tener dificultades para concentrarse, estados depresivos y desesperanzados, nervios e inseguridad" (Luo, Wang, & Luo, 2009, pp. 12-13). También se pueden producir reacciones fisiológicas como manos sudorosas, puños apretados, sensación de enfermedad, sequedad labial, palidez facial, produciendo que los estudiantes pierdan no solo su interés por la matemática sino también su seguridad por aprenderla (Luo et al., 2009).

Profesores que están ansiosos con la matemática frecuentemente traspasan su propia ansiedad a sus estudiantes, lo que puede generar una perpetuación del problema (Beilock et al., 2010; Sloan, 2010; Vinson, 2001). En efecto, Beilock et al. (2010) reporta que profesoras de Educación Básica que padecen ansiedad matemática, repercuten negativamente en el logro matemático de sus estudiantes mujeres. El estudio revela que mientras mayor sea la ansiedad con la matemática de la profesora, mayor es la probabilidad de que las estudiantes crean que los hombres son mejores para la matemática que las mujeres. Dejar de lado este "carga de matemático" es crítico para los profesores (Brown, Mcnamara, Hanley, & Jones, 1999).

Perspectiva 2: Investigación narrativa en profesores como un medio para crear *Conversations that Matters* en torno a la matemática.

Académicos han utilizado de manera exitosa la narrativa en el campo de la educación, esta provee un foco claro de cómo los profesores le dan sentido al proceso de enseñar, incluyendo la relación con su propia experiencia escolar. (Clandinin & Connelly, 2000; Carter, 1993; Doyle & Carter, 2003). La indagación narrativa genera un espacio para que los profesores hablen y escriban sobre sus historias de vida, conectándolas a sus experiencias de enseñanza y dándole sentido (Carter, 1993; Clandinin & Connelly, 2000). Con el fin de lograr una comprensión más profunda del rol y las consecuencias que tiene la ansiedad matemática en los profesores de Educación Básica, Stoehr (2013) desarrolló un programa de investigación narrativa para estudiar los procesos cognitivos y los significados personales que profesores le dan a la matemática a través de "conversar" respecto a su experiencias con la matemática, como estudiantes y como profesores. El uso de narrativas, oral y escrito, es un método de investigación poderoso que puede ser utilizado para desarrollar una nueva concepción de la ansiedad matemática en profesoras de Educación Básica. Para este trabajo se utilizó el instrumento narrativo de conversación colaborativa *Conversations that Matter* diseñado por Stoehr (2013).

Fuente de los Datos

Este estudio es parte de una línea de trabajo de una prestigiosa universidad en Chile, como parte de sus esfuerzos por desarrollar y fortalecer las capacidades disciplinares y pedagógicas para enseñar matemática, en establecimientos educacionales públicos del país. Este estudio se enfoca en aproximadamente 54 profesores de escuelas básicas, la gran mayoría mujeres, que estaban participando en un taller de desarrollo profesional enfocado en Ansiedad Matemática. Los participantes eran todos chilenos de entre 25 a 60 años de edad, promedio de edad 40 años y el promedio de años de servicio era de 10 años.

Método y Contexto

Los participantes de este estudio fueron ordenados en grupos de 10 profesores con un investigador que facilitaba el trabajo en cada grupo. Luego se le pedía a cada participante que de

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manera individual respondiera 4 frases del instrumento *Conversations that Matter* relacionadas con el proceso de aprender y enseñar matemática. Estas frases eran: En matemática soy bueno para…; Cuando era estudiante, una de mis experiencias más desafiantes con la matemática fue…; Para mi enseñar matemática es…; Pienso que la ansiedad o poca seguridad en matemática es causada por…

Luego de completar cada frase, los participantes compartían sus respuestas unos con otros. Ellos comentaban similitudes que veían entre las respuestas de unos con otros, como también comentaban cómo la respuesta de uno los llevaba a pensar algo que ellos consideraban que era interesante compartir. A su vez, conversaron de los diferentes aprendizajes y conclusiones que llegaban al comentar cada frase. La narrativa recogida en las conversaciones generadas por el instrumento *Conversations that Matter* fueron grabadas y duraron una hora aproximadamente.

Durante el análisis, la narrativa de las *Conversations that Matter* fueron revisadas y analizadas de manera cuidadosa en orden de demarcar e identificar elementos analíticos de la narrativa relacionados con la ansiedad matemática. Utilizando técnicas temáticas e iterativas propias del análisis cualitativo, incluyendo métodos de comparación constante de los análisis (Bogdan and Biklen, 2006), la atención fue puesta entonces en los detalles temáticos documentados en el proceso de recolección de datos.

Resultados

Los hallazgos centrales del análisis de las narrativas y sus implicancias se muestran a continuación.

Resultado 1: Yo soy bueno para algo en matemática

Todos los participantes reportan ser exitosos de alguna manera en matemática. Algunos profesores hablaron de tener competencias en la capacidad de cálculo, otros compartieron ser buenos para la resolución de problemas, para razonar, resolver problemas de lógica, y para estimar. Otros participantes reportaron ser competentes en temáticas específicas como geometría y álgebra. Un participante dijo: "Soy bueno para motivar a mis estudiantes para trabajar en problemas matemáticos y yo soy bueno para hacer cálculo mental".

Resultado 2: Experiencias desafiantes en el aprendizaje de la matemática

Los profesores comparten una diversidad de experiencias al momento de recordar el proceso de aprender matemática en su época de estudiantes escolar y universitaria. Una temática que se repite de manera constante es la mala base o la falta de conocimientos sobre la matemática elemental, la cual era causal de dificultades en niveles superiores. Otros participantes declararon que se sentían incapaces de entender contenidos específicos de la matemática como el álgebra, geometría, cálculo, mientras otros reportaron que tenían dificultades porque no encontraban la matemática una asignatura relevante para sus vidas. Algunos profesores recalcan haber tenido experiencias de presión y ansiedad para completar tareas matemáticas al frente de sus compañeros. Por ejemplo un participante reportó: "ir a la pizarra a resolver un problema me ponía muy nerviosa".

Resultado 3: Perspectiva enseñando matemática

La mayoría de los participantes en el estudio reportaron que el enseñar matemática en Educación Básica era una experiencia desafiante. Un participante compartió que para ella era desafiante porque cada vez que debía enseñar un contenido, tenía que primero reaprenderlo y luego buscar maneras para enseñarlo a sus estudiantes. Algunos declararon que tenían poco manejo de la matemática, otros sentir angustia al momento de enseñarla. Sin embargo, otros participantes compartieron que se sentían desafiados al enseñar matemática ya que tenían la oportunidad de desarrollar habilidades matemáticas en sus estudiantes. Otros participantes reportaron que enseñar matemática para ellos era

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importante porque era una disciplina necesaria para la vida diaria, por lo que lograr que sus estudiantes aprendieran y se sintieran seguros con sus conocimientos matemáticos era un desafío.

Resultado 4: ¿Dónde nace la poca seguridad y la ansiedad en matemática?

Las respuestas de los profesores respecto al origen de la ansiedad o poca seguridad en matemática se relaciona principalmente con las propias experiencias de los docentes al aprender matemática. Sus respuestas incluyen los siguientes ejes temáticos: (1) Sentimientos de vergüenza al momento de aprender matemática; (2) miedo a fracasar al momento de "hacer matemática"; (3) falta de conocimiento matemático; y (4) sentirse siempre "tonto" para la matemática. Más específicamente, algunos participantes hablaron del miedo y vergüenza al momento de equivocarse o cometer errores. Otros participantes declararon que la reacción negativa de la profesora y la falta de apoyo a los estudiantes con dificultades podía generar ansiedad matemática. Algunos participantes creían que los métodos tradicionales de enseñanza para enseñar matemática podían generar en algunos estudiantes sentimientos de ansiedad.

Significancia académica del estudio

Nuestro estudio sugiere que hay mucho por aprender sobre la ansiedad al enseñar matemática que sienten las profesoras de Educación Básica. La metodología *Conversations that Matter* abre una ventana para que investigadores en matemática puedan observar cómo se genera la ansiedad matemática, sus consecuencias y puedan trabajar en estrategias para redirigir la ansiedad que algunas profesoras de Educación Básica viven. Este estudio revela que la ansiedad matemática puede ser un tema que puede afectar a profesoras de Educación Básica y que puede perpetuarse durante décadas. De hecho, puede ser que algunos individuos nunca dejen de sentir ansiedad al realizar actividades matemáticas. Futuras investigaciones deberían buscar estrategias para que los formadores de profesores en matemática pudieran dirigir y trabajar con las historias y experiencias de ansiedad matemática de los profesores, teniendo en cuenta que probablemente dichos esfuerzos al menos en su etapa inicial solo van a hacer consiente a la población de la existencia de esta problemática por sobre prevenir la aparición de la ansiedad matemática. Además, futuras investigaciones pueden proporcionar a los formadores de profesores en matemática una visión más profunda de lo penetrante que es la ansiedad matemática para algunas profesoras de Educación Básica.

The purpose of this study was to gain a better understanding of the issues that surround Chilean women in-service elementary teachers' mathematics anxieties through Conversations that Matter. These conversations revolved around the participants' mathematics experiences as students and practicing teachers. These narrative writings provide a powerful voice for the degree to which mathematics anxiety shapes teachers' attitudes in this subject area as well as their ability to be successful in mathematics. The focused analysis for this paper was directed to impact the elementary mathematics education community with a specific focus on women who have had anxiety in learning and teaching mathematics.

Keywords: Teacher Education-Inservice/Professional Development, Teacher Beliefs

Background and Purpose

Mathematical anxiety in women elementary teachers is a subject that has captured the interest of mathematics educators. Women's experiences of mathematics anxiety in their own student learning days can impact how they approach mathematics instruction in their own classroom (Ball, 1988; Bursal & Paznokas, 2006; Stoehr, 2015). Previous research in mathematics anxiety in women elementary teachers question if teachers who experience mathematics anxiety can be successful in teaching mathematics (Beilock, Gunderson, Ramirez, & Levine, 2010; Bursal & Paznokas, 2006). Moreover, there is concern that teachers who have mathematics anxiety may pass their anxiety onto their students (Beilock et al., 2010; Sloan, 2010). There is agreement that a confident and competent mathematics teacher is a vital necessity in the classroom (Beilock et al., 2010; Oswald, 2008).

Recently scholars have examined anxiety and confidence in mathematics from a narrative perspective. Research on narrative work suggests that collaborative conversations among women can be a powerful means by which women can reflect on issues that are not only important to them but told from their point of view (Stoehr, 2013). Engaging women in conversations of specific topics of study affords opportunities for themes to emerge that can either move them forward or propel them backwards (Belenky, Clinchy, Goldberger, & Tarule, 1986). Engaging teachers in narrative based Conversations That Matter (Stoehr, 2013) offers a means by which to engage students in the intellectual, moral, and complex work of *thinking like a teacher* (Griffin, 1988).

The specific objective in this paper is to report findings from a narrative-based study aimed at understanding Chilean women in-service teachers' issues of mathematics anxieties through a researched based and instructionally focused Conversations That Matter narrative tool (Stoehr, 2013). These conversations revolved around in-service teachers' own mathematics experiences as students as well as their experiences of teaching mathematics.

Perspectives/Frameworks

Perspective 1: Women and Mathematics Anxiety

Mathematical anxiety is more than just not liking mathematics (Vinson, 2001). "Mathematics anxiety refers to such unhealthy mood responses which occur when some students come upon mathematics problems and manifest themselves as being panicky and losing one's head, depressed and helpless, nervous and fearful, and so on" (Luo, Wang, & Luo, 2009, pp. 12-13). Physiological reactions such as sweaty palms, tight fists, feeling sick, having dry lips, and a pale face can also occur which can result in students losing not only their interest in mathematics but in their confidence to learn mathematics (Luo et al., 2009).

Teachers who are anxious about mathematics often pass their own anxieties to their students, which can result in a perpetuation of the problem (Beilock et al., 2010; Sloan, 2010; Vinson, 2001). Indeed, Beilock et al. (2010) reported that mathematically anxious women elementary teachers often impact the mathematics achievements of the girls in their class. The study revealed that the more anxious the teacher was about mathematics, the more likely the girls in the class were to believe boys were better at mathematics than girls. Discarding this "mathematical baggage" is critical for teachers (Brown, Mcnamara, Hanley, & Jones, 1999).

Perspective 2: Narrative Research in Teacher Education as a Means to Create Mathematical Conversations That Matter

Scholars have successfully used narratives in the field of education as a research framework to provide a clear focus of how new teachers make sense of teaching, including how it relates to their own school experiences (Clandinin & Connelly, 2000; Carter, 1993; Doyle & Carter, 2003). Narrative inquiry creates a means for teachers to talk and write about their storied lives while making connections to teaching (Carter, 1993; Clandinin & Connelly, 2000). In order to move forward

toward a productive and meaningful understanding regarding the role mathematics anxiety plays in women elementary teachers, a narrative research agenda was developed by (Stoehr, 2013) to examine the cognitive understandings and personal-sense making strategies used by the participants to "converse" about their mathematical experiences, as students and teachers. The use of narratives, both oral and written, is a powerful research tool that can be used to develop new understandings of mathematical anxiety in women who are elementary teachers. For this work, (Stoehr, 2013) has termed this narrative based collaborative discussion as *Conversations That Matter.*

Data Sources

This study is part of a larger, multi-year and on-going effort at a large University in Chile aimed at strengthening and developing disciplinary and pedagogical capacities for teaching and learning mathematics in public schools. This paper focuses on approximately 54 urban public school teachers who were predominantly women and who were participating in a professional development workshop on mathematics anxiety. The participants were all Chilean between the ages of 25 - 60, with an average age of 40 years old. The average years of teaching were 10.

Methods and Context

Participants in this study were arranged in groups of ten teachers with one researcher facilitating each group. Each participant was then asked to respond individually to four Conversations that Matter prompts that were related to learning and teaching. These prompts were as follows: In math I am good at…; One of my most challenging experiences learning mathematics as a student was…; For me teaching mathematics is…I think anxiety and/or low confidence in mathematics is caused by….

After completing each prompt, the participants shared their responses with each other. They commented on the similarities they saw in each other's responses or how one person's responses prompted them to think of something else they found relevant to share. They also talked about the themes they shared with one another. The Conversations that Matter narrative data collection activity was video recorded and lasted under one hour.

During analysis, the Conversations that Matter narratives were reviewed and carefully analyzed in order to demarcate and identify analytical narrative elements related to mathematics anxiety. Using iterative and thematic qualitative analysis techniques, including constant comparison methods (Bogdan and Biklen, 2006), attention was then turned to a detailed documentation of thematic elements that were revealed during the data collection process.

Findings

This brief research paper reports on the narrative experiences of in-service teachers' mathematics anxiety as students and as teachers. Central findings from the analysis of the narratives and their implications are shared below.

Finding 1: I Am Good at Something in Mathematics

All participants reported being successful in some way in mathematics. Some in-service teachers spoke of having competent computation skills. Others shared they were good at problem solving, logic and reasoning, estimation. Some participants reported they were able to do well in specific content areas such as algebra and geometry. One participant stated, "I'm good for motivating my students to work on math problems, and I´m good for doing mental calculus."

Finding 2: Challenging Experiences Learning Mathematics

The teachers shared a variety of different types of challenging experience of learning mathematics as a student. One major theme included participants reporting a lack of a strong foundational mathematics background that made higher levels of mathematics difficult. Other

participants stated they felt unable to understand specific content areas such as algebra, geometry, and calculus, and while others reported that they struggled to find mathematics as being relevant in their lives. Some teachers recalled experiencing pressure and anxiety to complete mathematics tasks in front of their classmates. For example, one participant reported that "going to the board for solving problems made me very nervous."

Finding 3: Perspectives on Teaching Mathematics

The majority of the participants in the study reported that they found teaching elementary level mathematics to be challenging. One participant shared how each time she taught a content area she had to try and first relearn the content herself and then worried how she would teach it to her students. Some shared that they lacked expertise in mathematics content while others stated they felt distress when teaching this content area. However, other participants stated that despite feeling challenged in teaching mathematics, they were dedicated to creating mathematics understanding for their students. Some participants reported that teaching mathematics was an important content area for their students to learn and feel secure in, as mathematics is needed in everyday life.

Finding 4: Where Does Mathematics Anxiety and Low Confidence Come From?

The teachers' responses regarding the sources of mathematics anxiety and lack of confidence in this content area were derived mainly from their own experiences of learning mathematics. These responses included the following themes: (1) strong feelings of embarrassment while learning mathematics; (2) failure to be able to "do math"; (3) lack of mathematical understanding; and (4) always feeling mathematically dumb. More specifically, some participants spoke of the fear and shame that can accompany being wrong or making mistakes. Other participants stated that teachers' negative reactions or lack of support for students struggling in mathematics could create mathematics anxiety in students. Some participants believed that traditional methods of teaching mathematics (such as direct instruction) and the lack of opportunities to work with peers in the mathematics classrooms can lead to some students feeling anxious about mathematics.

Scholarly Significance of the Study

Our study suggests that much that there is much to be learned about women elementary teachers' mathematics anxiety. Conversations That Matter opens a window for mathematics researchers to peer inside of the issues that create mathematical anxiety for teachers as well as ways to address mathematics anxiety that some women elementary teachers experience. This study reveals that mathematics anxiety may be an issue or concern for women elementary teachers that may recur for decades. In fact, it may be that some individuals may never stop experiencing mathematics anxiety. Future research might address how mathematics educators can work with teachers' histories of experiences with mathematics anxiety, keeping in mind that their efforts might primarily serve to raise awareness and provide options rather than to prevent mathematics anxiety. In addition, future research may provide mathematics teacher educators with a more in-depth view of how pervasive mathematics anxiety is for some women elementary teachers.

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IMMERSING ELEMENTARY TEACHERS IN MATHEMATICAL MODELING AS CO-DESIGNERS THROUGH LESSON STUDY

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Using design research as a method of inquiry, we immersed twenty-four elementary teachers in mathematical modeling (MM) during a professional development institute and Lesson Study to co-design lessons and collaborate in collecting evidence of student learning. Results reveal that the design of the professional development and Lesson Study offered opportunities for teachers to become inspired by what their students were capable of doing as young mathematical modelers. Despite encountering challenges as they transformed their teaching approach to focus more on the cyclical process of MM, teachers found ways to rethink their mathematics instructional practices in order to promote higher ordered thinking skills in their elementary students.

Keywords: Teacher Education-Inservice/Professional Development, Modeling, Instructional Activities and Practices

Introduction

In the spirit of *Questioning Borders in Mathematics Education,* we attempt to identify the barriers that exists when teachers take up reform practice. Typically, reform in mathematics involves teachers moving from traditional approaches to teaching that is more problem-based in nature. The purpose of our study was to examine the challenges and support teachers needed as they implemented mathematical modeling in the elementary grades. We situated the professional learning in schools and classroom contexts through Lesson Study to provide the authentic classroom setting in which to consider the impact on student learning (Putnam & Borko, 2000). In addition, the mathematics educators worked with teachers in a *community of practices*, as teachers assumed the role of co-designers and implementers of this novel instructional approach in elementary grades.

Research Literature on Mathematical Modeling

Mathematical modeling (MM) is seen as a powerful tool for advancing students understanding of mathematics and for developing an appreciation of mathematics as a tool for analyzing critical issues in the real-world, that is, the world outside of the mathematics classroom (Greer & Mukhopadhyay, 2012). Traditionally, MM has been implemented primarily in secondary schools, but recent research examines using this approach with elementary students to promote their problem solving and problem-posing abilities (e.g. English, 2010). As students create and modify mathematical models to understand and solve real-world problems, they engage in a cyclical process of generating and validating their model and results. The teacher must be able to: (a) provide opportunities for students to acquire mathematical competencies and make connections between the real world and mathematics; (b) maintain the high cognitive demand of the MM process; and (c) provide classroom management that is learner-centered (Blum & Ferri, 2009). MM can be difficult for teachers to implement as they must be able to merge mathematical content and real-world applications while teaching in a more open-ended and less predictable way (Blum & Ferri, 2009).

Method of Inquiry

Using design research as a method of inquiry, we immersed twenty-four elementary teachers in mathematical modeling during a professional development institute so they would experience first hand the cycle of mathematical modeling before implementing MM through follow-up Lesson Study in their own classrooms. Through our professional development and research design, we wanted to understand and document how teachers developed and grew over time in their practice as they implemented mathematical modeling in the elementary grades. Specifically, we asked: (RQ1) What challenges do teachers face when implementing mathematics modeling in the elementary grades? (RQ2) What were some affordances of mathematical modeling that motivated teachers to continue to pursue it in their classrooms? (RQ3) What kinds of support do teachers need in order to implement mathematical modeling in the elementary grades?

To begin analyzing the themes, we used the document analysis technique using teachers' individual reflections, video transcripts of the Lesson Study debriefs, symposium presentations and researcher memos. We systematically analyzed the data by developing initial codes and used the method of axial coding to find categories in such a way that drew emerging themes (Miles & Huberman, 1994).

Results

In the results section, we present our analysis of the challenges teachers encountered when implementing MM, the affordances that inspired them to persist in using MM, and the supports needed as they continue to use MM in their classrooms (See Figure 1). The Mathematical Modeling Lesson Study tasks included planning a school fundraiser, designing and budgeting a butterfly garden, fighting hunger in their community, and being a smart consumer of goods. This analysis reflects the qualitative observations and data collected during the Lesson Study process that the teachers were engaged in.

Teacher Challenges

For the first research question we examined the challenges teachers face when implementing mathematics modeling in the elementary grades. The challenges that emerged from the data analysis included a) completing the modeling process, b) managing discourse, and c) school constraints. When implementing MM in the classrooms for the first time, teachers found that is was difficult to move students through the full process of creating and validating their mathematical models. For example, teachers commented that it was hard "…getting students to transition from collecting information to making a model", and "the students struggled moving past those assumptions because they wanted an answer to everything." As they implemented MM in their classrooms, teachers struggled with what type of support to provide to students during the MM process; they were unsure of when to intervene and when to allow students to productively struggle. Teachers "struggled answering and providing them with guidance" and noted that it was important, "not to give too many suggestions or impose too much control". During the debriefing of the Lesson Study, the participants acknowledged that MM takes time to implement in the classroom and that additional class time to implement these tasks would be helpful.

Affordances of Mathematical Modeling

For the second research question we examined the affordances of mathematical modeling that motivated teachers to continue to teach mathematics using this new method. The main affordances our teacher-researchers mentioned in our data emerged from mathematical modeling providing the a) opportunity for content to be covered without direct instruction, b) its interdisciplinary nature, and how MM provided c) mathematical relevance, and d) student engagement. When teachers implemented MM in their classrooms for the first time they were amazed at the amount of content

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that could be covered without direct instruction. By having students go through the process of creating and validating their model, teachers commented, "it was amazing to see them just automatically asking for this math. They were asking you to teach them this math until they could figure it out and they could solve this problem. I think that this was the coolest part of this math model." Another positive take-away from implementing MM in these teachers' classrooms for the first time was how MM created a space where content covered was interdisciplinary. During lesson study a host teacher noted, "our math modeling on choosing a school fundraiser was cross curricular. One of the biggest things in third grade is economics where you're talking about opportunity cost is what you have to give up when you choose something". Like this host teacher experienced, multiple lesson studies seamlessly incorporated economics, technology, geography, and science into their lessons and mathematics was used to solve problems and make decisions.

Support Teachers Need

For the third research question we examined the kinds of support teachers need in order to implement mathematics modeling in their classrooms. The three main areas of support that emerged from our teacher-researchers were access to a) MM resources and pictures of practice, b) time constraints and c) collaboration with like-minded teachers.

Teachers expressed a desire to see examples of "successful MM tasks" across various grade levels. Teachers indicated that these supports would be helpful as they continued to implement MM in their classrooms. During the debriefing part of the Lesson Study, a number of teachers expressed the need for more time to work through and become comfortable with implementing the modeling process in their classrooms. Teachers noted that it was only in working through multiple MM cycles that they felt comfortable with the process and felt that their students were able to understand the whole MM process. A number of teachers had implemented several MM tasks in their classrooms since the beginning of the year and they discussed the benefit of doing multiple MM tasks as a way to familiarize students with the MM process. Finally, teachers indicated a desire to continue to work with a cohort to build MM lessons as well as to collaborate with and observe other teachers implement MM in their classrooms. They expressed the desire to work alongside a colleague who valued MM and with whom they could share ideas.
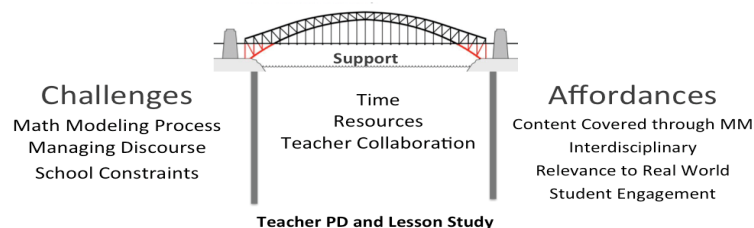

Figure 1. Bridging the barriers of implementing Mathematical Modeling in elementary grades during Lesson Study.

Conclusion

Reform practice in mathematics is a complex endeavor because there are many levels of support needed for change and many stakeholders who need to embrace the reform practices. In addition, teaching is a cultural activity that has many sociocultural norms. In order for reform practices, such as MM, to be taken up, teachers need to "relearn" mathematics through the reform practices so that they can understand how students will engage in that practice. Initially, our teachers were uncertain of their success with mathematical modeling with elementary students. It was after they were immersed as "mathematical modelers" and then co-designed and implemented a mathematical modeling lesson that the teachers gained more confidence about their ability to facilitate a

mathematical modeling lesson. Moreover, it was after seeing students engaged in the learning that teachers had the "proof of concept" needed for them to embrace this reform oriented teaching practices. Despite initial concerns about the time it would take to use MM in the classroom, the teachers learned that the modeling process provided a return on their investment of time because more content was covered meaningfully in the classroom using MM.

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OVERCOMING RURAL TEACHER ISOLATION AND PROMOTING CHANGE IN SECONDARY MATHEMATICS CLASSROOMS THROUGH VIDEO CLUBS

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The topic of rural mathematics education and the experiences of teachers working in these schools is one that has not been fully discussed within the literature. The relative isolation imposed by their circumstances create specific challenges, such as lack of resources, which is especially true for secondary teachers. This paper reports the results of a yearlong professional development project based on teacher noticing of student thinking involving the use of video club meetings at one rural school district. Findings show that while the project teachers developed deeper understanding of student thinking and task development/implementation they also changed their view of collaboration with their fellow teachers.

Keywords: Instructional Practices, Teacher Education-Inservice/Professional Development, Teacher Knowledge

Introduction

One of the largest shifts required for enacting the *Common Core State Standards for Mathematics* (CCSSM) (National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA & CCSSO], 2010) is the pedagogical considerations necessary for developing a new framework of classroom expectations (National Council of Supervisors of Mathematics [NCSM], 2014). Changes in the mathematical landscape motivate the need to analyze current pedagogical strategies and the implementation of available curricular resources (Lloyd, Remillard, & Herbel-Eisenmann, 2009). At the same time, the available literature on rural education topics, and more specifically mathematics education, is limited (Howley, 2008) despite these contexts representing more than 27% of schools in the United States (National Center for Education Statistics [NCES], 2015). In this paper we describe findings from a single-case study which examined the lived experience of three rural secondary mathematics teachers from one school district engaged in a video club based on *Teacher Noticing of Student Thinking*.

Background

Teachers who engage in conscious decision making regarding the impact of their own teaching have a greater chance of anticipating how students may respond to the tasks posed in the classroom (Sherin & Star, 2011). Through the process of planning lessons with several possible strategies in mind, and with set questions for each of these strategies at hand, teachers can be better prepared for addressing situations which arise (Smith & Stein, 2011). Reviewing teaching episodes provides teachers a necessary lens for analyzing student thinking and understanding opportunities the educator can draw on in the future. This was particularly important for the participants in this study, representing the secondary mathematics faculty at a rural, western school district, as they teach only one section of each class in a given day and do not have a means of clarifying content as the day progresses, as would secondary teachers in larger schools where individuals may have multiple sections of a single course.

This project assumed teacher noticing was a skill which could be learned (Jacobs, Lamb, & Philipp, 2010; Sherin & Star, 2011) and collaboration was an essential part of understanding one's own practice in the field of mathematics education (NCSM, 2014). The use of video clubs created a means for reviewing practice among groups of teachers and this reflection, when specifically focused on mathematical tasks and the instruction of those tasks, created a deeper capacity within the teachers

for analyzing instruction and planning for future lessons (Smith & Stein, 2011). Ideally the result would then be that teachers could develop better tasks, and these tasks would produce opportunities for students to learn content in meaningful ways.

Two primary research questions were created to address the problem of shifting teacher practice. These were (a) How does teacher noticing affect decision making around selecting and implementing classroom tasks? and (b) How does engaging in video club meetings focused on teacher noticing affect rural teachers' ability to identify and utilize pedagogical strategies which promote student thinking? Because of the nature of these questions, qualitative methodologies were used to collect and analyze data.

Methodology

Setting & Participants

The participants were three teachers representing all of the mathematical instruction from sixth through twelfth grade in one rural school district. Their teaching experience ranged from 15 ½ years to 23 years in the classroom and between 11 and 18 years in the particular district selected. Because of the small size of the district, none of the teachers taught the same course in a given day and all taught classes outside of the field of mathematics. The fact that no teacher taught the same course as their peer meant collaborative planning seldom occurred and each individual was responsible for preparing six different classes each day.

Data Collection

In line with creating a trustworthy study, multiple sources of data were collected throughout this project (Yin, 2009). Teaching observations were made on days when teachers conducted tasks that were filmed and later viewed during the video club (task days) and also on days when teachers were taught without being video recorded (non-task days). In addition, an initial and summative interview was conducted with each participant. These were semi-structured and served to create a deeper understanding of teacher thinking. Teachers also participated in five video club meetings. These meetings represented a type of group interview where participants would view clips of each member's teaching, reflect on their own pedagogy, and then discuss aspects of instruction and task selection; these meetings were recorded and transcribed for analysis. Several impromptu interviews occurred during the study as well and were video recorded. These were individual meetings, informally structured, and mostly focused as a discussion of instruction which had just occurred in the classroom. Finally, all observation notes, email exchanges between participants and the researcher or other teachers, teacher reflections, and classroom artifacts were also collected.

Data Analyses

Using a two-cycle coding method (Miles, Huberman, & Saldaña, 2014), data were first organized in clusters, which typified categories. These data were further refined using a system of pattern coding which sought to describe the clusters found during the initial cycle of coding and to better define the case as a whole. These were later condensed into categories, which represented commonalities in the coding focused on the researcher questions guiding this study. The three categories which emerged were: (a) beliefs about effective teaching, (b) role of classroom tasks, and (c) development and enactment of tasks. In addition, van Es' (2011) *Learning to Notice* framework (p. 139) was used to determine "what" teachers noticed and "how" the teachers noticed (van Es, 2011, p. 139). Determining the teachers' level of noticing provided a tangible means of evaluating the impact of the project on their beliefs about teaching, but equally important were the individual interview data as it allowed for deeper examination of views regarding the collaboration process.

Results

Teacher Noticing of Student Thinking

Following examinations of the video club meeting data, it became clear that teachers developed modified views of student thinking. Using van Es' (2011) *Learning to Notice* framework (p. 139), all three teachers advanced one level in both "what" they noticed and also in "how" they notice over the course of the study. None of the teachers were at the same initial level in either category, so therefore they ended the project with various depths of noticing. Data from the second and fifth video club illustrate these shifts (see Table 1).

Table 1: Noticing Levels Based on Learning to Notice Framework (van Es, 2011)

Teacher	Second Video Club		Fifth Video Club	
	What Teacher Noticed	How Teacher Noticed	What Teacher Noticed	How Teacher Noticed
Dean	Level 1–Baseline	Level 1–Baseline	Level 2–Mixed	Level 2-Mixed
Wilson	Level 2–Mixed	Level 2–Mixed	Level 3–Focused	Level 3-Focused
Larson	Level 3–Focused	Level 3-Focused	Level 4–Extended	Level 4-Extended

Curriculum Usage

As the teachers developed a deeper understanding of the role student thinking played in the learning process, their views of the curriculum, specifically regarding textbooks, began to shift.(See Figure 1.) Initially all three teachers were at various degrees of the offloading stage, which meant they used the curriculum materials and scope and sequence as intended by the curriculum developers. By the end of the project they all reflected different levels within the adapting stage where the teacher continues to use the textbook as a reference, but feels the freedom to make adjustments to content and sequencing. Mrs. Larson most closely approached the improvising stage, in which the teacher takes on authority for planning instruction and the curriculum is referenced periodically. All three teachers continued to use the curriculum, but they discussed how conversations with their peers began to have more significance.
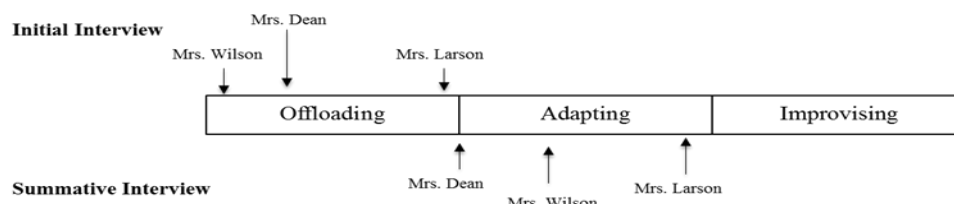
Figure 1: Teacher Curriculum Usage using Brown's (2009) Framework.

The teachers recognized shifts in perceptions regarding the power of tasks and the needs of students when this type of instruction was used. Collectively they began to notice the power of requiring students to discuss their mathematical thinking, which had not been a significant aspect of instruction prior to the beginning of the project. In the initial interview, Mrs. Dean described tasks as something fun for student to do, but in the summative interview she stated:

> Now that we have been together and done tasks together, I now see when I can tweak those things and where you get more out of them. It's one thing to do an application thing. It's another thing to squeeze out all of the thinking skills you want to get out there.

Throughout the project the teachers continually attempted practices they observed their peers using in the videos. The power of seeing these strategies applied locally increased the teachers' willingness to

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apply them while working with their own students. The video club meetings presented a forum for teachers to learn from each other and affect instructional changes.

Conclusion

Video clubs based on teacher noticing present an opportunity for teacher to grow in their own practice while removing barriers between colleagues who do not teach the same content. The use of this kind of professional development within rural contexts presents a valuable format for working with those who may otherwise feel isolated from their peers and complements the limited research on rural mathematics education (Howley, 2008). Further, the video club structure provided a context for the teachers to develop their noticing (van Es, 2011) and provided a system for the teachers to reflect on and make conscious decisions about their teaching in relation to students' mathematical thinking and the tasks posed in the classroom (Sherin & Starr, 2011). Although this research is limited to a case study for three practicing rural teachers, these data provide evidence of the importance of video clubs on developing teachers' abilities to notice and pay attention to students' mathematical reasoning. In this case, the data offer insight into how the teachers were implementing the CCSSM through tasks, while also emphasizing the students' mathematical strategies. Despite the fact that more research is needed, the findings of this project provide a resource for those working with rural educators.

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SUPPPORT SYSTEMS OF EARLY CAREER SECONDARY MATHEMATICS TEACHERS AND THEIR AFFECTS ON TEACHER RETENTION

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Significance/Purpose of Study

In partnership with the Mathematics Teacher Education Partnership (MTE-P), this study's main goal is to gather preliminary data on the nature and quality of professional support for early career teachers over time. Addressing the math teaching crisis meaningfully will require building a more cohesive system of teacher preparation, support, and development (Mehta, Theisen-Homer, Braslow, & Lopatin 2015).These data are part of the work of the *Secondary Teacher Retention and Induction in Diverse Educational Settings* (STRIDES) Research Action Cluster (RAC) of the MTE-P, toward efforts to increase the number of years that early career secondary mathematics teachers remain in the field and ultimately, increase their effectiveness at facilitating student learning.

Methodology/ Survey Details

The research question guiding this work is: *What is the perceived scope, nature and impact of professional support for early career math teachers, and how does this (a) change as teachers progress in their teaching career and (b) relate to how likely it is a teacher will remain teaching?*

The current data collection tool is a 20-item survey asking participants – secondary mathematics teachers in their first, second, or third year of teaching – to reflect on the degree to which the professional learning activities and communities they participate in (e.g., working with a mentor, attending a professional conference, being a Noyce Scholar) increases their enthusiasm for teaching mathematics and influences their ability to facilitate student learning. Additionally, participants are asked to describe the role of administrators, universities, and school structures (e.g., teaching load) on these self-reports, and their satisfaction with teaching and likelihood to continue teaching.

In order to better understand the degree to which early-career mathematics teachers are being supported by: 1) professional development, 2) professional learning communities and 3) administrators, the MTE-P STRIDES survey allows participants to specify activities that have helped them grow professionally, and the degree to which these activities were worthwhile to them. Additionally, since the survey is longitudinal, responses can be measured over time, allowing the researchers to understand how these teachers are supported throughout their early service (pre-service, 1st, 2nd, or 3rd) years. The survey ends with an estimation of: 1) their overall level of satisfaction in their teaching, 2) whether they would choose the profession again knowing what they have learned so far, and 3) how long they plan to remain in the teaching profession.

Possible Implementations of the Findings

The STRIDES RAC goal is to assure that, by July 1, 2022 at least 85% of the program's early-service teachers employed in partner districts begin a third year of employment as math educators.

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NEGOTIATING AN EQUITABLE MATHEMATICAL SYSTEM THROUGH PROFESSIONAL DEVELOPMENT

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We position the learning of mathematics as participation within a community, specifically, a mathematical system. In order to learn, one needs *access* to the community, in other words, they need the resources (tangible and otherwise) and practices that enable people to participate in ways that are viewed by other members in the mathematical systems as competent. *Agency*, for those within the community, is the ability to use such resources and practices both within and on the mathematical system. Finally, those co-invested in negotiating what equity means within the community are *allies*. These are the mathematics teachers, students, community members, mathematics teacher educators, administrators, etc. who help, support, or act in solidarity toward what has been negotiated as an equitable mathematical system. We seek to negotiate and work toward an equitable mathematics system by designing, facilitating, and studying professional development based around the principles of access, agency and allies. Within this poster we will illustrate the design around five strands: (a) rich mathematical (Horn, 2012) tasks and social justice (Gutstein, 2003) mathematical tasks; (b) discourse practices (Herbel-Eisenmann, Steele, & Cirillo, 2013), (c) cultural connections (Civil, 2007; Aguirre et al., 2013), (d) action research (Zeichner & Noffke, 2001), and (e) privilege and oppression (McIntosh, 1989). We will present our planned progression of these five strands across a two week teacher institute and describe activities designed to connect the strands to each other and to *access*, *agency*, and *allies*.

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SUPPORTING CULTURALLY RESPONSIVE TEACHING: WHEN MEETING TEACHERS WHERE THEY ARE MAY PERPETUATE THE STATUS QUO

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An enduring challenge in mathematics education is supporting teachers in meeting the needs of diverse students. One promising practice to meet this challenge is culturally relevant pedagogy (Ladson- Billings, 1995). With this in mind, we designed a course to support secondary mathematics teachers in building their knowledge to more effectively teach diverse students.

An issue we confronted from the inception of the course was how to bring the ideas of culturally relevant pedagogy into our particular context. We characterize our context by the large presence of Whiteness (of teachers and administrators) accompanied by the heterogeneous nature of the student population. In this poster, we explore our reflections on one instructional dilemma: how to include in the course a focus on critical consciousness (e.g., discussions of power and privilege) while also addressing a central tenet of culturally relevant pedagogy to meet students (in this case, teachers) where they are so as to engage them in learning. This dilemma troubles us because a tension seemed to exist between two recommended pedagogical strategies for culturally responsive teaching: meeting our teachers where they are and explicitly grappling with teachers on issues of power and privilege.

Following the work of Toll, Nierstheimer, Lenski & Kolloff (2004), we take a narrative research approach. First, we each wrote a response to the prompt "Can you appropriately/ authentically port culturally responsive pedagogy to our context?" as a way to identify some of the dilemmas we faced as mathematics teacher educators in this particular context. These initial narratives became the foundation of our research. Next, each of us read the three narratives and wrote a response to each other's stories (one response per author). Our data sources, then, included our initial stories and our responses to those stories. Data analysis consisted of open coding of the stories (Miles & Huberman, 1994) for recurring themes.

Our poster will share verbatim text from our stories that describe how we perceived what it meant to meet teachers where they are and whether or not this resulted in teachers developing an understanding of power and privilege. It will also highlight key questions that remain, such as questions about empathy for and empowerment of teachers, the potential for indoctrination and the use of power, and the role of our Whiteness in this project.

Like Toll et al, "we seesaw between exerting our power to shape teachers and withholding our power to make space for teachers' own exertion of power, and the result is our confusion and guilt" (p. 173). Our dilemma addresses important, unsolved issues in how to design and implement professional development for secondary mathematics teachers, in contexts such as ours, which result in more equitable mathematics education.

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EXPLORING CONDITIONS OF VIDEO-BASED PROFESSIONAL DEVELOPMENT: TEACHERS' CONVERSATIONS ABOUT AND REFLECTIONS ON INSTRUCTION

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The use of video clips of mathematics instruction for professional development purposes continues to grow. While exploring the research literature on video-based professional development (VB-PD), we mainly found studies in which groups of teachers experienced either facilitator- *or* teacher-directed sessions, and studies that used video clips from teacher participants *or* from unknown teachers. We found only one study that compared the type of facilitation (van Es & Sherin, 2006) and two studies comparing the type of video (Seidel, Sturmer, Blomberg, Kobarg, & Schwindt, 2011; Zhang, Lundeberg, Koehler & Eberhardt, 2011). Due to these findings, we aimed to answer the question: what is the optimal type of facilitation and the best type of video for mathematics VB-PD?

We designed a VB-PD program that would cross the conditions to create four types of professional development: (a) facilitator-led, stock-video, (b) facilitator-led, own-video, (c) teacher-led, stock video, and (4) teacher-led, own video. Twelve groups of teachers were assigned to these types of VB-PD and received the same training to use the *Mathematical Quality of Instruction* (MQI) instrument as a lens for watching videos. The MQI was also used to ensure that teachers had a common language for discussing mathematics instruction. Teachers then had ten professional development sessions in which they viewed, scored, and discussed video clips. In facilitator-led groups, facilitators directed teachers to focus on evidence from the video clip and use language from the MQI. In teacher-led groups, teachers were asked to lead the conversation about the clips. All groups viewed the same stock video clips for four weeks, then teachers in the own-video groups shared clips from their classrooms.

To answer our research questions, we audio recorded the groups' conversations, collected teachers' reflections on their own lessons, and developed coding schemes for the two sets of data. Based on our analysis, we found no significant difference in the quality of conversations in the comparison of facilitator- and teacher-led groups. In the comparison of own- versus stock-video conversations, we found teachers in the own-video groups more likely to come to consensus about MQI scores on their peers' clips, with more willingness to agree-to-disagree about the clips in the stock-video groups. With regard to their reflections, teachers in the teacher-led, own video groups were more likely to utilize the MQI when reflecting on their teaching.

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THE IMPACT OF PROFESSIONAL DEVELOPMENT ON HIGH SCHOOL ALGEBRA TEACHERS' KNOWLEDGE AND PRACTICES

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Algebra is a gatekeeper to advanced study in mathematics, yet many students do not successfully complete algebra and achievement gaps are large. A grant-supported professional development (PD) program worked to increase high school students' algebra achievement in a large, urban district (CA Dept. of Education, #10-703). The district, UC, is large (~80,000 students) and only 13% of students were considered proficient in algebra in 2009. In 2010, at least 24% of the algebra teachers were teaching outside their credentialed areas. The PD (3 years, ~250 hours) included: (a) intensive, week-long Summer Institutes (SI), (b) on-site monthly workshops, and (c) within classroom on-going coaching and support. Professional Learning Communities were developed to support teachers' learning and on-going communication and collaboration. The PD intentionally provided experiences that would assist teachers in learning new ways of thinking about mathematical content and the teaching of that content (Zwiep & Benken, 2013), specifically as it related to the new algebra standards required by the state (CCSS-M). Participants included 42 teachers at 6 UC high schools separated into two cohorts (experimental group, EG=26 teachers; control group, CG=16 teachers, ~40% of PD).

What impact did the PD have on teachers' algebra content knowledge and pedagogical practices? Analysis of survey content questions showed an overall increase by both groups ($p < 0.05$). Teachers in the EG made larger gains than teachers in the CG; these gains were statistically significant in years 2 and 3 ($p < 0.05$). Growth was most notable within strands addressing function content and on generated items based on SIs. Participants indicated that PD experiences allowed them to both enhance their understandings of relevant algebraic topics (e.g., solving equations) and improve their ability to communicate these conceptions. Following each SI, participants enhanced their capacity to problem solve both individually and in small groups, including developing multiple solutions/ways to conceptualize a given problem.

Analysis of practice questions on surveys, classroom observations, and PD evaluations support that participants in the EG statistically significantly developed their practice as follows: (1) increasing the use of different methods for students to perform mathematical tasks ($p=0.0321$), (2) integrating problems that have multiple approaches and answers ($p=0.0406$), (3) adapting content to differing learning and language levels ($p=0.0001$), and (4) facilitating more interaction within the classroom ($p=0.0091$). Thus, the EG showed significant gains in many areas; although the CG showed increases after their first 6 months in the PD, this growth was not yet statistically significant.

Results suggest that when teachers experience and are modeled strong long-term PD that is focused on integrating content-rich activities with pedagogical practices grounded on data and district need, they can expand their knowledge, which can then translate to changes in practice.

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MIDDLE GRADE TEACHERS' PERCEPTIONS ON COMMON CORE CURRICULUMS: EFFECTS ON TEACHING AND STUDENT LEARNING

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The launch of the Common Core State Standards (CCSS) in 2009 has influenced many aspects of K-12 mathematics education in the United States including areas such as content, pedagogy, curriculum, and assessment. School districts have looked to support these changes in varied ways, including the adoption of curriculum materials whose content is intended to align with these newly formed standards. However, adoption of standards and use of appropriate curriculum materials is only one step in the successful implementation of educational reform. The fidelity of implementation of mathematics curriculum reforms is strongly dependent on the way the standards and accompanied materials are implemented and interpreted by the teacher in the classroom (e.g., Hill and Ball, 2004; Roehrig and Kruse, 2005).

The objective of this study was to contribute to the understanding of role and influence of the current CCSS focused curriculum materials on teacher instruction, specifically on the middle grades level. Our focus on this level was driven largely by the increasing emphasis to move traditional high school level content to younger grades, thus adding to the complexity of integrating curriculum reforms in the middle grades. In gathering our data a mixed-method approach was used, which included interviews (n=18) and surveys (n=36) that used a Likert Scale. Questions from the two instruments focused on the ways in which in-service teachers perceived the impact of their CCSS curriculum materials on their own teaching strategies, classroom structure, use of technology, and content focus. The study also looked at the ways in which middle grades mathematics teachers perceived the adoption of these programs impacting student learning in their classrooms.

The results of the study indicate that the majority of teachers supported the use of a unified form of mathematics standards. However, the teachers in the study suggested that the implementation of the CCSS curriculum materials and associated assessments have widened the achievement gap for students. Participants found they are highly dependent on the mathematics curriculum materials for their planning and instruction, and they do not have ample opportunity for creativity and differentiation. Participants also discussed that they found the rigor of the step-by-step instruction detrimental to the students because it did not allow them to master certain skills before moving on to different concepts. There were areas of praise for the adoption of the CCSS curriculum materials including higher level thinking and integrating multiple modes of learning. Implications for professional development emerged from the analysis.

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THE IMPACT OF SUPPLEMENTAL INSTRUCTION ON THE LEADER

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Supplemental Instruction (SI) programs, based on the model pioneered at the University of Missouri – Kansas City, have long been shown to be beneficial to student participants. In this model, peer leaders conduct informal seminars in which students review notes, work together to solve problems, and learn study skills. However, evidence that SI programs also benefit the peer leaders of the SI workshops has been largely anecdotal.

One recent quantitative study (Malm, Bryngfors, and Morner, 2012) was conducted in an engineering program at a Swedish university. This study found that the peer leaders of the SI program benefitted from the program with improved communication, interpersonal, and leadership skills, as well as improved self-confidence and a deeper understanding of course content.

Our study is a similar quantitative study, conducted at a large public university in southern California, and largely confirms the study by Malm, et al. (2012). However, we also examine the difference in responses based on gender and under-represented minority (URM) status. Our study consisted of ten Likert-scale items sent to 153 SI leaders, of which 88 responded. The vast majority of respondents led SI sessions in STEM disciplines. Table 1 shows demographic information that we used in our study.

Using the Mann-Whitney U-test, we found that men were significantly ($p < 0.05$) more likely than women to report that leading an SI session helped them to be more effective when communicating with professors and students. We also found that URM students were significantly ($p < 0.05$) more likely than non-URM students to report that leading an SI session has had a strong influence on their career choice, helped them to become more aware of campus resources, and to deal with student conflict.

Our results show that it is important to encourage female SI leaders, in particular, to take advantage of leading an SI workshop to develop their communication skills with professors and students, especially in STEM fields, where under-representation of women is a long-standing problem.

Table 1: Demographic Information

Female	Male	Declined to State
44	43	1
URM	Non-URM	Declined to State
26	50	12

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USING THE INSTRUCTIONAL QUALITY ASSESSMENT OBSERVATION TOOL IN A PROFESSIONAL DEVELOPMENT CAPACITY WITH TEACHERS

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Research on teachers' use of high cognitive demand tasks (HCDTs) has included providing professional development (PD) on HCDTs and whether or not the PD impacted teachers' use of HCDTs (e.g., Stein, Grover, & Henningsen, 1996). Boston (2012) developed an observation instrument used to rate teachers' implementation of HCDTs, the Instructional Quality Assessment (IQA). In a previous study I used the IQA to rate teachers' implementation of HCDTs and after completion, I wanted to know if using the IQA in a PD capacity would give mathematics teachers a better frame to plan and implement HCDTs. I recently conducted a yearlong PD with 11 participants with the following question framing the study: *Does PD using the IQA as a frame for planning and implementing, impact teachers' use of HCDTs from their perspective and the perspective of the researcher?*

The IQA consists of six rubrics and is based on the premise that in order for students to engage in rigorous mathematics, they must have access to the type of problems that will allow such engagement (Boston, 2012). The rubrics of the IQA are based on the task implementation framework developed by Stein and colleagues (1996). Boston created the rubrics to comprise of four dimensions critical to assessing students' opportunities to learn mathematics with understanding. These four dimensions include potential of the task, implementation of the task, student discussion following the task, and the academic rigor of the teachers' expectations. When using the IQA, at least two raters observe a lesson and rate each aspect of the lesson with the given rubric.

The participants included 10 teachers in grades 3-8 and the mathematics coach from a charter school in a large Midwestern city. There were two teachers from each grade level to provide each with a collaborative partner, and the mathematics coach so there could be continued support. The PD included two half-day sessions before the start of the school year, six 2.5-hour sessions during the school year, and a concluding session at the end of the PD. Each session was based on different topics from the IQA (e.g., potential of the task, student discussions, teacher questioning) and how those topics related to maintaining demand of HCDTs. Data collection included video recorded pre and post observations of each teacher using the IQA to rate the lessons, one audio recorded individual interview, video of each PD session, and recordings of the teachers watching and rating each other using the IQA and their reflections of that process. The poster will share initial data analysis on the teachers' perspectives of the PD and how the IQA impacted their use of HCDTs and the researchers' perspective of how the PD and IQA impacted their practice.

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DEVELOPING MATHEMATICS TEACHER LEADERS THROUGH A PARTNERSHIP ENHANCEMENT PROJECT

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We describe a school district (Wren) involved in a partnership enhancement project (PEP), where school districts received micro-investments for professional development (PD) programs designed to provide support for locally identified and developed mathematics and science education projects. Kirschenbaum (2001) found school partnerships with universities were most effective when there was effective leadership and stakeholder involvement at the school level. A traditional top-down approach often fails because teachers do not implement innovations in the way intended by designers (van Driel, Beijaard, & Verloop, 2001). This qualitative study examined PEP challenges, outcomes, and collaborations in Wren. Data collection consisted of project proposals, midpoint conference presentations, and final project reports. The data were "analyzed to identify the recurring patterns or common themes" (Merriam, 2002, p. 6). This study addresses the research questions: How did participating in a partnership enhancement project enable the development of mathematics teacher-leaders? What evidence exists for the likelihood of sustaining partnerships and teacher leaders?

Prior to participating in the PEP, less than 50% of Wren students met mathematics benchmark scores for their age group on standardized assessments. Wren set goals to identify and address gaps in curriculum and in teacher and student content understandings. They planned to include all algebra teachers (grades 6-12) and to partner with instructional coaches and higher education. Wren held district-wide PD in year one for secondary mathematics teachers. Thirty-nine teachers participated in PD focusing on formative assessment, instructional and intervention strategies, and progression of Algebra skills. In year two, 22 middle school mathematics teachers attended PD along with 5 high school algebra teachers who acted as mentor teachers. Mentor teachers attended PD alongside the middle school teachers, providing support and instruction.

The PEP allowed teachers to take an active role in the improvement of district-wide algebra education, participating in planning and implementing professional development. Some teachers emerged as leaders who became mentor teachers and helped to build relationships between schools in addition to providing support for middle school mathematics teachers. The PEP was originally designed as a one-year project. When given the opportunity for a second year, Wren focused on bringing in sixth grade teachers, as they realized early preparation for algebra success was needed. This commitment to continually identifying factors in student algebra success in addition to finding time and funding for teachers to meet and plan suggests that the partnerships and teacher-leaders developed through the PEP will continue.

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TEACHER NOTICING OF STUDENT THINKING DURING VIDEO CLUB DISCUSSION OF A PROBLEM ABOUT PERPENDICULAR BISECTORS

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This study examines the effects of combining video clubs (Sherin & Han, 2004) and Lesson Study (Fernandez, 2002) to promote teacher noticing of students' prior knowledge in a professional development program funded by the National Science Foundation. Similar to the notion of noticing proposed by Jacobs, Lamb, and Philipp (2010), teacher noticing of students' prior knowledge requires teachers' attention to students' prior knowledge, interpretation of that prior knowledge, and anticipation of how to use that prior knowledge in instruction. The teachers planned and taught a problem-based lesson situated in the context of finding a fair location on a map to place a new building. We ask: *what notions of "fairness" do teachers notice during the video club discussions about student problem-solving strategies?* We investigate whether attention to students' problem solving strategies promoted a better understanding of the relationship between students' solutions and their prior knowledge.

We analyzed data from four video club sessions following the Toulmin model (1958). The lesson focused on the theorem: *If a point is on the perpendicular bisector of a segment, then it is equidistant from the endpoints of the segment*. The problem was situated within the context of finding three possible "fair" locations for building a sports complex between two schools. When planning the lesson, the participants had little discussion about connections between students' prior knowledge and their work on the problem.

During the video club, the participants noticed five notions of fairness. We labeled these notions of fairness as: (1) equidistant, (2) area of two intersecting circles, (3) symmetry, (4) close cluster, and (5) nearness. The first three notions of fairness are related to students' mathematical prior knowledge of equidistance, area, and symmetry. The other two notions of fairness are related to students' prior knowledge of the problem's context. There was a case where teachers did not notice a notion of fairness that we had hypothesized was evident in a video. The construct of noticing students' prior knowledge enables us to examine whether teachers pay attention to other sources of knowledge besides their mathematical knowledge.

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AN INSTRUMENT TO MEASURE TEACHER'S SELF-PERCEPTION OF THEIR TPACK: WRITTEN AND VALIDATED IN SPANISH

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The Instrument

The purpose of this study was to validate a Spanish version of the Secondary Mathematics Teachers TPACK survey developed by Zelkowski, Gleason, Cox, and Bismarck (2013). This new survey was named *Encuesta para Maestros de Matemática de Escuela Secundaria: Conocimiento Tecnológico, Pedagógico, Matemático (hereafter EMMES-CTPM).* The participants in the study were all Spanish speakers, Spanish being their mother tongue. For that reason there was a need for the original survey to be translated into Spanish. The survey needed to be validated given that participants in the study were (1) Spanish speakers, and (2) in-service teachers whereas Zelkowski et al. (2013) used preservice teachers as participants. The survey was administered in person to forty-nine secondary mathematics in-service teachers at the University of Puerto Rico at Mayaguez.

Findings

Although *a priori* information from Zelkowski et al. (2013) suggested there might be four factors (TK, CK, PK, and TPACK), the validated survey consists of three factors (TK, CK, and TPACK). The internal structure and the internal validity of the EMMES-CTPM survey was verified through an exploratory factor analysis using principal axial factoring (PAF) as the extraction method. Cronbach's alphas for the TK, CK, and TPACK constructs were .87, .91, and .95 respectively. Each factor has 5 or more strongly loading items (items loading is more than 0.50 each), and item communalities are within the recommended boundaries to be considered acceptable (between .40 and .70). The validated survey consisted of a total of 18 items, five within the TK construct, five within the CK construct, and eight within the TPACK construct.

The existence of a new instrument translated to the Spanish language and validated to be used with secondary in-service mathematics teachers provides researcher within the field with a new tool to measure teachers' self-perceptions of their TK, CK, and TPACK. Having a validated instrument offers researchers in countries where Spanish is the main language the opportunity to obtain reliable measures of teachers' self-perceptions of their TK, CK and TPACK since the items in the instrument are written in a language teachers can comprehend.

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REFRAMING MATHEMATICAL UNDERPARTICIPATORS: WHAT TEACHERS NOTICE IN STUDENTS' SMALL GROUP INTERACTIONS

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Noticing is defined to consist of attending to significant events, reasoning about events using knowledge from one's context, and making decisions based on connections between events and broader principles of teaching and learning (van Es, 2011; van Es & Sherin, 2008). Noticing is considered a skill of the expert teacher (van Es, 2011), however it is a skill that might need further development (van Es & Sherin, 2008). Status and its effect on student learning is one such construct that teachers might need further development to notice. Cohen (1994) defined status as a social ranking where it is better to be ranked high. Students of low social or academic status can lack self-confidence and become underparticipators in the classroom. Teachers cannot notice status issues occurring until they recognize that status exists within their classroom culture. Once a teacher has acknowledged that status does exist, they can begin to analyze how it affects everybody's access to the mathematical learning.

I adapted van Es's (2011) framework on learning to notice, shifting the focus from student mathematical thinking to noticing students' participation in small group interactions. The framework was used to code the annotations of what eight elementary teachers noticed when they observed small group interactions, the strategies teachers used to analyze what they observed, and the level of detail at which teachers discussed their observations of small group interactions in a Complex Instruction structured setting. For this poster, I focus on one teacher's annotations of one 8-minute video clip of a group of students working on a mathematical task.

On a scale of one to four the level of *what* the teacher noticed in terms of students' participation in small group interactions rated fairly high, meaning the teacher was a focused noticer. In the course of the 8-minute video-clip, the teacher's fourteen annotations could be coded as a Level 3: Focused, meaning the teacher attended to particular students' mathematical participation, or a Level 4: Extended, meaning the teacher attended to the relationship between particular students' mathematical participation. The level of *how* the teacher noticed in terms of students' participation in small group interactions rated low. All but one of the teacher's annotations was coded as a Level 1: Baseline, or a Level 2: Mixed. The teacher formed general impressions or highlighted noteworthy events, provided primarily descriptive and evaluative comments with some interpretive comments, and provided little or no evidence to support analysis. Further time in a notice-focused setting might continue to develop the teacher's ability of *how* they notice, and provide an opportunity to suggest pedagogical strategies to disrupt the issues of status which interfere with all students' learning experiences.

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DESIGNING AND DELIVERING A STANDARDS-BASED PROGRAM TO AUTHORIZE ELEMENTARY MATHEMATICS SPECIALISTS

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The purpose of this poster is to detail the development and implementation of an Elementary Mathematics Instructional Leader graduate program at Western Oregon University. In 2013, the Association of Mathematics Teacher Educators adopted the Standards for Elementary Mathematics Specialists (AMTE, 2013). Using the Standards as a framework, the State of Oregon authorized the development of the Elementary Mathematics Instructional Leader Specialization to be added to current teaching licenses. This specialization includes three components: three years of successful teaching at the elementary level, a passing score on the NES Multiple Subjects Examination, and demonstrated competency of the Standards for Elementary Mathematics Specialists (SEMS). The graduate program outlined in this study was the first approved program in the state and the first to recommend teachers for the specialization.

In November 2014, the university in this study was awarded a Oregon Department of Education grant to fund the development of up to 60 teachers through the program. The grant includes funds for graduate coursework, monthly webinars, and summer institutes. This poster will report the findings of the first two years of the program including program design, participant experiences, and initial assessment data.

Initial findings indicate some attrition, mostly due to the perceived difficulty of the mathematics coursework. Data indicate the students appreciate the structure of the program, support of the faculty, and the program activities that connect to their current teaching role in the schools. As leaders see their roles changing and growing, the project partners search for ways to advocate for and support the leaders after they finish the program.

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THE IMPORTANCE OF A LEADER IN PROFESSIONAL LEARNING GROUPS

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The Ontario Ministry, for one, has called for the use of professional learning groups to support teachers in developing new and effective practices for teaching (Ontario Ministry of Education, 2007). Personal experience and past research (e.g. Kajander & Mason, 2007) have shown professional learning groups in mathematics can have a varied effect on the teachers in the groups. Other research has suggested that the importance of having a shared and supportive leadership (Hord & Sommers, 2008) in order for the groups to be productive and meaningful.

This research focuses on the results of a three-year, narrative case study of a mathematics professional learning group of upper elementary (grades 6 to 8) and early secondary (grades 9 and 10) in northwestern Ontario, Canada (Holm, 2014). Over the three years, field notes, meeting recordings, and interviews comprised the data collection for the study. The current report focuses on data related to one aspect of these observations, that of the role of the leader.

The teachers included in the professional learning group varied in their knowledge of mathematics, their beliefs about teaching, and their understanding of teaching mathematics. Within the group there was a strong dichotomy between the secondary teachers who believed that mathematics was about teaching better rules and procedures, and the elementary teachers who supported the need for discussions and explorations within mathematics. In a number of observed instances, various members of the group were vocal about leading the group in non-productive directions or back towards a focus on only teaching rules and procedures earlier. Different stimuli were observed to steer the group back to a position of growth. At times these were another member of the group, or even several members. At other times it was a piece of literature, visiting presenter, or other external source that was used as a focusing piece for the group. What appears important based on this case study is that the leader has a strong knowledge of appropriate mathematics and supports a belief in using reform-based strategies within mathematics teaching.

In the end, we found that the various instances of this fluid idea of the leader seemed important to allow the group studied here to move forward in making reform-oriented changes. We saw the leader (in its various forms) as continually expanding the reach of the group and encouraging the ongoing growth of the group. This focus allowed for the direction of the group to be on strategies that were expanding the knowledge of the teachers instead of maintaining the status quo.

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TEACHERS' ACTIVE ENGAGEMENT IN SOCIAL MEDIA: ANALYZING THE QUALITY OF MATHEMATICAL PRACTICES WITHIN PINTEREST

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In a digital age, social media has become part of teachers' daily routines. In this paper, we focus on a photo-based personalized media platform that allows teachers to demonstrate their conceptualization of mathematics instruction through the act of pinning resources and ideas in visual forms.

Theoretical Framework

This work focuses on the potential of the written tasks that attract ECTs' attention, motivate them to conceptualize mathematics teaching, and construct knowledge of practices (Cochran-Smith & Lytle, 1999). The cognitive process levels the task requires of students are determined using one dimension of the Revised Bloom's Taxonomy (Anderson & Krathwohl, 2001) and its interpretation in the mathematics context. The six hierarchical cognitive process levels are 1) Remembering; 2) Understanding3) Applying; 4) Analyzing; 5) Evaluating; and 6) Creating.

Method

Using a sample of 19 Early Career Teachers (ECTs) in the 2014-2015 school year, identified through a larger study on planning and enactment of elementary mathematics, we use the above framework to code for the primary task embedded in ECTs' mathematical pins. For example, a pin features a poster with the Think Math activity that reads: "The answer is 9 penguins, what is the problem?" We coded this pin with a primary task *at Level 6: Creating*, because the task is about to ask students to *create* a word problem with the answer 9 penguins. It is an open-ended question, and students need to understand different operations and their meaning in order to create a meaningful word problem.

Results

Out of the 1,123 mathematical pins examined, 396 of them are content resources with a conglomerate of tasks, whose cognitive process cannot be determined because the primary nature of the task is not definite. For the remaining 727 pins with mutually exclusive mathematical tasks, the majority are at the level of *Level 2: Understanding* (n=315), followed by *Level 1: Remembering* (n=302). Also, ECTs within this sample present few mathematical pins requiring a higher cognitive process demand for their students.

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EMERGING LEADERSHIP ACTIONS OF ELEMENTARY TEACHERS
IN AN URBAN MATHEMATICS EDUCATION GRADUATE PROGRAM

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Educational improvement at the level of daily mathematics instruction necessitates leadership by teachers (Neumerski, 2013). This requires teachers to open up their own instruction to scrutiny, while navigating and building relationships and trust within school cultures (Lumpkin, Claxton, & Wilson, 2014). In addition, teachers must be seen as credible in regards to their mathematical knowledge for teaching at the intended grade levels (Manno & Firestone, 2008).

This study examined the effects of a three-year graduate program on the mathematics leadership actions that teachers pursued in their schools. Participants studied mathematical knowledge for teaching, examined equitable and high-leverage teaching practices, and developed coaching and facilitation skills. The research questions guiding this study were:

- What aspects of the program encouraged teachers to move beyond their own classroom instruction to support and influence the professional practice of others?
- What informal and formal opportunities did teachers recognize or generate to provide mathematics leadership at their school site?

We examined the evolution of 26 elementary and middle school teachers as they progressed through a graduate program in urban mathematics education. Periodically participants prepared written summaries and reflections of ways in which they provided leadership for mathematics within their schools and districts. Additional data sources included course assignments, online discussions, and verbal and email conversations. Qualitative data coding using constant comparison for both cross-case and within-case analyses surfaced several themes. Some of the major emerging themes among the emerging teacher leaders included the following:

- Teachers grew in their confidence as capable of providing leadership for peers.
- Teachers became initiators of leadership actions in prompting learning of colleagues.
- Powerful "sites" of leadership included informal (e.g., bulletin boards, lunch) and structured interactions (e.g., grade-level meetings, staff development, committees).
- Teachers lead on the content studied in the graduate program (e.g., equality, discourse, representations, unit fraction instruction, number talks, growth mindsets, role of mistakes).
- Several teachers were on a "mission" to improve the mathematics teaching and learning throughout their schools or across their school districts.

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CREATING OPPORTUNITIES FOR JUSTIFYING: AN EXAMINATION OF ONE TEACHER'S EVOLVING QUESTIONING STRATEGIES

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Research demonstrates the way students are engaged in mathematics tasks determines what they learn from the experience (Stein, Smith, Henningsen, & Silver, 2000). Evidence points to an increase in achievement when students are asked to explain their thinking on complex problems (Boaler & Staples, 2008). Teachers must not only present mathematics tasks to students, but also engage them in productive interactions that include opportunities to explain their thinking and participate in discussions of their peers' thinking (Stein et al., 2000). Making Mathematical Reasoning Explicit (MMRE) is a five-year NSF MSP professional development (PD) project focused on helping teachers increase their students' instances of justifying. This study reports how one teacher's questions changed over time as she elicited student justifications.

As we examined the ways in which Pam (a pseudonym) engaged her students in justifying, we collected and analyzed research notes and video recordings focused on Pam's classroom interactions with her math students over the course of five years. We found there was a clear difference in her questioning strategies before and after she engaged in PD activities.

Pam showed an affinity for engaging students in sense making during mathematics classes prior to her involvement with MMRE. Her desire to have students explain their thinking was evident throughout her involvement with MMRE. However, early on, Pam posed questions like "what did you do?" that led to students stating or describing a procedure. Over time, the type of questions Pam asked students shifted focus, and she began to ask questions such as "why did you choose to solve the problem this way?" These types of questions align with assertions from Stein et al. (2000) that teachers can maintain a high level of cognitive engagement by a sustained press for justification, which was demonstrated in student responses in Pam's classes.

As we reflect on the experiences Pam was offered throughout the PD, it is apparent that the PD activities had an impact on her ability to engage students in high level thinking. She showed clear changes in her ability to engage students in explaining their mathematical reasoning over time. However, the fact that Pam demonstrated this evolution over the span of five years serves in stark contrast to the perhaps too frequently accepted notion that sustainable educational change can and should occur quickly. We urge teacher educators and others associated with implementing teacher change to be patient as they work with teachers to alter practice.

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HOW ELEMENTARY MATHEMATICS TEACHERS BECOME MORE REFLECTIVE IN THE CONTENT-FOCUSED COACHING CONTEXTS

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While teaching has been considered as highly complex, recent reform-oriented principles and standards (e.g. CCSSI, 2010; NCTM, 2000) suggest teaching should be student-centered or responsive to student thinking. This new teaching practice makes teaching more complicated, uncertain, and dependent on teachers' ability to improvise as they encounter and respond to students' novel ideas during teaching. Thus it is critical for teachers to learn how to plan in ways that lead to being more prepared for the uncertainty and improvisation that accompanies teaching (Heaton, 2000). This study investigates how elementary teachers become more intentional and reflective in their lesson planning in the context of content-focused coaching, called *Math Studio*. The Math Studio, one model of teacher professional development with a coach and other members of a school community including a principal and classroom teachers, consists of three phases in a cycle: (1) lesson planning; (2) lesson implementation; and (3) lesson reflection. There are three to five cycles, across a year. This is a kind of job-embedded professional development for in-service teachers that has been happening for the last 4 years in an urban school district in Nebraska. In this article, we focused on three elementary teachers who were studio teachers in the during the 2014 - 2015 school year. Audio-recordings of each teacher's planning sessions over the school year were transcribed and analyzed using quantified qualitative methods. The analytical framework was developed building on an instructional triangle (Cohen, Raudenbush, & Ball, 2002; Lampert, 2001) consisting of three aspects, *teaching*, *student thinking and learning,* and *mathematical content.* In order to capture the patterns of teachers' deeper understanding of each aspect and how they became more intentional, planful, and reflective revealed in teacher-coach interactions, different levels of depth (level 1 – level 4) corresponded to each code. The full analytical framework will be presented on our poster in more details.

Preliminary analysis of multiple planning sessions across a year illustrates different paths of teachers' learning. Patterns of discourse revealed various and different levels of depth, with all teachers' discourse moving from lower to higher levels. Findings describe how individual teachers make more sense of student mathematical thinking and learning, more intentionally use and create lesson materials, and attend more to their own teaching practice with connections to student learning, particularly in their lesson planning. Detailed diagrams of results and examples will be described on the poster. This research adds to our understanding of how teachers learn and of the process through which teachers have opportunities to advance their understanding on the details and substance of student mathematical thinking and to intentionally choose the lesson materials. Our work has implications for curriculum developers and designers of professional development how to best support teachers to become more responsive to student thinking.

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SECONDARY TEACHERS' PROFESSIONAL NOTICING OF STUDENTS' MATHEMATICAL THINKING

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Professional noticing of children's mathematical thinking (PNCMT) requires expertise consisting of three interrelated component-skills: (a) attending to the details of a child's strategy, (b) interpreting a child's understanding, and (c) deciding how to respond on the basis of this understanding (Jacobs, Lamb, & Philipp, 2010). Jacobs et al. described this expertise as a part of a teacher's in-the-moment response to a student's explanation, and thus is a fundamental component of instruction that builds on student thinking. Hence, investigating the nature and development of teachers' noticing of children's mathematical thinking is important.

Many studies on PNCMT focus on preservice teachers (e.g. Sanchez-Matamoros, Fernandez, & Llinarez, 2014). Few studies have investigated K-12 practicing teachers' PNCMT and even fewer have focused on secondary teachers. Notably, Jacobs et al. (2010) found in their cross-sectional study that practicing K-3 teachers tended to demonstrate robust evidence of deciding how to respond on the basis of children's mathematical thinking only after completing *four or more years* of professional development (PD) around children's mathematical thinking. For many K-3 teachers, two years of PD did not fully support their development of this expertise.

In our longitudinal study we investigated the noticing expertise of 16 secondary mathematics teachers who we are supporting in a 5-year PD to improve their practice and become leaders of their teaching communities. Due to the rigorous selection process for the PD, we claim these teachers all (a) exhibit effective teaching practices, and (b) have a positive stance toward learning. Activities from the PD have included interviewing students, analyzing students' written work, working on challenging mathematical tasks, and collaboratively planning lessons.

We collected data on teachers' noticing expertise twice: prior to the PD, and two years into the PD. We collected and analyzed data using similar prompts and coding procedures as Jacobs et al. (2010). Both times, teachers watched and then responded to noticing prompts about an eight-minute video of middle school students working on a generalizing patterns task. Our findings indicated that initially our participants were able to attend to key details of students' strategies. However, they shared little evidence of interpreting students' understandings or deciding how to respond on the basis of those understandings. In our poster we will share the degree to which this expertise changed over the first two years of the professional development.

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HOW DOES A PROFESSIONAL DEVELOPMENT PROGRAM AFFECT MATHEMATICAL QUALITY OF ELEMENTARY TEACHERS' INSTRUCTION?

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The MQI is a research-based instrument developed by Learning Mathematics for Teaching Project and designed to assess the quality of mathematics instruction by evaluating several dimensions of instruction (Hill et al., 2008). In this study we used the dimensions of the MQI instrument as a lens to investigate the effects of PD program on elementary teachers' instructional practices in the case of four 4th grade elementary mathematics teachers. Specifically, we focused on answering the following research questions: (1) what changes do teachers demonstrate in terms of their teaching practice before and after the PD? and (2) how does the PD influence teachers' quality of mathematical teaching?

In this study, we used pre-and post- teaching videos as the main data sources along with materials used in workshops. At the beginning of PD, all four teachers were asked to video-record their mathematics lessons. These videos were used as the pre-teaching video. After a year-long PD, we asked teachers to plan a lesson and teach the lesson in their classroom, which happened almost the end of the PD. Teachers collaborated with a researcher to plan a math lesson and implement the lesson through the end of the PD. We used these videos as our post-teaching videos.

Two pairs of researchers who had completed a training to use the MQI instrument, analysed each teacher's 40 minute-teaching video after choosing an 8-minute video clip that is the densest with mathematical activity. After the initial analysis, the researchers reconciled their results, and then created a detailed summary about each teacher's teaching practice. Based on the summary of each teacher, we examined overall tendency of each major dimension from four teachers' teaching videos.

In the pre-teaching videos, the average of the four teachers' MQI scores was 36.25 but in the post-teaching videos, the average increased to 47.5. Every teacher showed the increase of MQI scores from 5 to 19. Teacher A demonstrated the most significant change and Teacher B showed a small amount of change. In terms of the five major dimensions, teachers showed the biggest improvement in the aspect of "richness of mathematics" and followed by "working with students and mathematics" Also, teachers demonstrated pretty small improvement in the aspect of "student participation in meaning-making and reasoning." Moreover, teachers showed the smallest change in "errors and imprecision" aspect because they tended to show high scores in both pre-and post-teaching videos.

The findings of this study shed light on the importance of PD programs integrating strategies that can be used to promote students' participation in meaning-making and reasoning. To do so effectively, it may require to establish socio-mathematical norms which include mathematical discourse, unpacking mathematical problems and evaluating different solution strategies.

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DOCENMAT: UN AMBIENTE INFORMAL EXPERIMENTAL PARA EL DESARROLLO PROFESIONAL DEL PROFESOR DE MATEMÁTICAS

DOCENMAT: AN EXPERIMENTAL AND INFORMAL ENVIRONMENT FOR THE PROFESSIONAL DEVELOPMENT OF THE MATHEMATICS TEACHER

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Palabras clave: Educación Informal, Capacitación Docente / Desarrollo Profesional, Conocimiento del Profesor

Existen múltiples investigaciones que reportan aspectos del desarrollo profesional del profesor de matemáticas, su relevancia y resultados de diversas experiencias formativas (Even & Ball, 2009; Paquay, Altet, & Charlier, 2012; Wood, 2008), pero es reducida la información sobre la actividad de desarrollo profesional del profesor de matemáticas en escenarios informales. La colaboración entre personas y grupos de personas empleando Internet, se transforma no sólo porque los participantes se encuentran geográficamente en lugares y contextos distintos, sino porque dicha interacción le cambia su naturaleza; es multimodal, pues permite interactuar combinando múltiples formatos que la potencian (voz, texto, música, video, fotografía) (Borba, Clarkson, & Gadanidis, 2013). El trabajo previo con profesores de matemáticas en modalidad a distancia, nos ha llevado a la construcción de una red social *DocenMat*, en una plataforma en línea, la que ha permitido experimentar diferentes tipos de interacciones entre sus miembros: profesores de matemáticas de distintos niveles educativos, estudiantes de posgrado, formadores de profesores e investigadores de la Matemática Educativa, tanto en México como de América Latina. Es este espacio se pone al servicio de los profesores, buscando que éstos reconozcan en la Matemática Educativa el campo de saber de referencia del profesor de matemáticas, así como a una comunidad de pares que ejercen la profesión y promueve su desarrollo profesional, haciéndole accesibles herramientas que lo pueden ayudar para mejorar su práctica docente. Además encuentran a una comunidad que investiga y que produce conocimiento para el profesor. En *DocenMat* los profesores pueden interactuar de múltiples maneras, grupos de trabajo especializados, diálogos académicos en foros, actividades que le permiten reflexionar sistemáticamente su quehacer. Hasta ahora no se cuenta con un protocolo que evalúe efectos en el desempeño profesional de los profesores a partir de su participación en la Red, se producen cotidianamente en la red, evidencias de sus entradas y acciones en ella. Esto se convierte en material de análisis. Así mismo se mantienen contactos periódicos con varios profesores y grupos de trabajo que nos permiten conocer y valorar el impacto de su adhesión a DocenMat. Tenemos evidencias para afirmar que *DocenMat* (http://www.docenciaenmatematicas.ning.com/) constituye un espacio de desarrollo profesional informal -libre, comunitario, basado en intereses específicos de los participantes- y proveedor de conocimientos formales de un campo de saber.

Keywords: Informal Educational, Teacher Education-Inservice/Professional Development, Teacher Knowledge

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

Multiple research studies have reported on aspects of professional development for mathematics teachers, as well as the importance and the results of different learning experiences (Even & Ball, 2009; Paquay, Altet, & Charlier, 2012; Wood, 2008), but there is little information about professional development with mathematics teachers in informal settings. With the internet, collaboration among people and groups of people changes not only because the participants are located in different geographical places and contexts, but because the nature of the interaction changes; it is multimodal, using a variety of formats that boost interaction (voice, text, music, video, photographs. (Borba, Clarkson, & Gadanidis, 2013). Previous work conducted with mathematics teachers in distance learning formats, has led us to create a social network in an online platform called *DocenMat,* which has allowed different types of interactions among the members including mathematics teachers of different educational levels, graduate students, teacher educators and researchers of mathematics education, from Mexico and Latin America. The aim of this online tool is to help teachers recognize the knowledge of mathematics teachers in the field of mathematics education, as well as to support a community of peers practicing the same profession, and to promote professional development by making tools accessible to help teachers in their daily teaching practice. The tool also provides a community that researches and produces knowledge for teachers. Through *DocenMat* teachers are able to interact in many ways, such as in specialized working group, academic discussion forums, and systemic reflection activities related to the work of teaching. So far there is not a protocol to assess the effects of participation in online professional networks on teacher performance, although evidence of participation and actions in online networks are produced daily. This material becomes data for analysis. This also allows maintaining regular contact with various teachers and working groups to understand and assess the impact of participation in DocenMat. We have evidence to establish that DocenMat (http://www.docenciaenmatematicas.ning.com/) is an informal professional development space - free, communal, based on specific interests of participants- and a provider of formal knowledge in specific fields.

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MEASURING FIDELITY OF IMPLEMENTATION IN A LARGE-SCALE PROFESSIONAL DEVELOPMENT EFFICACY STUDY

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Fidelity of implementation is the essential link between professional development (PD) experiences and changes in outcomes. Yet, as a field we have few measures available to directly look at implementation. In this poster, we present our preliminary work measuring teachers' implementation of a research-based PD.

Background

We are currently conducting a large study evaluating the efficacy of a studio model professional development (Foreman, 2013) in a mid-sized urban school district working with 3rd-5th grade teachers. The PD focuses on a set of mathematical habits for students and teachers that promote high-level reasoning and productive discourse in mathematics classrooms. In order to measure teacher fidelity of implementation, we identified these habits along with cognitive demand (Stein & Smith, 1998) and connection to learning target as the *critical components* (O'Donnell, 2008) of the PD.

The Measures of Implementation

In order to measure implementation, we developed a classroom observation tool aligned with the critical components discussed in the previous section. We triangulated this measure with both PD facilitator ratings, and teacher self-reports. We piloted the tool in 22 teacher classrooms. We found that the observation tool implementation scores were consistent with facilitator ratings, but diverged from teacher self-reports.

We also looked at outside measures: Mathematical Quality of Instruction (MQI) (Hill, 2010) and student achievement on the Smarter Balanced assessment. We found that teachers who scored high on implementation observation tool had higher overall MQI scores and higher percentages of students passing the standardized assessment than medium and low implementers.

Discussion

In this poster session, we share a model for measuring fidelity of implementation. We explore the development of a tool targeting implementation, the triangulation with other measures, and the correspondence with outside variables. We hope to contribute a process that can be leveraged by other researchers evaluating the impact of PDs.

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UNDERSTANDING THE WAYS GTAS APPROPRIATE AND TRANSFORM PROFESSIONAL DEVELOPMENT

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Students' experiences in Calculus I are a main contributing factor to students' decisions to leave the STEM (Science, Technology, Engineering, and Mathematics) disciplines (Seymour & Hewitt, 1997). One way departments have been working to address this is to offer professional development to the Graduate Teaching Assistants (GTAs) and to reorganize the structure of the GTA program. Research has been done on the varieties of professional development available for GTAs across the nation (e.g., Belnap & Allred, 2009), as well as on particular structures at specific universities (e.g. Burr, 2016). However, little research has been done on how the GTAs are appropriating and transforming the professional development activities.

In our study, the GTA program around Calculus I and II has been restructured into a lead TA format. In this, there is a GTA designated as the lead who observes the other GTAs and provides feedback on their teaching. Furthermore, the lead TA becomes the main point of contact for the other GTAs; they assist the coordinators of the courses in setting up weekly meetings. Additionally, the GTAs at this university participate in formal professional development (PD) with mathematics education faculty. Hence, GTAs engage with various forms of professional development: weekly meetings with the coordinator, supervision of the lead TA, and formal PD.

Since there are goals in the minds of the professional development leaders for the teaching done by the GTAs, we are interested in how exactly GTAs are appropriating the professional development in which they are engaged. Specifically, how do the GTAs appropriate the professional development as they participate in the meetings and how are they transforming it to support their own needs? Additionally, what role does the lead TA play in publicizing and the conventionalization of the professional development activities?

We are collecting and analyzing data for Calculus I TAs from the weekly meetings with the coordinator, their classrooms, the professional development sessions, and the debriefings conducted by the lead TA. Through the use of a framework known as the Vygotsky Space, we will be looking for ways in which the professional development topics are appropriated and transformed by the GTAs over time. The Vygotsky Space framework is used to describe the ways the individual appropriates ideas from the social plane into their individual plane, transform these ideas, and then makes the ideas public again to the social plane (Harré, 1983). In our poster, we will discuss the initial findings from our study using the Vygotsky Space framework.

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MATHEMATICS LESSONS AS STORIES: A REASON TO DO THE MATH

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Keywords: Curriculum, Teacher Education-Inservice/Professional Development, Teacher Beliefs

There is evidence that one way to improve the quality of mathematical instruction is to improve the curriculum decisions of teachers at varying scales, such as how to respond to a student's question during a lesson or the selection and sequence of tasks. Studies of how teachers read and plan with curriculum materials highlight the need for useful conceptual tools for making sense of the content of mathematics curriculum (e.g., Remillard, 1999). Teachers need multiple ways to analyze their curriculum beyond just attention to individual tasks and directions. One way to conceptualize how mathematical ideas unfold in a classroom is to interpret a math lesson as a story (Dietiker, 2015). Similar to a literary story, a *mathematical story* is the ordered sequence of connected mathematical events connecting a beginning with its end. Examining curriculum through the lens of the mathematical story framework enables teachers to see how the posing and resolution of mathematics tasks and mathematical questions over time could shape the experience of the learner.

As part of a three-day professional development in Summer 2015, teachers learned about the mathematical story framework and how it could be used to interpret their curriculum. We collected statements during the professional development and during the following year that captured how they were and were not using the story framework to frame their thinking around curriculum. These statements were analyzed for patterns that reveal how teachers make sense of the story framework and how it informs their practice. In particular, this poster reports our findings regarding how the narrative elements of this framework supported these teachers in thinking about the development of mathematical ideas and curricular goals. In addition, the poster presents examples of mathematical stories the teachers created. Our findings indicate that this framework supports the curricular insight of teachers into how even minor curriculum adaptations can potentially enhance student learning and engagement. Themes of teachers as authors of mathematical stories and the importance of sequence emerged. Several teachers also connected student motivation with mathematical story. One teacher noted that when the mathematical story can grip a student's attention, it "gives [the student] a reason to do the math."

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PROGRAMMING EXPERIENCES IN AN ALGEBRA CONTEXT: TEACHERS CROSSING BORDERS BETWEEN MATHEMATICS AND COMPUTING

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How do we engage a broader swath of U.S. students in STEM and computer science fields, especially for minorities and low-income students who have traditionally been underrepresented in STEM and computer science? This study is part of an NSF-funded STEM+Computing project to pilot a secondary algebra unit that integrates project-based learning, engineering, and computer science into an exploration of linear functions. The study examines the impact of a strategic professional development on algebra teachers' pedagogical content knowledge of functions, competency in basic computer modeling and programming, and engagement in computational thinking.

The professional development piloted in the study involves computational concepts, computational practices, and computational perspectives. The study's approach reflects the benefits of context-based experiences with mathematics (Heid, 1995) and incorporates project-based learning (PBL) as well as the *Guided Inquiry and Modeling Instructional Framework* (EIMA) proposed by Schwarz and Gwekwerere (2006).

The study: 1) employs design-based research methods that deliberately intertwine the design of innovative learning environments and the development of a theory of learning (The Design Based Research Collective, 2003); and 2) uses progressive refinement to revise both the learning environment and the theory of learning through cycles of design, implementation, analysis, and revision (Cobb, 2001). At the core of this process is a 10-day professional development designed to engage teachers as learners through an engineering-focused exploration of linear functions in a project-based unit utilizing computer modeling, programming, and problem-solving.

A brief intake survey will identify teachers' current level of programming knowledge and ability using a spectrum that runs from novice to advanced beginner, to competent, proficient, and expert. By complementing core exercises with collaborative problem-solving and mini-lessons that highlight characteristics of expert practice, the training aims to move teachers to the stage of advanced beginner or competent programmer. Tables will highlight findings from teachers' pre- and post-tests. Numerical data will be complemented by brief analysis of short video clips to illustrate changing engagement with computational thinking and programming approaches to teaching functions.

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ACCEPTED STUDENTS' ANSWERS TO HIGHER ORDER QUESTIONS AND CONCEPTUAL UNDERSTANDING

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Keywords: Teacher Education-Inservice/Professional Development, Classroom Discourse

Principles to Actions include Posing Purposeful Questions as one of the Mathematics Teaching Practices, allowing teachers "to assess and advance students' reasoning and sense making about mathematical ideas and relationships" (NCTM, 2014, p. 10). However, as Walsh and Sattes (2005) point out, teachers, when faced with incorrect or incomplete answers, rarely use prompts and questioning to guide students' reasoning to help them give correct answers.

Over the past two years we have provided professional development (PD) to mathematics teachers from three all-boys charter high schools in an urban area (98% African-American, 84% low-income households.) PD focused on the concepts from their curricula and Mathematics Teaching Practices (NCTM, 2014). Teachers solved tasks that allowed multiple entry points and solution strategies, promoted reasoning and problem solving, shared with each other, used manipulatives, worked in cooperative groups, and reflected on their experience, as recommended by Desimone (2011). We then observed teachers and provided feedback that included specific examples from their instruction to point out how they could use questioning to extend student understanding beyond procedural mastery.

To analyze our observation notes, we used a framework described in Principles to Actions (NCTM, 2014) for four types of questions: *gathering questions* ask students to recall facts, definitions, and/or procedures; *probing questions* require students to explain, elaborate, or clarify their thinking and communicate the steps in solution approaches or the completion of a task; *making the mathematics visible* questions ask students to discuss mathematical structures and make connections among mathematical ideas and relationships; and *reflection* questions encourage students to reflect on their reasoning and actions, and provide valid arguments for their work. Similarly to Walsh and Sattes (2005), we noticed that, although the teachers asked questions perceived to require higher order thinking, teachers accepted student answers at a lower cognitive level, such as recalling facts, definitions, and/or procedures. Our poster presents three examples of how students' answers and/or explanations to higher order questions accepted by their teachers as appropriate could hinder student conceptual understanding of the content.

We concluded that experiences and support we provided thus far in our PD were not sufficient for teachers to realize that asking a higher level question is not enough – they must also demand an appropriate higher cognitive level response from their students in order to gauge their students' conceptual understanding. Future PD will address this disconnection.

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OPPORTUNITES AND CHALLENGES AS TEACHERS PROBLEM-SOLVE

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Current reform practices in mathematics hold problem solving processes at the core of teaching and learning. However, bridging the gap between intentions and implementation of problem solving methods in the classroom remains a challenge. Enhancing teachers' content and pedagogical knowledge through professional development (PD) opportunities is critical to on-going teacher excellence, as those experiences can provide teachers with valuable dialogue that develops personal mathematical understandings and restructures instructional beliefs and practices (Crespo & Featherstone, 2006). Providing teachers with rich problem solving experiences has the potential to "help teachers reflect on their identities as mathematics learners and to understand their role in the development of their students' mathematical identities" (McCulloch, Marshall, DeCuir-Gunby, & Caldwell, 2013, p.1).

The purpose of this study was to immerse middle-school teachers in the problem solving process and examine their experiences as learners and teachers. The analysis addresses the following questions: (1) What opportunities and challenges do teachers experience as problem solvers and (2) what are their attitudes and beliefs about problem solving and how do they develop, if at all? Researchers investigated fifteen middle-school teachers who participated in a grant-based summer PD program about problem solving. The study analyzed data collected from (1) teachers' reflections during the PD, (2) artifacts from the PD, (3) researchers' field notes taken during the PD, (4) the PD facilitators' reflections, (5) videos of teachers' model-lessons during the PD, and (6) observations from teachers' classrooms. Data was analyzed using a phenomenological approach as it aligned with the study's intention of examining a group's shared experience: middle-school teachers' interactions with problem solving.

This poster focuses on an analysis of teachers' reflections in the context of the other data sources. Findings show that teachers experienced a range of opportunities and challenges in their interaction with problem solving in the PD. Teachers' comments fell under two overarching themes: their experiences as problem solvers and their potential implementations of problem solving in their own classrooms. Teachers revealed a variety of orientations towards problem solving, many of which changed over the course of the PD and were heavily influenced by teachers' knowledge of and beliefs about their students. Even as teachers worked through problems themselves as learners (employing a "learner lens"), they did not abandon their "teaching lens," which revealed contrasting perspectives about implementing such problems in their classrooms.

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TEACHERS' SELF-FACILITATED INQUIRY IN NOTICING CHILDREN'S THINKING

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School-based collaborative inquiry groups are increasingly recommended as a key feature of teacher professional development as they allow teachers to enhance their expertise through the investigation of and reflection on their own practice. In this study we explored how teachers in grades 3-5 worked together in self-facilitated, school-based collaborative inquiry (CI) groups to develop the practice of noticing children's mathematical thinking (Jacobs, Lamb, & Philipp, 2010) of fractions. The teachers were participants in a larger study of professional development and were expected to complete a total of 12 CI sessions over three years. Prior to each session, teachers posed a fraction story problem to their class and collected students' written work to bring to the session for discussion. Each CI session involved two teachers, and their conversation was supported by an online protocol that prompted teachers to discuss what they noticed about individual students' strategies.

Data was collected from three groups engaged in their ninth CI session during their third year of professional development. The teachers agreed to audio record their discussion using their own devices, copy the written student work, and mail the recording and images to the off-site researchers. To explore their collaboration, we considered two aspects: the form and the content of these discussions. Form was analyzed using Mercer's *ways of talk* (Mercer, 2000). Content was analyzed in terms of teachers' noticing of children's mathematical thinking, which was defined as the interrelated skills of (a) attending to a child's strategy details, (b) interpreting what the child understands, and (c) deciding how to respond based on that understanding.

Building on Mercer's work, we identified two ways teachers exchanged ideas*: show-and-tell* and *back-and-forth*. Most often teachers used *show-and-tell,* that is one teacher presented what he or she noticed about a student's thinking while the partner teacher listened to and acknowledged the speaker. During these exchanges teachers mostly either attended to the details of the strategy or interpreted the understandings, seldom making connections between the two. In contrast, teachers sometimes used a *back-and-forth* exchange in which they both shared details of what they noticed by building on or challenging one another. We found that in these exchanges teachers more explicitly made connections among attending, interpreting, and deciding how to respond. This exploratory study suggests the importance of looking at not only the content but also the forms of interactions to support teachers' engagement in all three skills of noticing. Further study with a larger sample is needed to confirm and extend these findings.

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DEVELOPING LEADERSHIP CAPACITY: THE IMPACT OF AN ELEMENTARY MATHEMATICS SPECIALISTS PROGRAM

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In response to a newly developed Elementary Mathematics Specialist (EMS) certification program in our state, we collaboratively developed six graduate courses to satisfy the EMS program and emphasized 1) mathematics content knowledge, 2) instructional and assessment strategies, and 3) leadership skills. There is a scarcity of evidence detailing the impact of this type of program on teachers and on their students and as such, the guiding research question for this study was: What is the impact of the EMS program on teachers' leadership capacity?

Desimone (2009) proposed a conceptual framework for exploring the impact of professional development on teachers and students. This model illustrates the interactive connections between "the *critical features* of professional development, teacher knowledge and beliefs, classroom practice, and student outcomes" (p. 184). EMS graduate coursework can be considered on-going targeted professional development. Program participants tend to apply what they are learning to their current classrooms and share those experiences during class time. This qualitative study involved 16 elementary teachers who completed pre-post short answer questions and semi-structured interviews that included questions focused on teachers' leadership activities and program aspects that had impacted them most. Initially, researchers independently analyzed data through an open-coding process. Once themes were determined, the data were analyzed again and coded in an effort to determine the impact of the program on teacher leadership capacity.

Results indicate several themes related to the impact of the EMS program on teachers' leadership: *increased empowerment*, *collaboration/networking*, *professional development*, and *increased content and pedagogical knowledge*. Several teachers indicated that they have an increased sense of *empowerment* as a teacher and their confidence in what they do and why they do it was impacted by their work in the EMS program. The importance of their collaboration and new teacher networks seemed to be prevalent in the teachers' beliefs about the impact of the program. Teachers pointed to their increased comfort level to share their newfound knowledge by conducting workshops and facilitating book studies. The teachers in this study overwhelmingly cited an increase in their content and pedagogical knowledge from their participation in the EMS program as a catalyst and support for their leadership activities.

"A growing body of research makes it clear poverty and ethnicity are not the primary causal variables related to student achievement … leadership, teaching and adult actions matter. Adult variables, including the professional practices of teachers and the decision leaders make can be more important than demographic variables" (Reeves, 2006, p. xxiii). The results of this study reveal that the experiences and coursework as part of the elementary mathematics specialist program have an impact on teachers' leadership skills, abilities, and activities.

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PERFORMANCES ON "TEST OF LOGICAL THINKING" AND "COGNITIVE REFLECTION TEST": PILOT RESEARCH WITH MATHEMATICS TEACHERS

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Keywords: Cognition, Metacognition, Teacher Knowledge

The Cognitive Reflection Test (CRT), designed by Frederick (2005), became a popular instrument for exploring the factors that influence decision taking and elementary economic behavior. The CRT consists of three well-known mathematical puzzles on which many people give erroneous answer, using "fast thinking" (Kahneman, 2011). It turns out that persons, who perform poorly on the CRT (no correct answer to any of three puzzles), also take erroneous economic and business decisions.

In this pilot research, 21 in-service Mexican mathematics teachers answered the CRT and the Test of Logical Thinking (TOLT). Spanish version of the CRT was formulated by López Puga (2012), while Acevedo and Oliva Martínez (1995) elaborated and validated the Spanish version of the TOLT. The TOLT has 10 items and measures, beside control of variables, four reasoning modes (proportional, probabilistic, correlational and combinatorial).

Research questions were: Are mathematics teachers better at solving mathematical puzzles than university students? Is there a relationship between the results on the TOLT and the CRT?

Average teachers' score on the CRT was *1.52*. Ten teachers (5 with no correct answer and 5 with 1 correct answer) were below, and 11 teachers (6 with 2 and 5 with 3 correct answers) were above that average score. Percentages of teachers with a particular score were: No correct answer (24%), 1 correct answer (24%), 2 correct answers (28%) and 3 correct answers (24%). Data for 3,428 university students, collected by Frederick (2005), are very similar. The average score is *1.24* and percentages of particular scores are: no correct answer (33%), 1 correct answer (28%), 2 correct answers (23%) and 3 correct answers (17%). So, mathematics teachers involved in this research are only slightly better than university students at solving well-known mathematical puzzles. This result calls for more presence of CRT-like puzzles in mathematics teaching

Teachers' average score on the TOLT was *8.38*. Eight teachers (scores 5, 6 and 7) were below and 13 teachers (scores 9 and 10) were above that average score.

Although the number of teachers was rather small, it seems that someone with good results on the TOLT is likely to have good results on the CRT, too. Namely, the teachers with 10 points on the TOLT got on average *2.11* points on the CRT, while those with 5, 6 and 7 points had average CRT result of only *1.25* points. Nevertheless, the fact that some teachers with 10 points on the TOLT get zero or one point on the CRT indicates that good performances on the CRT (3 correct answers) do not depend only on logical thinking, but on other cognitive variables, too.

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THE USE OF COACH-FACILITATED PROFESSIONAL DEVELOPMENT TO DEEPEN ALGEBRAIC THINKING ACROSS THE TRANSITION GRADES

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Using a coach-facilitated PD model, we used Lesson Study to focus on the teaching and learning of algebra in the transitions grades guided by these research questions: 1) How did the focus on the learning trajectories of algebra across the transition grades help teachers go beyond the borders of their grade level standards and their traditional instructional practices? 2) What were the affordances of using the coach-facilitated and school-based PD model?

Results from the analysis of teacher reflections, video analysis, and lesson study debriefs revealed that the design of the coach-facilitated professional development and Lesson Study offered opportunities for coaches and teachers to mutually develop in their content knowledge and pedagogy while deepening their understanding of students' learning progressions in algebraic thinking. School-based professional development and Lesson Study created a community of practice that provided opportunity for educators not only to co-design lessons but to bring together all their expertise and strength. This collective knowledge yielded more than the sum of their individual knowledge and provided opportunities for school teams to develop collective teaching agency. The coach facilitated PD provided teachers with the immediate trust in the work they were doing because they came with the school-based coach. Knowing that their coach endorsed the instructional practices and the problem-based approach validated our methods and gave them more impetus to put forth effort in their professional learning. Teachers with the various expertise were able to mutually offer and benefit from the vertical articulation and in essence provided a collaborative coaching environment. An added strength of a learning trajectories approach was that it emphasizes why each teacher, at each grade level along the way, had a "critical role to play in each student's mathematical development" (Confrey, 2012, p. 3). As we continue our work, we will continue to explore how coach-facilitated PD and Lesson Study can help sustain the PD efforts and make best practices "stick" with teachers.

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TEACHERS' ACTIVE ENGAGEMENT IN SOCIAL MEDIA: CONCEPTUALIZING MATHEMATICAL PRACTICES WITHIN PINTEREST

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Within schools, how teachers interact shape what and how they learn (Coburn, 2001; Spillane, 1999). Teacher professional communities happen both within and beyond school boundaries, with many exchanging resources or ideas, often informally, across virtual platforms including social media. These professional relationships may inform teachers' informal learning (Schugurensky, 2000) how one conceptualizes teaching, instruction, or their subject matter. This work examines teachers' conceptualization of mathematics instruction as evidenced by social media engagement.

Method
Pinterest, a widely used social media photo based platform, is a personalized virtual scrapbook of resources and interests. Using a sample of 19 Early Career Teachers (ECTs) in the 2014-2015 school year, identified through a larger study on planning and enactment of elementary mathematics, we identify and analyze teaching and mathematical pins. Relying on a process of retroduction (Ragin, 1994), existing frameworks for mathematics instruction, and mathematics education literature, we create a schema of mutually exclusive mathematical categories to represent the primary nature of ECTs' mathematical pins.

Results
Findings indicate ECTs vary in their use of Pinterest, pinning more content related to general teaching than mathematics. Among ECTs' mathematical pins, we find ECTs predominately pin resources related to visual mathematical representations, standard algorithm, and conglomerated content resources. Within Pinterest, ECTs most frequently conceptualize mathematics with less emphasis on problem solving or contextual mathematical tasks. This work contributes to an effort to understand the types of mathematical resources pinned and shared within a virtual network. Future work examines cognitive process rigor of mathematical tasks embedded in the pins.

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TEACHERS' PERCEPTIONS ABOUT *MATH SNACKS* SPANISH MATERIALS

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Mathematics has been a gatekeeper especially for diverse students (Martin, Gholson & Leonard, 2010). The implications of an increasingly diverse school population calls for the development of culturally and linguistically responsive materials, including support materials for mathematics in Spanish. Teachers of mathematics in the United States have limited resources available to them in Spanish to support bridging mathematics content and practices for English Learners (Moschkovich, 2013). The CCSS-M requires deep conceptual understanding in order for students to be successful in their formal K-12 education. Students, whose home language is other than English, need support and opportunities to success in mathematics classrooms. According to Wright (2010), a practice that could support these students is to learn new conceptual ideas in one's native language and then in English. Also, literature indicates the need of rich contexts to support learning content reducing cognitive demands due to language (Khisty, 1995). Educational games/animations could provide context for mathematics learning.

Math Snacks has created computer-based materials in Spanish to support mathematics learning. These free products, created initially in English, have been implemented in schools showing positive impact in students' learning. Due to the growing number of Spanish speakers in the school system, Spanish materials were created. Twenty mathematics teachers (3rd -7th grade), who had Spanish speakers in their classrooms, were invited to participate in a professional development session to learn about these materials. After the session, teachers were required to utilize Spanish materials (one animation and one game) in their classrooms. Then, teachers completed a survey regarding their experiences during the implementation.

In this poster we will share teachers' perceptions of these materials in their classrooms. Content analysis methodologies were utilized. Six themes emerged from this analysis. We found that teachers perceived that the materials (1) were helpful in understanding the mathematics concepts, (2) supported collaboration and communication among students, (3) enhanced Spanish speaker students engagement, (4) supported language development, (5) provided a visual representation that enhanced understanding, and (6) were played at home with family members. Our poster will include quotes from teachers' responses and more about data analysis.

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USING TECHNOLOGY TO DEVELOP SHARED KNOWLEDGE IN AND ACROSS GRADE LEVEL TEAMS

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A significant problem in mathematics education is the separation between what researchers know about the learning and teaching of mathematics and the instructional practices used by teachers. We report on our efforts to tap existing grade level team (GLT) structures to engage teachers in developing topic-specific knowledge for teaching mathematics across multiple schools, blurring the lines between researcher and practitioner. We see the GLT as an underutilized resource; while GLTs are currently present at many schools, they often do not attend to problems of instructional practice (Vescio, Ross, & Adams, 2008), and even if knowledge is developed, GLTs are isolated and lack structures for sharing across schools. Our project provides structure in which teachers use technological tools to collaborate with peers to investigate common problems of instruction.

In the project, we examined the implementation of two prototype modules with 17 third grade teachers in four elementary schools. For each module, each teacher administered the same equal sharing task in her class. The student strategies for these tasks were recorded using a screencast application on tablet devices. Then each teacher selected two or three recordings to share with the GLT, using a secure website to post and annotate the selected recordings. At face-to-face meetings, facilitators led discussions of selected screencasts, and each school collectively selected one or two annotated screencasts to share with all teachers in the project.

The initial analysis of the data revealed that teachers within the same school showed substantial variation in what they noticed and inferred when examining the same student work. Student screencasts stimulated rich discussions about student thinking within GLTs, and over time teachers showed increased attention to the details of students' solution strategies. Perhaps more importantly, this process resulted in set of rich examples of student thinking tagged with teachers' insights about the important mathematical ideas that were revealed at particular points in each recording. These artifacts can be preserved, shared with other GLTs, and refined in future iterations of the modules. In this way, teachers are not positioned merely as consumers of knowledge; they actively refine and update it as they work through the modules. We see this project as a first step towards the creation of a robust set of learning experiences for GLTs as well as the development of a repository of artifacts of teaching that can be improved by practitioners over time (Morris & Hiebert, 2011).

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HOW MIDDLE LEVEL INSERVICE MATHEMATICS AND SCIENCE TEACHERS VISUALIZE MOTION, SPACE, AND SCALE

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Keywords: Geometry and Geometrical and Spatial Thinking, Teacher Education-Inservice/Professional Development

This research focused on the alternative understandings that middle level teachers hold about spatial-scientific, lunar-related concepts. These alternative understandings were influenced by teachers' sense of scale and proportion, 3-Dimensional (3D) visualization skills, and mental rotation abilities. The term "alternative understandings" refers to the explanations learners construct that are different than the accepted scientific explanation (Danaia & McKinnon, 2008). The questions guiding this research were: (a) What spatial sense do middle level teachers possess with regard to mental rotation and visualization of 3D systems? (b) What alternative conceptions do middle level teachers hold concerning the scale and proportion of the Earth/Moon/Sun system and the cause of lunar phases? Results from investigating these questions can aid professional development educators in constructing the appropriate experiences and instruction to help teachers overcome their alternative beliefs and formulate accurate understandings of spatial-scientific concepts. Creating these instructional opportunities for teachers will strengthen their spatial-scientific identity and will help to prevent them from passing along their alternative conceptions to their current and future students.

This research study included 24 middle level mathematics and science teachers participating in a Professional Development (PD) workshop aimed to enhance teachers' spatial and content knowledge and to create a project-based instructional experience with an Earth-space unit for teachers as learners. For this study, we used pre-tests to determine teachers' pre-understandings of lunar-related concepts and spatial skills. Survey assessments included a Lunar Phases Concept Inventory (LPCI; Lindell & Olsen, 2002) and the Purdue Spatial Visualization-Rotation Test (PSVT-Rot) which assisted with diagnosing the level of teachers' mental rotation reasoning (Bodner & Guay, 1997).

Individual scores ranged from 20 – 95% correct on the overall LPCI and from 10 – 90% on the PSVT-Rot. Item analysis of LPCI results showed teachers displaying alternative conceptions concerning the cause of lunar phases. Alternative conceptions included a "blocking" notion, the Sun's shadow explanation, and the Earth's shadow explanation. One-third of the teachers showed a scientifically accurate explanation (phase due to the Moon's position relative to the Earth), 41.7% held the Earth's shadow explanation, 12.5% displayed the Sun's shadow explanation, and 12.5% chose an object-blocking notion. In terms of the LPCI spatial domains, teachers performed best on test items concerning periodicity (orbital, phases, etc.) with an average percent correct of 69.2%. The most difficult items for teachers were those items concerning cardinal direction with an average percent correct of 36.7%. Analysis of the LPCI test items displayed that not only did teachers not understand the cause of lunar phases, but they also had limited understanding of the apparent daily lunar motion (as a result of the Earth's spinning on its axis) where the Moon rises in the East and sets in the West.

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THE INFLUENCE OF MINDSET ON AN ELEMENTARY TEACHER'S ENACTMENTS OF MATHEMATICS PROFESSIONAL DEVELOPMENT

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Introduction and Research Question

An essential border exists in the translation of a teacher's professional development experiences into classroom practices. Although the influence of teachers' beliefs on their practices has been widely examined, there is little empirical research into the role of mindset as a mediator of classroom practice (Rattan, Good, & Dweck, 2012). Therefore, this study examined the research question: How do characteristics of the growth mindset influence an elementary mathematics teacher's enactments of her professional development experiences, if at all?

Theoretical Framework

Dweck and Leggett (1988) described a framework of implicit conceptions of the nature of ability that have come to be known as mindset. This research base and its connections to self-regulation theory (Burnette, Boyle, VanEpps, Pollack, & Finkel, 2013) formed the theoretical framework in which this study was founded. Additionally, contextual elements of the change environment posited in The Interconnected Model of Teacher Professional Growth (Clarke & Hollingsworth, 2002) offered a lens through which to consider the operationalization of mindset.

Methodology

To examine the role of mindset in professional development, I used a holistic, exploratory case study methodology (Yin, 2014) to examine the critical case of a teacher with strong characteristics of the growth mindset whose beliefs and practices regarding the teaching of mathematics were in transition. I performed a simple time series analysis (Yin, 2014) across four stages of data collection: participant selection, baseline classroom observations, a demonstration lesson, and the enactment of the demonstration lesson in the classroom.

Results

The preliminary results of this study indicated that a strong alignment exists between the tenets of self-regulation theory and operationalized characteristics of the growth mindset. Additionally, mindset appeared to act as a pathway for both enactment and reflection in the change networks predicted by the Interconnected Model of Teacher Professional Growth.

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Chapter 6

Mathematical Knowledge for Teaching

Research Reports

Brief Research Reports

Posters

COHERENCY OF A TEACHER'S PROPORTIONAL REASONING KNOWLEDGE IN AND OUT OF THE CLASSROOM

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In this exploratory study we considered how one teacher's understanding of proportional reasoning related to his teaching. We used Epistemic Network Analysis to consider the teachers' knowledge organization and connections between knowledge resources as a way to make sense of his understanding. Then, we examined how his understanding was reflected in his teaching. From our analysis, we found key aspects of the teacher's proportional reasoning from two interviews related to his lesson. We concluded that this teacher's knowledge organization influenced his ability to teach lessons in coherent ways. This has implications for professional development.

Keywords: Teacher Knowledge, Rational Numbers, Mathematical Knowledge for Teaching

Purpose and Background

Research in mathematics education is beginning to show that the ways in which teachers understand mathematics matter (e.g., Baumert et al., 2010; Hill, Rowan, & Ball, 2005). How teachers understand their content is associated with what they are able to do when they teach (Ma, 1999; Silverman & Thompson, 2008; Thompson, Carlson & Silverman, 2007; Thompson, 2015; Wilson & Berne, 1999), and how that content knowledge is organized shapes their ability to teach coherently (Thompson et al., 2007). Drawing from cognitive psychology, we could frame the mathematics education findings as contributing to research on the development of expertise. Thus, in this paper, we cross the borders between cognitive psychology and mathematics education research. We also cross the borders between research on how teachers understand proportional reasoning and what that knowledge means to their practice.

We have chosen to focus on proportional reasoning because it is a salient domain of middle school mathematics and teachers are expected to support students in developing deep understanding in this domain. Proportional reasoning has become a prominent area, being treated as its own content domain in the Common Core State Standards for Mathematics (National Governors Association & Council of Chief State School Officers, 2010). Interestingly, despite the importance and its role as foundational knowledge for advanced topics in mathematics (Lamon, 2007; Lobato & Ellis, 2010), little is known about how teachers understand proportional reasoning (Lamon, 2007). The limited research suggests that, like students, teachers struggle with proportional reasoning (e.g., Akar, 2010; Harel & Behr, 1995; Orrill & Brown, 2012; Orrill & Kittleson, 2015; Post, Harel, Behr, & Lesh, 1988; Riley, 2010).

Teacher struggle may be related to the dominance of rote algorithms, such as cross multiplication, to solve proportion tasks instead of focusing on the multiplicative nature of proportional relationships (Berk, Taber, Gorowara & Poetzl, 2009; Lobato, Orrill, Druken, & Jacobson, 2011; Modestou & Gagatsis, 2010; Orrill & Burke, 2013). Studies have also suggested that teachers hold naïve conceptions about proportions (Canada, Gilbert, & Adolphson, 2008; Lobato et al., 2011). For example, Canada et al. (2008) found that only 28 pre-service teachers out of a sample of 75 were able to reasonably interpret a unit rate (e.g., amount per dollar) as useful for determining which package was a better buy when comparing two different size packages of ice cream. Teachers'

proportional reasoning should include the understanding that a ratio represents a multiplicative comparison and not an additive comparison (Lamon, 2007; Lobato & Ellis, 2010; Sowder, Philipp, Armstrong, & Schappelle, 1998). This is a crucial understanding, as teachers need to be able to discern whether students are using additive or multiplicative reasoning (Sowder et al., 1998).

In this exploratory study, we examined how one teacher's understanding of proportional reasoning is related to his enactment of a lesson associated with proportional reasoning.

Framework

Knowledge in Pieces and Expertise

We rely on the knowledge in pieces theory (KiP; diSessa, 2006) to make sense of teachers' understandings. KiP asserts that our understandings are organized as fine-grained knowledge resources that can be drawn upon in a variety of combinations for a given situation. Learning occurs through perturbations that promote the development of new resources and the refinement of existing ones. Learning also includes developing connections between knowledge resources so they can be more readily drawn upon on in a variety of situations. KiP offers a unique lens for exploring the development of expertise, which is dependent, in part, on the extent of the *coherency of knowledge* (Orrill & Burke, 2013). We see coherency as meaning multiple knowledge resources connected in robust ways allowing for *in situ* access. Coherence, combined with a robust set of knowledge resources, allows teachers to deal with complex situations in more efficient ways. This is consistent with cognitive psychology research on expertise that shows that experts have both more knowledge than novices in their area of expertise and that their knowledge is organized differently than that of novices (e.g., Bédard & Chi, 1992). We also see our idea of coherency among knowledge resources as being consistent with Ma's (1999) concept of profound understandings of fundamental mathematics. We hypothesize that as a teacher's knowledge becomes more coherent (i.e., more knowledge resources are inter-connected), the teacher will be more flexible in supporting student learning of mathematics.

Epistemic Network

Epistemic Network Analysis (ENA; Shaffer et al., 2009) provides an analytical lens for identifying the connections between resources that a participant uses. We use ENA to focus on the connections participants make between knowledge resources, which are predefined using a coding scheme. Analysis is binary—thus each utterance either does or does not exhibit the presence of each pre-specified knowledge resource. ENA visually shows the relationships between the knowledge resources present (See Figure 1 for an example of a representation of an ENA equiload graph) in that it draws lines between those resources that co-occurred in a given utterance. We have interpreted these connections as being a way of determining connections between the resources. In this way, ENA provides a new alternative for measuring complex thinking and problem solving (Shaffer et al., 2009).
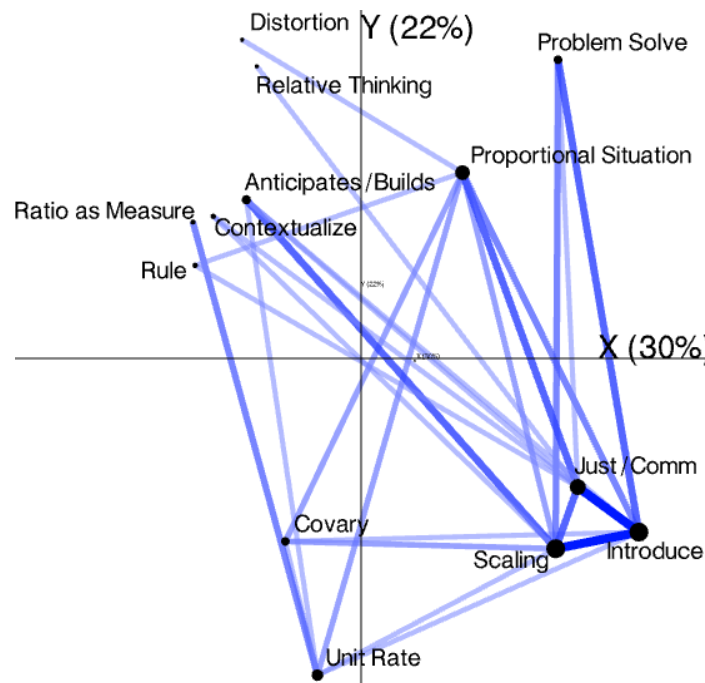
Figure 1. Matt's ENA equiload graph.

Methods and Data Sources

Data were collected as part of a larger project focused on teachers' proportional reasoning. Matt (pseudonym) was a certified 7th grade teacher with seven years of teaching experience. His classroom was in a K-8 school in an urban district in the United States.

Data were collected from two interviews and one classroom lesson. One interview relied on a paper-based protocol with 23 think-aloud prompts. Matt completed the protocol using a LiveScribe pen that recorded his voice as well as his written work. The second interview was a 90-minute videotaped clinical interview that included 18 items. The interview tasks were intended to elicit different aspects of reasoning about proportional relationships. The context of many items was in the work of teachers, asking participants to make sense of reasoning or work created by others. Both interviews were transcribed verbatim. The third data source was a video recording of Matt's 7th grade class during a single 90-minute lesson. We used two cameras: one focused on the primary speaker(s) and one directed on written work, when students were working in multiple groups a camera followed the teacher to document all teacher-student interactions.

We analyzed Matt's response to each interview task using a predefined code set for proportional reasoning knowledge resources (Table 1 shows all knowledge resources present in Matt's interview). We then created an ENA equiload graph for Matt's responses to all interview prompts (see Figure 1). In an ENA equiload graph, knowledge resources serve as vertices and the lines indicate those knowledge resources that co-occurred in the same response (see Table 1). The line thickness indicates the relative frequency of each co-occurrence across the interviews.

We relied on the same coding scheme to identify knowledge resources present in the classroom lesson. For the lesson, we coded knowledge resources present in each turn of Matt's talk. We then relied on qualitative analysis to compare Matt's understanding as portrayed in the ENA equiload graph to his understanding presented in the enacted lesson. (Note that we were unable to conduct an ENA-based analysis of the single class session due to mathematical limitations of ENA.)

Table 1: Knowledge resources present in Matt's interviews

Code	Description
Covariance	Recognizes that as one quantity varies in a rational number the other quantity must covary to maintain a constant relationship.
Ratio as Measure	Identifies an abstractable quantity created from the combination of the two quantities (e.g., flavor or speed) or discusses the effect of changing one attribute in terms of its effect on the ratio.
Unit Rate	Uses the relationship between the two quantities to develop sharing-like relationships such as amount-per-one or amount-per-x.
Scaling Up/Down	Uses multiplication to scale both quantities to get from one ratio in an equivalence class to another.
Relative Thinking	Demonstrates multiplicative reasoning about the change in a quantity relative to itself or another quantity. This includes re-norming.
Proportional Situation	Recognizes that a situation involves proportional reasoning.
Distortion	Describes "that things need to not get distorted" in similarity contexts.
Rules	Shares a verbal or written rule (e.g., blue = red + 2) stated in a way that conveys a generalizable relationship.
Anticipates or Builds from Others' Thinking	Talks about or builds from the mathematical thinking of others.
Contextualizing	Introduces a context for a relationship or anticipates impact of a context for students' reasoning.
Problem solving with Representation	Uses representation to support reasoning about the problem.
Justify or Communicate with Rep.	Justify or clarify a position already developed using the representation.
Introduce New Representation	Introduces a representation not implied or requested by task.

Results

In our analysis we found key aspects of Matt's proportional reasoning from interviews related to his lesson. Because of space limitations, our findings focus on Matt's knowledge organization and the ways it was reflected in his teaching.

Matt's Knowledge Organization

Matt's ENA equiload graph (Figure 1) showed strong connections between several pairs of knowledge resources. Strong connections indicate that Matt used those resources together in addressing particular tasks. We assert that co-occurrences of knowledge resources serve as indicators that the knowledge is linked in some way for the participant. Thus, we infer that connected knowledge resources indicate ideas that are conceptually tied together for the participant.

An example emphasizing strong connections is from the paper-based interview, in which Matt was asked to determine which was the better buy: a 16 oz box of Bites that costs $3.36 or a 12 oz box of Bits that costs $2.64. To solve this, he introduced a ratio table as a new representation (Figure 2) and used it to find a common multiple (link between *Introduce New Representation - Scaling Up/Down)*. Then, Matt communicated the idea that using ratio tables helps in "keeping it balanced and proportioned" (*Introduce New Representation – Justify or Communicate with Representation)*. Further, Matt emphasized that a ratio table helps to show how the proportion is maintained

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

(*Introduce New Representation - Proportional Situation*). In his solution, we see the 1s used by Matt to add 3.36 twice to get to 10.08 in his ratio table as opposed to him using multiplicative reasoning to solve the problem.

Matt was not limited to his own ratio table reasoning. He offered that students might try to solve Bits and Bites with long division and suggested that scaling is more efficient for them (*Scaling Up/Down -Anticipates or Builds from Others' Thinking*). He also explained that this task could be solved using unit rate (*Unit Rate*) when he said: "I could've also got down to unit rate—how much per ounce." Matt had access to this knowledge resource, but chose not to use it often in solving tasks. He explicitly talked about how unit rate can be difficult for students to use because of division. We assert that this special attention to *Unit Rate* is consistent with Matt's ENA equiload graph, which shows *Unit Rate* only weakly linked to *Scaling Up/Down* for Matt. This suggests Matt considers *Unit Rate* as an approach for only specific tasks.

Figure 2. Matt's written work for the Bits and Bites problem.

Matt's Classroom

Matt's pattern of knowledge resource use was echoed in his teaching. Matt described the concept of proportion in his lesson the same way he had in his interview. He emphasized the importance of keeping the equivalence by *adding* the same amounts. This way of describing how to maintain a proportion suggested additive reasoning rather than multiplicative reasoning.

In Matt's lesson he posed a task similar to Bits and Bites. Students were asked to compare two deals on pencils. Matt set a clear objective "students will be able to determine the better deal using proportional reasoning and ratio table". This parallels the strong connection between *Introduce New Representation* and *Scaling Up/Down* consistent with his own problem-solving approach.

In the whole class discussion, Matt emphasized the use of ratio tables to solve the pencils problem by recognizing that the situation involved proportional reasoning. This was consistent with the approach we saw him take in Bits and Bites. However, Matt was able to make sense of students' different representations and solution strategies as well as how they communicated their solutions. For example, one student suggested another strategy using the common multiple of 120. Matt took a paper and pencil and tried it out himself by representing the pencil deals in ratio tables and scaling up both packs to 120 to solve the problem (this parallels his pattern of connecting *Introduce New Representation - Scaling Up/Down*). Matt also used students' work in his instruction, for example he used some students' work to discuss scaling using a ratio table versus long division (*Scaling Up/Down -Anticipates or Builds from Others' Thinking*).

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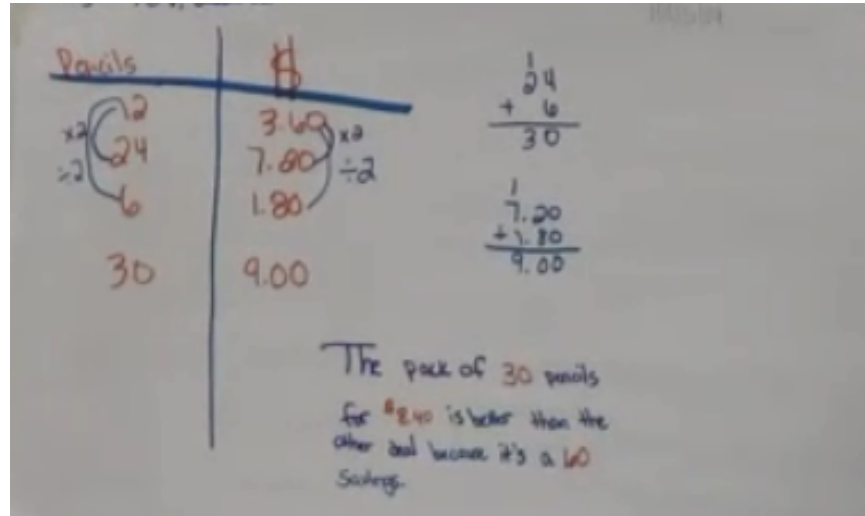
Figure 3. Student's work – using one ratio table to scale the 12-pack of pencils.

Consistent with his approach in the interviews, Matt did not rely on *Unit Rate* in his teaching. As students worked in small groups, they used ratio tables in different ways. Some built one ratio table (Figure 3) to scale only one pack of pencils and some used two ratio tables to scale both packs (Figure 4). And when students did attempt unit rate, Matt tried to steer them away. For instance, Matt approached one student who tried to use unit rate and asked if "there is a reason to get to one?" He then guided the student to think about getting to 60 instead. Further, when one group presented the solution using unit rate to the class (Figure 4), Matt said that finding unit rate is an efficient strategy "if the question asks you for the price per pencil". Matt seemed to prefer *Scaling Up/Down* using ratio tables in his teaching as well as in his problem solving. This was consistent with his interviews in which he mentioned that he worried about students relying on unit rate because it required strong division skills that his students often did not have.
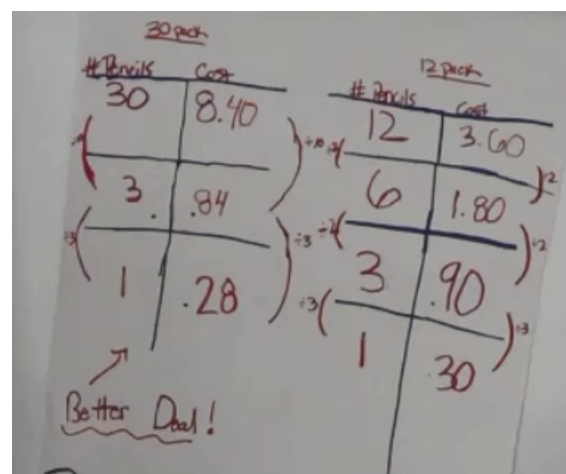
Figure 4. Student's work – using ratio tables to scale both packs of pencils to 1.

Conclusions

The knowledge resources and connections Matt used to solve problems about proportions were consistent with the ways in which Matt used the same resources and connections to guide his lesson. This aligns with the assertion that a teacher's relative level of coherence will shape their ability to teach lessons in coherent ways (Thompson, et al., 2007). We note that Matt's views of *Unit Rate*, his

definition of proportion, and his reliance on ratio tables were factors that shaped his interaction with his students. Additionally, his use of additive language as opposed to multiplicative language when describing a proportion was used with his students and in his interviews. Matt understood unit rate and had taught his students unit rate; but he reinforced other approaches for problem solving in his classroom. One important take-away from this study is that despite showing evidence of having particular knowledge resources, Matt did not draw on them in his teaching. Thus, demonstrating particular measurable knowledge may not be a sufficient measure for teacher knowledge as they may have access to an array of knowledge that is not used in their classroom teaching. This raises questions about how best to measure or research teacher knowledge as it relates to opportunities for student learning.

Scholarly Significance

This study explores what it means for a teacher to have coherent knowledge. While many researchers assert that mathematics teachers need a deep understanding of the mathematics they teach (e.g. Ma, 1999; Thompson et al., 2007), little work has been done to understand what this means and what difference it makes to students' opportunities to learn. This study contributes to research on teacher understanding by stepping away from focusing on quantifying the amount of knowledge a teacher exhibits to instead focus on how the organization of that knowledge was used to drive the enactment of a single lesson. Studying how teachers understand and use the mathematics they teach has practical implications on the design of teacher preparation and professional development programs.

The teacher, in our study, demonstrated a variety of knowledge resources about proportional reasoning and strong connections between some of those resources. Based on our experience watching teachers implement proportional reasoning lessons, we hypothesize that not all teachers have the same kinds of connections between their own understanding and their teaching. We will be conducting additional analyses to determine whether the similarities between personal knowledge and enacted knowledge are maintained.

Acknowledgments

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HOW TEACHERS IN CHINA AND U.S. RESPOND TO STUDENT ERRORS IN SOLVING QUADRATIC EQUATIONS

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To improve mathematics achievement, students' errors should be treated as a source to stimulate their understanding of the conceptual and procedural basis of their errors. The study investigated 20 Chinese and 20 U.S. high school teachers' interpretations and responses to a student's errors in solving a quadratic equation. The teachers' responses were analyzed quantitatively and qualitatively. Analysis results show that the Chinese teachers provided more negative evaluations toward students' errors and identified more students' errors than the U.S. teachers did. Responding to students' errors, the two groups of teachers highlighted conceptual explanations targeting students' mistakes. The U.S. teachers were more likely to provide general knowledge guidance while the Chinese teachers tended to go back to basic knowledge.

Keywords: Algebra and Algebraic Thinking, Teacher Knowledge, High School Education

Introduction

Algebra has long been regarded as a critical bridge to high school mathematics. National Council of Teachers of Mathematics (NCTM, 2000) highlighted the importance of algebra to all students. The content in school algebra mainly covers two major themes: equations and functions (NCTM, 2000; Drijvers, Goddijn, & Kindt, 2010). Quadratic equations take on an important role in the high school Algebra I curriculum. From straight lines to curves it is an essential transition that requires students' conceptual understanding and computational proficiency. Prior research reveals that many students are challenged with solving quadratic equations (Vaiyavutjamai, Ellerton, & Clements, 2005; Zaslavsky, 1997). For example, Didiş, Baş, and Erbaş (2011) found that 10th graders lacked conceptual understanding of the null factor law in solving quadratic equations. Additionally, when asked to solve a quadratic equation in the form $(x - a)(x - b) = 0$, many students who correctly found the solutions mistakenly held the concept that x in $(x - a)$ was equal to a, and simultaneously the x in $(x - b)$ was equal to b.

Helping students develop mathematical understanding, NCTM (2000) indicated that teachers should recognize and respond to students' errors appropriately. Students who figured out the misunderstandings under their mistakes can learn what they did not know and what they thought they knew. Rather than avoiding discussing students' errors, teachers are being called to use such errors as catalyst for stimulating reflection and exploration (Ashlock, 2006; Borasi, 1994). Taking good advantage of students' errors initiates the path of developing students' understanding of the conceptual and procedural basis of their errors.

The 2011 Trends in International Mathematics and Science Study (TIMSS) reported that both Chinese 4th graders and 8th graders outperformed their U.S. counterparts in mathematics remarkably (Provasnik et al., 2012). Teachers' knowledge has a long history of being identified as an essential factor that affects students' achievement (Ma, 1999; Hill, Rowan, & Ball, 2005). In this study, we investigated Chinese and U.S. high school algebra teachers' knowledge of interpreting and responding to students' errors in solving quadratic equations. The research questions that guided are: (1) How do Chinese and U.S. teachers interpret students' errors in solving quadratic equations?; (2) How do Chinese and U.S. teachers respond to students' errors in solving quadratic equations?; and (3) What are the similarities and differences between Chinese and U.S. teachers' knowledge of interpreting and responding to students' errors?

Theoretical Framework

Students' Conceptual Obstacles in Solving Quadratic Equations
The methods of solving quadratic equations are introduced through factorization, the quadratic formula, and completing the square by using symbolic algorithms. Of these techniques, Didiş et al. (2011) argued that students prefer factorization since it is much faster than the other two methods. This result aligns with that from Eraslan's (2005) study. However, while applying factorization to solve quadratic equations students tended to follow the procedural rules without paying attention to the structure and conceptual meaning (Sönnerhed, 2009). As a result, they tended to make some common errors. Didiş and his colleagues (2011) summarized that when attempting to solve quadratic equations presented in a factored form, students tended to expand the parentheses to get the standard form and then re-factorize. Also, students lacked conceptual understandings of the zero-product property that they used to miss the root $x = 0$ by doing simplification. Additionally, students mistakenly tried to transfer the zero-product property into a new context, for example, to solve $(x - a)(x + b) = 12$, they simply let $x - a = 3$ and $x + b = 4$. Moreover, students used "and" rather than "or" to combine two solutions of a quadratic equation. This finding aligns with those from Ellerton and Clements (2011) that 79% of the 328 preservice middle school teachers in the study did not know that $x^2 + 6 = 0$ has no real-number solutions and many of them thought two x's in $(x - 2)(x + 3) = 0$ hold different values.

Analytical Framework
Tables 1 and 2 present the analytical frameworks utilized in the study. Peng and Luo (2009) developed a framework to analyze teachers' knowledge of students' mathematical errors (see Table 1). They identified four analytical categories for the dimension of phrases of error analysis, namely, identify, interpret, evaluate, and remediate. The levels within each dimension of teacher knowledge of students' mathematical errors are sequential and hierarchical, with progress from one level to the next, and the different levels of analysis support and complement one another by giving a holistic and structured picture of teacher knowledge of students' mathematical errors.

Table 1: Framework for Phrases of Error Analysis (Peng &Luo, 2009)

Dimension	Analytical categorization	Description
Phrases of error analysis	Identify	Knowing the existence of mathematical error
	Interpret	Interpreting the underlying rationality of mathematical error
	Evaluate	Evaluating students' levels of performance according to mathematical error
	Remediate	Presenting teaching strategy to eliminate mathematical error

Referring to the description, the phrase of remediate actually is responding to students' errors. Analyzing preservice teachers' responses to students' errors of proportional reasoning in similar rectangles, Son (2013) developed a framework to analyze teachers' responses to students' mistakes (See Table 2). According to Son (2013), conceptual knowledge is defined as the explicit or implicit understanding of the principles that govern a domain and the interrelations between pieces of knowledge in a domain. Procedural knowledge is defined as the action sequences for solving problems. Form of address signifies whether teachers deliver verbal or non-verbal information for students to hear and see (this kind of responses usually uses the very words "show" or "tell") or for students to do something and to answer questions (this kind of responses usually uses the very words "give" and "ask"). Act of communication barrier refers to the difficulties students and teachers have

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in communicating about students' errors. In the over-generalization category, teachers tend to provide too general an intervention that doesn't directly address students' misunderstandings. By using a Plato-and-the-slave-boy approach, teachers assume that students actually know how to solve the problem correctly but simply have forgotten. Therefore, teachers plan to ask students questions in order to help them to recall the math facts and procedures to solve problems. Returning to the basics means simply leading students to return to underlying principle. This method is regarded as either introducing more problems for students or making students forget the original problem.

Table 2: Analytical Framework for PST's Responses to Students' Mistakes (Son, 2013)

Aspect	Categories
1 Mathematical/ instructional focus	Conceptual vs. procedural
2 Form of address	Show-tell vs. give-ask
3 Pedagogical action(s)	Re-explains, suggests cognitive conflict, probes student thinking, etc.
4 Degree of student error use	Active, intermediate, or rare
5 Act of communication barrier	Over-generalization, a Plato-and-the-slave-boy approach, or a return to the basics

Methods

Twenty Chinese teachers and twenty U.S. teachers who have taught Algebra I before or are currently teaching Algebra I participated in this study. While most of the U.S. teachers hold master degrees most of the Chinese teachers have bachelor degrees. The U.S. teachers took more college level math courses than the Chinese teachers did. The group of Chinese teachers are more experienced than the group of U.S. teachers. In terms of the time that students spent on learning Algebra, it seems that Chinese students do not take as many classes as U.S. students do, but Chinese students spend more than twice of the time that U.S. students spend in doing homework. All the participants are currently teaching at high schools that have characteristics typical of each nation's public schools with respect to the students' ethnic, economic, and cultural diversity.

Figure 1 shows the main task used for this study. This problem was developed by Ellerton and Clements (2011) to test teachers' knowledge of quadratic equations. The participants were asked to analyze and respond to Amy's errors. Their written responses were coded in terms of the analytical frameworks shown in Tables 1 and 2. While analyzing the responses, we expected new categories to come out, which would optimize the existing frameworks. We first coded the participants' evaluations of the student's performance on the math topic, then examined whether the participants discovered all the student's mistakes presented in the question scenario, and finally checked whether the participants identified any underlying mathematical concepts and principles that Amy lacked of.

The participants' responses in helping Amy to correct her errors were analyzed in terms of the five aspects as elaborated in Table 2. The conceptual versus procedural distinction was utilized first, followed by the identification of pedagogical actions. After addressing these global oriented characteristics of the teachers' responses, more detailed analysis was conducted with respect to teaching approaches: form of address, degree of student error use and communication barriers. Each participant's response might be assigned more than one code within each category since more than one teaching strategy might be applied.

Students were asked to solve $(x + 2)(2x + 5) = 0$, then to check their answer. One student,

Amy, wrote the following (line numbers have been added):

$(x + 2)(2x + 5) = 0$	Line 1
$\therefore 2x^2 + 5x + 4x + 10 = 0$	Line2
$\therefore 2x^2 + 9x + 10 = 0$	Line3
$\therefore (2x + 5)(x + 2) = 0$	Line 4
$\therefore (2x + 5) = 0 \text{ and } (x + 2) = 0$	Line 5
$\therefore 2x = -5 \text{ and } x = -2$	Line 6
$\therefore x = -\dfrac{5}{2} \text{ and } x = -2$	Line 7

Check: Put $x = -5/2$ in $(2x + 5)$, and put $x = -2$ in $(x + 2)$.

Thus, when $x = -5/2$ and $x = -2$, $(2x + 5)(x + 2)$ is equal to 0×0 which is equal to

0. Since 0 is on the right-hand side of the original equation, it follows that $x = -5/2$ and $x =$

-2 are the correct solutions.

Figure 1. Main task for the study.

Amy in Figure 1 did not have a clear understanding of the following four pieces of mathematical concepts and principles: (1) Rationale of the method of factorization; (2) Zero product property; (3) Difference between "and" and "or"; and (4) Meaning of solutions for quadratic equations. There were three mistakes from Amy's response.

- Mistake 1: Lines 2, 3, and 4 were unnecessary, since the left-side is already factored in Line 1.
- Mistake 2: In Lines 5 through 7, the word "or", and not "and", should have been used.
- Mistake 3: As for the checking process, each solution should have been substituted into both parentheses in the initial equation.

Results

Identify Students' Errors

Most of the Chinese and the U.S. teachers identified that Amy did some unproductive work. Around half of the Chinese and the U.S. teachers noticed that Amy mistakenly checked the solutions. While 80% of the Chinese teachers recognized Amy used "and" to combine the two solutions only 40% of the U.S. teachers identified it. So, the number of the Chinese teachers who found the second mistake was twice as many as that of the U.S. teachers. In addition, the Chinese teachers identified more of Amy's errors in solving the quadratic equation than the U.S. teachers did. Table 3 presents the distribution of US and Chinese teachers in identifying Amy's mistake.

Table 3: Identifications of Amy's Mistakes on Solving the Quadratic Equation

Categories	Chinese (n=20)	U.S. (n=20)
Mistake 1	14(70%)	14(70%)
Mistake 2	17(85%)	8(40%)
Mistake 3	12(60%)	11(55%)
No mistake	1(5%)	2(10%)
One mistake	3(15%)	6(30%)
Two mistakes	8(40%)	9(45%)
Three mistakes	8(40%)	3(15%)

Interpret Students' Errors

As it is shown in table 4, most of the Chinese and the U.S. teachers did not try to identify which mathematical knowledge that Amy lacked of. Among those teachers who interpreted the mathematical knowledge that Amy needed, the Chinese teachers emphasized the difference between "and" and "or" while the U.S. teachers focused on zero-product property.

Table 4: Interpretations of the Mathematical Knowledge that Amy Needed

Category	Chinese (n=20)	U.S. (n=20)
Rationale of the factoring method	2(10%)	1(5%)
Zero-product property	2(10%)	7(35%)
Differences between "and" and "or"	8(40%)	0(0%)
Meaning of solutions of quadratic equations	0(0%)	3(15%)
No interpretation	10(50%)	12(60%)
One interpretation	9(45%)	5(25%)
Two interpretations	0(0%)	3(15%)
Three interpretations	1(5%)	0(0%)

Evaluate Students' Performance

Evaluating Amy's performance, 90% of the Chinese teachers condemned Amy's performance while 10% gave a half and half comment that suggested Amy did something correct but also made mistakes. No Chinese teacher provided positive evaluations. Different from the Chinese teachers, 30% of the U.S. teachers did not evaluate Amy's overall performance. Almost half of the U.S. teachers gave half and half evaluations, whereas 15% of the teachers were positive about Amy's performance. None of the U.S. teacher gave negative evaluations. Thus, the U.S. teachers seem to be more tolerant than the Chinese teachers in front of students' errors.

Respond to Students' Errors

Around 50% of the Chinese teachers did not specifically address any mistake. 20% of the Chinese teachers demonstrated the first and the third mistakes respectively while 45% of them addressed the second mistake, that is, Amy used "and" to connect the two solutions. About one fourth of the U.S. teachers did not respond to Amy's mistakes (see Table 5). While more than fifty percent of the U.S. teachers addressed the third mistake, around 40% of them addressed the first mistake, the second mistake was neglected by most of them.

We found that the U.S. teachers differed from the Chinese teachers in terms of the number of teachers who addressed Amy's mistakes. The same number of Chinese teachers and U.S. teachers responded to two or three mistakes. In terms of Amy's three mistakes, the Chinese teachers highlighted using "or" but not "and" to connect the two solutions while the U.S. teachers emphasized how to check the solutions. Furthermore, it was found that Chinese teachers tended to address Amy's errors conceptually while the U.S. teachers favored conceptual and procedural explanations equally.

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Table 5: Mistakes Addressed by the Teachers

Category	Chinese (n=20)	U.S. (n=19)	Total (n=39)
Mistake 1	5(25%)	7(36.8%)	12(30.8%)
Mistake 2	9(45%)	3(15.8%)	12(30.8%)
Mistake 3	4(20%)	11(57.9%)	15(38.5%)
No mistake	11(55%)	5(26.3%)	16(41.0%)
One mistake	4(20%)	9(47.4%)	13(33.3%)
Two mistakes	1(5%)	3(15.8%)	4(10.3%)
Three mistakes	4(20%)	2(10.5%)	6(15.4%)

Table 6: Mathematical Knowledge Addressed by the Teachers

Category	Chinese(n=17)	U.S. (n=10)	Total(n=27)
Rationale of the factoring method	1(5.9%)	1(10%)	2(7.4%)
Zero-product property	13(76.5%)	10(100%)	23(85.2%)
Difference between "and" and "or"	7(41.2%)	1(10%)	8(29.6%)
Meaning of solutions of quadratic functions	9(53.0%)	1(10%)	10(37.0%)
One piece of knowledge	6(35.3%)	7(70%)	13(48.2%)
Two pieces of knowledge	9(52.9%)	3(30%)	12(44.4%)
Three pieces of knowledge	2(11.8%)	0(0%)	2(7.4%)

Since some teachers addressed more than one piece of conceptual knowledge, the percentage for each knowledge category in Table 6 was calculated out of 100%. As for the four pieces of mathematical knowledge which have been identified as the reasons for Amy's mistakes, most of the Chinese teachers addressed the zero-product property and around half of the Chinese teachers explained the difference between "and" and "or" and the meaning of solutions of quadratic functions. Only one Chinese teacher explained that the rationale of the factoring method was the zero-product property. Also, one U.S. teacher addressed this rationale. While all the U.S. teachers elaborated the zero-product property, the other three pieces of knowledge were overlooked by them. To conclude, the Chinese teachers outperformed the U.S. teachers in both the variety and the quantity of the addressed conceptual knowledge.

Table 7 summarizes local characteristics of the teachers' responses to Amy's errors. The Chinese teachers all applied "show and tell" strategy while some of them simultaneously asked Amy questions to likely include her in the teaching process. Almost half of the Chinese teachers did not employ Amy's mistakes in their responses while the number of the Chinese teachers who actively addressed Amy's errors and intermediately used Amy's errors are equally distributed. Additionally, the Chinese teachers tended to go back to basic knowledge.

Similar to the Chinese teachers, the U.S. teachers also emphasized "show and tell" approach. In terms of "use of student error," most of the U.S. teachers employed Amy's errors when responding to her. Moreover, they tended to hold the thought that Amy just temporarily forgot the knowledge required to solve the equation and she would perform well if they can ask questions to help her refresh the knowledge and procedures.

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Table 7: Local Characteristics of the Teachers' Responses to Amy's Errors

Aspect	Categories	Chinese(n=20)	U.S.(n=19)
Form of address	1. Show and tell	20(100%)	15(78.9%)
	2. Give and ask	7(35%)	6(31.6%)
Use of student error	1. Active use	4(20%)	7(36.8%)
	2. Intermediate use	5(25%)	4(21.1%)
	3. Rare use	11(55%)	8(42.1%)
With/Without communicative barrier	1. Over-generalization	7(35%)	5(26.3%)
	2. Plato-and-the-slave-boy	1(5%)	4(21.1%)
	3. Return to the basics	8(40%)	5(26.3%)
	4. Specific to student error	7(35%)	6(31.6%)

Discussion and Conclusions

We found that the Chinese teachers identified more of the student's mistakes than the U.S. teachers did and they are less tolerant to the student's mistakes than the U.S. teachers. Most of the teachers identified more than one of Amy's errors but they did not address all the identified errors when responding to Amy. Most of the teachers did not interpret the mathematical knowledge that Amy needed while they identifying her errors but they explained the knowledge that they believe Amy needed when responding to her. The Chinese teachers explained the mathematical knowledge conceptually and most of them demonstrated more than one piece of knowledge. In sum, the Chinese teachers outperformed the U.S. teachers in both the variety and the quantity of the addressed conceptual knowledge.

Interestingly, both the Chinese and the U.S. teachers intended to use teacher-centered pedagogical actions that highlighted "show and tell." More U.S. teachers than Chinese teachers seemed to believe that Amy simply needed help to recall all the needed mathematical knowledge so they actively used Amy's mistakes to deduce her lapses in knowledge about solving quadratic equations. The Chinese tended to go back to basic knowledge, maybe this practice is time-consuming but it is helpful for students to solve related problems correctly in the future. This study has implications to teacher educators and professional developers in both US and China.

First, both the Chinese teachers and the U.S. teachers showed the gap between the errors they identified and the errors they addressed. Since it is the errors that teachers addressed help students learn from their mistakes, teacher educators need to consider instructional interventions to help teachers develop strategies and knowledge to identify and address students' errors consistently. Second, both the Chinese teachers and the U.S. teachers applied "show and tell" approach. Teacher-centered instruction helps students to recall what they learned and provides students opportunities to learn what they missed. However, using teacher-centered instructions in front of students' errors can not probe why and how students made the errors. To learn from errors, students should know why and how they made such errors. Therefore, teacher educators need to train teachers to use multiple ways, including both teacher-centered approaches and student-centered approaches, to respond to students' errors. Third, given that the Chinese teachers outperformed the U.S. teachers in both the variety and the quantity of the addressed conceptual knowledge, professional developers may consider sessions to help in-service teachers in U.S. to construct deep conceptual understandings of certain mathematics topics that students usually are challenged by. Of relevance, teacher educators may also consider to adopt professional development sessions to help preservice teachers become sufficient in dealing with students' errors and in supporting students become mathematically competent. Last but not least, since Chinese teachers are more likely to give negative comments and less likely to employ students' errors when responding to students' errors, professional developments that help Chinese teachers build reasonable attitudes towards students' mistakes and develop flexible strategies to deal with students' errors should be considered.

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TEACHERS' KNOWLEDGE OF CHILDREN'S STRATEGIES FOR EQUAL SHARING FRACTION PROBLEMS

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In this exploratory study, we documented teachers' knowledge of children's mathematical thinking as they engaged in the task of anticipating children's strategies for an equal sharing fraction problem. To elicit an array of knowledge, 18 teachers were deliberately selected with a variety of numbers of years participating in professional development focused on children's mathematical thinking. We characterized the flexibility of teachers' knowledge by the number of valid and distinct strategies teachers anticipated and the degree to which these strategies reflected strategies research has shown children typically use. We argue that this flexibility is necessary for teachers to engage in instructional practices that are responsive to children's mathematical thinking.

Introduction

In this study we explored the knowledge that 18 upper elementary teachers displayed as they engaged in the task of anticipating children's strategies for the following equal sharing fraction problem: *A teacher has 4 pancakes to share equally among 6 children. How much pancake does each child get?* To elicit an array of knowledge, we selected for our analysis teachers with a range of numbers of years of participation in professional development focused on children's mathematical thinking. Our goal was to characterize teachers' knowledge of children's mathematical thinking in terms of its flexibility.

We view *knowledge of children's mathematical thinking* as a distinctive type of knowledge that is essential for teaching in ways that are responsive to children's mathematical thinking (Jacobs & Empson, 2016). We take the perspective that teachers' knowledge is situated (Putnam & Borko, 2000) and thus are interested in the knowledge teachers use as they engage in instructional practices that are part of responsive teaching. We focused specifically on the instructional practice of anticipating strategies, adopting what Stein, Engle, Smith, and Hughes (2008) defined as teachers' consideration of the array of strategies students would be likely to use for a given problem and the potential mathematics that could be learned with those strategies. In this study, we focus on teachers' anticipation of the array of strategies students would be likely to use.

Our analysis of teachers' knowledge was informed by the idea of flexibility. We considered the flexibility of teachers' knowledge in two ways. First, we used Whitacre's (2015) focus on the number of distinct and valid strategies that a teacher anticipated. Second, we considered to what degree did the teacher's anticipated strategies provide evidence of knowledge that was organized primarily around children's ways of making sense of the mathematics or around the teacher's own ways? In other words, to what extent was a teacher able to shift perspective and "think like a child"?

This study represents an exploratory effort to document teachers' knowledge of children's mathematical thinking in the domain of fractions as it was used to anticipate children's strategies for an equal sharing fraction problem. Teaching is inherently knowledge intensive. Documenting the

knowledge teachers use in instructional practices such as anticipating strategies is an essential step in promoting the development of expertise in teaching responsively on the basis of children's mathematical thinking.

Conceptual Framework

Teachers' knowledge of students and content has been conceptualized as a type of pedagogical content knowledge involving knowledge of how students learn a specific topic (Hill, Ball, & Schilling, 2008). It includes knowledge of students' strategies, common misconceptions, and typical developmental paths. Although its importance is widely acknowledged, teachers' knowledge of students and content has been the explicit focus of only a handful of studies (e.g., Bell, Wilson, Higgins, and McCoach, 2010).

In the current study, we focused on teachers' knowledge of students' strategies. We asked teachers to anticipate the range of strategies children might use to solve a story problem involving fractions and then to identify which strategies showed the most basic and advanced understanding. We then analyzed these strategies to explore the flexibility of teachers' knowledge. We were interested in two types of flexibility. The first type of flexibility involved the number of valid and distinct strategies anticipated by a teacher (Whitacre, 2015). We defined strategies as *valid* if a teacher's written response provided evidence about a process a child could use to solve the problem and reach a correct answer. Evidence of a process (vs. only an answer) was important because we wanted to elicit teachers' knowledge of children's thinking and not simply whether or not a teacher was able to solve the problem; if a teacher provided only an answer (e.g., "4/6 = 2/3"), the response was not counted as a valid strategy because we could not tell what reasoning the teacher thought the child might use to arrive at that answer. Strategies were further defined as *distinct* if they involved a qualitative difference in the process a child might use (e.g., partition pancakes into sixths vs. partition pancakes into thirds). Strategies were not considered distinct if they differed only in terms of a superficial feature that did not reflect a different process (e.g., partition into sixths and distribute pieces by numbering the pieces vs. partition into sixths and distribute pieces by drawing lines to connect to sharers).

The second type of flexibility involved the extent to which a teacher's anticipated strategies were organized around categories of strategies that children typically use, as documented in past research (Empson & Levi, 2011). The more flexible a teacher's knowledge is in this sense, the more the teacher is able to "think like a child." The less flexible in this sense, the less a teacher is able to see the problem the way a child might. Instead, teachers with less of this second type of flexibility may tend to anticipate strategies that are organized around instructed procedures and conventions—generalized, all-purpose methods that are introduced in school mathematics. Although the use of instructed procedures and conventions is a valuable goal of school mathematics it does not necessarily signify understanding on the part of the child or link to the child's informal strategies. In particular, when procedures and conventions are explicitly taught before children have had a chance to advance their understanding of fractional quantities, children's use of instructed strategies tends to be error-prone and procedurally-driven. Teachers who anticipate strategies mainly in terms of instructed procedures and conventions may be unaware of the role of children's informal strategies in the advancement of children's understanding of fractions.

These two types of flexibility provide a basis for describing the knowledge that enables teachers to be responsive to children's mathematical thinking: the greater the flexibility of teachers' knowledge, the greater their ability to make sense of and respond to children's ever evolving thinking. We therefore identified in teachers' anticipated strategies the degree to which there was a focus on children's typical strategies and noted how instructed procedures and conventions were used.

Focus on children's typical strategies. Children's strategies for equal sharing have been well documented (Empson & Levi, 2011). When invited to solve problems on the basis of what makes sense to them, children use a variety of informal strategies that are driven by their understanding of partitioning and equal distribution to create fractional quantities. There are three main categories of strategies typically used by children to share 4 pancakes among 6 children, distinguished by the progressive abstraction of fractional quantities, which reflect advances in understanding. In the most basic category, children draw models of some sort – often circles or rectangles – and partition them using familiar or easy-to-make partitions such as halves and fourths without consideration for the number of sharers. Pieces are distributed one by one to sharers. This category is called *non-anticipatory direct modeling*. As children's understanding of fractions develops, they continue to draw models and distribute pieces one-by-one, but they partition the models in a manner linked to the number of sharers. If there are six sharers, a child might begin by partitioning one pancake into sixths or two pancakes each into thirds. In both cases, six pieces are created, one for each sharer. This category is called *emergent anticipatory direct modeling*. In the most advanced category of strategies, children operate on mental models rather than drawn models; in particular they understand and are able to use the relationship that one pancake divided among six children is equivalent to one-sixth pancake per child in a variety of ways to figure how much pancake each child gets. They may use notation to support their thinking and express the strategy as $1/6 + 1/6 + 1/6 + 1/6 = 4/6$, $4 \times 1/6 = 4/6$, or $4 \div 6 = 4/6$. This category is referred to as *anticipatory*. Children may also use various combinations and transitional versions of these three main categories as their understanding develops.

Focus on instructed procedures and conventions. Children may also use instructed procedures and conventions while solving an equal sharing fraction problem. For example, they may "invert and multiply" ($4 \div 6 = 4/1 \times 1/6 = 4/6$), simplify fractions to lowest terms ($4/6 = 2/3$), or use long division. Children may use such procedures and conventions because their understanding of fractions has reached a level of fluency in which operations are routine and do not need to be directly modeled or decomposed into simpler computations, as in the informal strategies above. However, children also use such procedures and conventions because they think it is expected rather than because they understand fraction operations well. A procedure such as long division with a repeating decimal answer, for example, is an inefficient choice of strategy for this problem because of the number of steps involved and the difficulty of relating the answer to a fractional size of cake.

Methods

Data and Participants

The data analyzed for this study were taken from a sample of 71 teacher written responses collected to pilot an assessment of elementary teachers' knowledge of children's mathematical thinking about fractions. The teachers were all teaching grade 3, 4, or 5 at the time and reported a number of years of participation in professional development focused on children's mathematical thinking ranging from no participation to three or more years. In selecting our sample our goal was to maximize to the extent possible the variation in teachers' responses with respect to knowledge of children's mathematical thinking. We selected all teachers who reported 3 or more years of professional development focused on children's mathematical thinking and all teachers who reported no years of such PD. This selection resulted in a sample of 18 teachers (9 from each end of the range).

Task

> *A teacher gave this problem to the class:* A teacher has 4 pancakes to share equally among 6 children. How much pancake does each child get?
> a. *Provide 5 different valid strategies that represent the range of strategies that elementary students might use to solve this problem. Write out or draw each strategy the way a student might.*
> b. *Mark the strategy that indicates the strongest understanding of fractions.*
> c. *Mark the strategy that indicates the most basic understanding of fractions.*

Figure 1. The task posed to teachers in the study.

Teachers were given the task in Figure 1 to elicit their ability to anticipate a range of strategies and differentiate the understandings reflected in those strategies. We recognize that the written nature of the data limited the extent to which we could explore teachers' knowledge of children's mathematical thinking captured by this task. In particular, because we could not ask follow-up questions, we did not have in-depth information about the reasoning behind their choices.

Analysis

We began by coding each teacher's set of anticipated strategies for our two types of flexibility: (a) counting the number of distinct and valid strategies and (b) determining the degree to which the teacher's focus was on children's typical strategies. We also identified the specific strategies anticipated by each teacher to get a sense of his or her range focused on children's typical strategies and examined the details of these strategies for consistency with children's reasoning. We also coded each teacher's ability to differentiate the most and least understanding in the strategies. Finally, we synthesized all of this information to develop profiles that reflected teachers' knowledge and its flexibility. Three of the authors double-coded the data, with discussion and resolution of discrepancies.

Findings and Discussion

Teachers' knowledge of children's mathematical thinking as reflected in their anticipated strategies for equal sharing fell into three groups. We discovered that the number of distinct and valid strategies did not differentiate meaningfully between groups of teachers, and so in our profiles we focused on the degree to which teachers' anticipated strategies were consistent with children's typical strategies, both in terms of which strategies were used and the details within those strategies.

Profile 1: Robust evidence of knowledge of children's mathematical thinking

Each teacher in this group anticipated a set of three or more strategies in which the strategies and the details of the strategies were consistent with children's typical strategies. Teachers were also consistent in their assignment of least and most advanced understanding. Overall, these responses reflected teachers' ability to "think like a child." Figure 2 shows a set of these strategies from one teacher in which all of the strategies are consistent with the kinds of strategies children might use, as documented in research. The first strategy, non-anticipatory direct modeling, involved the use of repeated halving and did not exhaust the pancakes. The next strategy, emergent anticipatory direct modeling, involved the use of drawn models to partition the pancakes and then distribute the pieces one by one (by numbering them). The pancakes were partitioned into sixths — a common partition that children use for six sharers — and the final answer in the strategy was expressed in a manner generally consistent with how children might express it. The third strategy, another example of emergent anticipatory direct modeling, involved partitioning all four drawn pancakes into thirds, also a common partition that children use. The fourth, more advanced transitional anticipatory strategy

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used a drawn model to partition two pancakes each into thirds, then continued with an equation of the form $1/3 \times 2 = 2/3$, showing how a child could use an equation to represent an abstraction of their direct modeling solution.

The flexibility of teachers' knowledge was reflected primarily in this profile by the range of strategies anticipated within the strategy category of emergent anticipatory direct modeling. The majority of teachers' strategies in this group included two if not three different ways to partition and distribute pancakes using drawn models (into sixths, into thirds, and into halves and sixths). Interestingly, each teacher anticipated at most one anticipatory strategy and these tended to be transitional and contain elements of direct modeling strategies, such as the fourth strategy in Figure 2.
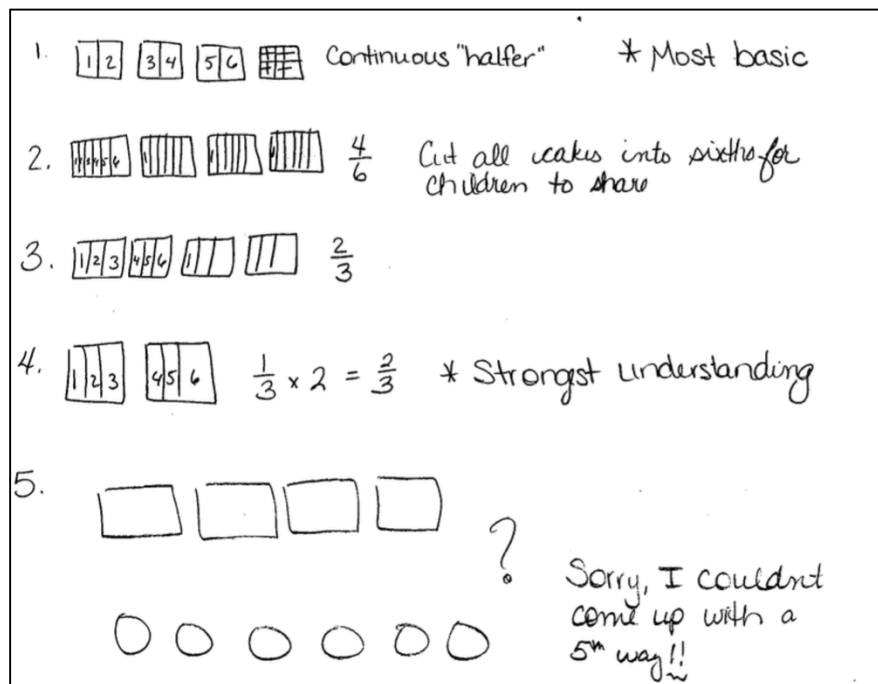

Figure 2. A response with strategies all consistent with children's mathematical thinking.

Profile 2: Limited evidence of knowledge of children's mathematical thinking

Teachers in this group anticipated a smaller number of children's typical strategies – one or two – and the majority of these strategies included details that were inconsistent with the way children might reason in the strategy. For example, virtually all of the direct modeling strategies in this group indicated that children would also simplify fractions to lowest terms or combine fractions with unlike denominators—actions that children who direct model are not likely to perform. The strategy on the left in Figure 3 provides an example that is consistent with what a child might do with a drawn model— the pancakes were cut into sixths and distributed one at a time—yet the simplification of the final answer of 4/6 to 2/3 is inconsistent with what a child using this strategy would likely understand about fractions. Similarly, another teacher anticipated that a child would draw a partition of the first three pancakes into fourths and distribute them, then draw the partition of the last pancake into sixths and distribute, and then combine the final amount, $1/4 + 1/4 + 1/6$, numerically by finding a least common denominator of 12. A child who needed to draw the fractional quantities to make sense of the problem, as in these strategies, would likely not reason mentally to express fractions in lowest terms or combine using least common denominators.

The strategies anticipated by teachers in this group that were in line with children's typical strategies encompassed a smaller range than those anticipated in the first group. For example,

teachers tended to anticipate only partitioning pancakes into sixths in their direct modeling strategies, and the rest of their valid anticipated strategies consisted of instructed procedures and conventions, such as long division or "invert and multiply" for fraction division (e.g., the set of strategies on the right in Figure 3). This combination suggests only a moderate amount of flexibility with respect to thinking like a child. Strategies in this profile were also not accurately assigned to least and most understanding. For example, one teacher indicated that an emergent anticipatory direct modeling strategy (involving sixths) represented the most understanding, whereas an anticipatory strategy that involved adding 1/6 four times represented the least understanding (when children's understanding of fractions actually develops in the opposite direction).
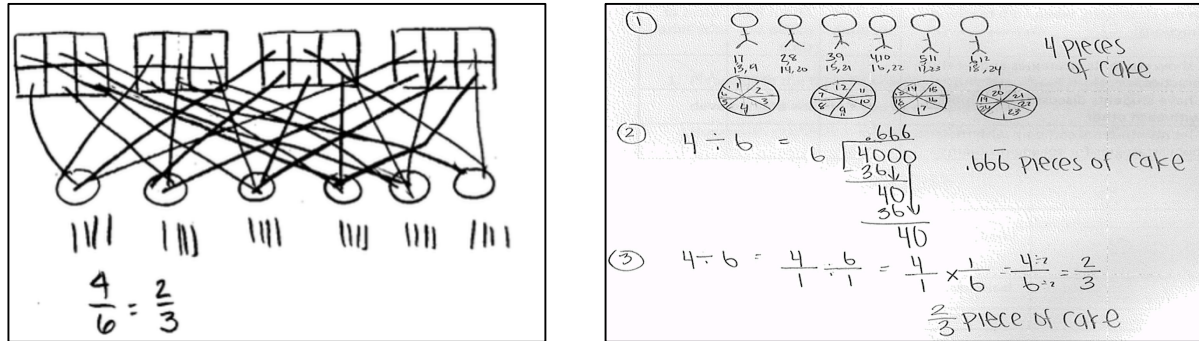

Figure 3. Responses with a mix of children's typical strategies and instructed procedures and conventions.

Profile 3: Lack of evidence of knowledge of children's mathematical thinking

The strategies anticipated by teachers in this group did not include any strategies that fully represented the reasoning children typically use in their strategies. Although each teacher thought that children would use drawn models to solve the problem, the details of how children would use these representations were inconsistent with how children would reason. For example, the strategy in Figure 4 on the left shows the pancakes partitioned into sixths, with the sixths distributed four at a time, an atypical process for a child relying on drawn models to solve this problem. Similarly, the strategy on the right shows the partitions distributed two at a time, also an atypical process for a child relaying on this drawn model. This distribution may suggests that the teacher solved the problem first and then used the answer as the basis for completing the drawn model, in contrast to distributing the pieces one by one, as a child who took the trouble to draw each pancake and partition it into sixths would likely do while solving the problem. Although these teachers had the intuition that children would use drawn models to solve this problem, there was a lack of evidence of knowledge of the typical informal strategies that children would use.
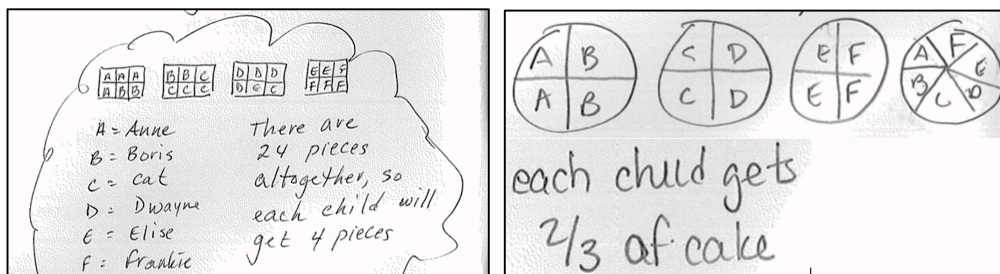
Figure 4. Strategies with details that are inconsistent with children's reasoning.

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

Similarly to teachers in the second group, teachers in this group did not accurately assign strategies to least and most understanding. They also included some instructed procedures and conventions, though not as consistently as teachers in the middle group. Thus, teachers in this group appeared to have the least flexible knowledge with respect to "thinking like a child."

Conclusions

Imagining how a child would solve an equal sharing fraction problem such as the one in this study requires a teacher to see the problem through a child's eyes. It can be difficult to suspend one's own knowledge of mathematics to engage in thinking the way a child might. Yet teachers must do this to teach in ways that are responsive to children's thinking – they need *knowledge of children's mathematical thinking* to anticipate the strategies children might use, notice children's mathematical thinking, and respond appropriately with supporting and extending moves (Jacobs & Empson, 2016).

The majority of teachers in this exploratory study anticipated at least one strategy that was consistent with children's typical strategies as documented in past research. Among these anticipated strategies, direct modeling strategies were the most common. A handful of teachers anticipated nothing but direct modeling strategies or strategies using drawn models; and among those teachers who anticipated a small range of strategies consistent with children's typical strategies, a direct modeling strategy was always included. The pervasiveness of direct modeling in teachers' anticipated strategies suggests the accessibility of direct modeling as a way for teachers to make sense of children's mathematical thinking and to imagine how a child would solve equal sharing fraction problems.

Teachers' anticipation of direct modeling and other strategies allowed us to identify three knowledge profiles which varied along a continuum and were distinguished primarily by their flexibility in terms of thinking like a child. Flexibility with respect to the number of distinct and valid strategies provided a less useful measure by which to distinguish groups, although we conjecture that as a teacher's knowledge of children's mathematical thinking advances, there would be a corresponding increase in the number of valid and distinct strategies consistent with children's typical reasoning. We also conjecture that this number would include instructed procedures and conventions, although not at the expense of the variety of informal strategies children are known to use to make sense of an equal sharing fraction problem.

In summary we found the idea of flexibility as the extent to which teachers seemed to be able to think like a child not only useful for characterizing teachers' knowledge, but also an important extension to Whitacre's (2015) characterization of flexibility in terms of the number of distinct and valid strategies. The greater this flexibility, we conjecture, the greater teachers' ability to respond to children's ever evolving mathematical thinking during instruction. This exploratory study can be extended in two ways: investigate the knowledge of children's mathematical thinking in a larger sample of teachers systematically selected on the basis of their documented participation in professional development focused on children's mathematical thinking and use methods such as clinical interviewing that elicit teachers' reasoning for the strategies that they anticipate.

Acknowledgements

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APPROPRIATENESS OF PROPORTIONAL REASONING: TEACHERS' KNOWLEDGE USED TO IDENTIFY PROPORTIONAL SITUATIONS

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In this study we explored to what extent middle school teachers were able to appropriately identify proportional situations when presented with various mathematical structures and if there were relationships between attributes of the teachers and their ability to identify proportional situations. Interestingly, there were no strong relationships aside from teachers' perception of their knowledge of mathematics and their ability to identify proportional situations. Teachers were also found to correctly identify proportional situations significantly more often than non-proportional situations. Nearly one third of the teachers misidentified non-proportional linear situations as proportional. Thirteen participants' responses to such a situation were analyzed qualitatively resulting in some common knowledge resources that they appeared to use when attempting to identify whether a situation was proportional or not.

Keywords: Teacher Knowledge, Rational Numbers, Mathematical Knowledge for Teaching

Purpose and Background

Proportional reasoning is an important content area that has gained prominence in middle school mathematics. One indication of this is that the Common Core State Standards for Mathematics (National Governors Association & Council of Chief State School Officers, 2010) have made "ratios and proportional reasoning" its own content domain for grades 6 and 7. Despite this recognition, there has been little focus on proportional reasoning in research in relation to its importance (Lamon, 2007), including research on teachers' knowledge of the domain. The research that is available on teacher knowledge of proportions indicates that, like students, teachers struggle with proportions (e.g., Akar, 2010; Harel & Behr, 1995; Orrill, Izsák, Cohen, Templin & Lobato 2010; Post, Harel, Behr, & Lesh, 1988; Riley, 2010).

One fundamental way of demonstrating proportional understanding is in the ability to identify when a situation warrants the use of proportional reasoning, which pertains to the mathematical structure of the problem rather than other identifiable aspects. Orrill et al. (2010) observed that middle school teachers had trouble identifying situations as appropriate or inappropriate for using proportional reasoning. For example, often when teachers were given a problem with three values and asked to find a missing fourth value, teachers tended to treat it as directly proportional even if the actual relationship was inversely proportional. Teachers also struggled to reason about proportions in a qualitative task that asked them to compare one pile of blocks to another pile similar to those tasks used by Harel, Behr, Post, and Lesh (1992).

These findings suggest that when teachers do not rely on a strong mathematical understanding of proportions to evaluate a situation, they draw on understandings that may be based on something other than mathematical structure when deciding whether a situation is proportional. In this paper, we explored the extent to which middle school teachers were able to identify proportional situations and whether there was a relationship between the teachers' backgrounds and their ability to identify proportional situations? To further explore this, we investigated whether there was a relationship between the underlying mathematical structure of the situation presented and the teachers' ability to

identify proportional situations. We followed this with an analysis of the knowledge resources our participants invoked when identifying a non-proportional, linear situation. We see this work as border crossing because we are looking at the between proportions and other relationships. Specifically, we are investigating middle school teachers' abilities to recognize the border between situations that are and are not proportional. We conjecture that without such recognition teachers will struggle to use appropriate reasoning to make sense of a given situation.

Theoretical Framework

We work from the knowledge in pieces perspective (diSessa 1988, 2006), which asserts that individuals hold understandings of various grain sizes that are used as knowledge resources in a given situation. These resources are connected, over time, through learning opportunities that lead to the refinement of the resources and the development of rich connections. Having a series of robust connections allows a knowledge resource to be available in more situations. This is parallel to the research on expertise that has shown that experts have both more knowledge and a different organization of knowledge than novices in their domain (Bédard & Chi, 1992). It is also aligned with Ma's (1999) interpretation of teachers' need for profound understandings of fundamental mathematics. By having a robust set of knowledge resources that are coherently connected, we posit that teachers will be more able to access their myriad understandings to apply them to a wider range of mathematics and teaching situations than others whose knowledge resources are less coherently connected. We refer to this richly connected collection of knowledge resources as being *coherent* and assert that more coherent teachers will be better able to support student learning (e.g., Thompson, Carlson, & Silverman, 2007). This approach differs from much research on teacher knowledge in that we are not trying to identify deficiencies in teachers' understanding of mathematics, rather, we are trying to understand how teachers understand the mathematics they teach and how different knowledge resources are drawn upon for solving problems and teaching.

Methods

This study is part of a larger project investigating teachers' knowledge of proportional reasoning for teaching. In this section we will describe the participants of the study as well as our data collection and analysis procedures and tools.

The participants included a convenience sample of 32 in-service, grade 5-8 mathematics teachers, whose teaching experiences ranged from 1 to 26 years. The participants were from four states. They taught at a variety of schools (public, private, and charter). Twenty-four of the teachers identified as female and eight identified as male. Six of the teachers identified as a race other than white.

The data analyzed for this study were collected through a multiple-choice assessment and clinical task-based interview. We selected those items designed to assess teachers' ability to differentiate between situations that are or are not proportional. The items (n=20) were drawn from three existing assessments designed to measure teachers' proportional reasoning abilities. We also collected data on attributes of teachers' backgrounds including: the number of years teaching, number of mathematics and methods courses taken, and teacher's self-efficacy. To do this, we relied on seven, five-point Likert scale items taken from the Learning Mathematics for Teaching assessments (Learning Mathematics for Teaching, 2007).
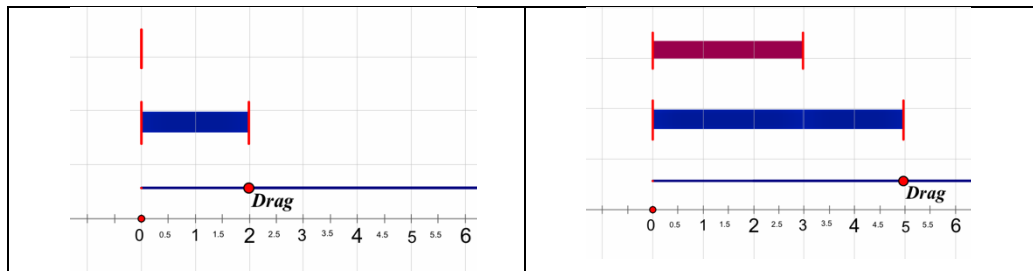
Figure 1. Screenshot of Thermometers task.

We carried out an exploratory data analysis of the proportional reasoning items, followed by a correlational analysis and a non-parametric comparison of group centers to investigate the first three research questions. The results of the analysis led us to analyze the Thermometers task from the clinical interview. Thermometers, a dynamic sketch, presented the participants with two thermometers, one red and one blue, whose lengths could be varied by dragging a point on a number line (as shown in Figure 1). Two scenarios were shown to participants (one at a time) and with each scenario participants were asked: (a) whether there was a relationship between the thermometers; (b) whether the relationship was proportional; (c) whether they could provide a rule and a story problem or real-world situation for that relationship; and (d) whether they see a scale factor involved in the situation. For this study we analyzed the participant's responses to the first scenario where the thermometers were designed to maintain a constant difference of two units in length of the lines as the point on the slider is dragged from left to right, shown in Figure 1. This situation represents a non-proportional linear relationship between the two thermometers.

Table 1: Codes of Knowledge Resources Used in Thermometers Task

Code	Description
Comparison of Quantities	States that ratio as a comparison of two quantities.
Covariance	Recognizes that as one quantity varies in rational number the other quantity must covary to maintain a constant relationship.
Unit Rate	Uses the relationship between the two quantities to develop sharing-like relationships such as amount-per-one or amount-per-x.
Equivalence	Describes proportion as a relationship of equality between ratios or fractions.
Between Measure Space	Asserts that the ratio between the quantities in a proportion stays constant.
Scaling Up/Down	Uses multiplication to scale both quantities to get from one ratio in an equivalence class to another.
Horizon knowledge	Demonstrates knowledge that extends into mathematics beyond proportions
Relative Thinking	Demonstrates multiplicative reasoning about the change in a quantity relative to itself or another quantity. This includes re-norming.
Proportional Situation	Recognizes that a situation involves proportional reasoning.
Rule	Shares a verbal or written rule (e.g., Red = Blue - 2) stated in a way that conveys a generalizable relationship.
One Unit at Time	Describes the relationship between the two quantities as increasing by one unit at a time

The qualitative analysis of the participant's responses was carried out by coding the participants' utterances using a coding scheme (Table 1). The scheme was developed using open coding (Corbin & Strauss, 2007) and refined across several interviews. It was specifically designed to consider knowledge resources related to proportional reasoning. We note that this specific task was non-proportional, so there were other resources that the participants drew on to engage with the thermometer scenario. Our coding relied on a binary approach in which each utterance was coded as a 1 or a 0 based on whether a particular knowledge resource was observed. Every interview was

coded by at least two researchers and 100% agreement was reached on all coding.

Results

Question 1: Extent of Correct Identification of Proportional Situations

The 32 teachers in this sample were largely able to correctly identify proportional situations. For the 20 situations analyzed, the mean number correct was 15.22 (SD=2.97). The range of correct answers was 7-19.

Table 2: Kendall's tau b Correlation Coefficients

	# Items Correct		Min	Max	Mean	SD
	Kendall's tau b	Sig. (2-tailed)				
Years Teaching Math	.144	.272	0	26	9.20	7.202
Math Courses	.135	.352	1	4	3.13	.92
Methods Courses	-.012	.932	1	4	2.59	1.01
I enjoy teaching mathematics	.000	1.000	3	5	4.72	.581
Mathematics isn't my strongest subject to teach	-.271	.065	1	5	1.69	1.120
I consider myself a "master" mathematics teacher	.255	.070	1	5	3.03	1.177
Overall, I know the mathematics needed to teach this subject	.278	.063	3	5	4.34	.653
I have strong knowledge of ratio, proportional reasoning, and rate	.483[**]	.001	2	5	3.81	.738
I have strong knowledge of all areas of mathematics	.281[*]	.050	1	5	3.09	.963
My knowledge of ratio, proportional reasoning and rate is adequate to the task of teaching these subjects	.343[*]	.017	1	5	3.94	1.105

Note. Math and Methods courses measured on a scale of 1 to 4 with 1 corresponding to no classes, 2 to one or two classes, 3 to three to five classes, and 4 to six or more classes. *$p \le .05$, **$p < .01$*

Question 2: Relationship between Attributes of Teacher's Background and Ability to Identify Proportional Situations

In response to the second research question a correlational analysis was done using Kendall's tau B (shown in Table 2), which is a non-parametric measure of correlation appropriate for small samples (Field, 2013). Interestingly there was no significant correlation between the number of items the teachers answered correctly and the number of college mathematics or methods courses. There was also no significant correlation between the numbers of years teaching and the number of correct responses. However, there was a significant correlation between the number of items answered correctly and the teachers self-rating of their knowledge of mathematics; of ratio, proportional reasoning, and rate; and of perceived ability to teach these subjects. Further investigation in this area is needed as these results are not generalizable given the sample size.

Question 3: Relationship between Mathematical Structure and Appropriateness

In response to the third research question we looked to see if there was a relationship between the underlying mathematical structure of the situation and teachers' abilities to determine whether it was proportional. Using the non-parametric Mann-Whitney U-test we found that the median number of teachers who correctly identified the proportional situations was 31 for each of the seven items that was proportional. The median number of teachers who correctly identified the non-proportional situations was 23 for each of the 13 non-proportional items. Using the Mann-Whitney U-test, we

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found that this was a significant difference (U=3.00, z=3.34 p=.0002, r=.75). This means that the teachers were better at appropriately identifying proportional situations than they were at identifying non-proportional situations. Further analysis revealed a clear pattern in the relationship between the mathematical structure of the item and the number of teachers able to correctly identify it as proportional or not (Figure 2). There is a clear stratification of the number of appropriateness items that teachers responded to correctly based on the underlying mathematical structure of the situation presented. There is also a statistically significant correlation (Kendall's tau-b -.889, p<.0001) between the mathematical structure and the correctness of teachers' responses.
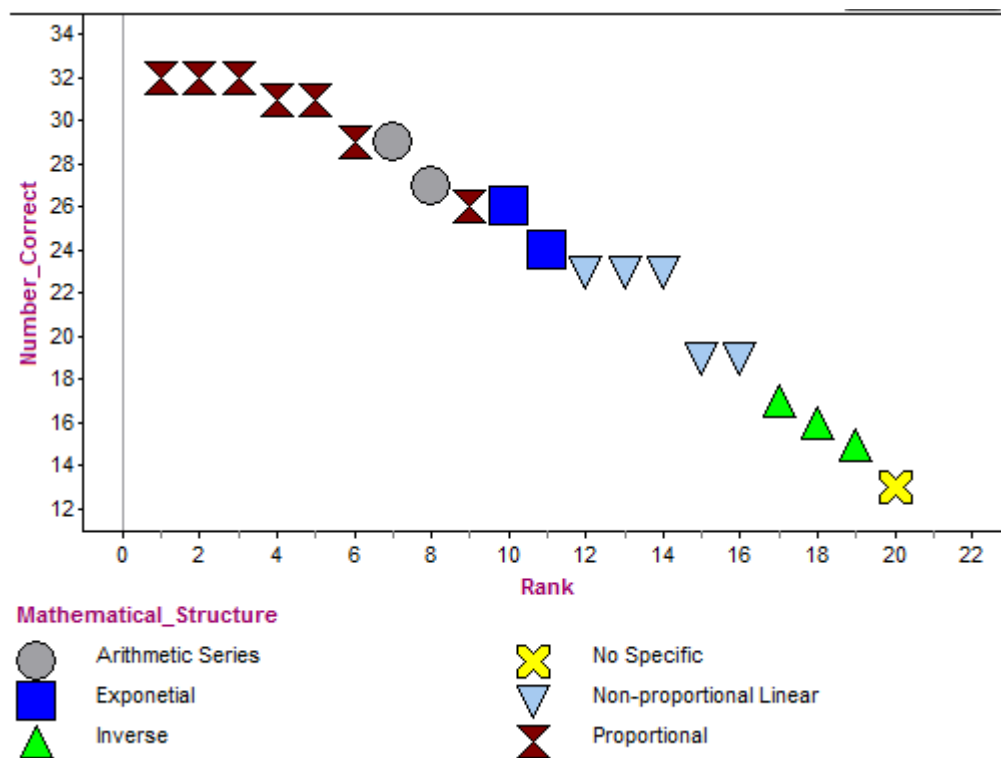

Figure 2. Structure of appropriateness items with number of correct responses.

Interestingly, nearly one-third of the teachers misidentified non-proportional linear situations as proportional. Linear functions are a significant and heavily emphasized topic in school mathematics and therefore are important for teachers to be able to differentiate from the more specific case of proportional situations. This finding led us to further investigate the knowledge resources these teachers used when evaluating the appropriateness of proportional reasoning in a linear situation, specifically the Thermometers task.

Question 4: Resources Used in Determining Whether a Situation Involved Proportion

Preliminary results from analysis of reasoning by thirteen of the participants on the non-proportional Thermometers task are shared here. Eight participants (Group 1 - Heather, Eileen, Ella, Matt, Alan, Magen, Larissa, and Tonya) correctly identified the situation as non-proportional. Three participants (Group 2 - Tori, Kathleen, and Allison), first identified the situation as proportional but changed their mind during the interview to identify the situation as non-proportional. In fact, Tori changed her mind twice and ended by identifying the situation as proportional. Two participants (Group 3 - David and Bridgett) identified the situation as proportional. All names are pseudonyms.

Proportional Knowledge Resources. Many participants in Groups 1 and 2 used rules, scaling up and/or down, and equivalence to appropriately identify this Thermometers task as non-proportional. All the participants in Group 1 were able to provide a clear *rule* that stated the generalizable relationship between the two thermometers. For instance, Ella claimed that "the blue equals the red plus two". On the other hand, the participants from Group 2 and Bridgett (Group 3) did not provide a rule even when they were asked explicitly to do so. David (Group 3) was able to provide a rule, however he incorrectly identified the relationship as a proportional one. Creating a generalizable rule was a knowledge resource that participants in Group 1 used when they correctly identified the situation as non-proportional from the beginning.

Three participants from Group 1 (Tonya, Larissa and Meagan) used the idea of *scaling up/down* to explain why the relationship between the two thermometers was not proportional. For example, Larissa stated, "if the red was at one and blue was at two and it was a times two and then two and then four, then yes, it would be proportional. But not in this case." Allison (Group 2) also used this knowledge resource to resolve the issue she had with the difference between the thermometers being "always just two". At first she claimed, "it's proportionally it's going up the same when you drag it". However, as she continued to move the thermometers, she said, "If they were similar it wouldn't always be two because if something's four and two, if I double it to eight, that would be four if they were proportionally the same. And that's not happening here." She used multiplication to determine the equivalent ratios that would be found in a proportional relationship (*Scaling Up/Down*), and then determined that the relationship was non-proportional.

Two participants from Group 2 (Tonya and Larissa) explained the situation as non-proportional by using the idea of *equivalence* "because it's add two, there's no equivalent; the fractions created wouldn't be equivalent" (Larissa). Surprisingly, David (Group 3), who incorrectly identified the relationship as proportional, shared an accurate definition of proportions but said he was "having a hard time putting that onto this [the thermometers situation]".

Additive Knowledge Resources. Consistent with previous research on the use of language that suggests additive reasoning (e.g., Lamon, 2007; Nagar, Weiland, Orrill, & Burke, 2015), nine out of thirteen participants (at least one from each group) drew on the idea of *One Unit at a Time*. These participants claimed that the thermometers move "up one unit at a time" (Heather, Group 1). The participants used this resource to determine whether the situation was proportional and/or to explain why the situation is not proportional. For instance, Eileen (Group 1) said, "both of the bars…are moving by one unit amount…which means they are not moving in proportion to each other." Tori (Group 2) at first determined that the situation is proportional, but then she explained that when she thinks "of that [the relation between the bars] as a fraction, three fifths" and then dragged "it to where the blue is at eight, the red is at six. That's not proportional". Interestingly, she continued to explore the relationship and found that there is a constant relationship where "for every increment for red, there's an increase of one for the blue" and determined it to be a proportion. Both Bridgett and David (Group 3) also used this knowledge resource. Like Tori, David found the idea of *One Unit at a Time* as related to proportion when he claimed that "talking about the slope, the rate, it is proportional, they're going up one unit". Bridgett did not mention this resource in the context of proportion.

Conclusions

Determining whether a given situation was proportional or not was most challenging for our participants when the situation was non-proportional. Interestingly no attributes of the teachers' background seemed to relate to their ability to identify a proportional situation. Participants were able to successfully use *Rule*, *Scaling*, and *Equivalence* to identify non-proportional situations. The use of additive reasoning was common across participants (nine out of thirteen) regardless of their ability to correctly identify the situation as proportion or not. Since proportional reasoning is multiplicative

this could suggest that teachers have the same tendency to rely on build-up strategies as their students (Lamon, 2007).

Surprisingly, we also did not observe participants relying on comparisons of quantities to determine whether the relationship was proportional. We find this interesting because the basic definition of a ratio is a multiplicative comparison of two quantities, something the participants did not draw upon in their reasoning. Instead, most of the participants referred to ratios but used additive language as they described their thinking. This suggests that teachers may not draw upon their understanding of ratio as comparison when they identify proportional situations. We noted that they did not seem to clarify whether the comparison was additive or multiplicative nor did they appear to rely on the definition of ratio which would have suggested comparing quantities rather than build-up strategies (Lobato, Ellis, Charles & Zbiek, 2010).

Our research suggests two main findings. First, teachers seem to understand their own mathematical abilities. This is important because it contradicts the widespread warnings about the suspect nature of self-reported data. At a large grainsize, these teachers had a relatively accurate assessment of their understandings. Second, there may be knowledge resources that are more useful for determining whether a situation is proportional. The teachers in this study had greater success with *Rule*, *Scaling*, and *Equivalence* than with other resources that were tried. These findings suggest that professional developers could rely more on teachers to provide insights into their own needs in content knowledge development. It also suggests that teacher development should potentially include explicit discussion of the use of different approaches to reason about the proportionality of a situation. Future research should also explore if these participants use of knowledge resources is representative of middle grades teachers.

Acknowledgment

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CONOCIMIENTO MATEMÁTICO PARA LA ENSEÑANZA DEL VOLUMEN DE PRISMAS EN PRIMARIA

MATHEMATICS KNOWLEDGE FOR TEACHING THE VOLUME OF PRISMS IN ELEMENTARY SCHOOL

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El estudio de objetos tridimensionales en el currículo mexicano se aborda en toda la educación básica. Sin embargo, hay pocos estudios sobre el conocimiento matemático para la enseñanza de los profesores al respecto. Este reporte da cuenta de un acercamiento a dicho conocimiento, a partir de la observación de las clases, que tres profesoras de sexto grado ponen en juego cuando enseñan a sus estudiantes a calcular el volumen de prismas en dos escuelas de la ciudad de México. Los resultados muestran evidencias respecto a ciertos subdominios del conocimiento matemático para la enseñanza. Si bien hay fortalezas en dicho conocimiento, también algunas carencias, lo que muestra la necesidad de construir espacios formativos (inicial y continua) donde el análisis y discusiones favorezcan el aprendizaje de conocimientos tanto geométricos como didácticos que requieren para su práctica docente.

Palabras clave: Geometría y Pensamiento Geométrico y Espacial, Conocimiento Matemático para la Enseñanza, Educación Primaria

Introducción

La geometría posibilita en los estudiantes el estudio de las figuras del espacio y sus relaciones. El espacio geométrico es construido a partir de la exploración empírica que parte de un espacio real y llevado hacia una abstracción geométrica. Su enseñanza en México, y en particular el estudio de la geometría tridimensional, permea el currículo de toda la educación básica (SEP, 2011). Los aspectos a trabajar son "la exploración de las características y propiedades de […] cuerpos." Y "el conocimiento de los principios básicos de la ubicación espacial y el cálculo geométrico." (p. 73), donde la mediación del profesor y por tanto, sus conocimientos matemáticos y didácticos, son esenciales.

El aprendizaje del volumen de cuerpos geométricos, en particular, presenta dificultades. El conocimiento del profesor respecto de este contenido tiene mayor relevancia, si cabe, que respecto de la enseñanza de otros contenidos de menor dificultad para el alumno. Parte de estas dificultades se relacionan con su estrecha relación con la capacidad (Freudenthal, 1993; Saiz, 2002; Zevenbergen, 2005), la distinción entre los conceptos matemático y físico de volumen (Saiz, 2002) y la importancia de la visualización en su comprensión (Gutiérrez, 1998).

El conocimiento que tienen los profesores de educación primaria sobre la geometría tridimensional es una problemática que ha sido poco estudiada (Aslan-Tutak & Adams, 2015; Tekin & Isiksa, 2013). Los estudios reportados en relación con dicho conocimiento del profesor en formación y/o en servicio ponen de relieve dificultades en aspectos conceptuales y de habilidades espaciales. Así, Zevenbergen (2005) señala que profesores en formación tienen obstáculos similares a los alumnos en primaria vinculados con el desarrollo de su sentido numérico, de medición y espacial. Carencias que limitan su competencia para identificar en sus alumnos errores asociados, por

ejemplo, con resolución de problemas de cálculo de volúmenes. Profesores para primaria en formación señalan que en sus cursos de matemáticas y su didáctica, no logran desarrollar una comprensión profunda sobre contenidos de geometría elemental. También reconocen la importancia de las representaciones y habilidades visuales en este tópico de las matemáticas (Aslan-Tutak y Adams, 2015). En cuanto a la comprensión de cuerpos geométricos, Dorantes (2008) mostró carencias respecto a los conocimientos de un grupo de profesores relacionadas con la identificación de características de poliedros en términos de sus aristas, vértices y caras; la exploración de éstos mediante la observación en diferentes perspectivas y el cálculo de su volumen. Estos resultados coinciden con los de otros estudios (Saiz, 2002; Bozkurt & Koç, 2012). En particular, Çakmak *et al* (2015) encontraron que las definiciones dadas por estudiantes y los ejemplos y maneras de resolver problemas, son análogos a los usados por sus maestros en las clases y en muchos casos presenta limitaciones.

El objetivo de la investigación en la que se enmarca este reporte fue identificar y describir el *conocimiento matemático para la enseñanza* evidenciado en profesores de sexto de primaria en la enseñanza del volumen de prismas (Moctezuma, 2015).

Visualización geométrica, representaciones y volumen de objetos tridimensionales

La percepción visual apoya a la geometría tridimensional puesto que las representaciones visuales constituyen un medio esencial de anticipación. Los alumnos tienden a depender de la información visual, según los resultados de investigaciones como la de Gal & Linchevski (2010). La atención inicial en geometría se centra más en los objetos que en los procesos debido a que el interés se enfoca en las propiedades figurales percibidas por medio de los sentidos e interpretadas por la reflexión mental mediada por una representación espacial. Sin embargo, toda representación particular a la vez que proporciona cierta información, oculta otra y puede ser difícil recuperarla para quien la está interpretando. Para Del Olmo *et al* (1993) las dificultades de los niños al medir el volumen se pueden originar por la falta de manipulación previa; no dominan la visualización espacial, al carecer de la habilidad de manipular mentalmente figuras rígidas.

Existen diferentes formas de representar los cuerpos geométricos para fines didácticos: en perspectiva, paralela, isométrica, por niveles, ortogonal y mediante desarrollos planos. Cada uno de ellos requiere el aprendizaje de convenciones para interpretarlas y el desarrollo de habilidades para reproducirlas (Gutiérrez, 1998). Los tipos de representaciones utilizadas en los libros de texto de quinto y sexto grado de primaria (SEP, 2012)[i] para proyectar cubos, prismas y pirámides son las representaciones paralelas, isométricas y mediante desarrollos planos. Sin embargo, como se mostrará más adelante, hay desconocimiento de los profesores sobre dicha diversidad que tienen impacto en el uso que se les dan a las mismas en clase.

Conocimiento matemático para la enseñanza de los prismas y su volumen en primaria

El conocimiento para la enseñanza toma su caracterización desde los conocimientos planteados por Shulman (1986) y detallados, para el caso de las matemáticas, en los subdominios del *Mathematical Knowledge for Teaching* (MKT) de Ball, Thames & Phelps (2008). Estos autores diferencian dos grandes dominios: Conocimiento del Contenido y Conocimiento Didáctico del contenido. El primero recoge el contenido que el profesor está enseñando y su fundamentación. El *conocimiento del contenido* se subdivide en Conocimiento Común del Contenido (conocimiento de una persona instruida en ese contenido) (CCK[ii]), Conocimiento Especializado del Contenido (conocimiento del contenido que distingue al profesor) (SKC) y Conocimiento del Horizonte matemático (HCK). El *Conocimiento didáctico del contenido* incluye los conocimientos que posee el profesor con respecto a la enseñanza del contenido, su aprendizaje, y el currículo escolar. El *Conocimiento del contenido y los Estudiantes* (KCS) es la unión de la comprensión del contenido y saber lo que los estudiantes pueden pensar o hacer en matemáticas. Involucra la identificación de los

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conceptos previos y anticipar probables dificultades de aprendizaje y concepciones erróneas respecto al contenido matemático a enseñar. El *Conocimiento del Contenido y la Enseñanza* (KCT) conjuga la comprensión del contenido y su familiaridad con los principios pedagógicos para enseñarlo. Finalmente, el *Conocimiento del Currículum* (KCC) incluye el conocimiento del profesor delas matemáticas como asignatura: su estructura, y los aprendizajes que se espera lograr con los alumnos al culminar su Educación Primaria.

En este reporte se dará cuenta sobre ¿Qué conocimiento matemático para la enseñanza, pone en acción un profesor al impartir una lección sobre el volumen de prismas en sexto de primaria?

Metodología

La investigación realizada tiene un enfoque cualitativo con un alcance descriptivo. Los datos principales provienen de lo que tres profesoras (que participaron de manera voluntaria) hacen en sus clases de matemáticas cuando enseñan el contenido de volumen de prismas.

La investigación se realizó entre 2014 y 2015 en dos escuelas primarias públicas de la ciudad de México, ubicadas en la periferia. Las profesoras impartían sexto grado y todas las asignaturas, entre las que se incluye Matemáticas. Las tres tienen más de 25 años de experiencia docente y son de formación normalista. Consuelo (C) es experta en quinto y sexto grado de primaria, mientras que Laura (L) y Rocío (R) han transitado de primero a sexto grado[iii] (nombres ficticios). Las tres, han impartido los temas de sexto grado durante dos años previamente a esta investigación. Este grupo de estudio fue tomado de manera intencionada, y se hizo un seguimiento a sus prácticas durante varias sesiones de clase, donde se abordaron temas vinculados con geometría tridimensional (en este reporte nos centraremos sólo en una lección referida al volumen de prismas).

El trabajo de campo se desarrolló durante trece meses en distintas fechas del año escolar. Las clases fueron videograbadas y transcritas. Para extraer información sobre los conocimientos movilizados por las profesoras se utilizaron tres instrumentos: observación no participante en clases, diario de campo y entrevista semiestructurada (esta última con el objetivo de completar la información obtenida de la observación).

Para el análisis de las clases observadas se utilizaron categorías prefijadas por el observador. Se partió de la descripción de los subdominios del MKT y de su concreción en los indicadores diseñados por Sosa (2011), adecuándose al nivel y contenido específico a analizar. Éstos se fueron refinando durante el proceso de investigación hasta obtener los indicadores definitivos.

En el libro de texto, la lección "¿Cuántos cubos hay en el prisma?" consta de cuatro actividades (SEP, 2012, pp. 162-163). Nos centraremos en las dos primeras, en las que los alumnos deben calcular el volumen de cuatro prismas rectangulares a partir de la cantidad de unidades cúbicas en las aristas (Figura 1.a). En A y B se identifican las tres dimensiones (largo, ancho y altura) y se muestran unidades cúbicas, mientras que en el prisma C se muestran marcas que permiten deducir la medida de longitud de únicamente dos aristas del prisma (Figura 1.b).

Un análisis a priori de esta lección permitió identificar una transición de un tratamiento unidimensional del volumen (usándose como unidad de medida un cubo) a uno tridimensional (calculándose a partir de las longitudes ancho, largo y alto), si bien el título de la lección destaca el primero. Además, en la actividad 3 se relacionan las nociones volumen y capacidad.

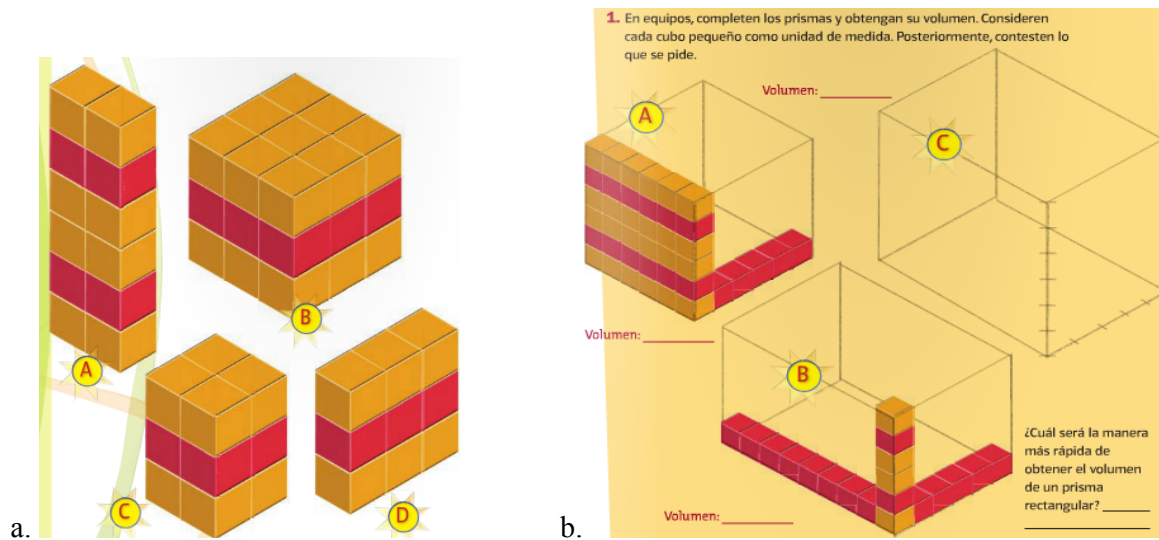Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

Figura 1. Actividades propuestas para el trabajo con volumen de prismas (SEP, 2012, pp. 162-163).

Conocimientos puestos en acción. Descripción de resultados y discusión

En el análisis de las tres sesiones observadas surgen dos elementos de interés sobre el conocimiento de las profesoras: el tratamiento uni/tridimensional del volumen y cómo calcular la medida no dada de una arista en el caso C de la actividad 2.

Una de las profesoras, Rocío, trata el volumen en todas las actividades desde una perspectiva tridimensional. De este modo, pierde sentido la pregunta final que plantea la actividad 2 ("¿Cuál será la forma más rápida de calcular el volumen de un prisma?"). Desde un inicio, Rocío ubica el largo, el ancho y la altura de cada prisma como paso necesario para el cálculo de su volumen:

> Rocío (R): Recuerden tomar en cuenta las tres dimensiones que habíamos hablado, saquen el volumen […] la cantidad de prismas que tienen esas figuras, la cantidad de unidades […]
> R: […] ¿Cuántas unidades son? Aquí [figura 1.b, apartado A] de base, de ancho 6, de largo 6 y de altura 6. Entonces multiplicamos 6 por 6 por 6 y nos da 216 […] En la figura B ¿Cuál es la base, el largo, el ancho? […] ¿Cuáles son las medidas? […] ¿Cuál es el largo?
> E8: 6 por 9
> R: […] Multiplicamos 9 por 6, […] 54… 54 por 6 […] 324. [La profesora escribe en el pizarrón 324 cm^3].

> (Fragmento 1 de la sesión de Rocío)

Rocío no hace una distinción explícita entre el tratamiento unidimensional y el tridimensional para el cálculo de volumen de prismas. Parece que no establece diferencias entre ambos tratamientos del volumen y ni percibe la necesidad de trabajar primero desde una perspectiva unidimensional para justificar la tridimensional por sus ventajas prácticas, ni la incidencia de dicha secuencia en la comprensión de la magnitud volumen.

Sin embargo, Laura, plantea la posibilidad de que los alumnos calculen el volumen de los prismas dados en la actividad 2 por el procedimiento que ellos consideren adecuado, surgiendo de este modo procedimientos usando una unidad de volumen y otros, haciendo uso de la longitud y tridimensionalidad del prisma.

> Laura (L): [Lee] completa el prisma, ¿Necesitamos completarlo? […]
> E5: Sí.
> L: ¿Llenarlos de cubitos?
> E5: No.

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L: ¿Tú qué opinas, E7, […]. A ver, ¿necesitamos completar los prismas?
E7: También podemos multiplicar.
L: Multiplicar, ¿tú qué estabas haciendo?
E7: Estaba multiplicando base por altura y después por ancho.
L: […] Y tú ¿qué estabas haciendo? [dirigiéndose a E8]
E8: Contándolos.
L: Contando los cubitos, […] Completen, cada quien tiene su método de sacar el volumen.

(Fragmento 1 de la sesión de Laura)

Los estudiantes refieren tanto el tratamiento unidimensional como el tridimensional para calcular el volumen; la profesora por su parte da libertad a los estudiantes para aplicar el método que prefieran. Parece que Laura distingue implícitamente entre el procedimiento unidimensional y el tridimensional para calcular el volumen de prismas (SCK) aunque no se observa que precise el tratamiento unidimensional de manera insistente como lo hace con el tridimensional.

Por otro lado, uno de los aspectos que resultan más problemáticos de la lección está vinculado con la actividad 2 (Figura 1.b) puesto que la representación del prisma C no permite inferir visualmente la longitud de una de sus aristas. La discusión que surge en la clase se relaciona con la unidad de medida. En las tres sesiones se observan formas diferentes de orientar esta situación, lo que pone de manifiesto distinto conocimiento matemático para la enseñanza de las profesoras. Este conocimiento se relaciona con el papel de la unidad en el proceso de medida, y su conceptualización (SCK); el conocimiento de la posibilidad de usar unidades de medida no convencionales (KCC); la diferencia entre medida, unidad, instrumento y procedimiento de medida (SCK) y las confusiones habituales de los alumnos al respecto (KCS); la dificultad de los alumnos para reconocer y usar unidades e instrumentos no convencionales (KCS); y el carácter aproximado de la medida (SCK).

Rocío no parece ser suficientemente consciente del papel de la unidad en el proceso de medida y del papel que juega conceptualmente en el concepto de medición (SCK), así como de las dificultades que esto origina en los estudiantes (KCS). Así, para solventar el problema que se plantea a la hora de calcular el volumen del prisma C (actividad 2), decide medir con una regla:

R: [Dirigiéndose a un estudiante] Préstame tu regla. [La usa para medir en su libro]
E7: 1 centímetro es de dos cubos.
E5: Son 5 centímetros.
E8: Son 6.
R: Si está de medio centímetro, entonces cada uno de los lados son medio centímetro. Entonces si yo mido…¿Quién decía que 11? Son 11 […]
R: ¿11 por ocho? [la maestra sigue midiendo] De altura son 10, fíjate bien: son 5 centímetros, cada una de las unidades está representando medio centímetro; son 10 unidades. Midan con su regla. Cada una de las unidades que nos está representando el libro mide medio centímetro, son cubos que miden medio centímetro.
[…]
R: Recuerden que en este cubo que está, en este prisma que está representado ahí, está el largo, el ancho y la altura. El largo es 11.
E8: Son 11 por 10 por 8, entonces… [La profesora mueve la cabeza en señal de aprobación]

(Fragmento 2 de la sesión de Rocío)

Rocío desecha las medidas de longitud que ya están dadas en el prisma (según la unidad que se indica) y mide con la regla cada una de las dimensiones (largo, ancho y alto). Usa el centímetro como unidad de medida de longitud, estableciendo la equivalencia entre el centímetro y la unidad de longitud dada en la actividad (medio centímetro).

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La representación dada en la actividad muestra que el prisma mide 5 unidades de ancho y 6 de altura; el largo no se sabe. El resultado de la medición de la maestra es 4cm para el ancho, 5.5 cm para el largo y 5cm para la altura. Ella lo traduce como 8 para el ancho, 11 para el largo y 10 para la altura.

Decide tomar una unidad de medida de longitud diferente a la que muestra el libro de texto y no precisa dejar claro a los estudiantes que ha cambiado la unidad de longitud para las tres dimensiones del prisma al calcular el volumen. Además, la elección de la unidad y el instrumento de medida de longitud que toma puede suponer no tomar en consideración la concepción errónea de los alumnos de que para medir (una cantidad de longitud) es necesario usar unidades del sistema métrico decimal (en este caso cm) e instrumentos convencionales (KCS).

En la sesión de Consuelo, también se recurre a una regla convencional (graduada en cm) para resolver la situación:

Consuelo (C): […] ¿Qué hicieron para tomar las medidas del prisma que está en la letra C?
Estudiante 9 (E9): Medí con una regla.
C: […] ¿Quién hizo algo diferente? [Señala a un estudiante y le pregunta] ¿Tú también mediste con una regla?
E5: Sí, pero medí lo de abajo para ver si […] tenía lo mismo de ancho.
C: Si tenía lo mismo de ancho… ¿Tú mediste el ancho, E7?
E7: Yo medí cuánto medía una rayita, pero se complicaba con la regla en los palitos y yo medí los seis cubitos con la regla y luego ver cuántos cabían en el ancho y en el largo.
C: Me puedes decir ¿qué mediste?¿Cuántos cubos están de largo en esa figura?
E7: De largo eran 7, de ancho eran 4 y de altura eran 6.
E5: De ancho son 5.
E3: Son 6.
C: 4, 5, 6… ¿Cómo vamos a saber?
[…]
C: Bueno, recuerden cuando lo hicieron por unanimidad, muchas veces nuestras reglas no son exactas […] [y se obtienen] diferentes medidas.
E7: A mí me dio 8 milímetros […]
E8: Midió un centímetro casi.
C: Un centímetro casi pero no exacto. […] Entonces vamos a tomarlas todas como de un centímetro; si es de un centímetro, E8 dime de cuánto va a quedar la medida a lo largo de ese prisma.

(Fragmento 1 de la sesión de Consuelo)

Como se ilustra en el diálogo, Consuelo considera la posibilidad de que el volumen que se está calculando no sea exacto sino aproximado y como unidad de medida, considera inicialmente el centímetro cúbico, aunque no lo explicita en sus acciones. La profesora conduce a los estudiantes hacia el uso de la regla para que por medio de un consenso, se defina la unidad de medida de longitud de la dimensión que está faltando, a partir de las longitudes conocidas y así poder calcular el volumen del prisma C.

Finalmente, Laura, hace un tratamiento diferente de la situación. Propone el uso de la unidad de medida que viene dada en la actividad y de un instrumento de medida usando dicha unidad.

L: […] ¿Cuántas medidas tiene […] la figura C?
E10: Una.
L: Nada más una ¿Qué podemos hacer para obtener la otra?
E2: ¿Multiplicando?

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L: ¿Multiplicando? Pero si no tenemos medida, ¿por qué lo vamos a multiplicar? [La profesora toma una hoja de cartulina, recorta una tira, la gradúa con la longitud de segmentos de una arista, la llama reglita y regresa con E10]. Vamos a medir la única medida que tenemos, la voy a marcar en esta reglita, esta va a ser mi reglita para medir las otras aristas. ¿Cuántas medidas tengo ahora?

E5: 3.

L: ¿Cuántas necesito para sacar el volumen? Pues nada más esas, entonces, ¿estaba muy difícil? Y ¿qué hicimos? Con nuestra reglita vimos cuántas veces cabe en cada una, ahora saquen el volumen. […] [E7 la coloca en su libro para medir una arista del prisma B, pero no sabe cómo utilizarla. La profesora muestra cómo hacerlo sobre una arista del prisma C diciendo:] Esta medida corresponde a lo que ya tienen ustedes, ¿Creen que les puede servir?

E7: No o ¿sí?

L: Ah, no sé, pues ve. […]

E7: 1, 2, 3, 4, 5, 6.

L: Ponle número para que no te equivoques. Ahora…¿Cuántas medidas te faltan? […]

E7: 1, 2, 3, 4, 5, 6, 7… ¿7?

L: 7 anótale ahí. Tienes tres medidas ¿Puedes sacar ya el volumen? […]

L: Si midió con regla, no se dio cuenta que esto no equivale a un centímetro, es menos de un centímetro, son como 8 milímetros [La profesora nuevamente saca su tira graduada] Por eso medí la misma que está marcada en el libro y la ocupé aquí y la ocupé acá. […]

<div align="right">(Fragmento 2 de la sesión de Laura)</div>

En contraste con Rocío, Laura usa otras estrategias para resolver el problema del cálculo de volumen del prisma C, construyendo un instrumento para hacer una medida directa de longitudes. Esto evidencia el uso adecuado de unidades de medida no convencionales (KCC).

Reflexiones finales

Al resolver problemas que involucran volumen emergen diferentes estrategias como uso de estimación, de dibujos, de representaciones con material concreto, uso de algoritmos, conteos, etc. Esta pluralidad de estrategias y su uso adecuado en la clase de matemáticas involucra conocimientos del profesor, tanto del propio contenido, y su relación con otros conceptos, como de su didáctica. Sin embargo, el trabajo realizado por Rocío y Consuelo evidencia un apresuramiento por una aproximación tridimensional del volumen, por el uso de la fórmula y de la unidad cm^3 (SCK y KCS), y evidencian dificultades para trabajar con unidades de medida no convencionales (KCC). Zevenbergen (2005) señala que este enfoque domina la mayor parte de las prácticas matemáticas escolares y en los programas de formación del profesorado, pues parece que "los adoctrinan en esta forma de trabajar" (p. 16). Además, como se mostró en los fragmentos de las tres profesoras, si no se tiene confianza en el conocimiento que se posee y cuando hay respuestas incorrectas, principalmente buscan resolverlas en un contexto aritmético y no geométrico. En este reporte se da cuenta de cómo analizar conocimientos matemáticos (o carencias) propios de su profesión como profesoras de matemáticas. Sin embargo, se requieren más investigaciones que fortalezcan la aproximación teórica para acceder a estos conocimientos.

A partir del análisis de las prácticas de las tres maestras, consideramos al igual que Zevenbergen (2005) y Aslan-Tutak & Adams (2015), que en los programas de formación inicial y continua así como de desarrollo profesional, se requiere de involucrar a los docentes en prácticas transformadoras donde la exploración y la compresión de los procesos sea el centro y no el obtener una respuesta correcta; donde el análisis de las actividades realizadas por los estudiantes sean analizadas a fin de reflexionar sobre sus aprendizajes; y tengan oportunidades para establecer conexiones entre distintos

conceptos geométricos (Bozkurt & Koç, 2012). Además de que desarrollen actividades en las que profundicen sus propios conocimientos geométricos.

The study of three-dimensional objects is undertaken in the Mexican curriculum throughout all levels of basic education. However, few studies relate to the mathematics knowledge needed to teach those ideas. This report represents an approach to the study of such knowledge. The approach is based on an account, developed from class observations, of three six grade teachers and their use of mathematics knowledge in teaching their students to calculate the volume of prisms, in two schools in Mexico City. The results provide evidence related to some sub-domains of mathematics knowledge for teaching. While there are some strengths related to this knowledge, teachers also showed that there is room for improvement. This speaks to the need to develop training spaces (both for pre-service and in-service teachers) where the analysis and discussions stimulate learning of both the geometric and pedagogical knowledge needed for teaching practice.

Keywords: Geometry and Geometrical and Spatial Thinking, Mathematical Knowledge for Teaching, Elementary School Education

Introduction

Geometry enables students to study figures within space and their relationships. Geometric space is built on the basis of empirical exploration of a real space that is taken to a geometric abstraction. In Mexico, teaching these ideas and, in particular, the study of three-dimensional geometry pervades the entire basic education curriculum (SEP, 2011). Aspects to be worked on include: "exploration of the characteristics and properties of [...] three dimensional shapes" and "knowledge of the basic principles of spatial location and geometric calculation" (p. 73). In these topics teacher mediation is key, thus the mathematics and pedagogical knowledge of teachers is essential as well.

Learning the volume of three-dimensional shapes is particularly difficult for students. Consequently, the relevance of teacher knowledge regarding this content is even greater, if possible, than that for teaching other content that is less difficult for students. Some of these difficulties are related to the close relationship between the notions of volume and capacity (Freudenthal, 1993; Saiz, 2002; Zevenbergen, 2005), the distinction between the mathematics and the physical concepts of volume (Saiz, 2002) and the importance of being able to visualize the concept in order to understand it (Gutiérrez, 1998).

Few studies have looked at elementary school teachers' knowledge of three-dimensional geometry (Aslan-Tutak & Adams, 2015; Tekin & Isiksa, 2013). Said studies underscore pre-service and/or in-service teacher difficulties related to conceptual and spatial skills. Thus Zevenbergen

(2005) states that pre-service teachers face similar obstacles as elementary school students, obstacles that are linked to the development of number sense, measurement and space. These are shortcomings that hamper their ability to identify errors among their students, associated for example with solving volume calculation problems. Teachers in training for elementary school state that, in their mathematics and teaching courses, they fail to develop a profound understanding of elementary geometry contents, while they recognize the importance of representations and visual skills in this area of mathematics (Aslan-Tutak & Adams, 2015). As for understanding 3D shapes, Dorantes (2008) points out the shortcomings in the knowledge of a group of teachers related to identifying characteristics of polyhedra, in terms of their edges, vertices and faces, exploring them through observation from different perspectives and calculating their volume. Such results coincide with those of other studies (Bozkurt & Koç, 2012; Saiz, 2002). In particular, Çakmak, Baş, Işık, Bekdemir, and Özturan (2015) finds that student definitions, examples and ways of solving problems are analogous to those used by their teachers in classrooms, and in many cases these methods are limited.

This report is part of a broader research project that seeks to identify and describe the *mathematics knowledge for teaching* evidenced by sixth grade teachers in teaching the volume of prisms (Moctezuma, 2015).

Geometric visualization, representations and volume of three-dimensional objects

Visual perception supports the learning of spatial geometry, because visual representations are an essential means of anticipation. According to research, such as Gal & Linchevski (2010), students tend to rely on visual information. The initial focus in geometry teaching relies more on the objects than on the processes, since the primary interest is on the figural properties as perceived by the senses and interpreted by mental reflection mediated by way of spatial representation. However, while any representation of a 3D shape highlights some features of the information, it hides others that may be difficult to retrieve for the person interpreting the representation. Del Olmo et al. (1993) explain that children's difficulties when measuring volume may originate from an absence of prior manipulation. They do not master spatial visualization given that they lack the ability to mentally manipulate rigid figures.

There are different ways of representing geometric objects for teaching purposes: in perspective, parallel, isometric, layered, orthogonal and nets of three-dimensional figures. Each of them requires learning conventions to interpret them, and developing skills to reproduce them (Gutiérrez, 1998). The types of representations used in the fifth and sixth grade textbooks (SEP, 2012)[i] for projecting cubes, prisms and pyramids are parallel and isometric projections, as well as nets. However, as will be shown below, teachers are unaware of such diversity between the different representations, which impacts their use in the classroom.

Mathematics knowledge for teaching prisms and volume in elementary school

Characterization of knowledge for teaching is based on the ideas of "Pedagogical content knowledge" (PCK) as presented by Shulman (1986) and later specified in greater detail in the case of mathematics, in the sub-domains of Mathematics Knowledge for Teaching (MKT) by Ball, Thames, and Phelps (2008). The latter authors differentiate between two domains: *subject matter knowledge* and *pedagogical content knowledge.* The former domain includes content that the teacher is teaching, as well as its foundations. It is subdivided into *Common Content Knowledge* (knowledge of a person instructed in that content) (CCK[ii]), *Specialized Content Knowledge* (content knowledge that distinguishes teachers) (SCK) and *Horizon Content Knowledge* (an awareness of how mathematics topics are related across the span of mathematics in the curriculum) (HCK). *Pedagogical content knowledge* includes knowledge possessed by the teacher with respect to teaching the content, learning and school curriculum. *Knowledge of Content and Students* (KCS) is the combination of

content comprehension together with knowing what students can think or do in mathematics. It involves identifying previous concepts and anticipating likely learning difficulties that students may face, as well as misconceptions regarding mathematics content. *Knowledge of Content and Teaching* (KCT) combines content comprehension and familiarity with pedagogical principles so as to teach that content. Finally, *Knowledge of Content and Curriculum* (KCC) include teacher knowledge of mathematics as a subject, which includes its structure, and the learning that students are expected to achieve upon completion of their elementary education.

The question addressed in this report is: What mathematics knowledge for teaching is put into practice by teachers as they give a lesson on the volume of prisms in sixth grade?

Methodology

This research follows a qualitative approach with a descriptive scope. The main data are derived from the classroom practice of three teachers (who participated voluntarily), as they teach mathematics content related to the volume of prisms.

The research was conducted between 2014 and 2015 in two public elementary schools located in the outlying areas of Mexico City. The teachers taught all sixth grade subjects, including mathematics. All three teachers had more than 25 years of teaching experience and were graduates of a teacher-training college. Consuelo (C) is an expert in fifth and sixth grade, while Laura (L) and Rocío (R) have taught first through sixth grades[iii] (all names are pseudonyms). The three teachers had taught the sixth grade subjects for two years prior to this research. The study group was chosen intentionally, and the subjects' practice was followed over the course of several class sessions, during which the topics addressed in class related to three-dimensional geometry (this report focuses solely on a lesson related to the volume of prisms).

Fieldwork took place at different times of the school year over a period of thirteen months. The lessons were videotaped and transcribed. Three instruments were used in order to collect as much information on teacher knowledge as possible: non-participatory observations of different lessons, a field diary and a semi-structured interview (the latter for the purpose of supplementing the information obtained during the observations).

The observer used preset categories for analysis of the lessons observed. The categories are based on the description of the subdomains of MKT and their solidification into indicators as designed by Sosa (2011). The categories were adapted to the elementary level and specific content to be analyzed. The researchers refined the categories during the research process until the final indicators were obtained.

In the textbook, the lesson entitled "How many cubes are in the prism?" consists of four activities (SEP, 2012, pp. 162-163). Here we will focus on the first two, in which the students have to calculate the volume of four rectangular prisms from the amount of cubic units on the edges (Figure 1.a). In prisms A and B, the three dimensions (length, width and height) are identified and cubic units are shown; while in prism C some ticks are shown that allow the students to infer the measurement of the length of just two of the prism edges (Figure 1b).

In an a priori analysis of the lesson, the researchers identified a transition from a one-dimensional treatment of volume (using a cube as the unit of measurement) to a three-dimensional treatment (calculated using length, width and height), even if the title of the lesson highlights the former. Moreover, in the third activity the notions of volume and capacity are addressed.

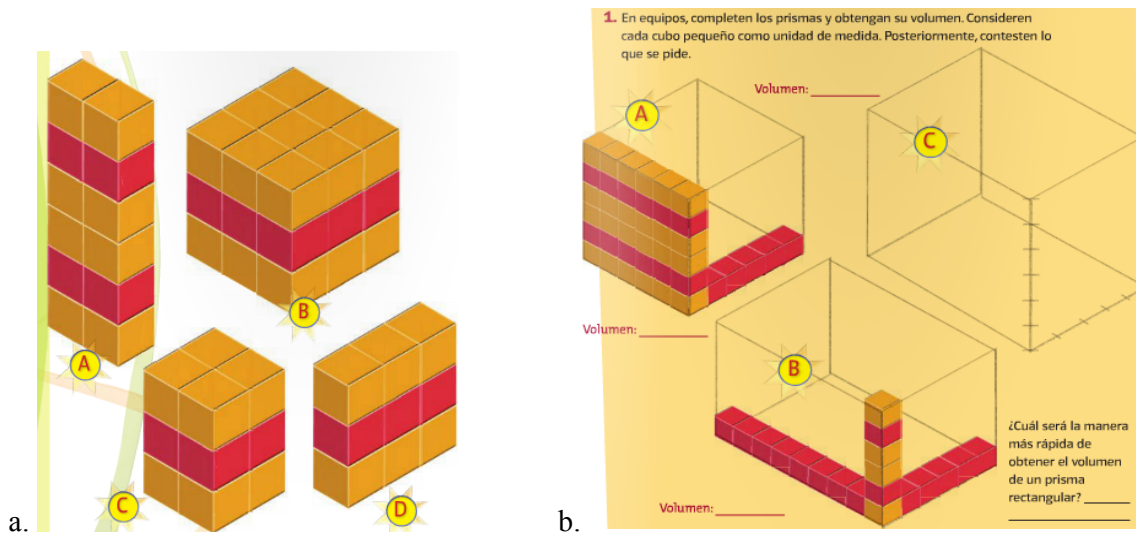Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

Figure 1. Activities proposed for working with volume of prisms (SEP, 2012, pp. 162-163)

Knowledge put into action. Description of findings and discussion

Two aspects of interest arose concerning teacher knowledge during our analysis of the three sessions observed, namely: a) the one/tridimensional treatment of volume, and b) how to calculate the unknown length of one of the edges in case C of activity 2.

One of the teachers, Rocío, treats volume in all the activities from a tridimensional perspective. As such, the last question raised in activity 2 becomes useless ("What is the fastest way to calculate the volume of a prism?"). From the very beginning, Rocío locates the length, width and height of each prism as a necessary step to calculate prism volume, as follows:

Rocío (R): Remember to take into account the three dimensions that we talked about, calculate the volume […] the number of prisms that those figures have, the number of units […]

R: […] How many units are there? Here [figure 1.b, section A] at the base. The width is 6, the length is 6 and the height is 6. So we multiply 6 times 6 times 6 and we get 216 […] In figure B, what are the base [sic], length, and width? […] What are the measurements? […] What is the length?

E8: 6 times 9.

R: […] We multiply 9 times 6, […] 54… 54 times 6 […] 324. [The teacher writes 324 cm^3 on the board].

(Fragment 1 from Rocío's session)

Rocío does not explicitly distinguish between the one-dimensional and the three-dimensional treatments for calculating prism volume. It would seem that she does not mark the differences between both treatments of volume, nor does she perceive the need for working with the one-dimensional perspective first so as to justify the tridimensional perspective, given its practical advantages; nor does she perceive the incidence of that sequence on comprehension of the volume magnitude.

However, Laura makes it possible for her students to calculate the volume of the prisms in activity 2, using the procedure the students deem appropriate. Hence, two different procedures arise. The first uses a unit of volume, while the second uses the length of the edge and the three-dimensionality of the prism.

Laura (L): [Reads] Complete the prism, do we need to complete it? [...]

E5: Yes.
L: Fill them with cubes?
E5: No.
L: What do you think, E7, [...] Let's see, do we need to complete the prisms?
E7: We can also multiply.
L: Multiply, what were you doing?
E7: I was multiplying the length of the base times height, and then times width.
L: [...] And what were you doing? [Addressing E8]
E8: Counting them.
L: Counting the cubes [...] Complete. Everyone has his or her own method of obtaining the volume.

(Fragment 1 from Laura's session)

The students refer to both the one-dimensional and the three-dimensional treatments for obtaining the volume, while the teacher gives them the freedom to apply the method they prefer. It seems that Laura implicitly distinguishes between the one-dimensional and the three-dimensional procedures for calculating prism volume (SCK), although she does not appear to specify the one-dimensional treatment with as much insistence as she does with the three-dimensional treatment.

One of the most difficult aspects of the lesson relates to activity 2 (Figure 1b). Representation of prism C does not enable a visual inference of the length of one of its edges. The discussion that arises in class relates to the unit of measure. In the three sessions, different ways of guiding this situation are observed, and this can be taken as different manifestations of MKT of these teachers. The knowledge relates to the role of the unit in the process of measuring and its conceptualization (SCK); knowledge of the possibility of using unconventional measurement units (KCC); the difference between measurement, unit, instrument and measurement procedure (SCK) and typical student misconceptions on the subject (KCS); student difficulty in recognizing and using unconventional units and instruments (KCS); and the approximate nature of the measurement (SCK).

Rocío does not seem to be sufficiently aware of the role of the unit in the measurement process and its conceptual role in the notion of measurement (SCK), as well as of the difficulties triggered amongst the students (KCS). Thus, to solve the problem that arises when calculating the volume of prism C (activity 2), she decides to measure using a ruler:

R: [Addressing a student] Lend me your ruler. [She uses it to measure in her book]
E7: 1 centimeter is two cubes.
E5: There are 5 centimeters.
E8: There are 6.
R: Yes, it is half a centimeter. So each of the sides is half a centimeter. So, if I measure… Who said that 11? It's 11 [...]
R: 11 times 8? [The teacher continues to measure] Height is 10, pay attention: it is 5 centimeters, each of the units represents half a centimeter; it is 10 units. Measure it with your ruler. Each unit represented in the textbook measures half a centimeter. The cubes measure half a centimeter.
 [...]
R: Remember that in this cube that is, that in this prism represented here, we have the length, width and height. The length is 11.
E8: It is 11 times 10 times 8, then… [The teacher nods to show approval]

(Fragment 2 Rocío's session)

Rocío discards the length measurements already given in the prism (according to the unit given) and measures each of the dimensions (length, width, and height) with a ruler. She uses the centimeter as a

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unit for measuring length, establishing equivalence between the centimeter and the unit of length given in the activity (half a centimeter). The representation given in the activity shows that the prism is 5 units wide and 6 units high; the length is unknown. The teacher's measurement results are 4 centimeters wide, 5.5 centimeters long and 5 centimeters for the height. She translates this into 8 for the width, 11 for the length and 10 for the height. She decides to take a different unit of length measurement than what is provided in the textbook and does not make it clear to the students that she has changed the length unit for the three dimensions of the prism in order to calculate the volume. Moreover, the choice of the unit and the length measuring instrument used may lead us to assume that she is not taking into consideration a possible student misconception, in which the student assumes that in order to measure (a length) it is necessary to use the metric system (in this case cm) and conventional instruments (KCS).

Consuelo also resorts to use of a conventional ruler (graduated in cm) to resolve the situation:

Consuelo (C): […] What did you do to take the measurements of the prism in C?
Student 9 (E9): I measured with a ruler.
C: […] Who did something else? [She points to a student and asks the student] Did you also measure with a ruler?
E5: Yes, but I measured the bottom part to see if […] it had the same width.
C: If it had the same width… Did you measure the width, E7?
E7: I measured the length of one little line, but it was complicated with the ruler on the lines, and I measured the six little cubes with the ruler and then saw how many of them fit into the width and the length.
C: Can you tell me what you measured? How many cubes are there lengthwise in that figure?
E7: Lengthwise, there are 7, 4 on the width and 6 on the height.
E5: The width is 5.
E3: There are 6.
C: 4, 5, 6… How are we to know?
[…]
C: Well, remember when you did it unanimously, often our rulers are not precise […] [and] different measurements [are obtained].
E7: I got 8 millimeters […]
E8: It measured almost a centimeter.
C: Almost a centimeter, but not exactly. […] So, let's take them all as [if they were] one centimeter; if it is one centimeter, E8, tell me how much it will be lengthwise for the prism.

(Fragment 1 Consuelo's session)

As illustrated in the dialogue, Consuelo considers the possibility that the volume being calculated may not be precise, but rather approximate; and initially, she considers the cubic centimeter as a unit of measure, although she does not make this explicit though her actions. The teacher encourages her students to use the ruler so that, by consensus, they define the unit of measure for the unknown length, based on the known measurements so as to calculate the volume of prism C.

Finally, Laura treats the situation differently. She proposes to use the measurement unit given in the activity and use of a measurement instrument that uses that same unit.

L: […] How many units does […] figure C have?
E10: One.
L: Just one. What can we do to get the other one?
E2: Multiplying?

L: Multiplying? But, if we do not have the measurement, what are we going to multiply it by? [The teacher takes a piece of cardboard, cuts a strip, graduates it with segments equal to the ones that are marked on one of the edges, and calls it "the little ruler". Then, she turns back to E10]. We will measure the only measurement that we have and I'll mark it on this "little ruler". It will be my little ruler to measure the other edges. How many measurements do I have now?

E5: 3.

L: How many do I need to calculate the volume? Just those ones; so, was it very difficult?

And what did we do? With our little ruler we saw how many times it fits into each one, now calculate the volume. […] [E7 puts it on her textbook to measure an edge of prism B, but does not know how to use it. The teacher shows the student how to do it on an edge of prism C, and says:] This measurement corresponds to what you already have; do you think this can be useful to you?

E7: No or yes?

L: Oh, I don't know, you have to check […]

E7: 1, 2, 3, 4, 5, 6.

L: Put numbers in so that you don't make a mistake. Now… how many units are you missing? […]

E7: 1, 2, 3, 4, 5, 6, 7… 7?

L: 7, write it down there. You have three measurements. Can you obtain the volume now? […]

L: If you measured with a ruler, you did not realize that this is not equivalent to one centimeter. It is less than a centimeter. It is about 8 millimeters. [The teacher again pulls out her graduated strip.] That is why I measured the one marked on the textbook and used it here and used it there […]

<div align="right">(Fragment 2 Laura's session)</div>

As opposed to Rocío, Laura uses other strategies to solve the problem of calculating the volume of prism C, by building an instrument to make direct length measurements. This demonstrates the proper use of unconventional units of measure (KCC).

Final Thoughts

When solving problems that involve the notion of volume, different strategies emerge, such as estimating, use of drawings, representations using concrete material, algorithms, counting, etc. This plurality of strategies and their appropriate use in mathematics class involve teacher knowledge, both of the content itself and its relationship with other concepts, as well as its didactics. However, the work done by Rocío and Consuelo shows a hastening to use a three-dimensional approximation of the volume, use of the formula and of cm^3 as the unit (SCK and KCS). Moreover, working with unconventional units of measure (KCC) proves difficult for them.

Zevenbergen (2005) points out that this approach permeates most school mathematics practices and teacher preparation programs, because it seems that they "indoctrinate them in this way of working" (p. 16). In addition, as shown in the three teacher fragments, if the teacher is not confident of the knowledge she possesses, then when wrong answers come up the teacher primarily seeks to solve the problems within an arithmetic rather than a geometric context. In this report, the authors illustrate how to analyze mathematics knowledge (or a lack thereof) *per se* that pertains to their profession as mathematics teachers. However, more research is needed to strengthen the theoretical approach to access that knowledge.

From the analysis of these three teachers' practices, we agree with Zevenbergen (2005) and Aslan-Tutak and Adams (2015) in that initial and lifelong teacher preparation programs and in professional development, teachers need to be involved in transformational practices where exploration and

understanding of the processes are the core, rather than finding the right answer; where student activities are analyzed in order to reflect upon their learning; and where they have opportunities to make connections between different geometric concepts (Bozkurt & Koç, 2012); in addition to their being able to carry out activities that deepen their own knowledge of geometry.

Endnotes

[i] In Mexico, primary education is free and compulsory. To this end, the Ministry of Public Education (SEP) generates materials such as free textbooks for the students and teacher guides, which are mandatory for all primary schools.

[ii] We will use the acronyms in English: SCK (Specialized content knowledge), KCS (Knowledge of Content & Students), KCT (Knowledge of Content & Teaching), KCC (Knowledge of Content & Curriculum).

[iii] The teachers' ages are: Rocío is 53, Laura is 55 and Consuelo is 59 years old.

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EXTENDING HORIZON: A STORY OF A TEACHER EDUCATOR

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We present a story of a teacher educator's response to a 'disturbance' and describe how her experience enhanced her personal mathematical knowledge and influenced her teaching. In our analysis we attend to different levels of awareness that support a teacher educator's work and illuminate the qualities of a teacher educator's knowledge, in particular, knowledge at the mathematical horizon.

Keywords: Mathematical Knowledge for Teaching, Teacher Knowledge, Advanced Mathematical Thinking

Numerous studies on mathematics teacher development have demonstrated that mathematics teachers learn through their teaching experiences (e.g., Leikin and Zazkis, 2010). This learning is multi-faceted and includes personal pedagogical growth, gaining further insights into students' thinking, learning about the feasibility of various instructional approaches, learning about implementing new curricula or technological tools, as well as enhancing personal mathematics. We extend this research on teachers' "learning through teaching" by focusing on a teacher educator, a teacher of teachers.

FOCUS ON TEACHER EDUCATORS

Research on the work of teachers of mathematics has devoted significant attention to teachers' knowledge and its various facets. Consequently, research on the work teacher educators follows suit (e.g., Jaworski, & Wood, 2008). This is evident, for example, in the working session at the International Group for the Psychology of Mathematics Education Conference on "Mathematics teacher educators' knowledge" (Beswick, Goos & Chapman, 2014.) Within multifaceted discussions on educators' knowledge, our focus is on personal mathematical knowledge in the work of teaching (Watson & Chick, 2013). In addition to knowledge of a teacher, personal mathematical knowledge of a teacher educator includes the ability to mobilize his/her knowledge in supporting teacher development. In particular, this support includes task design aimed at enhancing teachers' mathematical and pedagogical knowledge (Liljedahl, Chernoff & Zazkis, 2007). Particular qualities of teacher educators' knowledge that enable task design in support of teacher development, and how this knowledge develops and manifests, continue to be open areas for investigation.

THEORETICAL CONSTRUCTS

To gain insight into a teacher educator's personal mathematics knowledge, how this can develop through the act of teaching and the subsequent implications for teaching, we rely on an integration of two theoretical perspectives: Mason's *levels of awareness* (1998), and an extension of the notion of *knowledge at the mathematical horizon* (Zazkis & Mamolo, 2011).

Levels of Awareness

According to Mason (1998), awareness in and for teaching has 3 three different forms:

- *Awareness-in-action*: the ability to act in the moment. This level of awareness in teaching is recognized when a teacher poses a certain question, corrects a mistake, suggests an answer or selects a task, but is unable to justify or explain his/her choice.

- *Awareness-in-discipline*: awareness of awareness-in-action. This awareness is essential in order to articulate awareness-in-action for others. According to Mason, the one important distinction between the two kinds of awareness involves the ability "to do" in contrast with the ability to instruct others. Teachers who possess awareness-in-discipline are able to articulate the choices they make in instructional situations.
- *Awareness-in-counsel*: awareness of awareness-in-discipline. This awareness is essential in order to articulate awareness-in-discipline for others. It describes one's sensitivity to what others require for building or enhancing their awareness.

Each form of awareness may refer to being explicitly or potentially aware, where the former emerges from the latter by noticing change (shifts of attention). Thus, if we wish (and we do) that mathematics teachers are able to articulate the choices they make for pupils in instructional situations (awareness-in-discipline), then a teacher educator must develop sensitivity to what may foster awareness-in-discipline for her students (and so possess awareness-in-counsel).

Knowledge at the Mathematical Horizon (KMH)

As an extension of Ball and Bass' (2009) notion of horizon content knowledge, KMH is identified by Zazkis and Mamolo (2011) as a teacher's use of mathematical subject matter knowledge "beyond" the requirements of school curricula in a secondary or elementary school teaching situation. Ball, Thames, and Phelps (2008) describe horizon knowledge as "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (p. 403). This notion was developed by Ball and Bass who focused on teachers' knowledge of "students' mathematical horizons."

Zazkis and Mamolo (2013) extended the notion to teachers' horizons by making a connection to Husserl's philosophical description horizon, which relates to an individual's focus of attention. In particular, when an individual attends to an object (be it conceptual or physical), the focus of attention centers on the object and (some of) its properties, while the 'rest of the world' fades to the periphery and thus exists in the object's horizon (Follesdal, 2003). A teacher's horizon knowledge can therefore be interpreted as an awareness of a mathematical object's periphery, and can be characterized by a flexibility in focus of attention such that relevant properties, generalities, or connections, which embed the object within a greater mathematical structure, are accessed in teaching situations. A teacher's KMH is influenced by her propensity for exploring and studying new (for her) mathematics; teachers of different (school) levels may have vastly different objects of focus, different senses of the mathematical 'landscape', and thus different breadths to their horizons.

We draw a link between a teacher's KMH and Mason's notion of awareness-in-discipline, as knowledge of how mathematical topics are related as per their specific properties or underlying general structure provides a basis for understanding and articulating mathematical choices made in teaching. When considering a teacher educator's KMH, we note that both the objects of focus and the breadth of horizon extend beyond the awarenesses of the mathematics teacher. In what follows, we seek to refine the notion of KMH to suit studying the work of mathematics teacher educators.

Research question: How can the work of a teacher educator be explained in terms of levels of awareness and knowledge at the mathematical horizon?

THE STORY IN TWO ACCOUNTS

In presenting our story of Naomi – an experienced teacher educator, who has taught both content and methods courses in mathematics education – we follow Mason (2002) in distinguishing between *account-of* and *accounting-for*. The term *account-of* refers to a brief but vivid description of the key elements of the story, suspending as much as possible emotion, evaluation, judgment or explanations. This serves as data for *accounting-for*, which provides explanation, interpretation, value judgement

or theory-based analysis. Naomi, the main character of our story, is an amalgamation of the authors' experience. This is consistent with the narrative inquiry methodology, where "narrative inquiry is aimed at understanding and making meaning of experience" (Clandinin & Connely, 2000, p. 80).

Account of

Naomi's class of prospective elementary school teachers, working on conversion of square units, considered the following "area problem":

> *An architect is building a model of a park, in which every 10 meters are represented by 3 centimeters. There is a lake in a park. The area of the lake is 7200 square meters. What is the area of the lake in the model?*

The problem and its solution scheme paralleled a task that was previously discussed and resolved in class. The approach in class had been to build a proportion to solve the problem, and Naomi expected students to do the same in this case. The solution scheme for the task suggested:

> *Since 10 m are represented by 3 cm, then 100 m^2 are represented by 9 cm^2.*
> *Possible proportions useful for solving the problem include:* $\frac{7200}{x} = \frac{100}{9}$ *or* $\frac{7200}{100} = \frac{x}{9}$,
> *which yield as a final answer:* $x = 648$ *cm^2.*

While Naomi expected students to note the relationship that 100 square meters are represented by 9 square centimeters and use this for resolving the problem, not all students took that approach. Mikey, for instance, composed a different solution:

> *7200 = 80 × 90*
> *if 10 meters = 3 cm, then*
> *80 meters = 24 cm*
> *90 meters = 27 cm*
> *Then, 24 × 27 = 648, so the area of the lake on the model is 648 cm2.*

Naomi recalled the following conversation with Mikey:

Naomi: Mikey, please explain how you got your answer.
Mikey: If the lake had length of 90 meters and width of 80…
Naomi: Did you know the shape of the lake?
Mikey: So if it were a rectangle…
Naomi: Have you ever seen a rectangular lake?
Mikey: I pretended it was, and I got the correct answer.
Naomi: And what if it were a different rectangle, would you get the same answer?
Mikey: I have not checked, but this one worked.

Upon additional prompting, Mikey could not explain why his approach was appropriate; "it worked" seemed to be a sufficient reason to accept its correctness. Naomi, unsure at that moment how to explain the situation, diverted the issue, pointing out that Mikey's representations were inappropriate and that the equality sign in the presented solution was repeatedly misused (such as in claiming "*10 meters = 3 cm")*.

Upon reflection, Naomi realized why assigning random measures results in a correct answer. One can think of the shape in the model as an image of the shape of the lake under the transformation of dilation/scaling. This transformation preserves area-relations in a way that equal area shapes are always transformed to equal area shapes. This invariance under transformation is why the student's solution resulted in a correct answer: It calculated the area of a particular scaled shape, and this area is invariant across all the shapes that are images under dilation of the shapes with the same area. While this understanding of mathematical structures was accessible to Naomi, it was not immediate

and it did not seem evident in her teacher candidates, and as such she saw the 'incident' as an opportunity to develop related knowledge among her prospective secondary mathematics teachers.

Naomi developed a task in which she presented a problem and a solution, similar to that of Mikey's, and asked her students to imagine how a conversation between a teacher and student addressing the problem could continue, and to present it in a form of a scripted dialogue. Furthermore, Naomi invited her students to include commentary on how they personally understand the situation and to explain it to a "mathematically mature" colleague. These invitations were aimed at directing possible shifts of attention amongst teacher candidates' that could foster awareness-in-discipline as they attempted to articulate why specific instructional moves were made.

The scripts produced by prospective teachers served as a springboard to class discussion. This included consideration of a variety of pedagogical approaches, but focused on why the student's solution resulted in a correct answer. (A precise mathematical reason was absent in the scripts). Naomi introduced the concept of transformations of dilation/scaling using the dynamic geometry software Geometer's Sketchpad. (The software was familiar to the students, but the particular transformation was not). Her students used the software to explore the features of this transformation, including its invariants.

Accounting for (or Analysis)

Naomi's encounter with Mikey elicited for her a moment of disturbance – it was an unexpected solution and one for which she could not come up with an immediate and satisfactory, at least for herself, rebuttal. In response Naomi revisited the mathematics herself and in doing so, her awareness was broadened on two counts – first with respect to the aforementioned mathematical connection, then with respect to the pedagogical value of addressing such an interpretation as Mikey's.

As a teacher of mathematics, has Naomi learned mathematics as a result of her encounter with the unexpected student solution? After all, she was sufficiently familiar with the transformation of dilation in order to explain the phenomenon mathematically. The situation is similar to that described by Leikin and Zazkis (2010) in which teachers are hesitant to admit learning mathematics through teaching. Often their inability to use relevant mathematics in an instructional situation was explained as "I knew this, but have never thought about it". We agree with Leikin and Zazkis in their claim: "we consider anew "thinking about it" – when an instructional situation presents such an opportunity – as an indication of learning. In this case LTT [learning through teaching] can be thought of as transferring existing knowledge from teachers' passive repertoire to their active one" (p.19).

In Naomi's case she drew a connection between modeling tasks and particular geometric transformations, notions that were not explicitly linked for her before the experience of disturbance. This new connection highlights a change in Naomi's KMH: it broadens the periphery of the modeling task to include connections to the (initially) seemingly disparate topic of invariance under dilation. Knowledge of such connections corresponds to knowledge of major disciplinary ideas and structures and as such instantiates one of the elements of Ball and Bass's (2009) conception of horizon knowledge. Further, it embeds the particular mathematics originally at Naomi's focus of attention (conversion of square units) within a broader mathematical world. It was her awareness of the 'broader world' which allowed her to articulate for herself and her students why the particular features of the geometric transformation of dilation allow the random assignment of convenient measures to lead to a correct result.

This situation illustrates how knowledge beyond the specific curricular requirements is pertinent to mathematics teaching. A teacher's personal mathematical knowledge provided a deeper appreciation of the mathematics at hand, as well as a broader view of the salient features of the content. The technique of applying a transformation or mapping that preserves certain properties allows an individual to shift from working in a potentially unfamiliar space to one in which known properties aide in resolving the problematic situation. Such values and sensibilities, typically

acquired in advanced mathematics courses at the undergraduate and graduate levels, embed particular topics or concepts within a broader, more general, mathematical world, and as such are included within a mathematical object's outer horizon. These sensibilities correspond to the habits of thoughts and methods of resolutions that Mason attributed to *awareness-in-discipline.* They provide insight into why certain relationships are true and aide in the articulation of said truth.

The goals of the "area problems" were focused on appropriate conversion of square units, building proportions and developing proportional reasoning and computational relationships. The moment of disturbance provoked a shift in attention, redirecting Naomi's line of vision towards alternative mathematical ways of reasoning with the task. This shift enabled a formal resolution and it illustrated not only the facts and techniques of the discipline of mathematics, but also the "habits of thought, forms of fruitful questions, and methods of resolution of those questions" (Mason, 1998, p. 259) characteristic of awareness-in-discipline. However, in developing awareness-in-counsel, as a teacher educator, Naomi's aim is to share this interesting link with prospective teachers and in turn support their own development of awareness-in-discipline.

As a teacher educator, Naomi developed a task for prospective teachers, which has both mathematical and pedagogical value. We suggest that it was Naomi's awareness-in-counsel that invited her to reflect on the contingency of the situation as one which would be of benefit in preparing teacher candidates and in turn she developed an activity that could help broaden her students' perspectives. As Mason put it, awareness of awareness-in-discipline "is what supports effective teaching of that discipline" and allows "for structuring tasks and encounters from which [students] can learn" (1998, p. 260). In what follows, we offer an account-for that attends to Naomi's awarenesses of the discipline of mathematics-teaching.

In particular, clearly, the immediate goal for prospective teachers was to imagine an interaction with a student and present it in the form of a dialogue. In doing this, the prospective teachers considered their pedagogical response to a mathematical dilemma presented by the student's solution. However, Naomi's goal extended beyond involving teachers in playwriting. She considers the task as a vehicle for (1) extending personal mathematical experiences of teachers beyond the prescribed curriculum by drawing stronger connections between various topics in mathematics (e.g., units conversion and transformations); and (2) anticipating possible solutions and evaluating potential reactions to those. Both of these goals connect to fostering teachers' KMH – broadening the peripheral vision of teachers as they focus on particular mathematical objects by connecting the objects both to other mathematical topics, as well as to how students could interpret and reason with those objects based on their prior learning experiences.

The scripting task builds on growing research which attends to various pedagogical motivations for involving students in playwriting (e.g., Zazkis, Sinclair, & Liljedahl, 2013; Zazkis & D. Zazkis, 2014). The benefits of this approach were summarized for prospective teachers, for researchers and for teacher educators (ibid.). Focusing on the latter, the scripts or plays written by prospective teachers can be used as a tool for teacher development. In particular, the scripts can be used to discuss and highlight appropriate pedagogical approaches, to direct further attention to learners and their thinking, and to shift prospective teachers' thinking about preparation to instruction, beyond the traditional "lesson plan". These activities nurture shifts in attention that may raise teachers' awareness of the contingent nature of teaching while providing them with experiences from which they may later draw when responding to unanticipated scenarios.

In an attempt to foster awareness-in-discipline (of mathematics) for her students, Naomi set a task that could provoke a similar disturbance in others and thus trigger a shift in their awareness. Such tasks are intended to "provoke students into rehearsing or exercising skills, but which at the same time attract their attention away from the skill to be automated" (Mason, 1998, p. 259). The skill of responding to student interactions (with appropriate pedagogical sensitivities and strategies)

was to be rehearsed through the particular task developed by Naomi, but attention was meant to be attracted by the unexpected mathematics.

Naomi's use of scripting tasks exemplifies her awareness-in-counsel of the benefits for preparing teachers in this way – the essential awareness for enhancing the awarenesses of others manifested in the design and development of the task, as well as in the follow up task and discussion. In addition, this particular task developed by Naomi not only invites teachers to examine their approaches to a student solution, but also invites them to revisit their personal mathematics and think mathematically in an unfamiliar situation.

The scripting approach allows the discussion to revolve around the particular approaches presented by the group of prospective teachers. Without singling out a particular student, Naomi initiates a discussion by presenting several different approaches from the scripts and inviting a reaction. The 'scripting-first' tactic provides all students with an opportunity to think and rethink their reaction. This usually does not happen in a whole-class discussion, where the first speakers may influence the ways others consider the task. Further, this opportunity to think and rethink the reaction does not have time constrains. The students may consult any resource or each other, should they choose to do this. This an important difference between script-writing and role play, where the player has to "think on her feet", rather than provide a thoughtful and possibly adjusted response. The time to provide a thoughtful response to a moment of contingency, as a structural aspect of the scripting task, creates space for deeper mathematical engagement – and thus enhancement of teachers' KMH – than is afforded by role play activities. The opportunity to imagine and re-imagine a teaching scenario is geared towards enhancing teachers' noticing, both by completing the task as assigned as well as considering the approaches of classmates in the subsequent classroom discussion. Structuring a task such that students are provided with such an opportunity again shows Naomi's awareness of what are important encounters from which to develop knowledge at the mathematical horizon.

Thus, Naomi's response to a disturbance, coupled with her choice to draw to the attention of her prospective teachers the contingent nature of teaching, illustrates the awareness-in-counsel of a teacher educator.

AWARENESS OF HORIZON OF TEACHER EDUCATORS

When Mason (1998) identified different levels of awareness, the intended discipline in his descriptions was the discipline of mathematics. In our analysis of the work of a mathematics teacher educator we point to yet another discipline – that of mathematics-teaching. With this view, a teacher educator's awareness-in-counsel, which is the awareness that is employed to develop mathematics teachers' awareness-in-discipline (of mathematics), can also be described as the teacher educator's awareness-in-discipline of mathematics-teaching. This awareness results in the development of mathematically salient scenarios that could provoke a moment of disturbance for prospective teachers. We turn our attention now towards the mathematical knowledge required to notice and appreciate such saliency.

We suggest that it is the knowledge at the mathematical horizon of teacher educators (KMHTE) that includes knowledge of what mathematics will catch the attention of prospective or practicing teachers and provoke a disturbance that enables the shifts of attention required to develop their own horizon knowledge (KMH). The disturbance which shifts attention must also provide an invitation to extend the mathematical awareness of teachers in connection to what is applicable for students. It must provide opportunities for teachers to respond without readily available routines and thus incite a shift from awareness-in-action to awareness-in-discipline. At the focus of attention for Naomi is both the specific mathematics content, as well as its contextualization in student/teacher thinking. At the periphery of Naomi's attention is the horizon, accessible via her awareness-in-counsel.

Analogously to a teacher's KMH, we suggest that a teacher educator's KMHTE may be described as having both an inner horizon and an outer horizon. As before, we conceptualize horizon

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as the connections, features, and generalizable properties related to an object of thought and which embed that object within a greater structure. At the inner horizon are features of that object, which are specific to it, yet not at the focus of attention. At the outer horizon are the major disciplinary ideas, practices, and underlying structural components which situate the particular within the general. With respect to KMH, the object of focus is mathematics and as such the features, ideas, practices, and structures of the horizon are also mathematical.

For KMHTE, the object of focus is student/teacher thinking about mathematics, and as such, the features, ideas, practices, and structures of the horizon include a breadth and complexity not already encompassed by KMH. Awareness-in-counsel broadens a teacher educator's ability to direct (and retain) teachers' attention toward mathematical connections and structures, such that they enhance both their mathematical understanding, as well as their abilities to convey relevant aspects of this understanding to (and for) their future students.

CONCLUDING REMARKS

A mathematics teacher educator's disciplinary knowledge is multi-faceted and complex. It combines awareness of pedagogical practice, of mathematical content and processes, of the cognitive, social, affective, and environmental factors associated with thinking and learning. Indeed, an exhaustive list of the "required" knowledge for mathematics teacher education would be near impossible. We focused our attention on the personal mathematical knowledge of an experienced mathematics teacher educator and described how this knowledge was triggered and further developed. Inasmuch as mathematics may not be separated from mathematics pedagogical knowledge, mathematics may not be separated from mathematics education pedagogical knowledge – it is an essential component of the knowledge required to foster the professional growth of the individuals tasked with the responsibility of supporting students in their learning of mathematics.

Various researchers have suggested that teachers' personal mathematical knowledge makes a difference in their ability to plan for instruction (e.g., Watson & Barton, 2011) as well as respond to teaching situations in general, and situations of contingency in particular (e.g., Chick & Stacey, 2013; Rowland, Huckstep, & Thwaites, 2005; Watson & Chick, 2013). We extrapolate that this suggestion is also applicable for teacher educators. The main difference, however, is that the 'students' of a teacher educator are (in our case prospective) teachers, and therefore the teaching situations adhere to the learning of teachers. With this in mind, we illuminated some of the qualities of a teacher educator's mathematical knowledge for teaching – specifically, knowledge at the mathematical horizon – and her extended awarenesses.

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REHEARSALS OF TEACHING: ATTENDING TO MATHEMATICAL KNOWLEDGE FOR TEACHING

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We investigate how pedagogies of enactment that simulate the work of teaching, like rehearsals, afford practice-based work on mathematical knowledge for teaching (MKT). The analysis of 30 rehearsals from elementary methods courses shows that the teacher educator supported teachers' learning of MKT inside problem spaces emerging in the process of eliciting and representing student ideas, and through four types of interventions that helped novices consider aspects of their performance in relation to the mathematics and student learning.

Keywords: Mathematical Knowledge for Teaching, Teacher Education-Preservice

Research in mathematics education has established that high quality instruction requires mathematical knowledge for teaching (MKT). Ball, Thames, and Phelps (2008) describe MKT as the mathematical reasoning, skills, and understanding that are involved in various tasks of teaching. They identified a number of distinct domains of MKT, including: (a) common content knowledge (CCK), (b) specialized content knowledge (SCK), (c) knowledge of content and students (KCS), and (d) knowledge of content and teaching (KCT). CCK includes, for example, fluency with standard computational techniques. SCK is mathematical knowledge that is particular to teaching, such as choosing and using mathematical representations. KCS combines knowing about students with particular understandings about mathematics. KCT combines knowing about pedagogical issues that affect learning and specific mathematical understanding. In mathematics teacher education there has been increased awareness of the importance of MKT for novice teachers. Scholars have argued that to be effective, teachers not only need to *have* MKT but also learn it *for use* in the dynamic interplay of content with pedagogy in teachers' real-time problem solving (Ball & Bass, 2000). However, teaching MKT remains a challenge for teacher educators precisely because the nature of this knowledge is intertwined with practice, and requires supporting teachers in being able to use it adaptively in various situations.

In supporting novice teachers (NTs) to learn the complex work of teaching, some current approaches to teacher education employ pedagogies of enactment (Grossman & McDonald, 2008) that focus teacher learning on a set of principled instructional practices derived from research on student learning and professional standards. However, given that research on the nature of these pedagogies is in its early stages, it is still unclear how these practice-focused approaches to teacher learning can attend to the development of MKT. This study investigates this problem in the context of *rehearsals*—a teacher education pedagogy that approximates the actual work of teaching (Grossman et al., 2009) by engaging NTs in the deliberate practice (Ericsson, Krampe, & Tesch-Römer, 1993) of well-specified instructional activities with targeted coaching by the teacher educator (Lampert et al., 2013). We focus on how the teacher educator infuses considerations for MKT in the continuous back and forth that takes place in rehearsal.

Theoretical Framework: Problem Spaces and Guided Participation

Rehearsal as a teacher education pedagogy simulates the work of teaching where an NT leads an instructional activity while other NTs participate as students. Rehearsals are not scripted; they allow for improvised performance in response to student thinking and problems of practice (Lampert et al., 2013). Rehearsals open up *problem spaces* in which teacher educators (TEs) and NTs work together to develop new understandings. For Lampert (2010), problem spaces involve the relationships that

teachers manage as they support students' learning of content. They consist of various kinds of problems of practice that "arise in establishing and maintaining relationships with students and subject matter, and [where] the work that must be done to solve them is socially and intellectually complex." (p. 22). Lampert explains that problems of practice arise inside these relationships because of the constraints and possibilities that participants generate through their actions during instruction. In the context of teacher education, these problem spaces can provide TEs with opportunities to guide NTs participation (Rogoff, 1995), engaging them in a form of inquiry during which they actively seek to attend to these constraints and possibilities using available physical, social, and intellectual resources (Cook & Brown, 1999).

Study Context and Methods

The context of the study is mathematics "methods" courses for elementary NTs at three public universities in the US. The courses are part of a collaborative effort to design a model that enables NTs not only to develop knowledge about teaching and mathematics but also the ability to use the knowledge in actual interactions with children. Central to the model are instructional activities (IAs), designed as containers for the instructional practices, principles, and mathematical knowledge that NTs need to learn and be able to use in interaction with students (Lampert, Beasley, Ghousseini, Kazemi, & Franke, 2010). NTs rehearse the IAs publicly in front of their peers and the TE. Guiding participation in this setting, the TE acts as both coach and simulated student, enabling both the rehearsing NT and others in the group to study the range of actions a teacher might take in response to student mathematical thinking.

The data consist of 30 video-recorded rehearsals. The video-analysis software Studiocode© was used to create instances of video that were of two kinds: (a) instances where an NT was leading an IA as a teacher with other NTs and the TE participating as students (*NT Rehearsing*), and (b) instances where there were pauses in the simulation for exchanges between the TE and NTs (*TE/NT Exchange*). In order to understand how the TE supported opportunities for NTs' learning of MKT, we coded the above instances for the work that was being done in relation to the domains of MKT: common content knowledge (CCK), specialized content knowledge (SCK), knowledge of content and teaching (KCT), and knowledge of content and students (KCS). Concurrently, we analyzed for the way the teacher educator supported such participation by thematically coding the different interventions that she made during rehearsals.

Results

We start with a summary of the work that occurred during rehearsals in relation to MKT, and then provide a short vignette from one rehearsal to illustrate our findings. Our analysis revealed that novice teachers participated in two main ways in relation to MKT: (a) unpacking the mathematics as learners of the subject matter, and (b) enacting two types tasks of teaching that required mathematical insights and considerations: posing questions and using representations. These tasks were central aspects of the instructional activities that the novices rehearsed. Some key themes that emerged under these two tasks during either NT Rehearsing instances or TE/NT Exchanges included choosing representations, introducing a new representation, attending to the precision of representations, connecting representations, using representations to attend to mathematics, using questions to elicit answers or explanations, using questions to check on students' understanding, and using questions to emphasize a mathematical point. The analysis of the TE/NT Exchanges showed that the teacher educator supported opportunities to work on MKT by (a) providing insight into student thinking or the mathematics, (b) facilitating novices' unpacking of the mathematics, (c) noting affordances and constraints of instructional moves in relation to MKT, and (d) suggesting instructional moves based on MKT. The teacher educator provided this support inside problem spaces emerging in the context of enacting the tasks of teaching listed above. The teacher educator's interventions reflected

considerations for the development of novice teachers' MKT, and occurred in a continuous back and forth between novice teachers enacting particular practices and the teacher educator helping them consider aspects of their performance and their consequences in relation to student learning of the mathematics. The following excerpt of a rehearsal illustrates the kind of interactions that occurred between the TE and novice teachers, and depicts the way the teacher educator infused considerations for KCS and KCT in the work of eliciting and representing student thinking.

Vignette: Launching a Subtraction Problem and Attending to a Student Strategy.

Molly (All names are pseudonyms), the NT in the role of the teacher, starts launching a strategy sharing activity where students were to contribute in different ways to the solution of a problem. She states, "We are going to work on a problem today: 76 – 27." She then writes the problem on the board vertically, placing 27 under 76. As she prepares to continue with her instructions, the teacher educator intervenes: "Okay, actually, one thing I want you to do differently is, don't write the problem vertically like that. Write [the numbers] horizontally." The TE explains that stacking the numbers vertically encourages students to "imagine the standard algorithm for subtraction in their head," rather than producing a range of informal strategies, which is the goal of the instructional activity. Molly follows the TE's feedback and rewrites the problem horizontally.

Molly gives the students a little time to solve the problem and then asks them for their responses. She solicits three different answers: 49, 43, and 51. We focus on what happened when she elicited Nora's strategy for arriving at 49.

> *Molly*: How did you get 49 here? [Writes 49 at the end of the number sentence]
> *Nora*: I broke 27 up into 20 and a 7, and then I thought that 76 - 20 is 56. Then 56 – 7 is 49.
> *Molly*: [Looking skeptical] You just knew [that 56 – 7 is 49]? Like, how did you count it?
> *Nora*: I knew that, if I took a 10 from the 5, and I had a 4, I'd have a 16 minus 7 is 9, and then I
>   brought down my 4.

Molly represents Nora's strategy on the board. She decomposes 56 into 40 and 16, and then writes 16 – 7 = 9. Her action prompts another NT in the audience to exclaim, "That is a complicated way to do it!" In response, the TE intervenes, suggesting that 56 – 7 is, in and of itself, a good problem to give to students, because it taps into their invented strategies for subtracting across a decade, "You just go back 6 and then you go back 1 more." She also explains that most adults tend to use the traditional algorithm for such a problem, while kids often find more efficient ways to solve it. "Kids would count backwards, 55, 54, 53, …"

Guiding NTs' participation with considerations for MKT. Two problems of practice emerge in this vignette, opening opportunities for the TE to intervene and infuse the problem space with considerations for MKT. First, in her intervention following Molly's representation of the math task on the board, the TE signaled a problem of practice that could result from writing the subtraction problem vertically. Given the importance of allowing students to contribute in different ways to the solution of the problem, this vertical representation could limit the variety of solutions that students would present. The TE drew the NTs' attention to this problem by providing insight into how students might respond to such a representation (KCS), highlighting a general tendency in students to rely on the traditional algorithm when the numbers are stacked in this manner. Later when Nora said that she "just knew" that 56 – 7 is 49, Molly pressed her to explain her reasoning, and with her body language exhibited her perception of some constraint in relation to Nora's answer, suggesting that the leap from 56 to 49 is not trivial. The TE used this problem space to note how mathematical tasks that require students to subtract across a multiple of 10 on the number line could help support students' number sense (KCT). She also provided insight into how students may approach such a task: (a) counting back and (b) bridging across a multiple of 10 to subtract. Both of the TE's interventions supported the NTs in attending to both knowledge of content and students (KCS) and knowledge of

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content and teaching (KCT) *in* practice—in the moment to moment problem solving that teachers face in their work.

Discussion

Our findings suggest that pedagogies of enactment like rehearsals afford work on MKT. In the rehearsal of instructional activities, novice teachers have opportunities to consider and unpack mathematical ideas in the context of carrying out particular tasks of teaching like posing questions to launch a problem and using representations to attend to student thinking. The analysis of the vignette illustrates the way these opportunities play out with guidance from the teacher educator who uses her interventions during feedback exchanges to weave considerations for mathematics and pedagogy. When NTs participate in rehearsal, they have opportunities to unpack the mathematics from the perspective of both teachers and students. The TE strategically guides their participation toward multiple domains of MKT often simultaneously.

We end by highlighting two main contributions of this study. First, with content knowledge needs for pre-service teachers—especially at the elementary level—being so high, this study illustrates how MKT can be worked on in "methods" courses in ways that integrate the study of content with pedagogy. Second, this study illuminates how practice-based approaches to teaching MKT situate novices' learning about content in the face-to-face aspects of the work of teaching and in problem spaces where they actively engage this knowledge by attending to affordances and constraints that arise as a result of their experimentation with practice.

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A STUDY EXPLORING HOW TEACHERS ANALYZE AND USE FORMATIVE ASSESSMENT INFORMATION WITHIN INSTRUCTION

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This exploratory qualitative study describes how teachers in a yearlong professional development project conceptualized and used formative assessment within their instruction. The study focuses on how teachers interpreted formatives and the instructional decisions that teachers made. This paper documents the shifts in the lenses teachers adopted when interpreting student work and the instructional decisions that are possible based on the information teachers gain from looking at student work.

Keywords: Assessment and Evaluation, Instructional Activities and Practices, Learning Trajectories, Teacher Education-Inservice/Professional Development

Introduction

For decades, the research community has called for a shift in assessment practices towards assessment to inform instruction as opposed to evaluating learning (Black & Wiliam, 2009). Bridging formative assessment theory into practice is not a simple, straightforward endeavor (Allen & Penuel, 2015). Research has found that teachers' use of formative assessment practices remains limited (Otero, 2006). Dixon & Haigh (2009) point out that teachers experience difficulty translating what they know about assessment into practice. Therefore, it is important to understand how to support teachers to increase their effective use of this tool. This paper responds to the gap noted in the research (Dixon & Haigh, 2009) regarding the steps teachers take when interpreting formative assessments. It addresses the questions: (a) What qualities do teachers look for when analyzing assessments or observing students? and (b) How are teachers incorporating formative assessment information to make instructional decisions?

Theoretical Framework

For this paper, we use the definition of formative assessment provided by Black and Wiliam (2009), practice is formative when evidence is collected, interpreted, and *used* to enhance instruction with the purpose of improving instruction. The student information is the *feedback* that is used to tweak the instruction cycle to better adapt to the needs of the student. Teachers correlate two sets of information: the learning target (Where are they going?) and the student's current understanding (Where are they now?) to form an educated guess to plan the next best steps to support students achieving the latter.

Formative assessment is often described as a collection of practices and instructional techniques (Konrad, 2014). Underneath the instructional practices lies the three components of formative assessment: (a) collection of information, (b) analysis of student data, and (c) inclusion of analysis with instructional decisions.

Methods

Participants and Setting

This study was conducted in a western state. The thirty-nine teacher-participants constituted teachers from various elementary and middle schools across the district. During the summer session, teachers developed content knowledge (CK) and pedagogical content knowledge (PCK) through

active engagement with the Mathematical Practices. Follow-up sessions focused on supporting teachers' use of formative assessment.

Data Sources

This study drew on multiple data sources. Data was collected from teacher survey, field notes from each PD session, and focused reflections. *Survey.* Responses were marked on a five-point Likert-type scale. Teachers self-assessed the degree to which they were implementing the Whole Class Discussion Framework (*Whole Class Mathematical Discussions,* Lamberg, 2013). This framework was used to identify to what extent teachers report that they were implementing researched based practices in their classrooms. *Focused reflections.* Focused reflections provided researchers with direct input from the teachers. *PD field notes.* Field notes were typed for each PD session. This was ideal to develop an understanding of how teachers were tackling tasks that required use of formative assessment knowledge.

Data Analysis

Quantitative analysis of the Whole Class Discussion Framework pre- and post- questionnaire recorded changes in the group means. Qualitative analysis of reflections and field notes utilized the grounded theory approach (Strauss & Corbin, 1990). Using the grounded theory approach (Strauss & Corbin, 1990), transcripts from the focused reflections and PD field notes looked for patterns of responses. These were coded and grouped into concepts to understand and describe first, how teachers made sense of information gained from formative assessment and second, what instructional decisions arose from teachers' analysis of the data.

Results

Initial knowledge of formative assessment

All teachers indicated they engaged in formative assessment practices in their classrooms. Teachers employed thirty different types of assessment. The majority (not all) of the participants indicated a focus on right or wrong answers. Time was one challenge teachers felt affected their integration of formative assessment. Others revealed difficulty knowing what instructional steps to take based on analysis of data, stating that they can see the misconception but didn't know what to do with it. Overall, the group is representative of the research literature (Sondergeld, Bell & Leusner, 2010) in their commitment to using formative assessment and the challenges they express with implementing this instructional tool.

Several themes emerged surrounding shifts in how teachers thought about and used formative assessment.

Interpretive Lens: Determining What to Value. Prior to PD, the main purpose of analyzing student work was determining correct or incorrect answers. Post PD, teachers looked for the strategies and underlying thinking behind student answers as indicators of student understanding. This is evidenced in statements such as "you [the teacher] can't tell if a student understands just from the score."

Instruction Adjustments: Pacing to Planning. Pre PD, teachers mainly used the information to *pace* their lessons. The group transitioned from pacing lessons to building lessons that incorporated student understandings "to get them to the next level." This is evidenced by one teacher's statement that her goal is to "look at problems in the whole unit and try to sequence for student success based on what they bring with them or initial understandings."

Authority Change: Curriculum to Student. In the beginning, teachers "moved on" to the next lesson according to district or curriculum pacing guides or when a sufficient percentage of students

had "gotten it." The group began placing student mathematical development first, incorporating the students' current understandings to dictate next instructional moves.

Discussion and Conclusions

Overall, we found that teachers believed in the formative assessment process. Our findings of the teacher formative behaviors before they participated in the PD also support Riggan and Oláh, (2011) that intentions alone did not lead to effective use of formative assessment.

Comparative analysis of how teachers interpreted and used data before and after PD led to the construction of two parallel but different models that capture the relationship between the beliefs of teachers, what they prioritized as important, and the formative cycle outcomes that are associated with each (models adapted from Buck & Trauth-Nare, 2009). We employ the definitions of Wiliam (2010) to describe the two different paths, *monitoring* and *diagnostic*. These definitions include both the evaluative lens and possible instructional decisions.
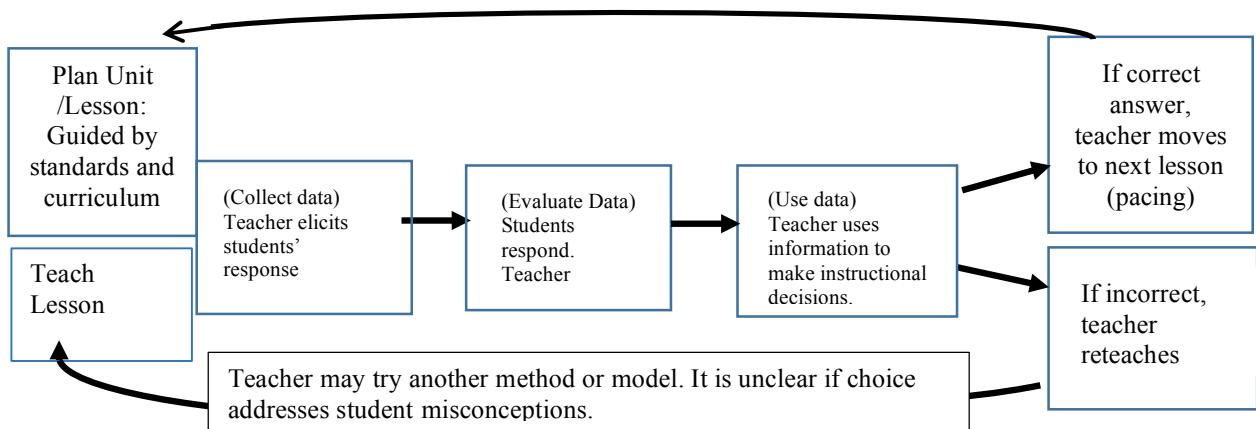
Figure 1: Monitoring Decision Cycle.

When teachers operated in a *monitoring decisions cycle* their evaluative stance was dialed in on whether answers were correct or the steps in an algorithm. The adoption of this lens limited their ability to professionally notice students' developing mathematical concepts. The information gained from adopting this lens also limited teachers' instructional choices. The information limited instructional decisions to the pacing of lessons.
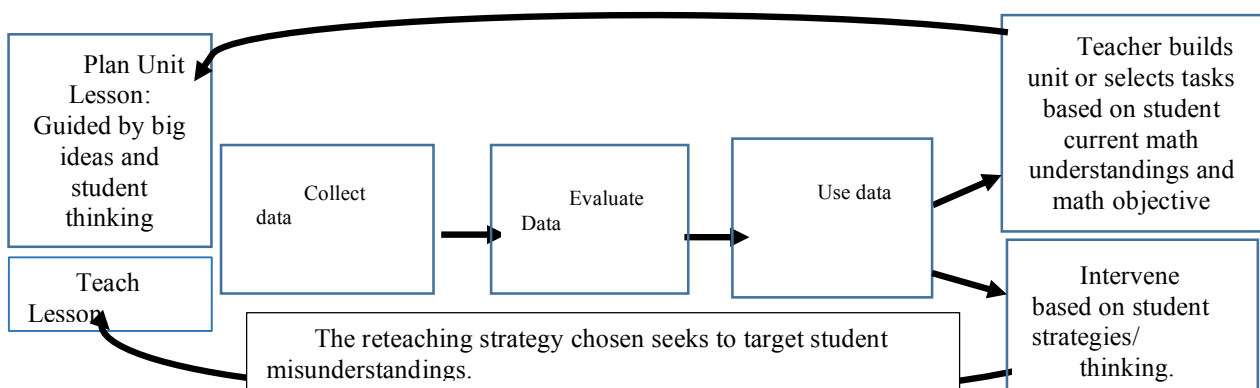
Figure 3: Diagnostic Decision Cycle.

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Teachers in the *diagnostic decision cycle* sought to pinpoint student thinking when evaluating data. Instructional decisions made from this evaluative stance mentioned students in relation to instructional next steps taken as opposed to curriculum pacing.

On the whole, the teachers required support at various stages within the process, building content knowledge, development of different lenses to interpret student work, and how to intertwine the two knowledge areas (mathematics and students) within lesson design. This reflects findings in the literature that teachers do not automatically connect assessment and instruction (Yeh & Santagata, 2015; Otero, 2006). The teachers did not shift their practices until provided facilitation and opportunity to connect and reflect. However, it is important to be mindful that professional development within these areas is not the only factor affecting teaching practices. Participants voiced frustration within their school context. Therefore, it is important to note that applying PD requires negotiation across contexts (Cobb, McClain, Lamberg, & Dean, 2003). Teachers also expressed gratitude for being provided the *time* during PD to engage and collaborate with colleagues, "we so rarely have time to do that!" This implies that teachers perceive their school context is not providing the *time* to interact with the formative assessment process, a factor expressed by teachers in the study by Allen and Penuel (2014) as well.

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TEACHERS' PERCEIVED DIFFICULTIES FOR CREATING MATHEMATICAL EXTENSIONS AT THE BORDER OF STUDENTS' DISCERNMENTS

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In this paper we describe bonusing as a strategy to continuously extend students' mathematical understanding from tasks initially offered in the classroom. While this strategy resembles enrichment activities often described in approaches based on Mastery Learning, there are fundamental differences in our use of assessment and the nature of the tasks we propose. After two years of implementing this strategy as part of the Math Minds initiative, we have found that teachers perceived bonusing as relevant and engaging for students. However, they experienced difficulties in the implementation of this strategy. In this paper, we draw from interviews with 14 elementary teachers to identify difficulties they perceive while attempting to implement bonusing. We propose the use of Variation Theory of Learning to inform the creation of bonus material and discuss connections to knowledge for teaching.

Keywords: Elementary School Education, Mathematical Knowledge for Teaching, Instructional Activities and Practices

Introduction

At the Math Minds initiative, a partnership that aims to improve mathematics instruction at the elementary level (Metz, Sabbaghan, Preciado Babb, & Davis, 2015), we have come to perceive bonusing as a strategy that fosters both a positive attitude towards mathematics and a deep mathematical understanding. However, teachers in this initiative have consistently indicated that creating and posing bonuses is challenging (Preciado Babb, Metz, Sabbaghan, & Davis, 2015). In this paper we extend our findings on these challenges and propose that Marton's (2015) Variation Theory of Learning could inform the creation of bonus material. We also discuss implications for the development of educative curricular material (Davis & Krajcik, 2005) that supports the learning of both students and teachers.

Similar to the enrichment activities in Mastery Learning (Guskey, 2007), the teachers' guides used in this initiative advise teachers to "be ready to write bonus questions on the board from time to time during the lesson for students who finish their quizzes or tasks earlier" (Mighton, Sabourin, & Klebanov, 2010, p. A-8). Suggested strategies include: changing to larger numbers or introducing new terms or elements; asking students to correct mistakes or complete missing terms in a sequence; varying the task or the problem slightly; looking for applications of the concept; and looking for patterns and asking students to describe them. In this paper we use the broader term bonus 'tasks'—instead of 'questions'—to include activities that are not necessarily questions. We concur with Bloom (1968) that these extension activities, or bonus tasks, should not just be a way to keep students busy while supporting other students. We believe that bonusing has a positive effect on students' attitudes and dispositions towards mathematics. Teachers are encouraged to pose bonus questions for all students, including those who perform lower in class, as suggested by Mighton (2007), the developer of the resources used in the Math Minds initiative.

We contend that the Variation Theory of Learning (Marton, 2015) has the potential to inform the selection or the creation of bonus tasks. Example spaces, as defined by Watson and Mason (2005), with sufficient variation can prompt learners to distinguish critical features of mathematical entities. For Marton "the secret of learning is to be found in the pattern of variance and invariance experienced by learners" (p. xi). Such patterns allow the learner to discern critical features required to learn a particular concept or skill.

Methods and Data Sources

Our data for this report were drawn from interviews conducted in 2015 with 14 teachers, as well as data from documented weekly classroom observations at two urban elementary research schools. The interview transcripts were analyzed with a focus on teachers' difficulties implementing bonusing—motivated by early research results showing that teachers found bonusing to be quite challenging (Preciado Babb et al., 2015). The research team, comprising two graduate students and four researchers, met every week to discuss and analyze data.

Findings

By comparing teachers' perceived difficulties from the interviews with our classroom observations, the following four categories of difficulty were identified.

Lack of experience and knowledge

Many teachers agreed that once they became more familiar with the resource, bonusing would become a less challenging task. The following excerpt is similar to many other teachers' comments:

> And just with working with you on bonusing and I think that the practice that I've had will definitely help me next year. I don't think next year will be as much of a struggle for me.

We documented many instances where students were asked to create their own bonuses. The interviews revealed that, in some cases, students created questions were difficult for the teacher to answer, as evidenced in the following comment:

> Usually if it's bonusing and students create their own bonusing or suggest something that is beyond the curriculum there have been times where I'm like, "I don't know the answer to that," and I don't know…how should I answer that question? Do I set them up to answer it on their own, or do I show them so that they don't get confused? That's when I hesitate. Usually with the JUMP questions I should know how to answer it because the teacher's book leads you.

The challenge of not being able to answer a student's proposed bonus could be due to the teacher's lack of mathematical knowledge or to a disposition that makes him or her reluctant to engage in unfamiliar situations. The last sentence of the excerpt suggests that the teacher relied on the "teacher's book" to deal with questions from the resource, reinforcing the idea that the difficulty was due mainly to a lack of mathematical knowledge.

Difficult to bonus without going beyond the concept to be learned

Some teachers found it difficult to bonus without moving away from the goal of the lesson or the concept to be learned. In the next excerpt, a teacher commented on the students who usually finish early and/or who understand the content of the lesson faster than others:

> They know they're doing well and they know what's coming next; they can anticipate that. And so, it's tricky for me to bonus them without moving away from the goal of the lesson. So, to give them bonuses that challenge them and that they find exciting without changing the curriculum for them.

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When teachers were asked to provide suggestions to the research team for improving the initiative, several teachers insisted that they needed more pre-made bonus tasks that they could implement in their classrooms:

> Anything that you can sort of give us that's like ready-made or ready to go on. … And bonusing is a tricky thing that you have to, I guess, get your head around.

This particular difficulty based on the goal of the lesson relates to the range of students' ability in the classroom, as explained in the next paragraph.

Difficult to be responsive and improvise in bonusing

Most of the teachers asked for "ready at hand" bonuses because they didn't know how to integrate flexible bonusing in their plans. They claimed that this was due in part to the wide range of ability between students. Teachers found it very difficult to create bonuses in the class or to modify planned bonuses in response to classroom situations. This is evident in the following:

> Just 'cause of the variety of students. And I'm not as confident in making those bonusing 'cause … when I think I'm bonusing with a small step, I find that I'm bonusing with a big step, and the kids look at me with blank faces and I'm like, "Okay, what did I do wrong there?" I think it puts the teacher in a tough position 'cause you have to bonus for each individual student and it's almost like you need to do it right there on the spot because if you plan it you have no idea.

In the previous excerpt, the teacher acknowledged that some bonus tasks turned out to be very difficult for the class, even though the teacher was "bonusing with a small step." Similar to the requests for 'ready-made' bonus questions, some teachers complained about the lack of bonus questions in the teachers' guides for bonusing at a level appropriate for the students.

Limited time to create and implement bonusing

Sometimes teachers indicated time constraints that made it difficult to create and implement bonusing. For example:

> When it [the teacher's guide] says, 'Advanced' or 'Bonusing' in the extra questions: They're nice for bonusing those students that need that type of bonus. For the class that I'm teaching right now, I find that I'm not using that as much. It's something that we don't get to just in terms of pacing but it is nice to have there just in case.

In this excerpt, the teacher acknowledged that students were not challenged with bonuses. This claim contrasts with the explicit directive of the resource to create bonuses, as well as with the shared perceptions from other teachers that bonusing is an important part of the program.

Time to prepare bonus material was a concern for some teachers. We observed that some of them made efforts to create bonus questions attractive to students:

> I find that just due to level of needs in my class sometimes I don't—I find myself struggling with the time to get around to them. And there's such a range of students that I find that being able to prepare ahead of time's not always a reality if you're planning five other subjects for the day.

It is not clear whether this teacher was creating the bonuses or just planning to use bonus tasks from the resources. In either case, planning for bonusing was perceived as time consuming.

Discussion

In the difficulties described above we perceived a need for particular knowledge required to create bonus material. Teachers consistently requested more ready-to-use bonus questions and even teachers with two years or more in the initiative indicated a need to improve the way they create bonus tasks.

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The difficulty in creating bonus tasks in response to students' range of ability also supports the idea that there is a particular knowledge teachers require for bonusing. We believe that a collection of ready-to-use bonus material, either from the resource or from the research team, is neither practical nor sufficient. In order to be responsive to students' needs in the classroom, teachers should be able to create bonus tasks on the spot with ease. Being able to improvise according to how the class unfolds would also reduce the time constraints for preparation.

Our analysis also revealed a particular perception of bonusing as something that has to be done at the end of the class, and only if time allows it. This perception of bonusing is restricting and might entail additional time for preparation.

We propose that Marton's (2015) Variation Theory of Learning holds great promise for supporting teachers in the creation of bonus tasks. Bonus tasks may be created by varying the original task in ways that allow new explorations and discoveries within the same concept or topic in a lesson, or in a way that represents an opportunity to apply (generalize) the same principle to different cases. Therefore, a future venue of research is how an understanding of this theory might inform teachers' implementation of bonusing in their classrooms.

We conclude with a potential implication for the development of and research on curricular material. Just as *educative curriculum materials* (Davis & Krajcik, 2005) can support teachers' learning, we believe that resources that teachers use in their classroom can facilitate the creation of bonus tasks and thereby further impact their knowledge for teaching mathematics. For instance, we have found that many tasks in the resources used in this initiative are presented in sequences of items with a clear pattern of variation, and very often with explicit suggestions for bonuses. Teachers can learn from the resource by paying attention to the patterns of variation related to key mathematical concepts and ideas in the examples and tasks presented to students. However, we need to better understand what teachers need to know to take advantage of the educative materials and what features the educative materials need to embody to draw teachers' attention to key mathematical ideas.

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THE IMPORTANCE OF QUANTITATIVE REASONING IN MIDDLE SCHOOL MATHEMATICS TEACHERS' PROPORTIONAL REASONING

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In this paper we describe a study involving middle school mathematics teachers who are participating in a large-scale professional development program designed to improve their conceptual understanding of mathematics. The design and implementation of this study were guided by the research question: In what ways do teachers reason through tasks designed to elicit proportional reasoning? Our analysis of the data was conducted through the lens of quantitative reasoning as described by Thompson. We highlight the issues that arose when some teachers performed numerical operations on values that they did not clearly connect to quantities.

Keywords: Number Concepts and Operations, Mathematical Knowledge for Teaching, Teacher Education – Inservice/Professional Development, Teacher Knowledge

Discussion of the Literature

Research in mathematics education over the last several decades has emphasized the importance of building rich conceptions of proportionality, specifically in the early grades (Kaput & West, 1994; Thompson, 1994). More recently, efforts have been made to institute a greater focus in the middle grades on proportional reasoning from a quantitative perspective through the adoption of the Common Core State Standards for Mathematics and the Standards for Mathematical Practice (National Governors Association, 2010), which calls for students to *reason abstractly and quantitatively*.

Thompson (1994) defines a *quantity* to be a measureable attribute of an object. Quantities are not objects themselves; they exist in the mind of the person who is imagining them. The *value* of a quantity is the numerical result of a measurement, and a measurement is a multiplicative comparison of the magnitude of the quantity to the magnitude of some unit-of-measure. Thompson (2011) describes a *quantitative operation* as the act of either combining two quantities (additively or multiplicatively) or comparing two quantities (additively or multiplicatively). A quantitative operation should not be confused with a numerical operation, which is the act of combining or comparing two numerals.

Reasoning with quantities is foundational to robust proportional reasoning. According to Lesh et al. (1988), proportional reasoning is the capstone of elementary school mathematics and the cornerstone of high school mathematics. Cramer et al. (1993) outlined several components involved in proportional reasoning, including understanding the multiplicative relationships that exist within proportional situations and being unaffected by the situational context or the types of numbers in the task. Kaput and West (1994) found that the context of the problem, the language of the task, the kinds of quantities involved, and the numerical values of the quantities all impact student thinking. This paper adds to these findings by discussing the impact on proportional reasoning when people disassociate measurements from the quantities they measure.

Methodology

This study focused on nine middle school mathematics teachers who were participating in a two-year professional development program led by mathematics educators and researchers at a community college in the southwest. Leveraging Goldin's (2000) principles, researchers used semi-structured, task-based interviews for investigating teachers' thinking while working through tasks involving proportional relationships. Each teacher participated in five one-hour videotaped

interviews during their participation in the program. The research team analyzed all interview sessions with the goal of characterizing teachers' thinking and reasoning as they grappled with the mathematical tasks. The design of this investigation and subsequent data analysis were guided by the following research question: In what ways do teachers reason through tasks designed to elicit proportional reasoning? In this paper we present data from one of the tasks used in this study: The Juice Task.

Suppose 3/5 of a cup of juice gives you 4/3 of your daily serving of vitamin C.

What amount of your daily serving of vitamin C is in 1 cup of juice?

What amount of juice is needed for a full daily serving of vitamin C?

Discussion of the Data

In this section we discuss data from two teachers, referenced as Adam and Kim. While working on the Juice Task, Adam produced the expected answer in part *a*, but not in part *b*. His written work is displayed in Figure 1.
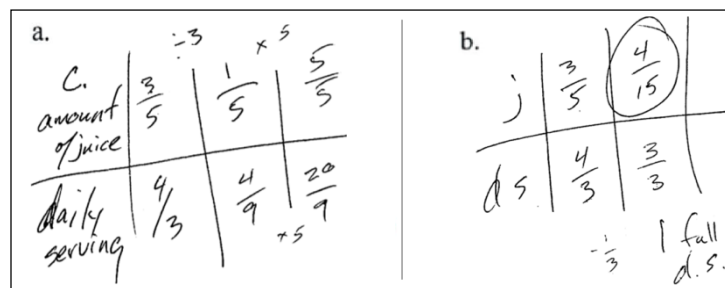
Figure 1. Adam's written work on part *a* and part *b* of the Juice Task.

The video data revealed that Adam used the operations of division and multiplication in part *a*, but resorted to the operation of subtraction in part *b*. For part *a*, Adam pointed to the 3/5 and 4/3 and stated "these are kind of a proportion or ratio where if I do something to [3/5] to get [1/5], then I do the same here (points to 4/3) to here (points to blank box), then it keeps the same ratio." Adam then divided 3/5 and 4/3 both by 3 to get 1/5 and 4/9. He then multiplied each of these results by 5 to get 5/5 and 20/9. He concluded that 20/9 of a daily serving of vitamin C is in one cup of juice. In part b, Adam did not use multiplication or division. Instead, he stated "I need to get [4/3] down to…um well, maybe I don't do the per unit…what I really need to do is get it down to 3/3 because that would equal one daily serving. So, I have subtracted 1/3." Adam then subtracted 1/3 from both 4/3 and 3/5, and concluded that 4/15 of a cup of juice would provide a full serving of vitamin C.

Adam may have used subtraction in part *b* because he was focused on *reducing* 4/3 of a serving down to one full serving. His objective in each part was to obtain the number 1 (either for cups in part *a* or for servings in part *b*) and he did not restrict which operations he could use. His guiding principle was to do to one value the same thing he did to the other value. This suggests that Adam was not mindful of the quantitative operations underlying his numerical operations. Removing 1/3 from 4/3 through the numerical operation of subtraction is sensible if these numerals are values of measurements in compatible units, i.e. if 1/3 and 4/3 are both measurements of servings of vitamin C. If Adam had been attentive to the unit of 1/3 perhaps he would have realized that removing 1/3 *of a serving* from 3/5 *of a cup* is not reasonably accomplished by the numerical operation he used (i.e., 3/5 – 1/3). Perhaps Adam was thinking that he was removing 1/3 *of a cup* from 3/5 of a cup, as well as removing 1/3 of a serving from 4/3 of a serving. In this case, because he was trying to do the same thing to both values, he would have had to reason that removing 1/3 of a serving is equivalent to removing 1/3 of a cup, which contradicts the premise of the task.

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We now consider Kim's thinking. While working on the Juice Task, Kim produced the expected answer in part *b,* but not in part *a*. Her written work is displayed in Figure 2.

Figure 2. Kim's written work on part a and part b of the Juice Task, in the order that she progressed through the task.

Although Kim produced the correct answer for part *b*, the video data provided limited evidence that she was sensibly reasoning about the quantities. When questioned about her choice to divide 3/5 by 4/3 she stated, "To figure out, uh, just how many groups of 4/3 fits into 3/5 and take it from there… And it was either multiply or divide and division made more sense." Note that her language, "how many groups of 4/3 fits into 3/5," is indicative of the quotitive division model (Simon, 1993). We now explore some possible ways that Kim was thinking using the quotitive division model. It is not clear to us from the data whether Kim was mindful of the quantities. Suppose she was. Since the amount of vitamin C and the amount of juice are directly proportional, one could effectively convert 4/3 servings to cups of juice and then wonder how many 4/3 servings (measured in cups) fit into 3/5 of a cup. This quantitative operation should lead Kim to conclude that *one* copy of 4/3 servings fits into 3/5 of a cup, because 4/3 servings corresponds to 3/5 of a cup. As another possibility, the quantitative operation associated with wondering "how many groups of 4/3 *servings* fit into 3/5 *of a cup*" would yield the numerical operation that Kim used (3/5÷4/3) only if *servings* and *cups* were interchangeable units (i.e. 4/3 servings corresponds to 4/3 cups); this contradicts the premise of the task. We conclude that despite her utterance, Kim could not have been thinking about the quantities as well as the quotitive division model. It is possible that despite her language, Kim was thinking using the partitive division model (Simon, 1993). If 3/5 of a cup were fit into 4/3 servings, partitive division would yield the numerical operation 3/5÷4/3, which would reveal the amount of juice per whole serving. However, the data do not indicate that Kim used the partitive division model. This analysis of the data suggests to us that when Kim said that division "made more sense," she was referring to the *result* of the division rather than the operation itself.

Kim's first attempt at part *a* also suggests that she was focused on evaluating the reasonability of results of numerical operations rather than attending to quantitative operations. After dividing and multiplying 3/5 and 4/3 she stated, "I'm trying to look at the numbers to see if they make any sense." In her second attempt, however, she sought to (1) determine the "value of each group," i.e. the amount of vitamin C in each 1/5 cup of juice; (2) subsequently determine the amount of servings in 2/5 cups of juice; and (3) combine this amount with the given amount of vitamin C in 3/5 cups of juice. She incorrectly determined that 4/15 servings were in 1/5 cups of juice by "dividing 4/3 of the daily serving into five equal groups because my denominator here (points to 3/5) is five." Using this information, she proceeded to add 8/15 [servings in 2/5 cups of juice] to 3/5 [original amount of juice]. She concluded there were 17/15 servings in one cup of juice. Had Kim been mindful of the referents associated with 8/15 and 3/5, she may have avoided the error of inappropriately using numerical addition to combine the two quantities.

Conclusion

This research adds to the body of work on factors that influence proportional reasoning by investigating the ways in which teachers reason through tasks designed to highlight proportional relationships. The initial data reveals that teachers in this study struggle to attend to quantities and quantitative operations in proportional situations. The data suggests that teachers disassociate measurements from the quantities they measure and simply operate on the numerals. We have found this issue to be a source of complexity as teachers grapple with tasks involving proportional quantities.

Acknowledgements

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ADVANCED MATHEMATICAL KNOWLEDGE FOR TEACHING: A CASE OF PROFESSIONAL TEACHING KNOWLEDGE INFLUENCING INSTRUCTION

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Teachers require some level of *advanced mathematical knowledge* (AMK) so that they can situate their mathematical work appropriately within a disciplinary context. There is little agreement, however, regarding how AMK can be acquired by and developed in teachers (e.g. Ball & Bass, 2009; Clay, Silverman, & Fischer, 2010; Davis, 2011). Further, the potential AMK that is useful for teaching may involve a great deal of tacit knowledge (Clay et al., 2010; Davis, 2011; Silverman & Thompson, 2008), making it difficult to detect through observations alone (Polanyi, 2009). Consequently, little is known about teachers' AMK used in practice.

This study was designed to illuminate the role of AMK in the practice of elementary mathematics teaching through observations of the mathematics teaching of a fourth grade teacher (Tasha) and, through conducting semi-structured, in-depth interviews, identifying the nature of the AMK she employed. In one particular episode, Tasha was teaching a lesson during which she made the decision that initiated a shift in the mathematical focus of the lesson from calculus concepts (e.g. discriminating between features of the graph that represent speed and changes of speed) to parametric relationships (i.e. representing the relationship between speed and distance as functions of time) due to a mismatch between the contextual information in curriculum materials and the goals of the lesson.

This case demonstrates that teachers may benefit from preparatory or professional development experiences that expose them to calculus ideas and support their interpretation of related curriculum tasks. More research would be required to identify if introducing calculus concepts to all elementary teachers would prove helpful on the wide scale. Alternatively, I propose that perhaps the construction of curriculum materials should account for the kinds of structural knowledge teachers may lack. In particular, it may be helpful to identify those mathematical structures with which teachers may have limited experience and incorporate teacher support into their curriculum materials where deviations are likely to occur.

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CALCULUS STUDENTS' IDEAS ABOUT FUNCTIONS: IDENTIFYING OPPORTUNITIES TO SUPPORT TEACHER LEARNING

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Keywords: Mathematical Knowledge for Teaching, Algebra and Algebraic Thinking

We describe the first phase of a study aimed at developing video-based instructional modules for secondary mathematics teachers. We began by consulting the literature on figural pattern tasks (c.f. Rivera, 2010) and teachers' ability to interpret student work (c.f. El Mouhayar & Jurdak, 2012). Interpreting student work on figural pattern tasks requires awareness of different problem solving strategies, such as recursive and constructive, and how students might use them with tasks that require different levels of generalization (El Mouhayar & Jurdak, 2012).

We conducted one-on-one interviews with 17 high school calculus students to capture authentic student work and reasoning on a figural pattern task. As students worked through each task, we asked students to document their work on paper and to explain their thinking aloud. Two video cameras were set up to capture the written work and discussion. Student interview data were initially coded according to strategy type: constructive, deconstructive, or recursive. Additional coding resulted in four categories for teacher learning (CTL): 1) Deciphering students' cryptic notations; 2) Recognizing what is correct in a slightly flawed strategy, 3) Recognizing a strategy that is "close" to a more advanced strategy, and 4) Making connections between different strategies.

The CTLs are a first step in designing video-based modules grounded in authentic student work with explicit learning goals for developing teachers' capacity to work with student's ideas. For example, Student A's response presents an opportunity for teachers to attend to a correct constructive strategy that breaks down in the final calculations (CTL2), while Student B's response presents an opportunity for teachers to attend to a transitional phase in writing explicit functions based on recursively defined patterns (CTL3). Both responses involve interpreting somewhat cryptic notations (CTL1), and together, they afford opportunities to make connections between ideas such as where the recursive addition of 8 can be observed in a constructive approach (CTL4).
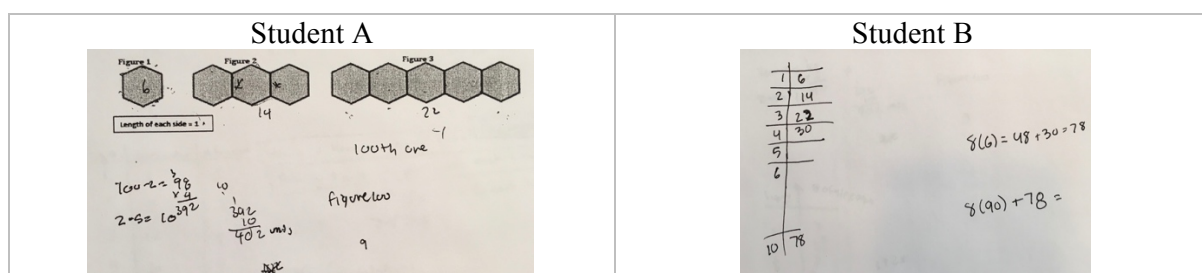

Figure 1. Work samples from Student A and Student B. Together they illustrate all four CTLs.

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FOSTERING TEACHER CANDIDATES ACROSS THE BORDER FROM DOERS TO TEACHERS

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One of the overarching aims of a teacher education program is to support "doers/students" of mathematics in becoming "teachers" of mathematics. This transition requires candidates to reorient from thinking about how they do mathematics to engaging with students and their work, understanding student representations, and planning instruction accordingly. To scaffold this transition, we developed a five-step Mathematics as Teacher Heuristic (MATH). This heuristic was designed based on Mathematical Knowledge for Teaching (MKT) (Silverman & Thompson, 2008) and the Noticing Framework (Jacobs, Lamb & Philipp, 2010). Our research question is "How do teacher candidates develop knowledge of teaching mathematics for conceptual understanding?"

Twenty-two teacher candidates completed the following five-step process using student work samples collected from authentic classroom settings: 1) solve a rich task as a "doer;" 2) assess/analyze authentic student work samples associated with the same task; 3) consider good questions to ask struggling students; 4) develop scaffolded instructional materials addressing student challenges, difficulties, and misconceptions; and 5) reflect on the process.

Through the analysis of the data, two major themes in the process of transforming were revealed: (a) teacher candidates building on what they noticed in the student work in the remaining parts of the MATH consistently; and (b) teacher candidates not building on or having gaps in what they noticed that then impacted on their work in the rest of the MATH. Teacher candidates with weak Key Developmental Understanding (KDU) (Silverman & Thompson, 2008) and/or unsuccessful noticing tend to be unsuccessful in decentering their own ideas. Consequently, they addressed what was lacking in the student solutions and posed more leading than scaffolding or imposed their own way of solving mathematics problems rather than empowering students.

We conclude that teachers need to be engaged in activities not only emphasizing the importance of separating teachers' understanding from the hypothetical understanding of the learner, but also recognizing the importance of self-reflection in the development and evolution of teacher knowledge, beliefs, and attitudes. The MATH model in this study includes reflection on the overall process of the activity including reflecting on preservice teachers' reflection on key developmental understandings of own and students and their plan of practice. For practicing teachers, reflection should be expanded to reflecting on students' learning and practice in the context of instruction.

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EXPLORING CONNECTIONS BETWEEN TEACHERS' MATHEMATICAL CONTENT KNOWLEDGE AND ITS RELEVANCE TO TEACHING K-12 MATHEMATICS

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Students learn the concept of linearity early in their school careers and frequent reinforcement makes them prone to apply it without discretion (De Bock, 2002). Consequently, they struggle to develop higher-level conceptual understanding of mathematical topics like differential equations because of their assumptions of linearity (Brabham, 2014). It is important therefore to introduce rate of change as it relates to both linear and non-linear models. We created an instructional unit with the goal to make connections between secondary and advanced mathematical topics. Specifically, our research question is: How does the knowledge of chain rule impact secondary school teachers' understanding and teaching of the rate of change?

Our project is informed by the idea of Mathematical knowledge for teaching (MKT), specifically, "horizon knowledge" which gives teachers a mathematical "peripheral vision" (Ball, Thames, & Phelps, 2008). It provides a larger view of mathematics that gives the teachers a sense of where to place the content that they are teaching and how it is connected to higher level mathematics (Ball & Bass, 2009).

Our unit was designed to engage the teachers in discourse by posing a series of thought provoking questions. This led the participants, experienced secondary school teachers, to examine the relationship between the instantaneous rates of change, the constant rate of change, and the chain rule. At the end of the lesson, the participants shared their reflection on the topic, both verbally and in writing. The class was audio taped and observation notes of the lesson were taken. Data was analyzed to find recommendations for content and pedagogy. We also recorded any connections between secondary and tertiary mathematics and student misconceptions on this topic. In the future, we plan to revise our unit and conduct this study with pre-service mathematics teachers.

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THE RELATIONSHIP BETWEEN TEACHERS' MATHEMATICS CONTENT KNOWLEDGE AND OBLIGATION TO THE DISCIPLINE

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Keywords: Mathematical Knowledge for Teaching, Instructional Activities and Practices

This poster focuses on mathematics teachers' *obligation to the discipline* (Herbst & Chazan 2012), which refers to the expectation that teachers will represent the discipline of mathematics appropriately. The study uses scenario-based assessments (Weekly & Polyhardt, 2006) to examine the extent to which teachers are disposed to alter "question the borders" of customary practice on account of this obligation and the mediating role of subject matter knowledge for teaching (SMK; Ball, Thames, and Phelps, 2008). We hypothesized that SMK would significantly predict the obligation to the discipline, because teachers with more SMK would be more willing to breach customary practice to represent their knowledge.

Methods

We collected responses from high school in-service mathematics teachers to three instruments. The study included participants from all states and distributed proportionally over five regions. The Justification of Actions instrument had 16 items that ask participants their agreement on a 6-point Likert scale with a teacher's decision to depart from an instructional norm and instead attend to their obligation to the discipline. The Decision instrument contained sixteen items, each of which required the participant to choose between four different decisions that differed in the extent to which they altered customary practice. The MKT-G instrument had 8 common content knowledge (CCK) questions and 20 specialized content knowledge (SCK) questions which we used to score SMK.

Two multiple regression analyses were conducted. The first used the average Justification of Actions rating as the outcome variable, SMK as the predictor variable, and controlled for number of college math courses. The second used the average Decision rating as the outcome variable, an average Justification of Actions rating as the predictor, and controlled for SMK. Because missing data was dropped and not all participants completed every instrument, the samples were slightly different for the two models.

Results

In the first regression (R^2=0.059, F(3, 318)=6.66, p<0.001), we found that CCK score significantly predicted the amount of obligation to the discipline, (B=-0.085, t(322)=-3.49, p=0.001). In the second regression, (R^2=0.208, F(3, 333)=29.20, p<0.001), we found that a higher level of disciplinary obligation predicted a higher tendency to decide to breach normative behavior (B=-0.261, t(337)=-8.17, p<0.001).

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PROSPECTIVE TEACHERS' ATTENTION TO REALISM AND CONSISTENCY IN A CHILD'S TEMPERATURE STORY

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Research tells us that prospective teachers (PTs) struggle to think conceptually about integer addition and subtraction, often focusing on procedures (Bofferding & Richardson, 2013). When PTs are asked to reason conceptually and apply contexts to integer addition and subtraction number sentences, they often do not utilize temperature contexts (Wessman-Enzinger & Tobias, 2015). When PTs do utilize temperature as a context, they often struggle to pose stories. Yet, in their work as teachers, PTs will need this skill, as well as the ability to move beyond posing stories. PTs will need to make sense of stories that children pose, judging the realism of the story and the use of mathematical operations supported by the story.

Seventy-seven elementary and middle school PTs participated in a study where they explored integer addition and subtraction tasks. This poster reports on The Sabrina Task, which engaged the PTs in sense making around a child's temperature story posed for the integer addition number sentence $-9 + -6 = \square$. The child's story: *It is -6 degrees 2 days ago. It was -9 yesterday. Now it is -15 degrees,* both did not contextually make sense (realism) and did not support the operation of addition (consistency). PTs' were asked whether the story made sense with the given number sentence and to justify their answers. Seventy-four PTs' written justifications were analyzed in relation to whether or not they believed that the story matched the number sentence, and what their justifications were. While 63 of the 77 (81.8%) PTs confirmed that the story did not match the number sentence, less than half of the PTs included realism in their justification, and only half of the PTs included consistency.

The results of this study imply that PTs need further experiences both in looking at integer operations in specific contexts, like temperature, and in making sense of integer stories, such as those posed by children. Despite instructional experiences including tasks such as these and the fact that task containing The Sabrina Problem appeared at the end of those instructional experiences, the PTs' justifications showed gaps in both their attention to realism and to consistency. These results indicate that we need to engage prospective teachers in these types of tasks more often. Specifically, PTs need more opportunities to reflect on realism and consistency in temperature problems, as well as problems in other contexts. Additionally, research is needed into how PTs' responses about realism and consistency are related, within temperature contexts and otherwise.

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EFFECTS OF A HISTORY OF MATHEMATICS PROFESSIONAL DEVELOPMENT COURSE ON TEACHERS' MATHEMATICAL KNOWLEDGE FOR TEACHING

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A consensus has emerged recognizing that mathematics teachers need mathematical knowledge that are specific to the work of teaching mathematics, yet much work remains on how to facilitate the development of this knowledge in teachers. Professional development is crucial, but the interplay between content courses and mathematical knowledge for teaching remains only loosely understood. While significant research has been done on professional development specifically designed to align with the development of certain types of mathematical knowledge for teaching, little work appears to have been done on how mathematical knowledge for teaching can develop indirectly. In this study, we examined the impact of participation in a four credit hour history of mathematics course on mathematical knowledge for teaching, as measured by the Learning Mathematics for Teaching (LMT) instrument (Hill, Schilling & Ball, 2004). Previous studies have reported that higher teacher performance on the LMT is correlated with the presentation of richer mathematics in the classroom (Hill, et.al., 2012) and with higher student achievement (Hill, Ball & Rowan, 2005).

Two subscales of the LMT were administered to 50 teachers over three course offerings across two years. The repeated measure ANOVA revealed a significant main effect of Test Form, $F(1, 63) = 108.31$, $p < .001$. Overall, Geometry scores were significantly higher than the Patterns Function and Algebra scores (M(SD) = .62(.75) and -0.07(.86), respectively), and this pattern was consistent across all three sites. The interaction of Test Form x Workshop Site was not significant, $F(2, 63) = .52$, $p = .60$.

There was also a significant main effect of Time Administered, $F(1, 63) = 15.17$, $p = <.001$. Posttest scores were higher on average than pretest scores (M(SD) = .39(.85) and .16(.85), respectfully). The interaction of Time Administered x Test Form was not significant with, $F(1,63)=.19$, p-.29. Further more, the interaction of Time Administered x Workshop Site was also not significant, $F(2,63) = 1.13$, $p = .29$, indicating that all three sites shared a pattern of differences between pre- and post-test administration and supporting the combination of data across sites for the planned comparisons tests. Overall, there was a larger difference in means for the pre- and post-test scores for Patterns Function and Algebra than for Geometry (M(SD) = .29(.59) and .18(.58), respectively).

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HOW EXPOSURE TO ADVANCED MATHEMATICS MAY IMPACT TEACHER'S UNDERSTANDING AND TEACHING OF SECONDARY MATHEMATICS

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Keywords: Algebra and Algebraic Thinking, Teacher Knowledge, Advanced Mathematical Thinking

It has long-been accepted that effective teaching practice stems not only from well-developed pedagogical practice, but also from advanced subject matter knowledge (Cochran, King, & DeRuiter, 1991). Despite evidence that stronger teacher knowledge of mathematical subject matter might have a positive impact on classroom instruction and student achievement, our understanding of how teachers' knowledge of advanced mathematical content impacts instruction and student achievement is still lacking. This study examines (1) how does secondary mathematics teachers' exposure to advanced mathematics impacts their understanding of secondary algebra? (2) How does exposure to abstract algebra impact secondary teachers' understanding of inverses? This study employs Mathematical Knowledge for Teaching (MKT)(Ball, Thames, & Phelps, 2008) as a means of improving teaching practice and student outcomes, according to which the connection between secondary and tertiary mathematics utilizes both mathematical content knowledge as wells as pedagogical content knowledge. The aspect of MKT on which we primarily focus is 'Horizon Content Knowledge', which helps develop an inclusive vision regarding how advanced tertiary mathematical content knowledge can be linked back to secondary mathematics (Ball et al., 2008). In this study, we investigate teachers' current understanding of operations used in solving equations and how attention to advanced algebraic structures and their properties may impact conceptual understanding and future instructional practices.

Participants were 14 in-service and pre-service teachers in a master's level mathematics education course. The researchers first provided participants with a mathematical content knowledge questionnaire to gain insight into the participants' conceptual understanding of algebraic operations and inverses. Then, the researchers taught a short abstract algebra lesson with a focus on group structures and modular arithmetic. During the lesson, participants were engaged in small and large group discussions, which were videotaped and all written artifacts were collected for future analysis. After initial analysis and coding of data, changes in participant's understanding of inverse were observed. Participant's original understanding was that inverses were operations that "undo" each other. Exposure to group structures as a component of the abstract algebra lesson prompted participants to consider identities with respect to a binary operation and consider the operation-element duality of inverses.

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Chapter 7

Mathematical Processes

Research Reports

Brief Research Reports

Posters

STAGES IDENTIFIED IN UNIVERSITY STUDENTS' BEHAVIOR USING MATHEMATICAL DEFINITIONS

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This paper reports and describes some of the observations and conclusions drawn from a study developed to find information on undergraduate students' spontaneous actions and reactions to mathematical definitions that are new to them. There were 23 participants from a transition-to-proof course. They were interviewed individually on a particular mathematical definition. The analysis of the interviews was iterative and consisted mainly of three phases. During the first two phases students' behaviors were observed and identified in order to categorize commonalities of their responses and actions. In the third phase, a search-back through the data was done for additional supporting evidence of the commonalities previously observed. This third phase allowed making a more general classification of the various behaviors observed. Participants were classified into five different stages in using mathematical definitions according to their behaviors when working with a particular definition.

Keywords: Reasoning and Proof, Advanced Mathematical Thinking, Mathematical Knowledge for Teaching

Introduction and Research Questions

The role of definitions in mathematics is fundamental. Professors apparently expect students to be able to grasp definitions and then proceed to do something with them (Alcock, 2010). Nevertheless, "Many students do not categorize mathematical definitions the way mathematicians do; many students do not use definitions the way mathematicians do, even when the students can correctly state and explain the definitions; many students do not use definitions the way mathematicians do, even in the apparent absence of any other course of action." (Edwards & Ward, 2004). This study aims to shed light on how undergraduate students proceed when they are presented with new (to them) mathematical definitions by addressing the following questions: How do students make use of mathematical definitions new to them? What are their spontaneous reactions? What contributes to their difficulties with "unpacking" and using abstract mathematical definitions? How do they use a definition in three different settings: examples, proofs, and true/false questions? This research falls within the scope of a framework developed by Selden and Selden (1995, 2008). I particularly focus on the part of the framework referring to operable interpretations of statements and formal and informal forms of a mathematical statement.

Literature Review

Research has revealed that students have a variety of difficulties understanding and using definitions, many of which could be attributed to the "structure of mathematics as conceived by mathematicians and the cognitive processes involved in concept acquisition" (Vinner, 1991). According to Tall (1980), "concept image is regarded as the cognitive structure consisting of the mental picture and the properties and processes associated with the concept…Quite distinct from the complex structure of the concept image is the concept definition which is the form of words used to describe the concept." A mismatch between the concept image developed by an individual and the actual implications of the concept definition often leads to obstacles in learning. The work of several researchers has confirmed this. Parameswaran (2010) has addressed how mathematicians approach new definitions; her research shows that examples and non-examples play a very important role in the process of learning a new definition. However, students are not frequently asked to generate

examples, most of the time they are provided with a worked-out example or an illustration (Watson & Mason, 2002). Although there are some studies addressing students' and mathematicians' use of definitions in the construction of proofs, there seems to be a lot more investigate in respect to students' perceptions of mathematical definitions.

Methodology

I conducted a series of semi-structured task-based interviews with voluntary participants taking a transition-to-proof course. There were five definitions: function, continuity, semigroup ideal, isomorphism, and group, spanning most of the course. I interviewed 23 volunteer students individually. For each definition, 4-5 students were interviewed approximately two weeks before that particular definition came up in the course, to assure they had not seen it before (in class). The interviews were audio-recorded, and the students used LiveScribe pen in order that their real-time responses could be analyzed. I wanted the interviews to address the following four main points. First, I wanted the students to be presented with a definition for the first time, that is, they were interviewed about a definition that they had not yet seen in class. Second, I also wanted to test their ability to interpret the definition, so I asked the participants if they were able to come up with some examples that could illustrate the definition. Third, I looked for information on their ability to make use of the definition in the construction of a proof that required no more than the definition itself. And fourth, I was interested in the way they could reason about true/false statements involving the definition. These four points were addressed by the design of five handouts, presented one after the other to each student individually in a 60 to 90 minute interview. This design was partly inspired by the work of Dahlberg and Housman (1997) and the work of Housman and Porter (2003). The first phase of the analysis was done considering each of the five definitions separately. I concentrated on only one definition at a time, analyzing all the data on the five handouts for the four to five respective students considering that definition. The second analysis was done by handout. I looked at the general performance across all handouts, considering one handout at time, across all participants.

Results

From a detailed observation and analysis of students' actions working with definitions newly introduced to them, a general perspective on students' behaviors has been developed. There seem to be various stages that the participants were in during their attempts to use mathematical definitions in the three different settings of this study; evaluating examples, constructing proofs, and answering true/false statements presented in five handouts. Students were initially provided with a handout containing only a definition, and then other four handouts with the different tasks were given. As a result of the analysis five stages (0-4), listed and described in detail in the following sections, were identified. These stages are not intended to be a definitive set of steps through which a student must pass in order to use definitions appropriately, but they describe and categorize the different behaviors observed amongst the 23 participants of this study. Each one of the participants worked on only one of the five definitions (function, continuity, group, ideal, and isomorphism) and was classified in one of the five stages according to their performance during the 60-90 minutes of the semi-structured interview. Further explanation and details about the design of the whole study (such as the mathematical definitions and the handouts) can be found in Holguin (2015). This paper exhibits only one of the main results.
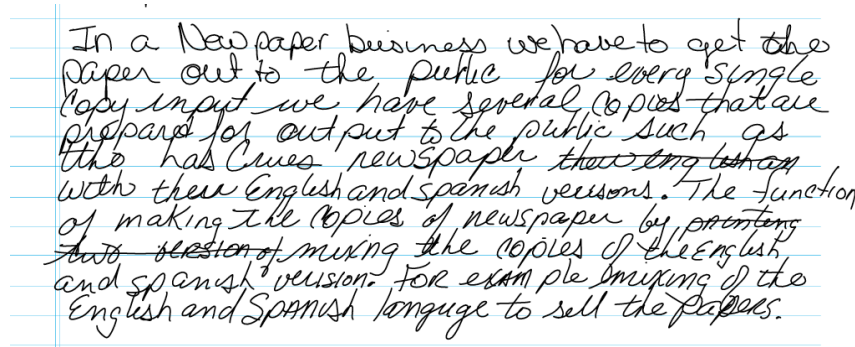
Description and evidences of each of the stages

Stage 0: Unawareness. Students at Stage 0 see and treat mathematical definitions as everyday words. A student is in Stage 0 if he/she does not see mathematical definitions as having meanings separate from the everyday linguistic meaning of the words used in them. Such students relate the words in a mathematical definition to words used in everyday language or to everyday situations.

They make connections to their previous knowledge, but those connections are not necessarily mathematical. Mathematical symbols are often hard to interpret perhaps because in the everyday linguistic context definitions rarely have symbols involved as mathematical definitions often do. This behavior was observed in two of the 23 students, Fay and Gaby. Fay was working with the definition of function. She wants to become a secondary mathematics teacher. When she was asked to explain the definition in her own words she said:

Fay: …a function is like a machine; you put something in to get something out. Like a machine that makes copies of a newspaper in English and Spanish…

Then the interviewer asked her if she could write down her thoughts. Figure 1 shows what Fay wrote.

Figure 1. Fay's thoughts about function.

Fay was classified in Stage 0 because the example she came up with exhibits a lack of awareness of the distinction between mathematical and everyday definitions. At best, she understood function as a process (machine) that gives a final product when some inputs are provided. But she gave an example of a newspaper copy machine that produces both English and Spanish versions. One can interpret this as indicating that she might be trying to express the mathematical property of a function of having one and only one output for every input, but has expressed it the wrong way around. She was assigning several copies to a single input, which violates the second condition of the given definition of function. In addition, she was blending her somewhat fuzzy mathematical ideas with an inappropriate real life situation.

Gaby was an engineering major; she was also asked to explain her understanding of the given definition. She read the definition of a group and she replied the following.

Gaby: A group is not a group if there is not different components or… you know, several elements. So if it were only one element you wouldn't call it a group. I think that's why [it] is telling you that if there is a g element of G and then there is a g' which is an element also of G, and those things comb... you know, make a group. To be a group I think you need to have more than one element."

One can see that Gaby is using the everyday meaning of the word "group". Later in the interview Gaby was asked to provide an example:

Interviewer: Can you think of a particular example? Can you tell me the properties that an object would need to have in order to be a group?

Gaby: OK, so you have to have an element, or actually each… you have to have a semigroup, each element has to have a subset... for example, I don't know, this is what I can picture, like the school. The school is a [with emphasis] school, as a whole, but it has different colleges, the university has colleges, there is colleges that belong to the university and each college has

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departments, like for example like a g', so departments, colleges, you know, are composed for and there... you know, the university, which equal university. That's the way I see it.

Gaby's example, as Fay's, is a blending of a real life situation with pieces of mathematical knowledge (some ideas from set theory). In Gaby's case, I find it harder to speculate which mathematical properties might be involved in her explanation. The only thing I see is that she might be thinking that the prefix "semi" before the word "group" implies that a semigroup is something smaller than the group but contained in it. While explaining the definition in her own words, she also drew the diagram in Figure 2.
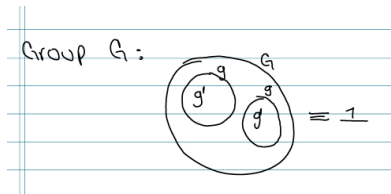
Figure 2. Gaby's diagram about group.

From this diagram and the conversation we had about it, I see that she might have been thinking of *g* (by itself) as a subset of *G* instead of as an element. And perhaps she was trying to illustrate the property of having an identity element, that is to say, the part of the definition of group stating that for each *g* in *G* there is a *g'* in *G* such that *gg'=g'g*=1. For the aforementioned reasons, Fay and Gaby were classified in Stage 0.

Stage 1: Awareness. Some students seemed able to recognize the differences between mathematical definitions and everyday definitions. They seemed to be aware of the importance of mathematical definitions, but they seemed to find them too complicated to make use of them or to think of examples of them. They seemed to make strong connections to definitions previously seen in other mathematics classes or to any mathematical concepts that seemed familiar to them, or that they thought were related to the particular presented definition. Students in Stage 1 did not demonstrate a recognition of how, why, what, or when to use mathematical definitions; there five out of the 23. The following excerpts illustrate these students' behaviors.

Carlos was majoring in engineering technology and was interviewed on continuity. He was told at the beginning of the interview, as every participant had been, that he was being given a definition. He read it and immediately afterwards he started talking. The following is a piece of what he said.

Carlos: This is like a weird... a weird problem
Interviewer: Have you seen this definition before?
Carlos: I have seen some of the symbols... in this class... in Algebra, Precalculus, Technical Calculus. I haven't seen absolute value symbols in this class yet, or continuous.
Interviewer: How about other classes?
Carlos: Continuous? Oh yeah! yeah! in calculus! Like natural log e be continuous… or they already gave a problem like saying here is a chemical that grows continuously at so and so, yeah that's the kind of problems I had.
Interviewer: And what was the meaning of continuity in those cases?
Carlos: The symbol e, then is e like decaying continuously or growing continuously, so yeah that's where. But for this problem, I don't know, is kind of like... I never... I don't how to approach it.
Interviewer: Well I am not giving a problem; it is just a definition, all I am asking is to read it and I'm just trying to see what you can see in it. So do you get some meaning from it? Do you get what it is telling you?

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Carlos: Yeah, just slightly... well you are telling there is a function from R to R, so something from R has to be in R. And the a is an element of R which is in f function continuous and... I don't really know about continuous 'cause we never did continuous so... or either greater than... I'm like what is that? Even delta... I know what is going from here, [pointing to the function] but I get knocked out here...[pointing to the inequalities] I'm like what is going on? I don't even know.

Interviewer: Well, but you said you have seen or heard the word "continuous" before, can you come up with some examples of something that has to do with continuous functions?

Carlos: Yeah, OK. I can just make it up, right?

Interviewer: Yes, yes, definitely.

Carlos: Mmmm... should I use like a chemical? or just say a dead body decays...

[Carlos continues to write what is shown in Figure 3]

Figure 3. Carlos' continuous function example.

It seems interesting that Carlos' first reaction was to think there was a problem to solve. His first words after reading the definition were "this is like a weird problem", when there was no problem at all. Every participant of the present study was provided with the same explanation and instructions about the procedure of the interview. It is not clear to me why Carlos thought of the definition as a problem. It might be due to the fact that, as stated by him, he did not understand much about what was being stated.

Another behavior worth noticing is the connection with previous knowledge from other mathematics classes about a function that grows or decays continuously. This once more demonstrates that students use their concept images whether or not they match the concept definition. These behaviors identify Carlos in Stage 1; he used mathematical terms to communicate his thoughts but he didn't get much from the information in the definition, he basically ignored it.

Stage 2: Contextualization. By contextualize, I mean being able to identify when and where to use a certain mathematical definition. Some students seemed to understand that a given definition makes sense within a particular situation of a particular area of mathematics, and that it is in such context that the definition should be used in the way it is stated. They were aware of their previous mathematical knowledge, but they seemed to understand that their previous knowledge might not always be helpful. There were 3 participants presenting this behavior.

Fred was majoring in mathematics. He worked with the definition of function.

Fred: I have never seen this definition before ... I have heard the word function from previous math classes, Calc I, II, III, trig and Precalc… It helps with graphing, but we are not going into graphing here.

One could see that, from looking and reading the definition, Fred had noticed a difference. He said that he hadn't seen this definition before, but he had definitely heard the word "function" before. He distinguished, without being told to, that this was a different context compared to others where he had used the definition previously.

Frank was an engineering major. He considered the definition of function. The following excerpt shows what he said after reading it.

Frank: So… what do I do?

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> *Interviewer*: Well… there is nothing to do now…It is only a definition, for now it's just reading it… Have you seen this definition before?
>
> *Frank*: It does sound like a derivative, because of the prime. [Clarification provided.]
>
> *Interviewer*: Now, have you heard the word "function" before?
>
> *Frank*: Yes, in Calculus, 191, 192, Differential Equations, I don't think I heard of function when I did Statistics, but yeah, in almost every math class … A function is like machine. And so a function is basically like a, this is the way some people do it, like here is a machine [drawing] and then so you have *x*, which is the independent variable and it goes through this machine.

Frank was classified in Stage 2 because although he was not strictly following the given definition, he seemed to know that some mathematical concepts are common to different mathematics courses. He was also using his knowledge from previous mathematics classes.

Stage 3: Implementation attempts. In Stage 3, a student is able to recall, look for, and attempt to use/follow mathematical definitions, not necessarily with success. Some participants appeared to understand the context in which the given definition was relevant; they tried to stick to it as much as possible. This does not necessarily imply that the student had acquired/developed the ability to use the definition correctly, but at least he/she appeared able to understand the importance of considering the explicit and/or implicit details of the definition. Students in this stage seemed to know there is something embedded in each part of the definition that needs to be unpacked. But still they didn't fully unpack the definition and use it on the tasks included in the handouts. There were 9 students identified in this stage.

Candy worked with the definition of continuity at a point. She was a graduate student in the Department of Curriculum and Instruction. This excerpt from the interview shows her explanation of how she was trying to prove a statement that required only the use of the definition, namely, that f:R→R defined by $f(x)=2x+3$ is continuous.

> *Interviewer*: So what is your approach? What are to trying to prove?
>
> *Candy*: OK, so I'm trying to prove this function is continuous. So you just let out your specifications, let *R* be the real numbers, let it be a function defined by the *2x+3*...and so... there is going to be... I want to use this definition... the definition for continuous and just sort of work that, so I didn't know if I should... suppose... that there is... but...[whispers reading the definition] I don't know if I need to prove that there is an 'a' also in the reals or... or not, or should that... yeah, I guess I'm not really quite sure... we have the reals over here and they are following this function 2*x*+3, and there is an element in here [R] that is mapping to this element in here [R] and... so that... I want to see...oh! I guess that would be, I guess I should use this... OK... so to me, maybe I wanna assume that there is an "*a*" in *R* such that this and that is true but... yeah... I... I'm not quite sure… I don't know I'm going around in circles…

Above we can notice that Candy was trying to follow the definition, she even stated that. Candy was the participant that best represented Stage 3. In her interview she seemed to be trying very hard to use and understand what the definition stated.

Stage 4: Accomplishment. This last stage can be described as the stage of successful manipulation of the definition. Students at this stage were able to use the definition appropriately and according to the particular mathematical setting. These students had passed, perhaps implicitly, through all previous stages and they were, somehow, able to determine whether or not a mathematical object satisfied the given definition. Please notice that I am not claiming that students at this stage had achieved conceptual understanding of the definition, however they demonstrated the ability to stick to the definition and used it appropriately during the interview. Conceptual understanding might have occurred but I'm not accounting for it in this study. The following excerpts

show some of the students' work that I considered a successful usage of the given mathematical definition. There were 4 students in this stage.

Scott worked with the definition of isomorphism. Scott was a mathematics major. In the following portion of his work (see Figure 4) one can see that he was able to unpack and use the definition appropriately, in particular, in attempting a proof.
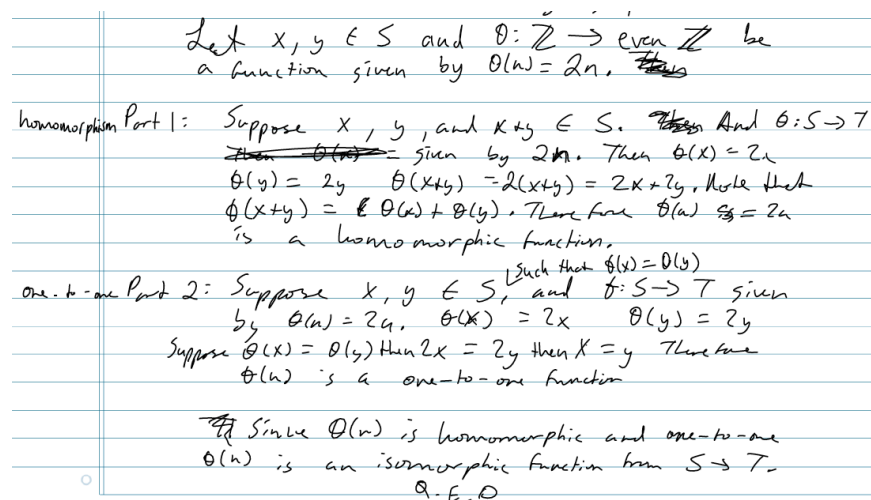
Figure 4. Scott's proof attempt.

Scott was using the proof framework suggested in class. This proof could have been written better, but Scott's attempt let me see that he was able to use and unpack a definition that had been newly introduced. For this reason, he was classified in Stage 4.

Summary and Conclusions

After identifying each participant as in one of the five stages, I noticed that these stages appear to be nested. Thus it is a conjecture that the process of getting to use a mathematical definition appropriately is teachable. One can attempt to learn how to be competent at it. Many opportunities for student success might be hidden behind the quality of communication between the teacher and the learner. However, to test this conjecture further research would be needed. The present study provides evidence that using mathematical definitions can be a difficult task for students, often expected to do it straightforwardly. I find it important to remark that conceptual understanding of the mathematical definition is not necessarily involved in these stages. These stages describe solely students' behavior reading and using mathematical definitions. Although to account for conceptual understanding was not the aim of this study, it is plausible that knowing in which of these stages a student is situated could be a fundamental early step towards the development of conceptual understanding. Being aware of the stage in which a student is, can make us more sensitive to their behaviors and the obstacles they face in attempting to consolidate their formal mathematical skills.

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THE ROLE OF DIAGRAMMATIC REASONING IN THE PROVING PROCESS

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The paper focuses on student-teachers' geometric diagrams to mediate the emergence of different proofs for a geometric proposition. For Peirce, a diagram is an icon that explicitly and implicitly represents the deep structural relations among the parts of the object that it stands for. Geometric diagrams can be seen as epistemological tools to understand explicit and hidden geometric relations. The systematic observation of and experimentation with geometric diagrams triggers abductive, inductive, and deductive reasoning which allows for the understanding of the conditions given for a geometric construction and its necessary logical consequences. We adopt Stjernfelt's model of diagrammatic reasoning to analyze two proofs for a geometric task posed to student-teachers who participated in a four-month classroom teaching experiment.

Keywords: Geometry and Geometrical and Spatial Thinking, Reasoning and Proof

Introduction

Peirce conceptualizes *sign* as a holistic triadic entity (*object*, *sign-vehicle*, *interpretant*). The word *sign* is sometimes used to refer to the *object* itself and, other times, to the mode of representation of that o*bject*. Peirce (1906) makes reference, through the paper, to the sigh being a general as to its object and as to its matter. The reader, then, is left with the task of interpreting either meaning from the context in which the word *sign* is used. His addition of the *interpretant*, as the third component of the sign, is one of his many significant contributions to semiotics. This component takes into account the effect of the sign-vehicle in the mind of the Person who interprets, uses, or produces it.

Peirce also classifies sign-vehicles as icons, indexes, and symbols according to their relation with the *object* they stand for. Fisch (1986) argues that this triad is not an autonomous species of sign-vehicles as if it were dogs, cats, and mice. Rather, it is a nested triad in which more complex sign-vehicles contain and involve specimens of simpler ones. Symbols typically involve indices which, in turn, involve icons. In other words, icons are incomplete indices which are, in turn, incomplete symbols.

The icon is a sign-vehicle that bears some sort of resemblance or similarity to its *object*. Peirce subdivides the icons into three types: images, diagrams, and metaphors. He argues, icon-diagrams have structural similarities with the structure of their *Objects*. This enables the observation, experimentation, and the emergence of inferential reasoning. He calls this amalgamated thinking *diagrammatic reasoning*. The index, instead, has a cause-effect connection to its *object*, and it directs the attention to its *object* by blind compulsion that hinges on association by contiguity (CP 1.558, 1867). The symbol, instead, hinges on intellectual operations, cultural conventions, or habit (CP 3.419, 1892).

> By diagrammatic reasoning, I mean reasoning which constructs a diagram according to a precept expressed in general terms, performs experiments upon this diagram, notes their results, assures itself that similar experiments performed upon any diagram constructed according to the same precept would have the same results, and expresses it in general form. (CP 2.96, 1902)

Diagrams as Tools for Inferential Thinking

Peirce defines icons far beyond their merely perceptual aspects: "A great distinguishing property of the icon is that by the direct *observation* of it other properties concerning its *object* can be

discovered than those which suffice to determine its construction" (Peirce 1895, Quoted in Stjernfelt, 2007, italics added). He clearly establishes diagrams as icons and as the only sign-vehicles from which more can be learned about the *object* beyond the grammar and syntax of their construction. While physical diagrams remain in the field of perception, new relations among their parts can possibly emerge by means of thought-experimentation and imagination. A diagram, then, can be characterized in one's mind in a variety of ways, "…as a token, as a general sign, as a definite form of relation, as a sign of an order in plurality, i.e., of an ordered plurality or multitude" (Robin 1967, Catalogue number 293, p. 31). The diagram, being an icon, has some kind of similarity with its *Object* in the sense that it displays the interrelations between the parts of the *object* in a skeleton-like sketch (Stjernfelt, 2007).

Peirce also argues that "the iconic diagram and its Initial Symbolic Interpretant constitute what… Kant calls *schema*, which is on one side an object capable of being observed while on the other side is a General" (NEM, p. 316) and that more can be learned about its *object* by contemplation of the explicit and implicit relations hidden in the diagram. In fact, he considers that diagrams are *epistemological tools* for inferential thinking. According to him "all necessary reasoning is diagrammatic" (Robin 1967, Catalogue number 293, p. 31).

Being a student of Kant's, Peirce adopts and adapts Kant's concept of geometric construction: "such a construction is formed according to a precept furnished by the hypotheses; being formed, the construction is submitted to the scrutiny of *observation*, and new relations are discovered among its parts, not stated in the precept by which it was formed, and are found, by a little *experimentation*, to be such that they will always be present in such a construction" (CP 3.560, italics added). This operational definition entails that once an empirical diagram is constructed what follows is some kind of mental experimentation and inferential experimentation.

A classic example of inferential manipulation and experimentation is Euclidean geometry. "Euclid first announces, in general terms, the proposition he intends to prove, and then proceeds to draw a diagram, usually a figure, to exhibit the antecedent condition thereof" (NEM, p. 317). Nowadays, given the dragging mode of dynamic geometry environments, the manipulation of geometric figures is expedited and, with it, the possibility of intentional experimentation. The observation of variant and invariant relations among the elements of the figure facilitates the conjecturing of its properties as well as the process of proving or disproving them. Stjernfelt, a semiotician, who has dedicated articles and books to the analysis of Peirce's diagrammatology, extensively argues that his definition of icon is non-trivial. This definition, he argues, avoids the weakness of most definitions of similarity because of its connection to the notion of observation and inferential experimentation to discover additional pieces of information about the *object* it stands for.

> …all deductive reasoning, even simple syllogism, involves an element of *observation*; namely, deduction consists in constructing an icon or diagram the relations of whose parts shall present a *complete analogy* with those of the parts of the object of reasoning, of experimenting upon this image in the imagination, and of observing the result so as to discover unnoticed and hidden relations among the parts (CP 3.363, italics added).

Diagrammatic Reasoning Process

Stjernfelt captures the essence of the process of diagrammatic reasoning in Figure 1. This is to say, a process which is rooted in perceptual and mental activity to produce chains of inferences. This figure is especially useful for thinking about proving and problem-solving processes. This skeleton-like figure, which is an icon-diagram itself, synthesizes a manifold of relationships that amalgamate the construction of a diagram, the observation of structural relations among its parts, and the physical and mental manipulation to produce a chain of deductions so as to attain a conclusion.
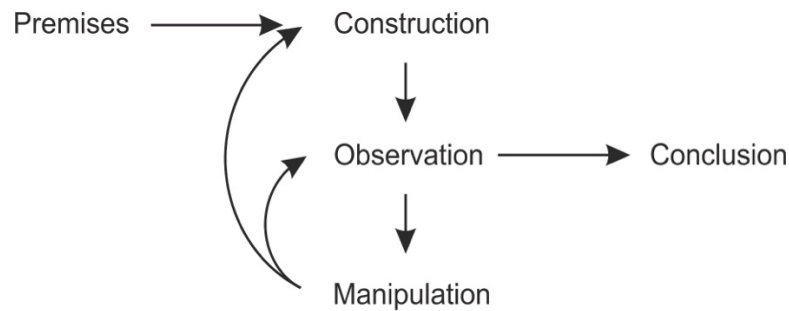
Figure 1. Diagrammatic reasoning process (Stjernfelt model, 2007).

In addition, Stjernfelt also describes this process in terms of the emergence of evolving interpretants generated during the transformation of diagrams. In this transformation, the implicit aspects of the *object* of the diagram are unveiled by means of the analogy between the relations among the characteristic properties of the object and the structural relations among the parts of the diagram. That is, the interpreting Person transforms icon-diagrams into sign-vehicles that have more and more symbolic aspects that hinge on mental operations and inferential reasoning. It is in this sense that "symbols grow," as Peirce says, because the meanings of their *objects* grow deeper and more general in the mind of the interpreting Person. A sequence of interpretants generated in this process is described by Stjernfelt (2007, p. 104) as follows:

a. Symbol (1)
b. Immediate iconic interpretant: Initial pre-diagrammatic icon-token that is rule-bound
c. Initial interpretant: (a+c) constituting the <u>initial transformand diagram</u>, the 'Schema' diagram-icon
d. Middle interpretant: the symbol-governed diagram equipped with possibilities of transformation (with two sources, **a** as well as **c)**
e. Eventual, rational interpretant: Transformate diagram
f. Symbol (2): Conclusion
g. A post-diagrammatical interpretant (different from **b**): This interpretant is an interpretant of **a**, but now, the diagrammatic reasoning is enriched by the <u>total interpretant of the concept **a**</u> [represented by Symbol (1)].

It is important to note that transformate diagrams are substantially contained in the transformand diagram with all its significant features. That is, diagrammatic reasoning is the process by which the interpreting Person intentionally endeavors both in the observation and manipulation of initial diagram-tokens (transformand diagrams) to mentally enrich and transform them (transformate diagrams) so that hidden relations among the parts of the *object* can be unveiled. These transformations facilitate the inference of the hidden structure of the *object*.

Methodology

A constructivist four-month classroom teaching-experiment on the teaching-learning of geometry was conducted with nine pre-service and in-service mathematics student-teachers who were taking a geometry methods course using the Geometer's Sketchpad (GSP). The main goal of this experiment was to improve student-teachers' ability to conjecture and to prove geometric propositions in plane Euclidean geometry using the GSP. An inquiry approach was used in which tasks were posed, drawings were constructed and manipulated by the students, conjectures were made, and proofs were generated. Student-teachers proved geometric statements in class and in homework assignments using this inquiry approach. They completed weekly homework assignments of at most seven tasks using the GSP. At the beginning of the semester, student-teachers were given a pre-test with twelve

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tasks to be solved using pencil and paper. At the end of the semester, they were given a post-test with thirteen tasks to be solved using the GSP. The purpose of these tests was to observe the influence, in the student-teachers' proving process, of the dynamic diagrams constructed, observed, and manipulated in the GSP environment.

Here we analyze, from the diagrammatic reasoning point of view, two proofs that pre-service student-teachers produced for task#1 in homework #9. This task could be proved in different ways: without using auxiliary lines and using auxiliary lines. The core of the geometric argument for each of the proofs was the conceptualization of congruent triangles embedded, implicitly or explicitly, in the diagram.

Data Analysis

In homework #9 student-teachers were given seven tasks. Task#1 was the following (see Figure 2):

> **Construct an isosceles triangle △ABC (AB ≅ AC). Extend the congruent sides BA and CA from the common vertex A and take AD = AE respectively. Let point M the midpoint of the base BC.**
> **Prove that the triangle △MDE is an isosceles triangle.**

Figure 2. Task#1 given to student-teachers.

There are many different ways to prove this task but only one without the use of auxiliary lines. Only one student-teacher completed the proof this way. Four student-teachers constructed auxiliary lines EB and DC and completed the proof. Two student-teachers used the property that the median from the vertex-angle in an isosceles triangle is also perpendicular bisector. The other two student-teachers were unable to complete the proof because they constructed an incorrect drawing by extending the sides AB and AC from vertices B and C. The teacher wrote her proof and then presented it to the class.

First Proof

When the student explained her proof to the class, she showed on the computer screen the sequence of diagrams (Figures 3a, 3b, & 3c). This sequence conceals a sequence of interpretants, in the mind of the student-teacher (the interpreting Person), that allowed her a mental transformation of the same physical diagram (transformand diagram, i.e., Figure 3a) into transformate diagrams (Figures 3b & 3c) to conceptualize geometric relations. Finally, she presented the written proof of the given statement.

She constructed a robust isosceles △ABC using two radii of a circle centered on the vertex-angle A. In the extensions of sides AB and AC, from vertex-angle, she constructed congruent line segments AD and AE respectively. She continued with the construction of the midpoint M on base BC and of the △EMD (see Figure 3a). Then the student-teacher wrote down the given information $\overline{AB} \cong \overline{AC}$ and $\overline{AD} \cong \overline{AE}$ (see Figure 3b) as well as the proof that △BDM is congruent to △CEM by SAS. For the congruence of the triangles she explained that $\overline{MB} \cong \overline{MC}$ because point M is midpoint of base BC; $\overline{DB} \cong \overline{EC}$ because $\overline{DA} + \overline{AB} \cong \overline{EA} + \overline{AC}$; and $\angle ABC \cong \angle ACB$ as base-angles of isosceles triangle △ABC. The congruence of triangles △BDM and △CEM implies $\overline{DM} \cong \overline{EM}$ proving that triangle △DME is isosceles (see Figure 3c).

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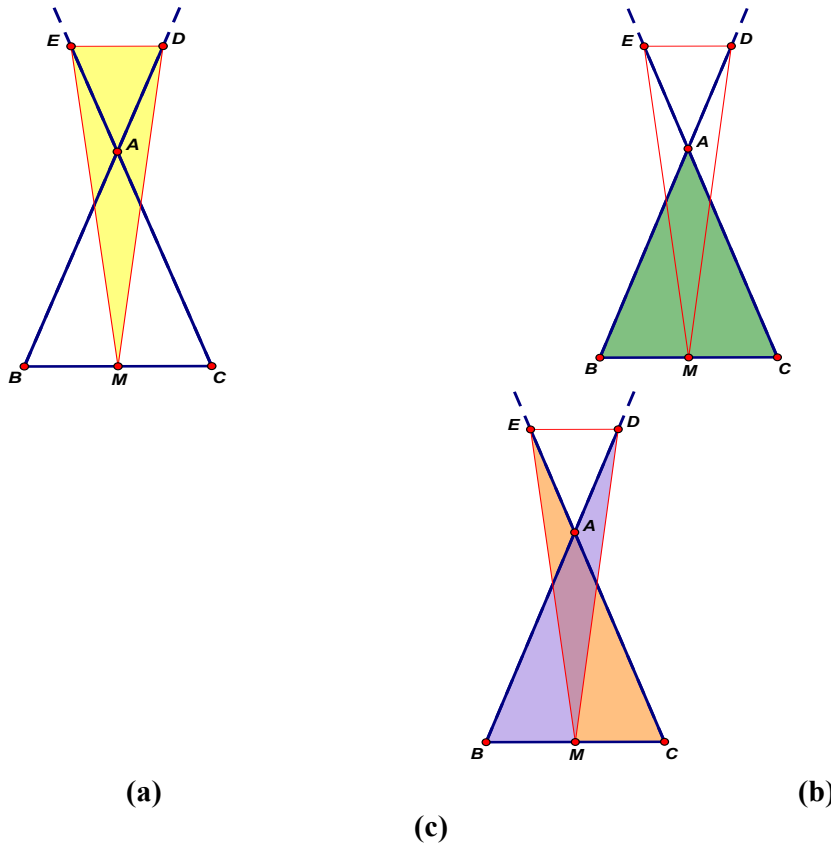
Figure 3. (a) Initial construction; (b) Focus on isosceles triangles ΔABC and ΔADE; (c) Comparison of triangles ΔDMB and ΔEMC.

Following is also the description of the sequence of interpretants generated:

- *Inmmediate iconic interpretant:* Visual perception of two isosceles triangles ΔABC and ΔADE and the triangle ΔEMD (Figure 3a, i.e., transformand diagram).
- *Initial Interpretant*: A realization that adding congruent line segments ($\overline{AB} \cong \overline{AC}$ and $\overline{AD} \cong \overline{AE}$), by parts, would generate new congruent line segments ($\overline{DA} + \overline{AB} \cong \overline{EA} + \overline{AC}$ then $\overline{DB} \cong \overline{EC}$) (Figure 3b, i.e., transformate diagram).
- *Middle Interpretant*: ΔDMB and ΔEMC have two congruent sides ($\overline{DB} \cong \overline{EC}$ and $\overline{MB} \cong \overline{MC}$) and the respective angles between them are also congruent ($\angle ABC \cong \angle ACB$) being the base angles of isosceles ΔABC (Figure 3b, i.e., transformate diagram).
- *Rational interpretant*: Given that ΔABC is isosceles with $\overline{AB} \cong \overline{AC}$, the congruence of the base angles is implied ($\angle ABC \cong \angle ACB$). From the fact that point M is the midpoint of side BC, the congruence of BM and MC is also implied ($\overline{BM} \cong \overline{MC}$). Also using the congruence $\overline{DB} \cong \overline{EC}$ triangles ΔDMB and ΔEMC are congruent by SAS. (Figure 3b, i.e., transformate diagram).
- *Eventual rational interpretant*: Line segments $\overline{DM}$ and $\overline{EM}$ are the third sides of congruent triangles ΔDMB and ΔEMC; thus these line segments have to be congruent (Figure 3c).
- *Post-diagrammatical interpretant*: Consider triangles ΔEMC and ΔDMB. How are they related?
- $\overline{BM} \cong \overline{MC}$ (point M is the midpoint of $\overline{BC}$)

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- ∠ABC ≅ ∠ACB (base-angles in the isosceles triangle ΔABC)
- $\overline{EC} \cong \overline{DB}$ ($\overline{EA} + \overline{AC} \cong \overline{DA} + \overline{AB}$ because $\overline{EA} \cong \overline{DA}$ and $\overline{AC} \cong \overline{AB}$ adding by parts)
- Therefore, ΔEMC ≅ ΔDMB by SAS and the corresponding sides $\overline{EM}$ and $\overline{DM}$ are congruent making triangle ΔEMD an isosceles triangle (Figure 3c, i.e., transformate diagram).

Her written proof certainly corresponds to the inferred intepretants from her diagrams (see Figure 4):

> **Given M is the midpoint of BC, ∴ CM = BM**
> **Given isosceles △ABC, ∴ AC = AB and ∠C = ∠B**
> **Given AE = AD, then AE + AC = AD + AB, ∴ EC = BD**
> **Therefore, triangles △EMC and △DMB are congruent by SAS.**
> **This implies that ME = MD making triangle △MED isosceles triangle.**

Figure 4. First proof of Task#1 without auxiliary lines.

In the above proof, the student-teacher used direct relations between congruent corresponding sides and angles of isosceles triangles in the diagram to prove the congruence of triangles ΔEMC and ΔDMB. Then she implied the congruence of line segments $\overline{EM}$ and $\overline{DM}$. Thus, she proved that ΔEMD is isosceles.

Second Proof

The four student-teachers who used the auxiliary line segments $\overline{EB}$ and $\overline{DC}$ gave essentially the same proof (see Figure 5).
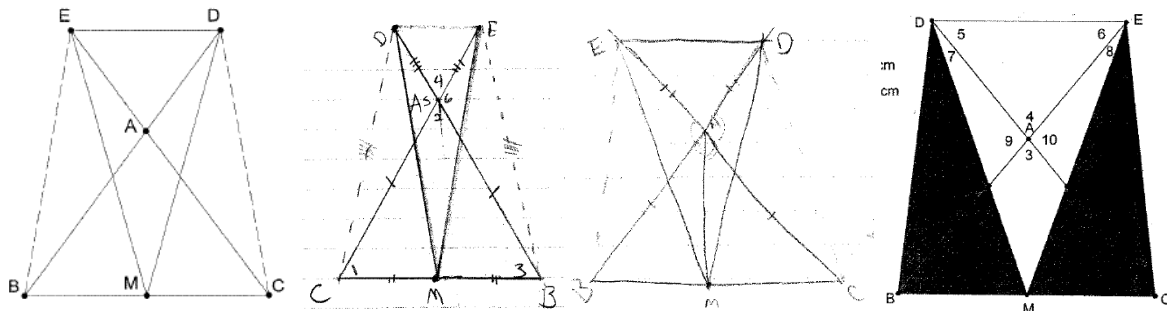
Figure 5. Drawings using the auxiliary line segments EB and DC.

Figure 5 presents the diagrams that four student-teachers gave using auxiliary lines EB and DC (or DB and EC). Below is the proof written by one of the four students, which corresponds to the first diagram on the left in Figure 5. The other student-teachers wrote similar proofs presenting the same argument (see Figure 6).

> AB ≅ AC, AD ≅ AE, and BM ≅ CM through the construction. Also, ∠CAD ≅ ∠BAE (Task 1). Thus, by the Side-Angle-Side Theorem △ABE ≅ △ACD. Since △ABE ≅ △ACD and BE corresponds to CD, BE ≅ CD and ∠ABE ≅ ∠ACD = a. By Homework 8 ∠ABC ≅ ∠ACB = y. Thus ∠EBM = x+y = ∠DCM. Thus by the Side-Angle-Side Theorem, △EBM ≅ △DCM. Since △EBM ≅ △DCM, ME ≅ MD. Therefore △MDE is an isosceles triangle since two of its sides are congruent.

Figure 6. Second proof of Task#1 using auxiliary lines.

This student-teacher started with the relations $\overline{AB} \cong \overline{AC}$ and $\overline{AD} \cong \overline{AE}$ according to the construction of the isosceles triangles in Figure 5 and he also used the congruent vertical angles ∠CAD and

$\angle BAE$. He proved that triangles ΔABE and ΔACD are congruent by SAS. From this he implied that corresponding sides and angles are congruent ($\overline{BE} \cong \overline{CD}$ and $\angle ABE \cong \angle ACD$). Using the congruence of the base angles $\angle ABC \cong \angle ACB$ in the isosceles triangle ΔABC and the congruence $\angle ABE \cong \angle ACD$, he proved that $\angle EBM \cong \angle DCM$ as sums of congruent angles. Combining the relations $\overline{BM} \cong \overline{MC}$, $\overline{BE} \cong \overline{CD}$, and $\angle EBM \cong \angle DCM$, he proved the congruence of triangles ΔEBM and ΔDCM by SAS. An implication of the congruence of triangles ΔEBM and ΔDCM is that $\overline{EM} \cong \overline{DM}$. Thus, he concluded that ΔEMD is isosceles.

The above description can be unfolded into a sequence of diagrams that conceals a sequence of interpretants in the mind of the student-teacher (the interpreting Person). These interpretants allow for mental transformations of the same physical diagram (transformand diagram, i.e., Figure 7a) into transformate diagrams (Figures 7b, 7c, & 7d) to conceptualize other geometric relations and, finally, the proof of the given statement.

(a)   (b)

(c)   (d)
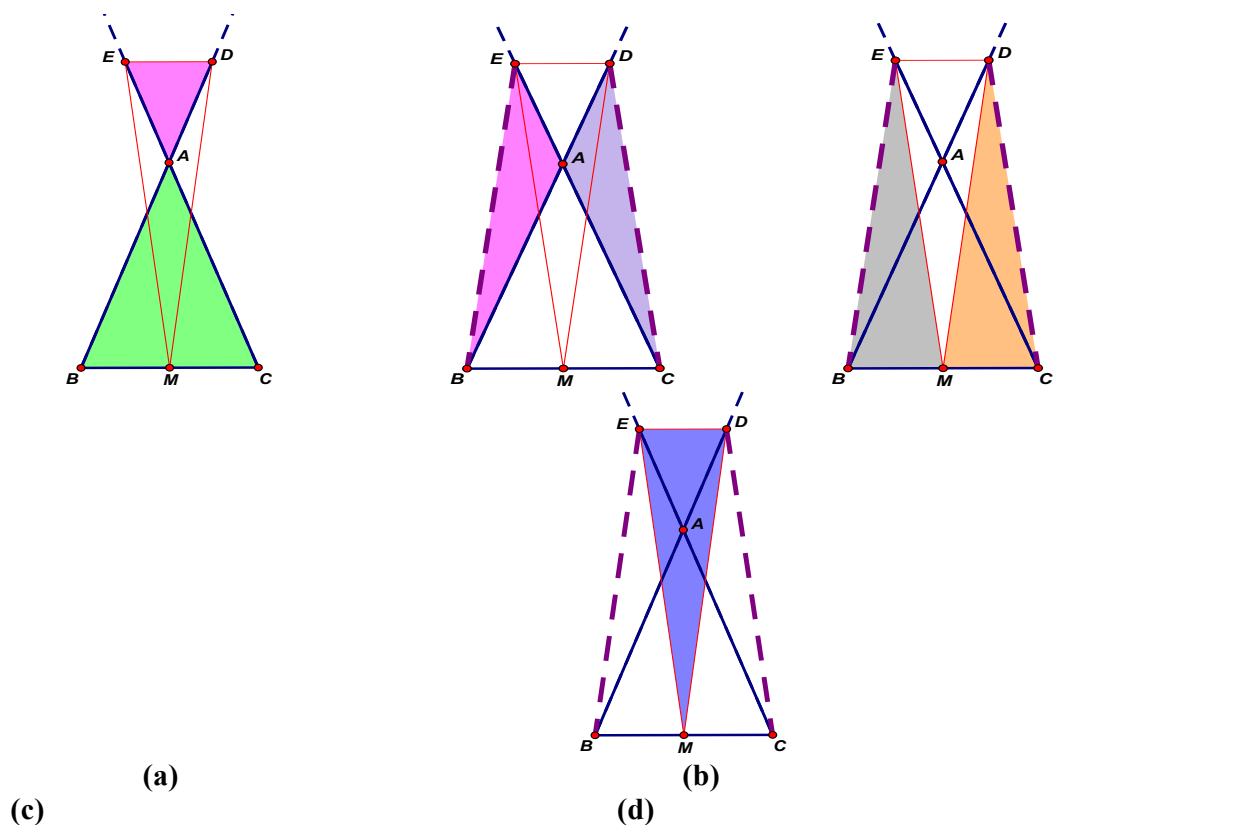
Figure 7. (a) Initial construction; (b) Comparison of triangles ΔABE and ΔACD created by auxiliary lines EB and DC; (c) Comparison of triangles ΔEBM and ΔDMC; (d) Conclusion triangle ΔEMD is isosceles.

- *Immediate iconic interpretant:* Visual perception of triangles (transformand diagram) constituted by two isosceles triangles (ΔABC and ΔADE) and also ΔEMD (Figure 7a, i.e., transformand diagram).
- *Initial Interpretant* A realization that drawing line segments $\overline{EB}$ and $\overline{DC}$ would generate two new sets of triangles: ΔEAB and ΔDAC & ΔEBM and ΔDCM (Figure 7b, i.e., transformate diagram).

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- *Middle Interpretant*: The segments $\overline{EM}$ and $\overline{DM}$ which are sides of ΔEBM and ΔDCM are also sides of ΔEMD. (Figure 7c, i.e., transformate diagram)
- *Rational interpretant*: A realization that $\overline{AB} \cong \overline{AC}$ are congruent sides of the given isosceles ΔABC. $\overline{AD} \cong \overline{AE}$ is a structural relation of the task. $\angle EAB \cong \angle DAC$ are vertical angles. Thus, ΔEAB and ΔDAC are congruent by SAS. Then both, the congruence of line segments $\overline{EB}$ and $\overline{DC}$ and the congruence of angles $\angle EBA$ and $\angle DCA$ are implied. (Figure 7b, i.e., transformate diagram)
- *Eventual rational interpretant*:
- How the line segments $\overline{EM}$ and $\overline{DM}$ could be compared? To which other triangles do they also belong? Triangles ΔEBM and ΔDCM are the triangles with the line segments $\overline{EM}$ and $\overline{DM}$ as sides (Figure 7c, i.e., transforme diagram).
- *Post-diagrammatical interpretant*:
- Compare triangles ΔEBM and ΔDCM
- $\overline{BM} \cong \overline{MC}$ (point M is the midpoint of $\overline{BC}$)
- $\overline{EB} \cong \overline{DC}$ (implication from the congruence of triangles ΔEAB and ΔDAC)
- $\angle EBM \cong \angle DCM$ (adding by parts congruent angles $\angle EBA \cong \angle DCA$ and $\angle ABM \cong \angle ACM$)
- Therefore, $\Delta EBM \cong \Delta DCM$ by SAS and the corresponding sides $\overline{EM}$ and $\overline{DM}$ are congruent making triangle ΔEMD an isosceles triangle (Figure 7d, i.e., transformate diagram).

In the above proof the student-teacher used auxiliary lines EB and DC. Then he used direct relations between congruent sides and angles to prove, first, the congruence of ΔEAB and ΔDAC and, then, the congruence of ΔEBM and ΔDCM. From the last congruence of triangles he implied the congruence of line segments $\overline{EM}$ and $\overline{DM}$. This proves that ΔEMD is an isosceles triangle.

Conclusion

The significance of *diagrammatic reasoning* in the teaching-learning of geometry during the proving process is analyzed in this paper. The Stjernfelt's model of *diagrammatic reasoning*, based on Peirce's own definition, was adopted. Essentially, diagrammatic reasoning consists of the systematic observation of a geometric diagram, the experimentation with the geometric diagram, and the inferential reasoning emerging from the observation of unveiled relations among the elements of the geometric diagram. Observation allows the visual perception of the explicit relations between the elements of the diagram. Experimentation with the diagram verifies these relations and facilitates the investigation of further relations. Finally, inferential reasoning emerges mediated by prior geometric knowledge and it makes possible the completion of the proving process.

Given the spatial and visual nature of Euclidean geometry, thinking and proving without diagrams seems to be an impossible task. Thus gaining awareness of diagrammatic reasoning as an epistemological tool appears to be useful for teachers to direct not only their own thinking but also the thinking of their students.

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THE EFFECT OF WORKED EXAMPLES ON STUDENT LEARNING AND ERROR ANTICIPATION IN ALGEBRA

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The present study examines the effectiveness of incorporating worked examples with prompts for self-explanation into a middle school math textbook. Algebra 1 students (N=75) completed an equation-solving unit with reform textbooks either containing the original practice problems or in which a portion of those problems were converted into correct, incorrect, or incomplete examples. Students completed pre- and posttest measures of conceptual understanding, procedural problem-solving skill, and error anticipation. Results indicate the example-based textbook assignments increased students' procedural knowledge and their ability to anticipate errors one might make when solving problems. Differences in students' anticipation of various types of errors are also examined.

Keywords: Algebra and Algebraic Thinking, Cognition, Middle School Education

Introduction

Over the past few decades, a significant body of work in cognitive science has provided evidence for the Worked Example principle, which argues for having learners spend some of their practice time studying worked out examples of problem solutions rather than solving all of the problems themselves (e.g., Sweller & Cooper, 1985; Sweller, 2006). Several studies have confirmed the effectiveness of studying worked examples in computer-based classrooms (Booth, Lange, Koedinger, & Newton, 2013; Kim, Weitz, Heffernan, & Krach, 2009; Paas, 1992) and in traditional mathematics classrooms (Booth, Cooper, Donovan, Huyghe, Koedinger & Pare-Blagoev, 2015; Booth, Oyer, Pare-Blagoev, Elliot, Barbieri, & Koedinger, 2015), though none to date have examined its effect in reform classrooms. Further, studies have also expanded on the worked example principle to suggest that having students self-explain the examples (Renkl, Stark, Gruber, & Mandl, 1998), gradually fading the support provided in correct examples (Atkinson, Renkl, & Merrill, 2003), and having students also consider incorrect examples (Siegler, 2002) can have enhanced learning benefits for conceptual and procedural knowledge, both of which are critical in mathematics (NMAP, 2008).

Despite this solid body of support, these practices are not typically undertaken in real world classrooms; the incorporation of incorrect examples is looked upon as a particularly foreign practice. In the United States, teachers tend to shy away from talking about errors (Lannin, Townsend, & Barker, 2006) at least in part due to the fear that their students will adopt the errors in their own problem solving (Santagata, 2004). However, students can learn a great deal from considering errors. Two established theories provide explanations for how studying errors can be useful for learning. Ohlsson's (1996) theory suggests that explaining why an error is wrong can help learners identify the particular features of the problem that make the solution incorrect; this can lead to refinement of problem solving skills and remediation of misconceptions. In addition, Siegler's (1996) overlapping waves theory maintains that individuals know and use a variety of (correct and incorrect) strategies for solving problems, and those strategies compete for use each time a problem is encountered. Studying errors is thought to be an effective way of helping learners accept that those particular strategies are wrong and prompting them to construct and strengthen other, correct strategies (Siegler, 2002).

Consistent with these theories, recent research has found that error reflection is indeed beneficial to learning. For example, having students think about and correct their own errors can lead to greater engagement and improved problem solving skill (Cherepinsky, 2011; Henderson & Harper, 2009);

studying the errors made by others may be even more effective (Yerushalmi & Polingher, 2006), in part because it exposes students to multiple perspectives other than their own (Siegler & Chen, 2008). Incorporating incorrect examples into class assignments and prompting students to explain why they are incorrect has been found to be particularly beneficial for improving conceptual understanding (Booth et al., 2013)

Thus, there is growing consensus that students can learn effectively from explaining errors. However, textbooks don't often include incorrect examples, and creating materials and lessons that include incorrect examples can be very time consuming for teachers. Some curriculum materials have recently been developed and tested and are available for use (e.g., see Booth et al., 2015). However, these efforts are only useful for the content they have explicitly been developed for—the considerable work that goes into such efforts does not translate directly for other content areas. One possible solution would be to ask students to think about the kinds of errors that might be made and explain to themselves why those solutions would be incorrect. But are students able to anticipate the types of errors other students might make? And might explaining worked examples improve students' ability to anticipate errors? These questions are a main focus of the present study.

The Present Study

There were two main purposes in this study. First, it aimed to replicate and extend previous results showing that the use of worked examples with self-explanation prompts is effective for students learning mathematics in real world classrooms. Prior results have found that in traditional mathematics classes, where procedural knowledge is typically emphasized and conceptual knowledge may be lacking, assignments containing correct and incorrect examples to explain along with problems to solve were more effective for improving both conceptual and procedural knowledge than assignments in which students solved all of the problems themselves. In the present study, correct, incorrect, and partially completed examples were incorporated into reform mathematics classrooms, where conceptual knowledge is typically emphasized through rich problem-based activities, but procedural knowledge may suffer (NRC, 2004). We hypothesize that worked examples will still be effective in a reform classroom and investigate differences in effectiveness for conceptual vs. procedural knowledge.

The second purpose of this study was to examine students' ability to anticipate the types of errors other students might make when solving equations, and determine if this ability is improved after experience explaining correct and incorrect examples, which highlight features of problems for which students might have misconceptions and demonstrate incorrect methods for solving the problems. We will examine the types of errors that students anticipate and determine how the anticipated types of errors change after instruction in general and/or the worked examples intervention.

Methods

Participants

Seventy-five 8th grade Algebra I students from an inner-ring suburban middle school in the Midwestern United States participated in the study (55% female; 59% African American, 21% White, 15% American Indian/Alaskan, 4% Asian, and 1% classified as other ethnicities). The study employed a quasi-experimental design where students were assigned to the treatment and control groups according to their rostered section of Algebra I. The four sections of Algebra I were taught by two teachers, with each teacher having one treatment and one control class. In all, 37 students (49%) participated in the experimental group while 38 students (51%) participated in the control group. All of the Algebra I classes utilized the Connected Mathematics Project 2 Curriculum (CMP2; Lappan, Fey, Fitzgerald, Friel, Phillips, 2006) CMP2 includes rich, problem-based investigations

during classroom lessons, and provides a variety of practice problems for students to solve afterward as classwork and/or homework. This study took place during the *Say It With Symbols* unit, which focuses on understanding symbols in algebraic equations.

Measures and Coding

A single, experimenter-designed paper-and-pencil test was utilized for this study and administered as both pretest and posttest. The test included three types of items: conceptual knowledge, procedural knowledge, and error anticipation.

Conceptual knowledge. To examine students' conceptual knowledge about problem features, we used 21 items focused on the meanings of different terms in an equation, identification of equivalent expressions, and categorization of functions as linear, quadratic, or exponential. The percentage of these items answered correctly was computed for each student at pretest and at posttest.

Procedural knowledge. To examine procedural skill, we used 9 items which asked students to solve multi-step equations, simplify expressions using the distributive property, and evaluate formulas at given values. The percentage of these items answered correctly was computed for each student at pretest and at posttest.

Error anticipation. To evaluate ability to identify errors that others might make when solving multi-step equations, we utilized one item which asked students what mistakes they thought a seventh-grader might make in solving the equation $5x - 2 = 8$. In total, they were asked to identify two potential errors. Student responses were coded first in terms of whether or not the provided responses were reasonable, and then by the type of error referenced: mistakes involving variables (e.g., handle the coefficient separately from the variable), like terms (e.g., subtract 2 from 5x), negative signs (e.g., subtract 2 from both sides instead of adding 2), equals sign (e.g., perform an operation to one side and not the other), operations (e.g., adding two numbers instead of multiplying), other reasonable errors, or unreasonable errors. For each type of error, students were scored in terms of whether at least one of their responses fit in that category.

Procedure

Prior to beginning the *Say It With Symbols* unit, all students took the paper-and-pencil pretest. Instruction for the treatment and control classrooms was kept constant within teacher (e.g., Teacher A provided the same lesson to both her treatment class and her control class). The only difference between conditions is that when students were to work on their practice problems, treatment classes were given an adapted version of the *Say It With Symbols* book. In the adapted book, approximately 26% of the practice problems were replaced with a correct, incorrect, or partial example of a solution to that problem. Teachers could assign as many or as few practice problems as they desired as long as the same items were assigned to both their treatment and control groups. When the unit was complete, teachers administered the paper-and-pencil posttest to all students.

Results

To examine the effectiveness of the treatment for improving students' conceptual and procedural knowledge, we conducted a 2 (condition: treatment vs. control) x 2 (time: pretest vs. posttest) x 2 (measure: conceptual vs. procedural) RMANOVA, with repeated measures on time and measure. The analysis yielded a main effect of time ($F(1, 73) = 23.50$, $p < .0.001$, $\eta_p^2 = 0.24$), with students performing better at posttest (M = 52%) than at pretest (M = 41%). The main effect of measure was also significant ($F(1, 73) = 52.97$, $p < .0.001$, $\eta_p^2 = 0.42$), with students performing better on conceptual items (M = 52%) than on procedural items (M = 41%). There was a significant time by measure interaction, $F(1, 73) = 15.43$, $p < .0.001$, $\eta_p^2 = 0.17$), with students improving more from pretest to posttest on procedural items (31% to 48%) than conceptual items (51% to 55%). Finally,

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there was a significant interaction between time and condition, $F(1, 73) = 4.66$, $p = .0.034$, $\eta_p^2 = 0.04$, revealing that students in the treatment group improved more from pretest to posttest (42% to 57%) than students in the control group (40% to 46%). No other main effects or interactions reached significance.

To examine the types of errors anticipated by students and how this anticipation changes with instruction and worked examples intervention, we conducted a 2 (condition: treatment vs. control) x 2 (time: pretest vs. posttest) x 6 (error type: variable, like terms, negative signs, equals signs, operations, and other reasonable errors) RMANOVA, with repeated measures on time and error type. The analysis yielded a main effect of time, $F(1, 73) = 23.92$, $p < .0.001$, $\eta_p^2 = 0.25$), with a greater likelihood of anticipating reasonable error responses at posttest (23%) than at pretest (14%). There was also a significant interaction between time and condition, $F(1, 73) = 4.77$, $p = .0.032$, $\eta_p^2 = 0.06$), with the likelihood of anticipating reasonable errors increasing more from pretest to posttest for the treatment group (12% to 24%) than for the control group (17% to 22%). The main effect of error type also reached significance, $F(5, 69) = 10.51$, $p < .0.001$, $\eta_p^2 = 0.43$). Post-hoc tests with Bonferroni correction revealed that variable, like terms, and negative sign errors were more likely to be anticipated than equals sign and other reasonable errors; variable errors were also more commonly anticipated than operations errors (see Figure 1). Finally, there was a significant interaction between time and error type $F(5, 69) = 2.47$, $p = .0.041$, $\eta_p^2 = 0.15$). Follow-up paired-sample t-tests revealed that the likelihood of anticipating three types of errors increased from pretest to posttest: Like terms (12% to 39%; $t(74) = 4.37$, $p < .001$), operations (8% to 19%; $t(74) = 2.04$, $p = .045$), and other reasonable errors (4% to 15%; $t(74) = 2.38$, $p = .020$).

Discussion

Results from the present study replicate and extend prior studies on the effectiveness of worked examples in mathematics learning in two important ways. First, they demonstrate that incorporating worked examples into students' practice in reform classrooms is a useful practice. Even though such classrooms are more focused on conceptual understanding than traditional classrooms, studying and explaining worked examples leads to improved learning over problem-solving practice alone. In particular, improvement in procedural knowledge as a result of worked examples may be especially helpful in reform classrooms.

The second type of extension provided by the present study is that practice containing worked examples, many of which prompt students to reflect on incorrect procedures, leads to an increased likelihood that students will be able to anticipate errors that other students might make. If teachers want to make learning from errors a more prominent part of their classrooms but do not have access to—or time to create—relevant error-centered lessons, it may be desirable to have students that can think about potential errors on their own and reflect on why those anticipated errors are problematic. Introducing assignments with incorrect examples earlier in the process may train students to anticipate such errors on their own. However, further research is needed to examine the mechanism underlying this improvement and determine whether and how error anticipation skill transfers from one particular type of content to another. For example, if students' error anticipation skill improves after worked example assignments in a linear equations unit, would they be more likely to anticipate errors for solving quadratic equations? For graphing linear functions? For geometry? The answers to these questions are likely different depending on whether error anticipation improves because students get used to reflecting on errors in general or because they are exposed to examples of errors specific to that content area.

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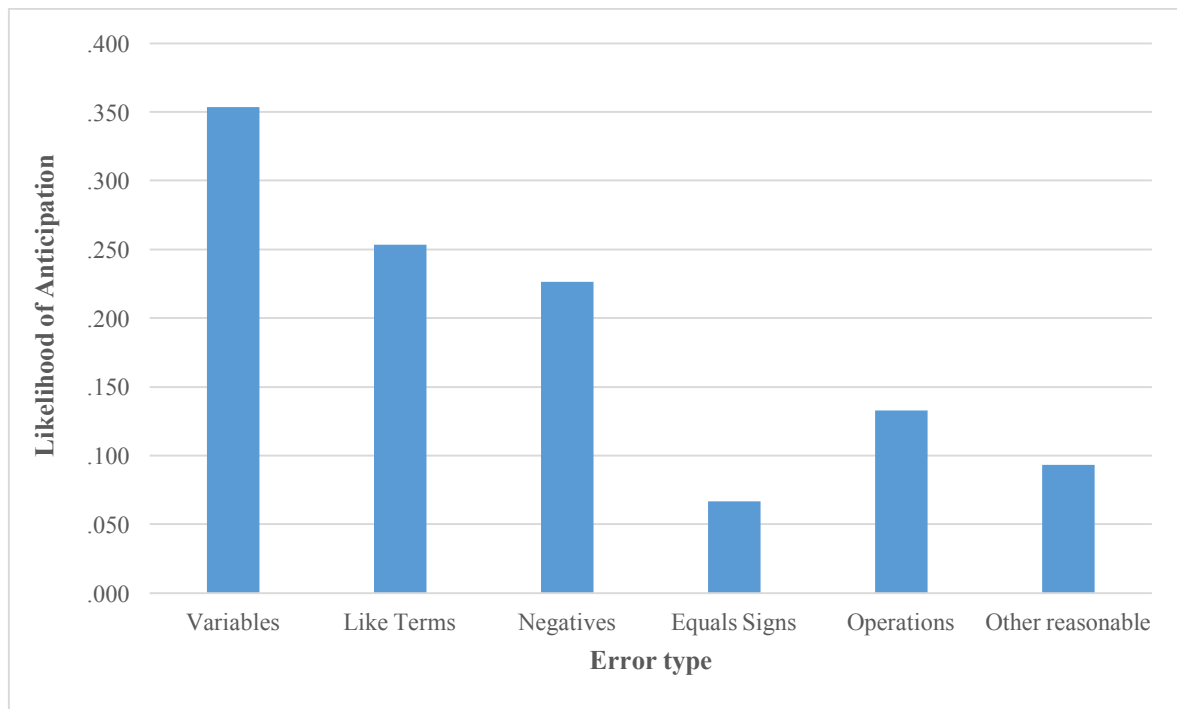
Figure 1: Likelihood of anticipation by error type.

The present study also revealed differences in the types of errors students tend to anticipate. Errors dealing with variables, like terms, and negative signs were the most frequently anticipated across time points, and anticipation of like terms, operations, and other reasonable errors were most likely to increase after students gained more knowledge about the content area. This is interesting for several reasons. For instance, prior research has identified the types of algebraic errors that are the most prevalent for Algebra I students (Booth, Barbieri, Eyer, & Pare-Blagoev, 2014). This work suggests that one of the types of highly-anticipated errors in the present study—those involving negative signs—are highly prevalent in equation-solving activities, but that the other two highly-anticipated error types—variables and terms errors—are not among the most prevalent in that content area. This indicates that students may be likely to anticipate negative sign errors because they see (or make) them frequently. However, it is not clear why students would be likely to anticipate variables or terms errors if they are not frequently made. Further study, perhaps with think aloud data collection, is necessary to determine how students come up with the errors they anticipate. A combination of think aloud and classroom observation data would also enable us to examine differences in these anticipations at pretest vs. posttest and determine why certain types of errors become easier to recognize after instruction on the topic. Knowing what the teachers are highlighting in their lessons may help explain why anticipation of certain error types increases while other types do not. If a teacher aims to get students to anticipate a wider variety of errors, further intervention targeted at helping students notice less anticipated errors may be necessary.

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"RELATIONAL" EQUITY: ELEMENTARY STUDENTS CO-CONSTRUCT A SOCIAL-MATHEMATICAL POWER DYNAMIC DURING COLLABORATIVE ENGAGEMENT ON EQUIVALENCE TASKS

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We report a qualitative analysis of elementary school students engaged in collaborative problem solving involving mathematical equivalence tasks. We build on previous research showing that students often use strategies based on either operational or relational understandings of the equal sign. We closely analyze three cases and identify nuanced aspects of the social interaction that influence whether and how students develop and use operational or relational strategies toward a final solution. Students' demonstrated understandings of the equal sign during collaboration aligned with those identified in past research. We argue that a social-mathematical power dynamic was co-constructed in each of the dyads, and the ways students navigated that dynamic affected the quality of individual engagement and therefore learning.

Keywords: Cognition, Elementary School Education, Equity and Diversity, Problem Solving

Previous Literature on Mathematical Equivalence and Study Objectives

Understanding that the two sides of an equation must represent the same quantity is crucial for the development of algebraic reasoning. However, many children struggle to understand the equal sign as a *relational* symbol that connotes this equality relationship. Instead, children aged 7-11 often define the equal sign in *operational* terms, describing it as a cue to add all the numbers in the problem (McNeil, 2007; Perry, Church & Goldin-Meadow, 1988). Additionally, when children of this age solve equivalence problems such as $3 + 4 + 5 = 5 + \_\_$, they often add all of the numbers together and mark 17 as the final answer (McNeil, 2007; Perry et al., 1988).

One key to solving mathematical equivalence problems correctly may be noticing the atypical location of the equal sign in these problems. When children aged 7-11 are asked to reproduce equivalence problems from memory, they often place the equal sign at the *end* of the problem rather than in the *middle* (e.g., a child will recall the problem above as $3 + 4 + 5 + 5 = \_\_$; McNeil & Alibali, 2004). This particular pattern of errors may be due to the fact that children in the United States rarely practice arithmetic problems that do not take the typical form $a + b = c$ (McNeil et al., 2006). Extensive practice with this problem format may lead children to expect that the equal sign will always appear at the end of a problem (McNeil & Alibali, 2004). Accurate encoding of the location of the equal sign in mathematical equivalence problems is closely related to using strategies based on a relational understanding of the equal sign to solve these problems correctly (e.g., Crooks & Alibali, 2013; McNeil & Alibali, 2004).

Although the association between accurate encoding of the equal sign and use of correct strategies to solve equivalence problems is well documented, no research to date has investigated how noticing the location of the equal sign leads to the development of improved problem-solving strategies within a naturalistic setting. The current study is a secondary analysis of video data from a larger, ongoing study of children collaborating to solve equivalence problems (Brown & Alibali, 2015). Our goal was to identify nuanced aspects of social interaction that affected whether noticing the location of the equal sign influenced how children solve the problems. We present a qualitative micro-analysis of a series of selected transcript segments.

Methods

The data reported here were drawn from a larger project involving 38 pairs of second and third grade friends. Each pair participated in one 45-minute session in which they completed a pretest, collaboration episode, and posttest. During the pretest, participants worked individually to solve four mathematical equivalence problems and to complete tasks assessing their understanding of the equal sign as a relational symbol. Following this pretest, the pair worked together to solve two additional equivalence problems. The collaboration episode was filmed for later transcription and analysis. The posttest was similar to the pretest and was used to assess whether the children had developed any new strategies during the collaboration episode.

This report focuses on the collaboration episodes of three pairs of children. None of the six children solved any of the pretest problems correctly. Most of the incorrect strategies they used at pretest were consistent with an operational understanding of the equal sign. Our goal is to explore what happens when children notice the location of the equal sign in the problems. We selected one pair in which the children noticed the equal sign and began to generate a relational strategy, but did not use this correct strategy to solve the problems. We also selected two pairs in which the children noticed the equal sign, generated a relational strategy, and used this correct strategy. However, the latter two pairs differ greatly in the quality of their collaboration and in the amount of learning. Comparing these pairs illuminates how social-mathematical asymmetry can influence the outcomes of peer collaboration (Gutiérrez, in press).

Results

In the following sections, we present data from three pairs of children. Each pair was instructed to work together to solve two mathematical equivalence problems, but to write down their final answers and indicate their certainty about those answers by themselves. Children had a large sheet of paper on which they could write during their collaboration, and the problems were presented on smaller sheets. For each case, we first provide a brief overview to highlight certain points of the interaction, anticipating the transcription and line-by-line analysis that follows. (Note: due to space constraints, we provide partial transcriptions for some of the pairs.)

Pair 1: The case of Elsa and Morgan

Problem 1 (8 + 5 + 4 = 4 + ___). In this excerpt, Elsa attempts to influence the pair's work and control the environment from the outset. For example, in transcript Lines 1–4, Morgan moved the problem sheet closer to them but Elsa moved the problem to a different location. Morgan momentarily spoke up (Line 5), suggesting that she wanted to argue about the placement of the problem or about Elsa's edict, but she then acquiesced so they could move on. This tiny slice of interaction mirrors the broader episode, in which Morgan puts forth an idea and temporarily resists Elsa's opposing idea before giving in and adopting Elsa's strategy.

From a mathematical perspective, Elsa's actions strongly suggest that she did not perceive Problem 1 as an arithmetic problem at all. She argued that the problem was actually a numerical "pattern" (i.e., a sequence of numbers governed by an unknown rule that determines each entry), thus she ignored the plus signs and the equal sign as well. Morgan momentarily bought into the idea of a "pattern" (Line 9) but then quickly moved to a strategy based on a different conception of the problem—one that recognizes the plus signs and the equal sign (Lines 11–17).

In our analysis below, we argue that Morgan's strategy contained the seeds of an emerging relational strategy that both children could have profited from exploring further; however, these seeds ultimately went untilled due to Elsa's influence on the collaboration (Lines 18–21). Thus, an asymmetric power dynamic was co-constructed by Morgan and Elsa and an opportunity to enter into dialogue and productively explore a new (relational) strategy was missed.

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Morgan: [moves problem closer, orients it so that it's visible to them both]

Elsa: Can—can you not put that [Problem 1] on the paper? [moves Problem 1 to the center of the table]

Morgan: Yeah.

Elsa: Or how about there on the paper? That's good, on the paper.

Morgan: Wait! [glances at Elsa then looks back down at the problem; indicating each number on the left side in the problem with her marker as she speaks] eight plus five plus four equals— and that's the middle [indicates the equal sign], but [indicating the blank space "__"] four plus five plus eight wont FIT.

Elsa: No, what would be after that?

Morgan: A five?

Elsa: [briefly glances in the interviewer's direction, then back to the worksheet] Wai—ooh yeah! Would—it's a pattern!

Morgan: Ay, yeah!

Elsa: And we have to figure it out. It's eight, five, four [writes "8 5 4" on the big sheet]. Ooh three! I think.

Morgan: Wait, no! //Eight [writes "8" on the big sheet]..

Elsa: //I think it's three.//

Morgan: Five [continues writing, "8 5"]..wait!

Elsa: Four.

Morgan: Plus [writes a plus sign in her inscription, "8+5"], //plus four [continues writing, "8+5+4"]..

Elsa: //It's counting down.// [adjusts in her seat, leans in farther] It's counting down!

Morgan: Equals four plus five plus eight. [completes inscribing her equation as she talks, "8+5+4=4+5+8"]

Elsa: No we have to, we have to—the answer to the problem is [writes "3" at end of her string of numbers, as "8 5 4 3"] three.

Morgan: [gazes at Elsa's string of numbers "8 5 4 3"] Oh! [scribbles over and completely blacks out her equation]

Elsa: *Because* it's counting down. [writes "3" on her answer sheet]

Morgan: Oh! Eight, five, four, three. [writes "8 5 4 3" above her scribble, then puts down marker, picks up pencil, writes "3" on her answer sheet]

Morgan begins by reading the problem aloud, and pauses to note that the equal sign is in the middle of the problem, suggesting that the equal sign's location struck her as unexpected (Line 5). In this same turn, she expresses confusion, because she wants to balance the equation by repeating the numbers from the left side on the right, but there was only room for one number in the answer space (Line 5). This reveals her nascent understanding of the equal sign as a relational symbol; this idea could have led them to a correct strategy had they pursued it.

While Morgan is contemplating her emerging strategy, Elsa offers an alternative: that the problem is a "pattern" and they need to find the next item in the sequence (Lines 8, 10). Thus, Elsa does not view the string of characters as an arithmetic problem. At first, Morgan accepts this idea (Line 9), but then temporarily returns to her own strategy (Lines 11–17). Morgan's actions (Lines 15, 17) indicate a strong commitment to her emerging idea and getting it down on paper; she does not acquiesce to Elsa's attempts to get her attention and completes writing her equation as "8+5+4=4+5+8." However, this brief moment of agency is broken by Elsa's statement that the answer is three (Line 18). Morgan quickly agrees to Elsa's suggested answer, using her marker to cross out all the written work she had produced (Lines 19, 21). Morgan appears to be on the cusp of

understanding the equal sign as a relational symbol, but because Elsa guides the interaction through her comments and actions—and because Morgan allowed her to do so—Morgan was persuaded to abandon her emerging understanding in favor of Elsa's pattern strategy which, in this context, was less effective.

Problem 2 (9 + 7 + 5 = ___ + 9). On the second problem, Morgan made statements that assert her status as an equal collaborator (Line 23), yet her contribution still went unrecognized by Elsa. Thus, the emerging social relation that was being co-constructed remained asymmetric. From a mathematical perspective, they proposed a final solution that was based on the same "pattern" game as for Problem 1 and again ignored both the plus signs and the equal sign.

> *Elsa*: Nine... uh. [giggling] I have no idea.. What's nine plus seven plus—
> *Morgan*: Wait! This is question number two. Since the other one was going *down*, this one might be going UP.
> *Elsa*: No, because see [indicating numerals on the problem sheet] nine, seven—oh counting by twos? No counting by *odd* numbers.
> *Morgan*: Yeah!

Interestingly, this time it is Elsa who initially suggests a summation strategy (Line 22), and Morgan reminds her that they are looking for a pattern. Once reminded, Elsa quickly returns to the pattern approach, but also quickly rejects Morgan's suggestion that the pattern "might be going up" (Lines 23 & 24). Both have fully adopted Elsa's idea that the goal is to find a pattern within the sequence of numbers, and neither child references addition or the equal sign. Elsa continues to drive the interaction, suggesting a series of possible patterns, and proposes a solution of "3"—the next odd number, counting down—that Morgan immediately takes up.

After the collaboration episode, when the children solve equivalence problems individually on the posttest, neither child used a correct, relational strategy. Thus, neither seems to have benefited from the collaboration, despite the fact Morgan appeared to have the seeds of a relational strategy. We argue that the nature of their collaborative interaction made it impossible for this nascent correct strategy to fully emerge and be beneficial for either child.

Pair 2: The Case of Dylan and Shawn

Problem 1 (8 + 5 + 4 = 4 + ___). Dylan and Shawn represent another case of an asymmetric power dynamic. Just prior to the excerpt below, both children made statements about potential solutions, however the conversation flowed mostly in one direction. Dylan acknowledged Shawn's propositions but Shawn did not give any indication that he considered Dylan's ideas. Shawn kept his gaze on the problem, whereas Dylan made several attempts to make eye contact, suggesting that he wished to check in with Shawn as they went along, but this never occurred. Shawn and Dylan essentially worked separately (as indicated by overlapping speech and unacknowledged turns of talk), each developing his own understanding of the problem. Shawn eventually articulated a solution based on a relational understanding of the equal sign (Line 26), placing Dylan in a position to "buy in" to Shawn's strategy and abandon his own approach (Line 27). The only time Shawn looked away from the problem was to assert his proposed solutions.

> *Shawn*: [referring to the left side of the equation] So that's 17. [leaves tip of pencil on left "8"; glances in Dylan's direction] Four plus *what* equals 17?
> *Dylan*: Four plus what? Four plus... well// [looks at Shawn] it's four so of course there's ten.
> *Shawn*: //Four.// [quickly glances up in Dylan's direction] Four plus three!
> *Dylan*: Wait! What?! [lifts gaze up, staring out at nothing in particular]

Shawn: [glances up at Dylan] Four plus 13! [reaches for problem, orients it so that he can write "13" on it]

Dylan: Four plus—oh yeah! Yeah! Plus 13.

At the outset of their collaboration, Shawn immediately begins adding the numbers on the left side of the equation, while Dylan's attention is drawn to the middle of the equation, suggesting that the structure of the problem is unfamiliar to him. When Shawn offers a strategy based on a relational understanding of the equal sign (Line 26), Dylan leaves his train of thought and attempts to follow Shawn's (Line 27). Dylan's final utterances (Lines 31) suggest that he agrees with Shawn's relational strategy; however, he may simply be appropriating Shawn's method without the relational understanding of the equal sign that undergirds it (Line 31). As Dylan records his final answer, his facial expressions, posture, and tone suggest that he is still uncertain about their final solution, despite the fact that he acquiesces to Shawn.

Problem 2 (9 + 7 + 5 = ___ + 9). The opening moments of Dylan and Shawn's interaction with Problem 2 showed great promise of authentic and equitable collaboration. They worked in tandem for several turns of talk, and successfully determined that the left side of the equation sum to 21. However, in approaching the right side, this emerging intersubjectivity broke down, and they began to work separately again. Shawn used the relational strategy they had used on Problem 1, whereas Dylan offered a strategy, based on an operational conception, of adding all the numbers: "Oh! That would be.. that would be 21 [referring to the left side] and then plus nine [referring to the right "9"]. So.. hmm." This cognitive–conceptual divergence resulted in a communication break-down that was not successfully repaired. For the remainder of their interaction, Dylan attempted to make sense of what Shawn had proposed, but to no avail. In the end, the pair offered the final answer of "12" that was proposed by Shawn. Despite his agreement, Dylan's speech and nonverbal behavior suggest that he was not convinced; he appeared dejected when he gave in to Shawn. Even as they wrote down "12" on their worksheets, Dylan quietly says that he thinks the answer might be something different.

In both excerpts involving Shawn and Dylan, they both reached correct solutions using a relational strategy. However, the social power dynamic that was simultaneously co-constructed, in and through their discursive productions, was asymmetric and, moreover, was not beneficial for both children. When Dylan and Shawn solved additional problems individually on the posttest, Shawn solved all four problems correctly, while Dylan solved only one correctly. This outcome suggests that, although Shawn gained insight into the underlying structure of the problems and how to approach them effectively, the *quality* of his engagement with a peer left much to be desired from a *relational equity* perspective (Boaler, 2008) (see below).

Pair 3: The Case of Marie and Jenny

Problem 1 (8 + 5 + 4 = 4 + ___). Marie and Jenny both immediately interpreted the first problem as involving arithmetic, and together they added the left side to arrive at 17. Moving beyond this point proved to be challenging; they worked in tandem to add the left side but reached an impasse when they arrived at the equal sign. They could not easily reconcile how the left side, which they both agreed was 17, was equal to "4", and the extra "blank" on the right side was also lost on them. They seemed on the cusp of giving up, but instead reexamined the problem. This process led Marie to see the equation in a new way and she proposed a relational solution strategy. The transcription below begins as Marie launched excitedly into an elaborate explanation in which she used both speech and gesture to communicate her newfound conceptualization to Jenny. Jenny, in turn, attended to Marie's speech and gestures and took up what Marie was attempting to explain (Lines 32–34). Together they arrived at the correct final solution based on a *shared* relational understanding. They worked mostly in tandem, sometimes interrupting one another and in some cases finishing one another's sentences.

Marie: Four. No.. Maybe it's like [leans back], so that's 17 [indicates left "8"; at this point, her speech speeds up and she speaks excitedly] so four [indicates the "4" on the right side] plus [indicates plus sign] what [indicates "blank"] would make it seventeen [both hands open, palms facing up and slightly towards each other, gesturing to left side of gesture space] also [shifts her hands to right side of gesture space], which makes it equal together. [Brings hands together, palms toward her chest, tips of fingers of the two hands touch]

Jenny: I don't get what you're saying.

Marie: So like.. that's 17 together [gesture underlines left side with pencil] so four [points with her whole hand to the RS "4"] plus what [point to the blank] would make that 17 [fingers and thumb bunched together in a "wide pinch" position, gestured at the right side] so that these two [index points to the left "4" then the right "4"] are equal together [hands open, fingers spread, palms toward each other as if holding something]? So since this is 17 together [drags hand across entire left side of equation, her hand in a pinch shape, with her thumb underlining the equation and her other fingers tracing above the equation]..

Jenny: Oh.. [releases tension she was holding in her posture then leans back]

Marie: Four [indicates right "4"] plus what [uses same dragging gesture as before, on right side] makes that equal together? Like, 17 [indicates left side with pencil] plus four [indicates right "4"] makes 17 [indicates blank],// makes it equal [waves hands, both open and facing down, back and forth above the whole equation]?

Jenny: //So should we// write 13? [Poises pencil above the blank, looks at Marie]

When Jenny expresses confusion about Marie's emerging relational strategy (Line 33), Marie offers an in-depth explanation. Jenny understands the strategy after this explanation and even offers the answer before Marie does (Line 37).

Problem 2 (9 + 7 + 5 = ___ + 9). Unlike Shawn and Dylan, the strategy that was articulated by Marie and Jenny during Problem 1 was robust enough that they were able to maintain it as a shared strategy and use it to solve Problem 2. Marie and Jenny were now acting in concert, sharing the discursive space, and this resulted in an efficient, authentic, and equitable interaction centered on a shared task. They approached Problem 2 enthusiastically and expeditiously, both stating at the same time, "Like we did it last time." Marie again used complex gestures and speech to articulate a relational strategy that was now shared with Jenny (Line 38).

Marie: Well, let's count it up. So seven plus five [covers up left "9" with her left thumb, and points to left "7" and left "5" with her right hand], what does that [cup shape under the left "7 + 5," grouping the numbers together], cause the nine is already used [indicates left "9" then the right "9," then she covers both nines, one with each hand]. So [indicates left "7 + 5"] seven plus five equals?

Both Marie and Jenny went on to solve all four individual posttest problems correctly after the collaboration episode.

Summative Comments Across All Three Pairs

All three dyads notice the location of the equal sign in the problems, and one child in each pair shows at least partial knowledge of the equal sign as a relational symbol. Despite these similarities, the three dyads differ greatly. Elsa does not consider Morgan's relational strategy, but rather asserts her own pattern strategy—one that lives outside of the realm of arithmetic and mathematical equivalence. She ultimately persuades Morgan to abandon her line of thinking in favor of the pattern strategy. In contrast, Shawn and Dylan eventually agree to use a relational strategy to solve the problems during the collaboration episode. However, both of these dyads experience a similar type of asymmetry in their interactions. While Elsa persuades Morgan to abandon a correct line of thinking,

Shawn forges ahead with his relational strategy and Dylan eventually accepts Shawn's answer despite not understanding it. Finally, although Marie and Jenny have a brief moment of asymmetry when Marie first offers her relational strategy and Jenny expresses confusion, the way these children navigate the asymmetry leads to both children learning. Marie works to help Jenny join her in her understanding so they can move forward together. This is in contrast to Shawn, who seems to drag Dylan along despite his protests. Marie and Jenny both make an effort to get on—and *stay on*—the same page, something that was missing in the other dyads, in which one of the two children was always leading the way.

Discussion and Implications for "Relational" Equity

In the spirit of the conference theme, "without borders" (sin fronteras), the analysis presented here is a first step toward bridging educational psychology and mathematics education research, with a focus on both the conceptual challenges that elementary students face when dealing with a certain genre of mathematical activity (equivalence problems), and the social aspects of collaboration with a peer. Our main finding, in broad strokes, is that students navigated an emergent power dynamic, and this dynamic in turn affected the quality of engagement and consequently, their learning. From a psychological perspective, this finding bears on classic questions about where new ideas come from and why they are (or are not) taken up. Our findings support the hypothesis that noticing the atypical location of the equal sign in mathematical equivalence problems can lead to the development of relational strategies, but our findings also suggest that the social context in which the act of noticing occurs may partially determine whether students develop and use a relational strategy.

From a mathematics education perspective, this finding bears on issues of equity and inclusion. Equity in mathematics education can be conceptualized in terms of issues related to status hierarchies, participation, and identity. Most relevant to this report is Boaler's (2008) notion of *relational equity* (where "relational" refers to "social relationships" not "relational understandings of the equal sign"—but we appreciate the coincidence in terminology). Boaler defines relational equity as "equitable [social] relations in classrooms; relations that include students treating each other with respect and considering different viewpoints fairly" (pg. 168). Boaler proposes that we focus on the *quality* of students' interactions during collaboration. We find her notion of relational equity useful for articulating the implications of our findings.

Boaler (2008) argues that when students come together to collaborate on mathematical tasks, they are not only learning content and concepts, but they are also learning and reifying values such as respect and responsibility. Specifically, Boaler assumes "that the ways students learn to treat each other and the respect they learn to form for each other will impact on the opportunities they extend to others in their lives in and beyond school" (pg. 168). We agree.

It is important to note that Boaler's notion of relational equity was developed in the context of a diverse, urban high school and refers to the ways in which students interacted with others from different social classes, cultural groups, and ability levels. Our data come from an educational psychology study involving clinical interviews, and our participants were demographically homogeneous. That said, we nevertheless propose a corollary to Boaler's definition: we argue that relational equity is contingent on an emergent social-mathematical power dynamic that is co-constructed, in situ, via students' actions and discourse (Gutiérrez, in press). Boaler, too, sees that (asymmetric) power dynamics play a role in mathematics learning:

> A common problem in the enactment of group work is an uneven distribution of work and responsibility among students, with some students doing more of the work and others choosing to opt out or being forced out of discussions." (Boaler, 2008, pg. 171)

This definition applies to the ways the children in our study responded to one another's mathematical perceptions and actions. Morgan's and Dylan's discursive contributions, for example,

were not taken up and discussed with their partners. In a sense, their personal meanings of the equal sign were "forced out" of the semiotic space. The lack of relational equity in these two cases is striking when compared to the case of Marie and Jenny. In this pair, Marie sensitively responded to fact that Jenny was falling behind. Moreover, when Marie viewed the equal sign as a relational symbol before Jenny, there was a possibility of another social-mathematical hierarchy emerging, yet Marie demonstrated a strong commitment to Jenny's learning, as indicated by her use of an elaborate array of communicative means. This kind of complex communication, commitment, and respect was not found with Dylan and Shawn, nor with Morgan and Elsa, which leaves us with two very different versions of "relational" equity.

Acknowledgments

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WHAT DO STUDENTS ATTEND TO WHEN FIRST GRAPHING PLANES IN R^3?

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This paper considers what students attend to as they first encounter R^3 coordinate axes and are asked to graph y = 3. Graphs are critical representations in single and multivariable calculus, yet findings from research indicate that students struggle with graphing functions of more than one variable. We found that some students thought y = 3 in R^3 would be a line, while others thought it would be a plane. In creating their graphs, students attended to equidistance, parallelism, specific points, and the role of x and z. Students' use of these ideas was often generalised from thinking about the graphs of y = b equations in R^2. A key finding is that the students who thought the graph was a plane always attended to the z variable as free.

Keywords: Post-Secondary Education, Cognition, Advanced Mathematical Thinking

Introduction

The inclusion of multivariable topics in K-12 mathematics has been proposed as a way to increase mathematical competence for all students (Ganter & Haver, 2011; Shaughnessy, 2011). Because multivariable topics share similarities with their single variable counterparts, many researchers studying student learning of multivariable topics focus on how students generalise across these contexts (e.g., Dorko & Weber, 2013; Jones & Dorko, 2015; Kabael, 2011; Yerushalmy, 1997). Graphs are critical representations in calculus, yet students struggle with creating graphs of multivariable functions (Kabael, 2011; Martinez-Planell & Trigueros, 2012) and finding the intersection of multivariable functions' graphs and fundamental planes (Trigueros & Martinez-Planell, 2010). Additionally, students' correct understandings about graphs in R^2 may interfere with their learning about graphs in R^3. Some students graph $f(x,y) = x^2$ as a parabola rather than as a parabolic surface or may draw $f(x,y) = x^2 + y^2$ as a cylinder or a sphere because they are accustomed to $x^2 + y^2$ representing a circle in R^2. These examples illustrate that as students think about the graphs of multivariable functions, they generalise the ways they think about graphs in R^2. We sought to further explore this, with the hypothesis that learning more about what students attend to when graphing can help instructors emphasize the productive connections students see across situations and address the sorts of overgeneralisations described above. Toward that end, we focus on the following research question: *what do students attend to as they first think about graphing a particular fundamental plane (y = 3) in R^3?*

Background Literature and Theoretical Perspective

Graphing in three dimensions requires students to coordinate three quantities, as well as shift from thinking of *y* as a dependent variable to considering a *z* that is dependent on *x* and *y*. This is a difficult generalisation to make. Students may give an *(x,y,z)* tuple as an element of the domain or range (Kabael, 2011) or may give the range as *y* values (Dorko & Weber, 2013), indicating that they have not reconceptualised *y* as an independent quantity, and do not necessarily think of *f*(x,y) as an output or the height of the graph at a particular *(x,y)*. As another example, Martinez-Planell and Trigueros (2012) described a student who drew $f(x,y) = x^2$ as a parabola on the *xz* plane and insisted that although the point (2,1,4) satisfied the equation, it was not on the graph. Students also have trouble determining the intersection of a surface with fundamental planes (a plane of the form *x = a, y = b, z = c* for a constant *c*). Trigueros and Martinez-Planell (2010) found that students who had taken multivariable calculus knew that these were planes, but weaker students had trouble placing such planes in a set of manipulatives and drawing the planes on a 2D image of a multivariable graph.

Stronger students could place the planes, but had difficulty determining the intersection of such planes with a multivariable surface.

Martinez-Planell and Trigueros' research has been in the context of developing a set of activities to help students learn how to graph multivariable functions. They found that notation may hinder students' multivariable graphing attempts. One of Martinez-Planell and Trigueros' (2012) students drew $f(x,y) = x^2$ as a parabola on the xz plane and was unsure whether or not she was in two or three dimensions. She gave the intersection of $f(x,y) = x^2$ and $y = 1$ as "two points," which the researchers interpreted as thinking of the graph in two dimensions. The same student drew a correct three-dimensional graph for $z = x^2$. Familiar notations, such as $x^2 + y^2$ (a circle in R^2) may result in students thinking that the graph of $f(x,y) = x^2 + y^2$ is a cylinder or a sphere (Martinez-Planell & Trigueros Gaisman, 2013; Trigueros & Martinez-Planell, 2010). The researchers subsequently altered the activity sets to avoid familiar notations, so it is unknown if students are able to use such notations productively. We build on these' authors work by considering how students' conceptions of single-variable functions' graphs interact with their conceptions of graphs of multivariable functions. Findings from other studies indicate that students can often successfully leverage their single-variable knowledge to make sense of multivariable topics (e.g., Dorko & Weber, 2013; Jones & Dorko, 2015; Kabael, 2011; Yerushalmy, 1997), and we wanted to study whether this was also the case when students graph in three dimensions.

Theoretically, we consider that using knowledge from a single-variable context to make sense of a multivariable context is an example of generalisation, or "the influence of a learner's prior activities on his or her activity in novel situations" (Ellis, 2007, p. 225). We think about generalisation from an actor-oriented transfer perspective (Ellis, 2007; Lobato, 2003). This perspective privileges what students see as similar across situations, even if the similarities they perceive are not normatively correct.

Data Collection and Analysis

We interviewed 11 differential calculus students who had not yet received instruction regarding R^3. We felt this would let us observe students' initial sense-making and generalisations in real time. This paper focuses on data from two tasks: students' graphs of $y = 2$ in R^2 and $y = 3$ in R^3. Before giving students the second task, we showed them an image of xyz axes and explained that the xy plane was flat with the z axis perpendicular to it. We used a tabletop (xy plane) and a pen (z axis) to show students what these axes looked like in 3D[1]. We asked follow-up questions such as "why did you draw a [line, plane] here?" We audio and video recorded the interviews and used LiveScribe pens. We transcribed the interviews and used the transcripts in data analysis.

We chose to focus on these problems because of reported difficulty students experience with multivariable functions' graphs, and also because fundamental planes can help students complete graphing and other tasks in calculus (e.g., visualising graphs; the boundaries in multiple integration). Hence it seemed valuable to explore how students might think about equations of the form $y = c$ in R^3 (and we specifically chose $y = 3$).

We used the constant comparative method (Strauss & Corbin, 1998) for data analysis. We first observed that some students had drawn $y = 3$ in R^3 as a line, others had drawn it as a plane, and two drew a line but then thought the graph might be a plane. We hence coded students in two categories: (1) students who drew a plane or said they were unsure whether the graph was a line or plane, and (2) students who drew a line. We then looked at the data a second time, asking how students might have arrived at their answers. Students' reasoning involved words and phrases such as parallel, equidistant, "all x points," "z can be any value," "x can be any value," "I don't think that x and z really have like any effect", "all values of x and z", variables as "not mattering," x and z being "any value," and "no matter what x or z is." We also observed students think about specific points, such as "if you say $x = 2$ and $z = 2$, it's going to be 3." We noticed that these utterances seemed to group

themselves into three broader, non-mutually-exclusive categories: reasoning using equidistance and parallelism, reasoning about *x* and *z*, and considering specific *(x,y)*, *(y,z)*, and *(x,y,z)* tuples.

Finally, we looked for patterns in how students had thought about the graphs, and whether their graphs were planes or lines. That is, we looked specifically to see if there were something common to all of the students who drew the graph as a plane, and all the students who drew the graph as a line. We noticed that the difference between the graphs seemed related to whether or not the students explicitly attended to *z* as a free variable.

Results

In this section, we first focus on the seven students who thought the graph was a plane or might be a plane, and then on the students who drew a line. All of the seven plane students thought about *x* and *z* values in some way. Three began by thinking about all values of *x* and *z*, then thought about specific *(x,y)*, *(y,z)*, and *(x,y,z)* tuples. The other four focused on all values of *x* and *z*, but did not think about specific points. We contrast these students' work with the four students who drew a line. Two of the line students thought about parallelism, one thought that the lack of *x* and *z* in the equation meant the graph would be "flattened down" and hence a line, and one seemed to have difficulty orienting himself to R^3. In this section we contrast data from the plane and line students.

Plane: Thinking about *x*, *z*, and specific points

Three students reasoned that the graph was a plane by first thinking about all *x* and *z* values, and then considering specific *(x,y)*, *(z,y)*, and/or *(x,y,z)* tuples (Table 1). Thinking that these points had to be on the graph of the function seemed to help S3 and S7 figure out that the graph was a plane. In particular, S3 began with two lines that looked like a plus sign, intersecting at (0,3,0). She drew the horizontal line to represent that *y* equaled 3 no matter what *x* was, and the vertical line to represent that *y* equaled 3 no matter what *z* was. Testing points not on those lines such as *(x,y)* = (2,3) and *(z,y)* = (1,3) allowed S3 to conclude, "I guess I drew a plane." Note that while S3 thought about two variables at a time, S7 first thought about two variables at a time when she said "*x* evaluated at any point will be *y* = 3" and then gave the 3-tuple (2,3,2) as a point on the graph. S13, who was unsure whether the graph was a line or a plane also used a 3-tuple, commenting "So I feel like you can be given like *x* = 0, but *y* = 3, and *z* equals something, and I feel like that could correlate." We speculate that in thinking about specific points, the students were focused on something concrete, and this afforded their realisation that the graph was a plane. Notably, none of the students who considered specific points said that the graph was a line. This suggests that having students consider specific *(x,y,z)* tuples may be a powerful tool for helping them create correct graphs.

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Table 1: Considering Specific Points and Drawing a Plane

S3	No matter what x or z is, y is always going to equal 3… I want to draw a line like this [indicates a line following the gridline for $y = 3$ on the xy plane] and a line like that [indicates a gridline parallel to the z axis and going through $(0,3,0)$]. So what it's saying here is if x were 1, y equals 3, or if $x = 2$, $y = 3$. And here too if z were 1, y is always going to equal 3 here... I guess I drew a plane... [a plane makes sense] when it is drawn out like that.
S7	So here's 3, we've got x, or the y, but then it's also going to be for all the z values, since z evaluated at any point will be $y = 3$. So I guess it would come out to be a plane… I kind of just thought since x evaluated at any point on the graph equals 3, since the function is basically saying all, it's saying $y = 3$ at all points on the graph, any point you evaluate, so if you say $z = 2$ and $x = 2$, it's going to be 3.
S13	So it would be, $y = 3$ would go, I want to say it would be like right here. [Draws a line at $y = 3$ that is parallel to the lines on the notebook; Figure 1]. I guess I'm just trying to relate it back to this, these equations [gestures to $y = 2$] and it just goes straight through…Here if it continues, it goes through x [meaning if she extends the line in Fig. 1, it will go through the x axis]. Then you would have an x and y value, which is the only reason why I feel like it's wrong. [using makeshift actual 3D axes]… I think it would go this way [Figure 2]. Just cuz it can't go this way [points to a line that would roughly go through $(0,1,0)$ and $(1,0,0)$] because then it will hit x, this way it doesn't hit any other points because z goes, negative z goes straight down. So I feel like this way it would go, it would hit the 3 …it could be like a line or it could be like a sheet but I feel like a sheet just makes more sense because then you can do, I was just given x and y, but you can be given, I am assuming you can be given like $x\, y\, z$ and plot those points. So I feel like you can be given like $x = 0$, but $y = 3$, and z equals something, and I feel like that could correlate.
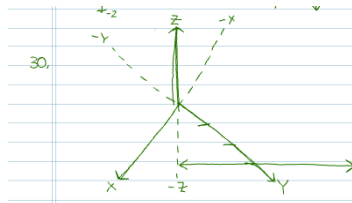
Figure 1: S13's first graph of $y = 3$ in $\mathbb{R}^3$.
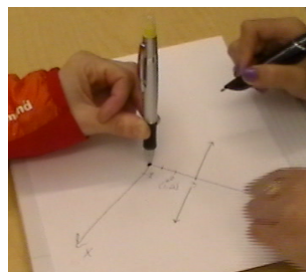
Figure 2: S13's second graph.

Plane: Thinking about x and z (but not specific points)

Three students determined that the graph was a plane by focusing on x and z as able to take on any value, though they did so in different ways. S8 thought about equidistance and parallelism, while S5 and S12 thought about the graph "stretching out" in the direction of a free variable(s) (Table 2).

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Table 2: Thinking about *x* and *z* and Drawing a Plane

S8	$Y = 3$ would be something like this, where this distance right here between each, between *y* and each of these axes would be 3, I think. I'm thinking that because if you take like this thing, and that would be everything except for *y* [shades *xz* plane]... I'm thinking of this plane in relation to *y* and having *y* be every distance that is 3 away from that plane. Actually, *y* = 3 wouldn't just be this circle, ... it would be an entire plane... it has to be parallel to *x*, and this has to be parallel to *z*, so it would be this plane right here that is 3 away from the plane that *x* and *z* creates... like for the last question when *y* is equal to 2, that is every value that is 2 away from *y* = 0, right? So I'm thinking that like *y* = 0 would be the same as this [*xz* plane]. So it's 3, it's 3 in the positive [*y*] direction it's going to be parallel to *x* in the same way that this line right here [draws $y = 2$ in R^2] is parallel …to the *x* axis.
S5	So if *y* = 3, *z* can be any value so it extends in the *z* direction like that, and *x* can be any value, so it extends in the *x* direction like that, and it forms a plane on the, like that… I just thought about this sort of logic, that if there's a variable that doesn't affect the equation then it just kind of stretches out into that direction.
S12	So maybe, maybe it would be like this [draws and shades plane]. Well this would kind of just be like just a flat sheet of paper on the *y* = 3, because all *x* values are 3, and then I guess you assume that all *z* values, since it's only, the only variable in the equation is *y* = 3, then it would have to be *y* = 3 for all *x* and *z* values. It's kind of just like a, I think it's supposed to be like a flat sheet kind of, like a piece of paper, and it's on *y* = 3, so it's supposed to encompass all the *x* values for negative and positive, and all the *z* values for *z*, positive *z* and negative *z*. They're all on *y* = 3... Well, I just thought like since *y* = 2 it should be like this, so if it's *y* = 3 it's like that, like all *x* values are *y* = 3. And *z* is going this way, so it must be, since there's no *z* in the equation, then it must be covering all this area.
S10	I'm going to extend that like that, so then that's 3. And this is on that like *x* plane, *xy* plane… I mean it looks pretty linear. I mean like it might be a sheet, but… like if it weren't a line, it would definitely be a sheet that extends into the *z* plane... because a line is like infinitely small points, like a line of points, and then if it extended into the next dimension, it would be, it would just be another line going in the other, perpendicular to that line of points.

S8 began by thinking about *y* = 3 as "three away" from the *x* and *z* axes, and drew a circle centred at (0,3,0), parallel to the *xz* plane. Importantly, the act of drawing the circle seemed to make him realise that the *x* and *z* axes lie in the *xz* plane, and hence *y* = 3 would be a plane that is "every distance that is 3 away from that plane." He appeared to confirm this idea by thinking about the plane as "parallel to *x* and… parallel to *z*", and referred back to the *y* = 2 question where the line was "2 away from *y* = 0" and parallel with the *x*-axis.

S5 generalised that the graph would stretch out in the direction of a free variable from having previously graphed $f(x,y) = x^2$ and noticing that because *y* can take on any value, the graph is a parabolic surface[2], which he saw as "stretching out" a parabola into a trough shape. Similarly, S12 realised that *y* would be 3 for all *x* and *z* values, so the graph would need to "cover" all *x* and *z* values. For these students, noticing that *x* and *z* were free allowed them to realise that the graph was a plane. S10, in contrast, did not explicitly say that *z* could take on any value, but her comment that the graph might be a "sheet" seemed to indicate that she knew, like S5 and S12, that *z* was free. Her reasoning that the graph could be a plane, however, seemed to come from finding a 3D analogue of a line. We infer this from her comment "if it extended into the next dimension." Such an extension to

R^3 seemed to come from thinking of $y = 3$ as a line on the xy plane composed of an infinite number of points, and drawing a line through each of those points, perpendicular to the $y = 3$ line. This reasoning is somewhat similar to S3 (above) who drew two perpendicular lines, then tested points to see if the graph were a plane.

In summary, the commonality to students in the *Thinking about x and z (but not specific points)* category is that realising that z could take on any value allowed them to draw a plane. To summarize the plane students more broadly, we speculate that attending to z, whether in the form of z as a free variable taking on any values or by considering specific points, allowed these students to create correct graphs.

Line

Of the four students drew $y = 3$ as a line, two of them did not mention z. Both of these students explicitly referred back to the graph of $y = 2$ in R^2, and generalised that since $y = 2$ is parallel to the x axis, $y = 3$ should also be parallel to the x axis. S6 described this as "following the x axis," which seemed to us to be attending to parallelism (Table 3).

Table 3: Attending to Parallelism to Draw a Line

S6	So $y = 3$. So on an xy graph [draws R^2 axes] at 3, would be going this way. So on the y, following the x. So [switches to R^3 axes] this would be on the y, this is the 3 point on the y, and it's following the x axis.
S9	So if this is, if this is y is equal to 3, then I'm wondering if it will just go parallel with the x axis, but just at, but just at the y is equal to 3 point… just kind of like looking here [goes back to $y = 2$] it's like parallel to the x axis at whatever y point because it's just that straight line, so if this is like y is equal to 3, I'd assume just for all x points that you know y would equal to 3. [I think the graph is a line] just cuz when it's just equal to you know 3, it just means no matter what x value there's like nothing, it's just 3. So it would be a line, there's no, you know, changing or anything because your y is only 3. It's not like interacting with, with anything else.

Unlike S8 (above), these students did not mention the z variable. We believe that S8's attention to z, which came in the form of shading the xz plane, allowed him to see that $y = 3$ would also be parallel to the z axis, which in turn helped him see that $y = 3$ was a plane. If we contrast the three students who used parallelism, two of them drew a line but did not attend to z, and one drew a plane and attended to z. This suggests to us that thinking explicitly about the z variable as free supports students in drawing correct graphs.

The final two students thought about z, but rather than realizing that z is free, these students thought z and x "can't change" or that they "don't have any effect" (Table 4). S11 mentioned that the symbols z and x were absent from the equation, and she then referred back to the $y = 2$ graph and said "it'd still just be like this," though she recognised that she was in 3D. We infer that because the $y = 3$ equation and $y = 2$ equation look the same (both having no x and no z), S11 generalised that $y = 3$ in R^3 would be a line just like it is in R^2.

S1 had trouble orienting himself in R^3, which may have contributed to his drawing a line. That is, he seemed to have so much trouble figuring out the structure of the three axes that he would have been unable to think of specific points, or coordinating parallelism in multiple directions. The student also assumed that, rather than being free, the z and x variables could not change at all. This may have resulted in his drawing a line contained on the xy plane, because such a line never intersects the x or z axis. Thinking about x and z as unable to change directly contrasts thinking of them as able to take on any value, as S3, S5, S7, and S12 (who all drew planes) did. This suggests that part of students' success in graphing $y = 3$ depended upon their ability to think of z as free.

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Table 4: *x* and *z* as Not Changing and/or Having No Effect

S11	It stays on like that like *xy* plane because there's no like *z* and there's no *x*, so it'd still just be like this [points to *y* = 2 in R2], but flattened down…because it's still just like a constant number, so no matter like, no matter like what the other ones are, it's just going to be that one number, which is just like a straight line across the thing... I don't think that *x* and *z* really have like any effect to *y* = 3 because there's no *x* and *z* variable in there, but it just like makes it like 3D... But it's, the function's still just like one line on the *y* axis.
S1	So the *z* direction, the *z* can't change and the *x* can't change, and the *y* would kind of have to be like at a diagonal [*the line y=3, when drawn on standard axes on paper looks to be at a diagonal*], but staying at *y* = 3. I can't really like see how to do it, but you'd find the spot where *y* = 3 for both *x* = 0 and [*the interviewer interrupted to ask the student to draw on the paper*] this might be a struggle, I'm not sure if I know - so this is going up, this is going this way, *x, y, z*…Yeah, *z* is the vertical here. The *x* value, so yeah. I'm not really sure which one I should mark as *y* = 3, though... so this would be *y* = 0, I'll just say this is the positive direction, 1, 2, 3, so *y* = 3 here, and I guess I would just use this line for *y* always equal to 3.

Discussion and Conclusions

We found that when students first try to graph *y* = 3 in R^3, some draw a line and some draw a plane. The students who drew planes seemed attend to *z* in a way that allowed them to realise that *z* is a free variable. Some students did this by thinking that *z* could take on any value, and hence the graph would extend in the *z* direction. Others tested particular points, indicating that they knew they could pick any value for *z*. Realising these points satisfied the *y* = 3 criteria seemed to help these students see the graph as a plane. Finally, students attended to parallelism and equidistance from the *y* = 2 case. Nearly all of the students referred back to the *y* = 2 in R^2 question, and their ways of thinking about the multivariable graph generalised properties from this single-variable case. For instance, some generalised *y* = 2 for all values of *x* in R^2 so *y* would equal 3 for all values of *x* and *z* in R^3, and others generalised that *y* = 2 is parallel to the *x* axis in R^2 so *y* = 3 is parallel to the *xz* plane in R^3.

That some of the students in this study drew *y* = 3 in R^3 as a line, as it would be in R^2, supports Kabael (2011) and Martinez-Planell and Trigueros' (2012) findings that students' knowledge of the shapes of graphs in R^2 may interfere with their learning about graphs in R^3 and Trigueros and Martinez-Planell's (2010) finding that students have difficulty with fundamental planes. However, other students in our study productively leveraged properties of *y* = 2 in R^2 to correctly graph *y* = 3 in R^3. This suggests that instructors can build on students' intuitive notions, perhaps first eliciting what students think a graph might look like and then asking what role *z* plays in such a graph. Finally, because some of our students found thinking about specific points to be helpful, we agree with Martinez-Planell and Trigueros (2012) that instructors can help students think about 3D graphs by asking whether a particular *(x,y,z)*-tuple is on the graph. Further, we agree with Martinez-Planell and Trigueros (2012) that students should have the experience of constructing a 3D graph point-by-point. This affords learning that an *f(x,y)* value is the function height at a point *(x,y)*, and we think it primes students with the strategy of thinking of specific points as being useful for determining the shapes of graphs.

The setup of the *y* = 2 in R^2 and *y* = 3 in R^3 tasks is a possible limitation of this study. One could argue that the order of the questions prompted students to refer back to the *y* = 2 task. While this is true, the fact that many students referenced back to it in productive ways (e.g., equidistance,

parallelism) implies that instructors might consider pairing graphs like $x = a$ and $y = b$ in R^2 with an introduction to fundamental planes.

Endnotes

[1]Such a demonstration does not guarantee that students understood how such a coordinate system works; in fact, researchers have found that students often must develop a schema for R^3 through specific actions like plotting *(x,y,z)*-tuples and working with fundamental planes (Martinez-Planell, & Trigueros, 2012; Trigueros & Martínez-Planell, 2010).

[2]S5 was the only student to have graphed this multivariable function before the $y = 3$ task analysed here.

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PLOTTING POINTS: IMPLICATIONS OF "OVER AND UP" ON STUDENTS' COVARIATIONAL REASONING

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In this study I investigate Saldanha and Thompson's (1998) claim that conceptualizing a coordinate pair in the Cartesian coordinate system as a multiplicative object, a way to unite two quantities' values, supports students in conceptualizing graphs as emergent representations of how two quantities' values change together. I presented three university precalculus students with an animated task showing varying values of two quantities along the axes and asked each student to sketch a graph of how the two quantities changed together. In this paper I document the difficulty students encountered when they did not conceptualize a coordinate pair as a multiplicative object. I address why the convention of "over x and up y" inhibits students from constructing a coordinate pair as a multiplicative object and I provide recommendations for supporting students in constructing coordinate pairs as multiplicative objects.

Keywords: Cognition, Modeling, Instructional Activities and Practices

Researchers continue to provide evidence that students have difficulty interpreting and constructing graphs (e.g., Monk, 1992; Oehrtman, Carlson, & Thompson, 2008). Specifically, researchers suggest that students do not typically think about graphs as representations of how two quantities' values change together (e.g., Dubinsky & Wilson, 2013; Thompson, 1994). Instead, as Moore and Thompson (2015) described, many students conceptualize graphs as shapes and curves and reason based on their perception of the shape of the graph. Moore and Thompson called this *static shape thinking* and explained that a student who engages in static shape thinking might, for example, understand slope as the property of the line that determines whether the line falls or rises as it goes from left to right.

An alternative way of thinking about graphs is what Moore and Thompson (2015) called *emergent shape thinking*. They explained,

> Emergent shape thinking involves understanding a graph *simultaneously* as what is made (a trace) and how it is made (covariation). As opposed to assimilating a graph as a static object, emergent shape thinking entails assimilating a graph as a trace in progress (or envisioning an already produced graph in terms of replaying its emergence), with the trace being a record of the relationship between covarying quantities. (p. 4)

Central to this conception of graphical representations is an understanding that a point in the Cartesian coordinate system represents the projections of two quantities' values whose measures are represented on the axes (Figure 1). This intersection point in the plane is the object the student then imagines tracing while engaging in emergent shape thinking.

In this paper I extend Moore & Thompson's work by examining how a student's conceptualization of points in the Cartesian coordinate system might inhibit or support her in engaging in emergent shape thinking. In particular, I address why the conventional activity of plotting points by going over *x* units and up *y* units does not support students in engaging in emergent shape thinking.
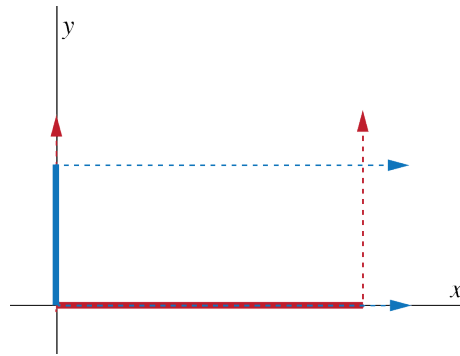
Figure 1. A point as a projection of two quantities' values represented on the axes.

Background

When one engages in emergent shape thinking she is engaging in covariational reasoning; activities involved in reasoning about how two varying quantities change in relation to each. Saldanha and Thompson (1998) provided one conception of covariational reasoning[1]. They explained,

> Our notion of covariation is of someone holding in mind a sustained image of two quantities' values (magnitudes) simultaneously. It entails coupling the two quantities, so that, in one's understanding, a multiplicative object is formed of the two. As a multiplicative object, one tracks either quantity's value with the immediate, explicit, and persistent realization that, at every moment, the other quantity also has a value. (Saldanha & Thompson, 1998, pp. 1-2)

This suggests that for Saldanha and Thompson engaging in covariational reasoning involves three mental actions: (1) conceptualizing quantities, (2) imagining quantities' values varying simultaneously, and (3) coupling two quantities through a multiplicative object.

For Saldanha and Thompson, multiplicative objects do not necessarily involve the numerical operation of multiplication. Instead, Thompson and Saldanha extend the work of Inhelder and Piaget (1964) and conceptualize multiplicative objects as mental constructions an individual makes when uniting two or more attributes simultaneously (Thompson, 2011b). Thompson provided the following examples of multiplicative objects:

1. A student can construct a rectangle's area as a multiplicative object that unites the rectangle's length and width.
2. A student can construct a point in the Cartesian plane as a multiplicative object that unites the distance of the point from the horizontal axis with the distance of the point from the vertical axis. (*ibid*, p. 47)

As Saldanha and Thompson (1998) explained, when a student constructs a multiplicative object he organizes his thoughts about the relationship between two quantities' varying values. As a result, whenever he imagines variation of one quantity he necessarily imagines the other quantity also having a value. For example, suppose a student imagines the values of x and y varying together. If the student constructs the point (x, y) in the Cartesian coordinate system as a multiplicative object then as he imagines the value of x varying he understands that the value of y necessarily has a value as well. With this conception of a point in the plane, the student can conceptualize graphs as emergent representations of how quantities' values change together.

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Methods

I conducted one-on-one task-based interviews with three university precalculus students, Sara, Carly, and Vince. All three students were in their first year of university and had declared a major in a STEM field. Thus, these students were taking precalculus to fulfill a pre-requisite for a required calculus course. The interview consisted of two phases. The first phase was a clinical interview (Clement, 2000). I engaged the students in tasks I anticipated would support me in understanding their meanings for tabular and graphical representations. The second phase of the interview was a task-based-teaching interview (Steffe & Thompson, 2000). My primary teaching goal was to support students in conceptualizing a graph as an emergent representation of how two quantities change together.

The main task in the teaching interview was based on an item from a diagnostic instrument (Thompson, 2011a). This animated item was originally designed to support researchers in better understanding in-service secondary mathematics teachers' meanings for covariational reasoning. For the purpose of this interview, I intended for this task to help me understand the nature of the multiplicative object a student constructs when she engages in covariational reasoning.

I showed each student a video that depicted a red bar along the horizontal axis and a blue bar along the vertical axis. As the video played, the lengths of the bars varied simultaneously in such a way that each bar had one end fixed at the origin. (See Figure 2 for selected screenshots from the video). The horizontal (red) bar's unfixed end varied at a steady pace from left to right while the vertical (blue) bar's unfixed end varied unsystematically. I explained to each student that the length of the red bar represented the varying value of Quantity A and the length of the blue bar represented the varying value of Quantity B. I asked each student to sketch a graph that represented how the lengths of the two bars changed together. I anticipated that students would be successful sketching a graph if they could imagine placing a point in the plane as a way to simultaneously represent the values of two quantities. Thus, this task assessed whether students saw the conventions of graphing in the Cartesian coordinate system as a way to simultaneously represent two quantities values.

The video played repeatedly until the student completed the task. The student had the opportunity to pause the video at any point. While students chose to pause videos used in previous tasks of the interview, none of the students chose to pause this video while completing the task. I engaged each student in three versions of this task. From the student's perspective, in each version of this task the length of the red and blue bars varied in different ways with respect to experiential time. From my perspective, this meant that each version of the task represented a different continuous functional relationship between the varying values of two quantities.

After I completed the interview process I engaged in retrospective analysis by transcribing each of the teaching sessions. While watching the videos and reviewing the transcriptions I identified instances that provided insights into the students' conceptualizations of graphs, points in the Cartesian coordinate system, and variation of a quantity's value. I used these instances to generate tentative models of each student's thinking that I then tested by searching for instances that confirmed or contradicted my tentative model. When I found evidence that contradicted my tentative model, I developed a new model that accounted for the student's mathematical activity. With this new model in mind, I reviewed all of the student's mathematical activity to either modify my previously constructed hypotheses or to document shifts in the student's ways of thinking.
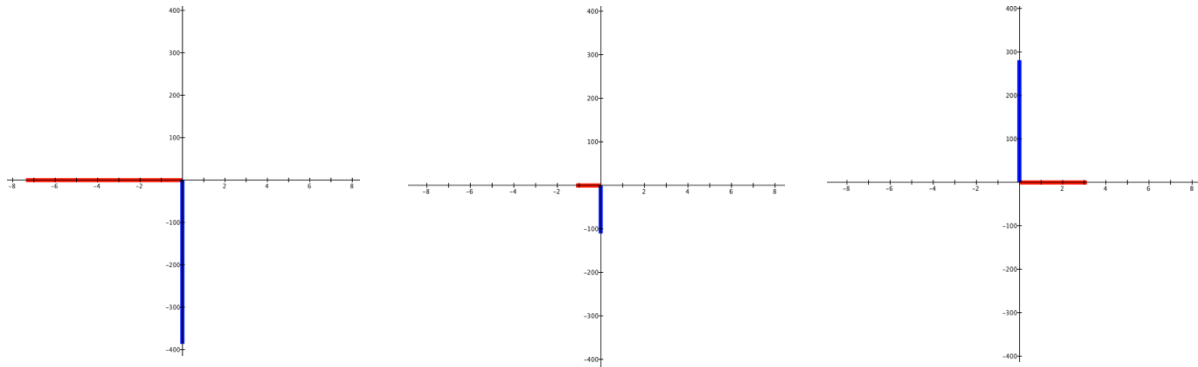
Figure 2. Three screenshots from animated task where values of Quantity A and B are represented along the horizontal and vertical axes with red and blue bars respectively.

Results

After reviewing videos and transcripts of each interview I found that while all three students plotted points from a table of values and interpreted the meaning of a point on a graph in a contextual situation, only one student independently constructed a point as a way to simultaneously represent two quantities' values when the values were represented on the axes.

At the beginning of each interview I asked the student to complete a set of conventional graphing tasks such as graphing a relationship described by a table of values and interpreting a point on his/her graph in terms of a contextual situation. Each student appropriately plotted points according to the conventions of the Cartesian coordinate system by going "over *x* and up *y*". For example, Carly explained that she plotted the point (4, -1) by going "over 4 on the horizontal and then down 1." When given a conceptual situation each student interpreted the meaning of the point in the plane in terms of the contextual situation. For example, Sara explained the point (1, 2) represented "at 1 second Susie was like 2 feet from her house."

The last task in the interview was the animated item I described above; I presented each student with a video where the length of the red bar along the horizontal axis represented the varying value of Quantity A and the length of the blue bar along the vertical axis represented the varying value of Quantity B. As the video played the lengths of the red and blue bars changed together so that each bar had one end fixed at the origin. While there were numerical values labeled along the axes, I anticipated that students would reason about the magnitude of each bar and not attend to the associated numerical values.

I presented each student with three versions of this task. I will focus on each student's first attempt at this type of problem in order to understand how students come to conceptualize a point in the Cartesian coordinate system as a multiplicative object. Figure 3 shows an accurate graph for the first version of this task.
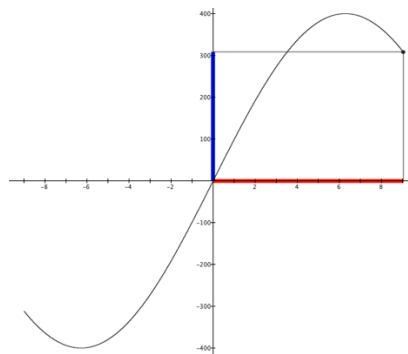
Figure 3. Accurate graph for animated item presented in Figure 2.

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Sara

As Sara watched the video, she appropriately described how the two quantities' magnitudes changed together. She explained, "As x was increasing at the beginning y was decreasing. But as x comes closer to 0 y also approaches 0 and they both increase for a little bit and as y keeps increasing or as x keeps increasing y starts to decrease." Although Sara described how the quantities' magnitudes changed together, she struggled to represent this graphically. In the following excerpt Sara explained her approach to constructing a graph from the video.

> *Sara*: In my head I like know like as that one is increasing you have to like. I try and like think of the shape of the line or the point or whatever to get to the line. Or yeah.
> *Int*: What do you mean you think of the shape to get to the line?
> *Sara*: So like for this like I have to see how like. Since that [value of x] is like increasing (*gestures left to right*) and that [value of y] is decreasing (*gestures up and down*) like what I am thinking in my head. Like I am like trying to figure out which way it needs to go.
> *Int*: Which way what needs to go? The graph?
> *Sara*: I don't know. That is why it takes me so long when I am just staring at the graphs.

Sara appeared to abandon her thinking about changing quantities when constructing a graph. Instead of conceptualizing the graph as a trace of how the value of x and y change together, she broke the graph up into chunks based on whether the value of y increased or decreased. Then she determined a shape that depicted the appropriate behavior of y as x increased. For example, if the value of y increased as the value of x increased then she knew the graph had to go up and to the right. While Sara appropriately described how the values of x and y changed together, her tendency to engage in static shape thinking prevented her from leveraging her reasoning about how the two quantities were changing together to construct a graph.

Carly

After Carly watched the video she sketched a graph by moving her pen up and down the vertical axis and then left to right on the horizontal axis (Figure 4). Carly was trying represent the dynamic nature of the video but in the moment of acting she did not construct a way to unite the variation of the two quantities' values. When explaining her graph Carly said, "All the x values are at zero and all the y values are at zero." It seemed Carly was attending to the red bar being on the axis and not the length of the red bar. To confirm this hypothesis, I paused the video and asked Carly to determine the value of x at that moment. She told me the value of x was 0 because the red bar was on the axis. When I asked Carly to consider the length of the red bar she determined the length of the red bar was 8 and independently concluded that meant the value of x was 8. With the video still paused, she reasoned similarly about the value of y and determined the point (8, 350) represented those two values.

Attending to the length of the red and blue bar represented a critical shift in Carly's thinking. After I supported Carly in conceptualizing the length of each bar Carly described, "plotting all the points" and sketched an appropriate graph of the behavior represented in the video.

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Figure 4. Carly's initial graph representing the behavior of two quantities' varying values.

Vince

After three minutes of puzzling about the task, Vince successfully represented the behavior in the video by sketching a graph in the plane. When explaining his approach, Vince described a point that he imagined as the "intersection" of the red and the blue bars and he described keeping track of this intersection as the video played. Although Vince described this approach when he first viewed the video, it took more than three minutes of reasoning for him to believe that this activity would represent how the lengths of the red and blue bars changed together.

I conjecture that Vince was able to construct the intersection point in the moment of acting because he had constructed a multiplicative object in thought that united the quantities' values. Throughout the entire interview when Vince talked about a value of x he also talked about the associated value of y. This was true whether Vince was referencing a table, graph, or contextual situation. This suggests that Vince had a way of thinking about relationships between quantities that united the values of x and y. When I presented Vince with this last task he focused on developing a way to graphically unite the values of x and y. He constructed a point as a multiplicative object in order to satisfy his need to unite the values of x and y. After conceptualizing the point as an "intersection" – a multiplicative object, he was able to represent the behavior in the video as a graph in the Cartesian coordinate system.

Discussion

All three students successfully plotted points from a table of values and gave contextual interpretations of points on their graphs. However, only Vince – with much hesitation – constructed a graph by conceptualizing a point as a way to unite two quantities' measures. These results suggest that the years of graphing practice that students endure in grade school do not necessarily prepare students for conceptualizing values in a table as measures represented along the axes nor does it prepare students for constructing a point in the Cartesian coordinate system as a m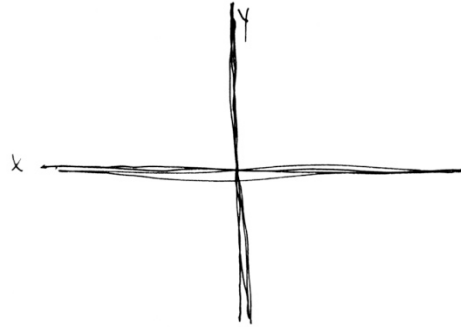ultiplicative object. In the following paragraphs I hypothesize why plotting points as "over x units and up y units" does not support students in conceptualizing a point as a multiplicative object. I also provide recommendations for how educators might support students in constructing points as multiplicative objects.

When the three students in this study plotted points from a table they enacted the activity "over x and up y". The point the student plotted was the product of this activity. It is likely that as soon as the student plotted the point he/she no longer thought about the activity that produced the point he/she constructed. For a student engaged in this type of activity, it is as if there is a place called (x, y) and the student is being asked to find it. As a result, the values of x and y are tied to a single place in space, as opposed to tied to two measures on the axes. Without conceptualizing attributes to unite, the student will never necessitate constructing a multiplicative object. Thus, it is not surprising that these students had difficulty using their meaning of points – locations in the plane – to construct a

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point in the Cartesian coordinate system as a way of uniting two quantities' values that are represented on the axes.

This suggests that educators need to support students in developing an entirely new conception of graphs that is rooted in conceptualizations of quantities' varying values being represented along the axes. Carly and Sara's work provides evidence of two difficulties students need to overcome in order to develop this conception of graphs.

When Sara first engaged with the animated task she explained how the two quantities' values changed together – she engaged in covariational reasoning. However, she abandoned this way of thinking when she went to sketch a graph and instead focused on the shapes that would represent the way the quantities' values changed together. While Sara was able to construct a correct graph, she did not construct the graph by keeping track of *how* the quantities were changing together. This suggests that conceptualizing a graph as an emergent trace of two quantities' values requires more than imagining how two quantities change together. While Sara was able to successfully complete this task, her thinking was entirely dependent upon breaking up the graph into chunks where she could appropriately determine the shape of the graph. This way of thinking is constrained to situations where Sara can imagine breaking up the behavior into graphs and limits the power of Sara's ability to represent and interpret graphs as emergent traces of covarying quantities.

Carly's difficulty stemmed from not differentiating between the length of the bar and the location of the bar on the axis. In order to construct a multiplicative object, the individual must first construct two properties to then unite. I conjecture that by focusing on the aspect of the bar that remained the same – its location on the axes – instead of attending to the aspect of the bar that varied – the length of the bar, Carly was unable to conceptualize two distinct properties. When I paused the video and asked Carly to determine the length of the bar, Carly constructed two properties - the length of the red bar and the length of the blue bar, which she could then think about uniting. While the numerical values likely helped Carly coordinate the animated task with her conception of values represented in a table, once Carly constructed the point as a way to unite two numerical values on the axes, she was able to imagine keeping track of all of the points. This suggests that Carly's thinking about points as uniting two measures on the axes was not dependent on knowing the values of the coordinates.

While Carly experienced difficulty conceptualizing two attributes to unite, Vince experienced difficulty conceptualizing the point as a way to unite two properties, namely the value of x and the value of y. This suggests that constructing a point in the Cartesian coordinate system as a multiplicative object is a nontrivial activity; conceptualizing two attributes to unite, and then conceptualizing how to unite these values graphically are both cognitively demanding activities.

By the end of the interviews, both Carly and Vince were able to construct a graph by imagining "all the points" and engaging in emergent shape thinking. Looking at the shifts in each student's thinking we see that attending to the measure of each bar's length can help students conceptualize two attributes to unite. Additionally, if a student conceptualizes relationships between quantities as a way unite values of two quantities in thought, then the student has the opportunity to construct representations of this relationship be it through tables, graphs, or formulas. Thus, educators should encourage students to unite their conception of two quantities' measures when reasoning about contextual situations, tabular representations, formulas, and graphical representations.

From a researcher's perspective, the coordinate pair (x, y) necessarily unites values of x and y in the Cartesian coordinate system. However, this study provides evidence that when plotting and interpreting points in the Cartesian coordinate system, students are not conceptualizing the coordinate pair (x, y) as a way to simultaneously represent values of two quantities. Additional research is needed to better understand how students construct coordinate pairs in the Cartesian coordinate system as multiplicative objects and how students leverage their conceptualization of coordinate

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pairs as multiplicative objects when engaging in covariational reasoning and reasoning about dynamic situations.

Endnotes

[1]See Carlson, Jacobs, Coe, Larsen, and Hsu (2002) and Confrey and Smith (1995) for other conceptions of covariational reasoning.

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THE BENEFITS OF FEEDBACK ON COMPUTER-BASED ALGEBRA HOMEWORK

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Math homework is transforming at a rapid rate with the continuous advances in educational technology. Computer-based homework, in particular, is gaining popularity across a range of schools, with little empirical evidence on how to optimize student learning. In the current study, middle school students (N = 143) solved a set of challenging algebraic problems on a computer-based homework assignment and received (1) no feedback, (2) correct-answer feedback, (3) try-again feedback, or (4) explanation feedback after each problem. Feedback resulted in higher posttest scores than no feedback, and this was true regardless of feedback type. On transfer items, correct-answer feedback has positive effects for low-knowledge students, but neutral effects for higher-knowledge students. Results suggest the provision of basic feedback on computer-based homework can benefit novice students' mathematics problem solving.

Keywords: Problem Solving, Algebra and Algebraic Thinking, Instructional Activities and Practices, Technology

Introduction

Modern advances in educational technology and increasing access to computers gives math teachers a wide range of tools for assigning homework and assessing students' problem-solving skills. Indeed, intelligent tutor systems and computer-based homework are quickly gaining popularity and prevalence in math classrooms across the world, necessitating research that crosses the borders of math education, cognitive science, and educational technology. Many of these computer systems are designed to enhance student learning through the use of targeted problem solving and feedback. However, recent evidence suggests there may be potential consequences of providing feedback during problem solving, especially for learners with higher prior knowledge (e.g., Fyfe & Rittle-Johnson, 2016). The goal of the current research was to evaluate the effects of different types of feedback on computer-based algebra homework for middle-school students. The selection of algebra was motivated by the recognition that algebra is a gatekeeper to future educational and employment opportunities (Adelman, 2006), and by concerns about students' inadequate understanding of and preparation in algebra (NMAP, 2008).

Theoretical Framework

In general, math education research supports the use of feedback during problem solving. Meta-analyses show that, on average, feedback has positive effects on learning outcomes relative to no feedback (Hattie & Timperley, 2007). Indeed, Hattie and Timperley (2007) claim that "feedback is one of the most powerful influences on learning and achievement" (p. 81). They even identify feedback as one of the top ten influences on student achievement, along with direct instruction and reciprocal teaching. However, the effects of feedback vary widely (Kluger & DeNisi, 1996) suggesting that certain types of feedback may be more effective than others. In fact, during mathematics problem solving, students can sometimes learn just as much, if not more, when no feedback is provided (Fyfe & Rittle-Johnson, 2016; Nussbaumer et al., 2008).

A growing body of evidence suggests that some of the variability in feedback effects is due to students' prior knowledge (e.g., Fyfe, Rittle-Johnson & DeCaro, 2012; Krause et al., 2009). Specifically, feedback often has strong, positive effects for students with lower prior knowledge, but neutral or negative effects for students with higher prior knowledge.

Theoretically, there are several reasons why feedback may hinder problem solving relative to no feedback. These reasons are related to the students' perception of the feedback, including their affective and cognitive processes. For example, feedback may draw attention to the self and elicit affective reactions that interfere with learning (Kluger & DeNisi, 1996). For example, feedback on incorrect answers can produce ego-threat (i.e., a threat to one's positive self-image), which may reduce one's confidence or motivation to continue. Students with higher prior knowledge may be especially affected by ego-threat as they likely have some expectation of performing well. Another possibility is that feedback overloads cognitive resources simply by providing additional information that needs integrated with the students' existing knowledge (Sweller, van Merrienboer, & Paas, 1998). For example, feedback can disrupt a student's internal processing of the task and ultimately hinder his ability to learn from it.

One key factor to consider may be the type of feedback provided. Dempsey et al. (1993) outlined a hierarchy of feedback types based on the information provided:

1. *No feedback*: provides no information about the student's response.
2. *Verification feedback*: informs the student if the response if correct or incorrect.
3. Correct-answer feedback: informs the student what the correct answer is.
4. Elaborated feedback: provides some explanation for why a response is correct.
5. Try-again feedback: allows one or more additional attempts to try again.

One possibility is that providing feedback with more information will have positive effects for both low- and high-knowledge students because it provides helpful information for moving forward. However, providing more information may also have consequences because it is more likely to overload cognitive resources. There is some consensus that effective feedback should at least provide the correct answer (Kluger & DeNisi, 1996). But, the benefits of extra information are less clear (see Mory, 2004), particularly in the realm of computer-based homework.

Computer-based math homework has several potential advantages relative to traditional paper-and-pencil homework, including the provision of immediate feedback to students on their performance. The goal of the current study was to evaluate the effects of this feedback using a particular system, ASSISTments.org (Heffernan & Heffernan, 2014). ASSISTments is a computer tutor system that can provide scaffolds and feedback to assist student. The use of computer-based homework offers several advantages for understanding the effects of feedback.

First, computer-based homework provides an ecologically valid context in which to evaluate the role of feedback on problem solving. Many prior feedback studies have been conducted in laboratory contexts in the presence of a researcher. Here, we test the effects of feedback in an authentic learning setting on homework assignments given to students by their teachers. Second, computer-based homework represents a learning setting that may reduce the negative effects of feedback. As mentioned above, one condition under which feedback may hinder learning is when it draws attention to the self as opposed to the task (Kluger & DeNisi, 1996). Attention on the self can invoke evaluations of one's abilities that interfere with the task. Computer-generated feedback is often viewed as a less evaluative source of information than person-generated feedback (Karabenick & Knapp, 1988), and may help decrease attention on the self and increase attention on the task. Computer-based homework may also reduce overload of cognitive resources by giving students control over when and how they process the feedback.

Current Study

The current study tested the effect of different types of feedback for middle school students solving algebra problems on computer-based homework via the ASSISTments system. Teachers and students who were already using ASSISTments were recruited so that the system was familiar to the

students and part of their regular classroom experience. Students were assigned to receive no feedback, correct-answer feedback, explanation feedback, or try-again feedback. Based on previous research, feedback was predicted to interact with prior knowledge such that feedback would have stronger, positive effects on problem solving for students with lower prior knowledge than for students with higher prior knowledge.

Method

Participants

All students from two sixth-grade teachers' classrooms and two seventh-grade teachers' classrooms were invited to participate. The teachers taught at three different schools and were using ASSISTments.org as part of their regular classroom experience. Of their 160 students, 17 students were not included in the study as they did not complete all required sessions. The final sample contained 143 students (65 in sixth-grade and 78 in seventh-grade).

Materials

All materials were presented using ASSISTments.org. The *pretest* included six algebraic equations to solve (see Table 1). There were four different problems types: $ax + b = c$, $b + ax = c$, $a(x + b) = c$, and $a(x + b) + c = d$. The *homework assignment* contained two worked examples at the beginning to re-familiarize students with correct problem-solving solutions (see Figure 1 for an example). The remaining problems were equations for students to solve on their own. Students solved 12 or 16 problems (i.e., three or four of each type of problem presented on the pretest). Whether students solved 12 or 16 problems reflected natural variation in teacher preference as two teachers opted for the 16-problem assignment ($n = 65$ students) and two teachers requested a shorter 12-problem assignment ($n = 78$ students). Percent correct scores were calculated for each student based on the number of items he or she was assigned. The *posttest* included eight equations to solve (see Table 1). Four items were isomorphic to the pretest problems (i.e., learning items) and the remaining four were challenge problems with novel problem structures (i.e., transfer items). Percent correct for each subscale was calculated.

Table 1: Problems Presented on the Pretest and Posttest

	Pretest	Posttest
1.	$2x + 3 = 23$	$5x + 13 = 73$
2.	$10 + 5x = 30$	$3 + 6x = 99$
3.	$3(x + 1) = 9$	$7(x + 4) = 63$
4.	$7(x + 3) + 2 = 51$	$2(x + 3) + 4 = 16$
5.	$5(x + 3) + 14 = 64$	$x/2 + 3 = 13$
6.	$8(x + 2) = 56$	$4(x + 2) + 3(x + 2) = 35$
7.	--	$x/9 + 31 = 34$
8.	--	$6(x + 4) + 2(x +4) = 48$

Note. On the posttest, problems 1 through 4 are learning items and 5 through 8 are transfer items.

Design and Procedure

The experiment had a pretest-homework-posttest design. Students completed the pretest during class or at home. Within three school days, students completed the homework assignment on their own. For the homework assignment, students were randomly assigned to one of four conditions: no-feedback ($n = 25$), correct-answer feedback ($n = 44$), explanation feedback ($n = 41$), or try-again

feedback ($n = 33$). Finally, students completed the posttest. All teachers assigned the posttest the same day students finished the homework, but three teachers had students complete it in class and one teacher had students complete it at home.

Figure 1. Worked example. An example given at the beginning of the homework assignment.

In the *no-feedback* condition, students did not receive any feedback during the assignment. After submitting each answer, the computer provided an "answer recorded" message and students clicked a button to move on to the next problem. In the *correct-answer feedback* condition, students received immediate, correct-answer feedback after each problem. If students typed the correct answer, a green check mark appeared with the word "Correct!" If students typed an incorrect answer, a red X appeared along with the words, "The correct answer is __" (with the correct answer filled in). In the *explanation feedback* condition, the feedback message included the correct answer, an explanation of why it was correct, and a worked example. For example, for $3x + 12 = 24$, the feedback message stated:

> The correct answer is 4 because when x is 4 both sides of the equal sign have the same amount. Let's plug 4 in for x and simplify to show that both sides have the same amount.
>
> 3x + 12 = 24
>
> 3*4 + 12 = 24
>
> 12 + 12 = 24
>
> 24 = 24

In the *try-again feedback* condition, the feedback message stated, "Sorry, try again. __ is not correct" (with the student's answer filled in). Students could continue inputting responses until they entered the correct answer or they could obtain the correct answer by clicking on a button.

Data Analysis

To examine the impact of feedback and prior knowledge, regression analyses were used for each outcome measure. Condition was dummy coded with correct-answer feedback, explanation-feedback, and try-again feedback entered into the models (with no-feedback as the reference group). Thus, each regression model included three condition variables, pretest score (mean centered), and three condition by pretest score interactions.

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Results

Pretest

On average, children solved 85% of the pretest problems correctly (SD = 23%). Scores did not differ significantly by condition, $F(3, 139)$ = 1.09, p = .36. Performance tended to be similar across all six items with percent correct on each item ranging from 80% to 92%.

Homework

The regression predicting homework scores from condition and prior knowledge was significant, $F(7, 135)$ = 11.64, p < .000, R^2 = .38, but the only individual predictor to reach significance was the main effect of prior knowledge, B = 0.53, SE = 0.16, p = .002. Students with higher prior knowledge on the pretest exhibited higher homework scores than students with lower prior knowledge. There were no main effects of feedback or feedback by prior knowledge interactions, ps > .10. Thus, feedback had little effect on performance during the homework.

However, qualitative evidence suggested that children were learning over the course of the assignment. For example, four problems were of the form a(x + b) = c. Only 77% of students solved the first of these problems correctly, but 88% solved the fourth of these problems correctly. Similarly, four problems were of the form a + bx = c. Only 80% of students solved the first one of these correctly, but 92% solved the final one correctly. Further, on earlier problems, errors tended to reflect common misconceptions about variables, whereas errors on later problems were more diverse, suggesting that students were at least attempting correct strategies later in the assignment. For example, the very first problem was 3x + 12 = 24. Two of the most common incorrect answers were 1.6 or 9. Students added the coefficient (3) and the isolated number (12) to get 15x on the left side. Then, they either calculated 24 divided by 15 or 24 minus 15 to get their answer. However, one of the last problems was 8 + 2x = 40, and no student showed evidence of making the mistake of adding the 8 and the 2. Thus, students appeared to improve over the course of the homework assignment, but not differentially by condition.

Posttest Learning Items

Students did well on the posttest learning items, solving nearly 90% correct. Indeed, on 3 of the 4 problems students were near mastery. The fourth problem, 2(x + 3) + 4 = 16, proved somewhat difficult with only 79% of students solving it correctly. Estimates from the regression predicting percent correct on the learning items are presented in Figure 2. The overall regression was significant, $F(7, 135)$ = 9.22, p < .000, R^2 = .32. There was a significant, positive effect of prior knowledge, B = 0.36, SE = 0.18, p = .04. However, there were also significant main effects of correct-answer feedback, B = 8.91, SE = 4.33, p = .04, and try-again feedback, B = 13.51, SE = 4.58, p = .004, as well as a marginal effect of explanation feedback, B = 7.40, SE = 4.35, p = .09. Although prior knowledge did not significantly interact with any feedback type, ps > .10, an examination of Figure 2 suggests that the effect of explanation feedback was only marginal because it was primarily effective for high-knowledge children, but not low-knowledge children. Descriptively, we also compared the percent of children at mastery by condition. Fewer children in the no-feedback condition (36%) solved all four learning items correctly compared to children who received correct-answer feedback (66%), explanation feedback (66%), and try-again feedback (72%). Thus, feedback boosted learning on the posttest relative to no feedback.

Figure 2. Posttest scores. Scores on the learning items by condition and prior knowledge. Unstandardized regression coefficients are plotted at ±1 standard deviation from the mean.

Posttest Transfer Items

Students struggled on the posttest transfer items, solving only 64% correct on average. This is expected given that these were novel problem types, which students did not practice. Although all four problems were difficult, the problems with two separate terms in parentheses, like $4(x + 2) + 3(x + 2) = 35$, were the most challenging. Estimates from the regression predicting percent correct on the transfer items are presented in Figure 3. The overall regression was significant, $F(7, 135) = 5.10$, $p < .000$, $R^2 = .21$. There was a significant positive effect of prior knowledge, $B = 1.12$, $SE = 0.33$, $p = .001$, but no main effects of feedback condition, ps $> .20$. However, there was a significant interaction between prior knowledge and the no-feedback vs. correct-answer feedback contrast, $B = -0.79$, $SE = 0.40$, $p = .05$. To examine the interaction, pretest scores were centered at one standard deviation below the mean in one model (i.e., low prior knowledge) and one standard deviation above the mean in a separate model (i.e., high prior knowledge). For low-knowledge students, there was a significant positive effect of correct-answer feedback, $B = 28.35$, $SE = 12.54$, $p = .03$. The effects of explanation feedback, $p = .14$, and try-again feedback, $p = .35$, were positive, but not statistically significant. However, for high-knowledge students, there were negative, but non-significant effect of correct-answer feedback, $p = .49$, explanation feedback, $p = .40$, and try-again feedback, $p = .40$ (see Figure 3). Thus, the effects of feedback on transfer to novel problems depended on children's prior knowledge on the pretest. Feedback (particularly correct-answer feedback) resulted in better transfer than no feedback for low-knowledge children. But, all three types of feedback resulted in slightly lower transfer scores relative to no feedback for high-knowledge children.

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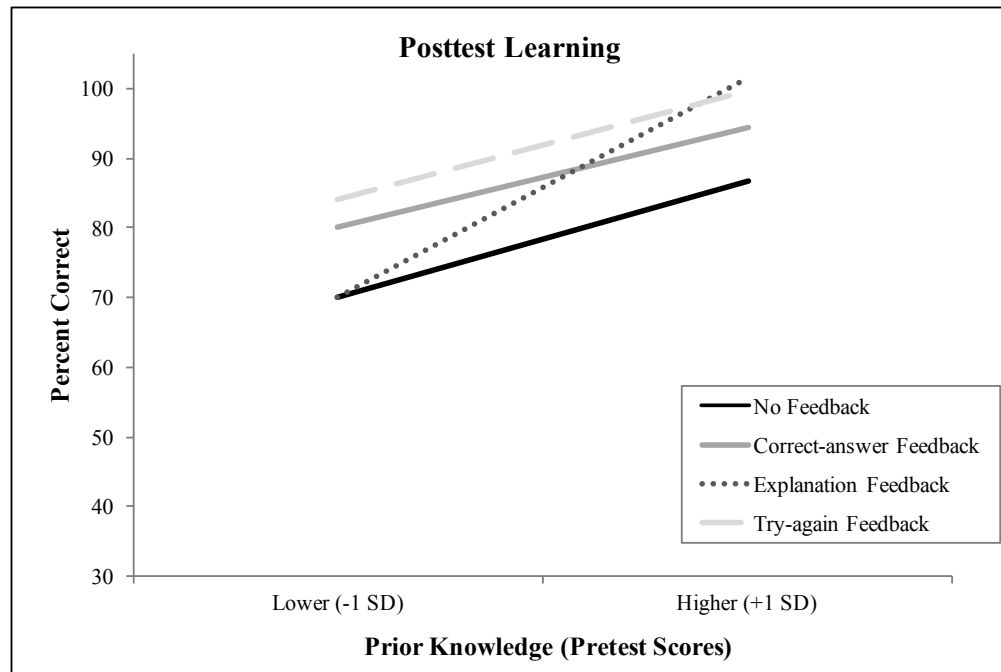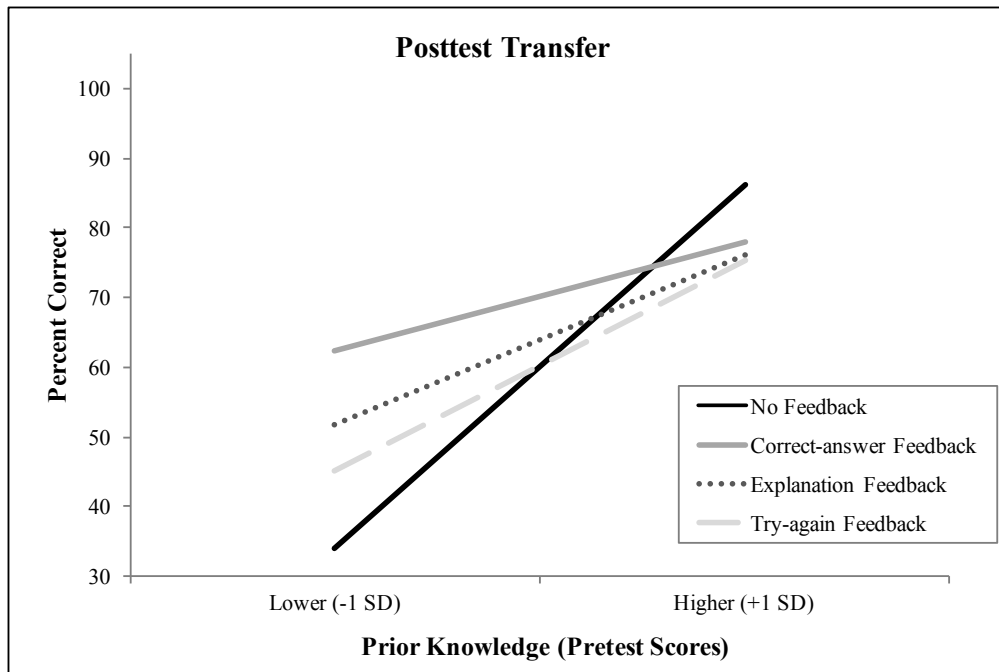
Figure 3: Posttest scores. Scores on the transfer items by condition and prior knowledge. Unstandardized regression coefficients are plotted at ±1 standard deviation from the mean.

Discussion

The goal of this study was to test the effects of feedback and prior knowledge on computer-based algebra homework using the ASSISTments system. Middle school students were assigned to receive no feedback, correct-answer feedback, explanation feedback, or try-again feedback during problem solving. After a single assignment, feedback resulted in higher posttest learning scores than no feedback, and this was true regardless of feedback type. On transfer items, feedback interacted with prior knowledge, such that correct-answer feedback has positive effects for low-knowledge students, but neutral effects for higher-knowledge students.

The findings from the current study make at least three contributions to research in mathematics education. First, the results demonstrate the benefits of three different types of feedback on problem solving. Further, they suggest that providing additional information or attempts does not always increase the efficacy of feedback. In fact, basic correct-answer feedback resulted in the best transfer for low-knowledge students, suggesting a possible threshold account (Phye, 1979). That is, when more information or support is provided beyond what is needed, it does not provide any additional advantage. Second, the results indicate that the benefits of feedback are strong for low-knowledge students, but that high-knowledge students sometimes do just as well without feedback during problem solving. Indeed, on posttest transfer, high-knowledge students tended to exhibit higher scores in the no-feedback condition, which is consistent with recent research (Fyfe & Rittle-Johnson, 2016; Krause et al., 2009). The third contribution is to introduce an exciting new method to conduct experimental research in an ecologically valid classroom. The ASSISTments project is a system bringing researchers and teachers together to better assist and assess student learning (Heffernan & Heffernan, 2014).

Future research is needed to test different types and schedules of feedback that are more dynamic and that adjust based on the student response. Indeed, one of the benefits of computer-based homework is the possibility of adapting the provision of feedback to students' needs. Further, more work is needed to enhance the provision of feedback for high-knowledge students. The high-knowledge students in the current study did not benefit from feedback, but still had room to grow.

One potential solution is to give high-knowledge students more control over the feedback (Hays et al., 2010), allowing them to skip unnecessary feedback and spend more time on challenge problems. Future studies should explore these and other possibilities.

In general, the current research is at the border of mathematics education, cognitive science, and educational technology. The specific results highlight the power and variability of feedback effects on computer-based homework. On the one hand, researchers and educators can marvel that such basic feedback can improve problem solving for novice students. On the other hand, these results challenge the intuition that feedback is always helpful and suggest that some students, under certain circumstances, can do just as well without it.

Acknowledgments

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UNDERGRADUATE STUDENTS' PERCEPTION OF TRANSFORMATION OF SINUSOIDAL FUNCTIONS

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Trigonometry is one of the fundamental topics taught in high school and university curricula, but it is considered as one of the most challenging subjects for teaching and learning. In the current study Mason's theory of attention has been used to examine undergraduate student's perception of the transformation of sinusoidal functions. Two types of tasks – (A) Recognizing sinusoidal functions and (B) Assigning coordinates – were used in this study. The results show that undergraduate students participating in this study experienced difficulties in identifying a period of a sinusoid, especially when it was a fraction of π radians.

Keywords: Post-Secondary Education, Algebra and Algebraic Thinking

Background

Trigonometry has a long history. Ancient people used trigonometry for different purposes. For example, Egyptians applied trigonometry to determine the correlation between the lengths of the shadow of a vertical stick with the time of day. Astronomers also used trigonometry to find the longitude and latitude of stars, as well as the size and distance of the moon and sun. However, trigonometry was not an essential part of mathematics textbooks until a Persian mathematician named Khwarizmi introduced trigonometric functions to the world. Since then, trigonometry has become one of the main topics in high school and university mathematics books and students are required to assign time for learning trigonometry, especially trigonometric functions. This is the case since a strong foundation in trigonometric functions will likely strengthen their learning of various mathematical topics, such as Fourier series and integration techniques (Moor, 2010). It is also shown that understanding calculus and analysis is dependent on the learning of trigonometric functions (Hirsh, Weinhold and Nicolas, 1991; Demir, 2011). However, learning and understanding trigonometric functions is a difficult and challenging task for students, compared to other mathematics functions, such as polynomial functions, and exponential and logarithmic functions. While other functions (e.g., logarithmic functions) can be computed by performing certain arithmetic calculations expressed by an algebraic formula, trigonometric functions involve geometric, algebraic and graphical concepts and procedures, simultaneously (Weber, 2005, Demir, 2011).

Despite its importance and its complexity, research on trigonometry is sparse and quite limited. In the literature, only a small number of studies concentrate on students' learning of trigonometric concepts, and in particular trigonometric functions (e.g. Gray and Tall, 1991; Brown, 2005; Weber, 2005, Moor, 2010). Challenger (2009), Moore (2010) and Weber (2005, 2008) indicated that students often have difficulty using sine and cosine functions defined over the domain of real numbers. Thompson (2008) also noted that students are unable to construct understanding of the trigonometry of right angles and the trigonometry of periodic functions. In a study of undergraduate students, Weber (2005, 2008) agreed that students could not rationalize various properties of trigonometric functions or reasonably estimate the output values of trigonometric functions for various input values. Kendal and Stacey (1997) concluded that students had difficulty interpreting trigonometric functions in the unit circle, recognizing that x and y coordinates of a point on the unit circle are cosine and sine values of corresponding angles compared with other determined trigonometric functions in terms of a right triangle.

In spite of all the research efforts in the area of teaching and learning trigonometry, especially trigonometric functions, there are still gaps in the literature. There is no research study that focuses on the concept of the transformation of sinusoidal functions; the current research attempts to fill this gap.

In order to deal with the transformation of sinusoidal functions, students need to understand the notion of the 'period of a function.' The period of a function is the distance (x value) in which function values repeat themselves. In the case of the canonical sine function $f(x) = sin\ x$, the period is 2π, the circumference of the unit circle. Considering the standard format for the sinusoidal function $f(x) = Asin(Bx \pm C) \pm D$, students are required to identify the relationship between the coefficient of x (B in the function) and the period when dealing with the transformation of sinusoidal functions. As such, the research questions are: How do undergraduate students identify period? How do they recognize the period on the graph of the sinusoidal functions?

Data Collection and Analysis

This study is part of a bigger project which examines undergraduate students' perception of the transformation of sinusoidal functions. In the larger study, seven undergraduate students from a large North American university participated They were selected from among students who had either completed a Calculus I course and were enrolled in Calculus II (3 students) or they were in a Calculus I course (4 students) in the Mathematics Department. Participants volunteered their time to contribute in the study right after I made a general request from all the classes (Calculus I and II). For the purpose of this research report, I focus only on the performance of one of the participants, Emma. She was studying Applied Science and was enrolled in a Calculus II course at the time of her interview.

A 60-minutes task-based interview was conducted and Emma was required to complete two types of interview tasks: **A)** Recognizing sinusoidal functions and **B)** Assigning coordinates. Both types of tasks were presented with the help of the Dynamic Geometry software, Sketchpad. For the 'Recognizing sinusoidal function' tasks, the sketches indicating the sinusoidal graphs were given and the student was asked to identify the sinusoidal functions represented in the given graphs (see Figure 1). For the 'Assigning coordinate' tasks, a wavy displace (see Figure 2) along with the sinusoidal functions were given and Emma was required to assign coordinates on the wavy curve such that it described the given functions. Type A tasks comprised of Task 1: $f(x) = sin(2x)$, Task 2: $f(x) = sin(\frac{2}{3}x)$ and Task 3: $f(x) = cos(\frac{2}{5}x - \frac{\pi}{5})$. Type B tasks included Task 4: $f(x) = sin(4x)$ and Task: 5 $f(x) = cos(3x - \frac{\pi}{4})$.

Theoretical Framework

The collected data in this study were analyzed and interpreted using the *theory of shifts of attention* (Mason, 2005). Mason's theory provides opportunity to study the critical role of attention and awareness in learning and understanding mathematics and in particular the concept of the transformation of sinusoidal functions. Mason (2005) distinguishes five different structures of attention: 1) Holding wholes; 2) Discerning details, 3) Recognizing relationships; 4) Perceiving properties; and 5) Reasoning on the basis of agreed properties. Mason's framework of shifts of attention is appropriate for analyzing the collected data in my research. Applying this framework supports me in gaining insights not only into '*what*' Emma attended to when completing mathematics tasks related to the transformation of sinusoidal functions, but also '*how*' she shifted her attention in identifying the period of sinusoids. Mason's terms for different structures of attention also provide a language for analyzing students' work. For example, when a student considers a particular graph and recognizes its shape as representing a sinusoidal function, s/he is *holding holes*. A student who looks for particular details from the given sinusoidal functions or the given sinusoidal

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curve (e.g., she is seeking for the point the graph intersects the y-axis), she is, in fact, *discerning details*. The student is *recognizing relationship* when she able to find a connection between the graphical representation of the sinusoidal functions and their symbolic representations. When a student determines the particular parameters that determine the given sinusoidal curve by considering its periods, she is *reasoning based on perceived properties*. To investigate how the participant realized the transformation of sinusoidal functions, and in particular, how she identified period from the given graphs/functions, I reviewed the student's answers and the transcripts several times.

Please note that in all the five interview tasks the participant was required to connect the period of the given sinusoidal function or the given sine curve to a coefficient of x in the standard formula for sinusoidal functions (considering the sinusoidal function in the standard form: $f(x) = Asin(Bx \pm C) \pm D$)). For brevity, we refer to this connection as 'recognizing the period' (see Figure 3).
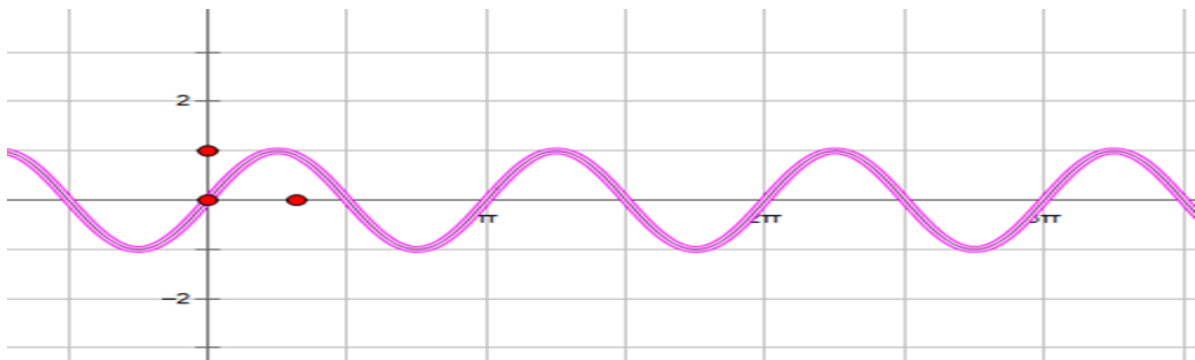
Figure 1. Graph presented in Task 1.
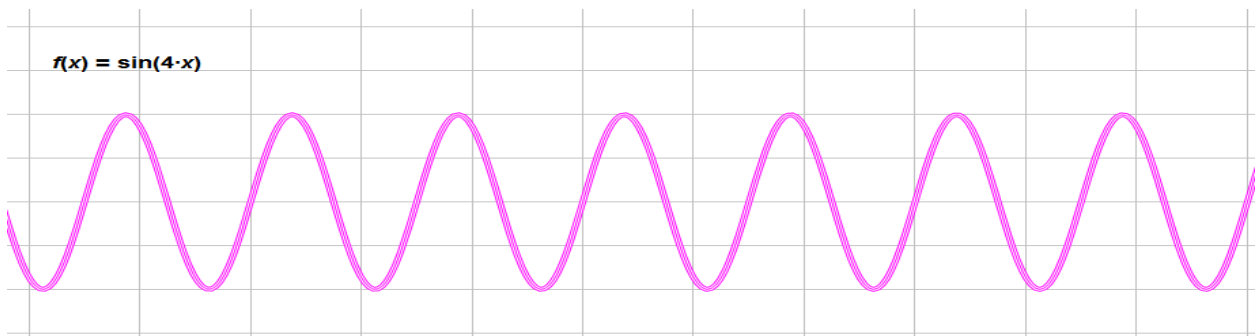
Figure 2. Graph presented in Task 4.
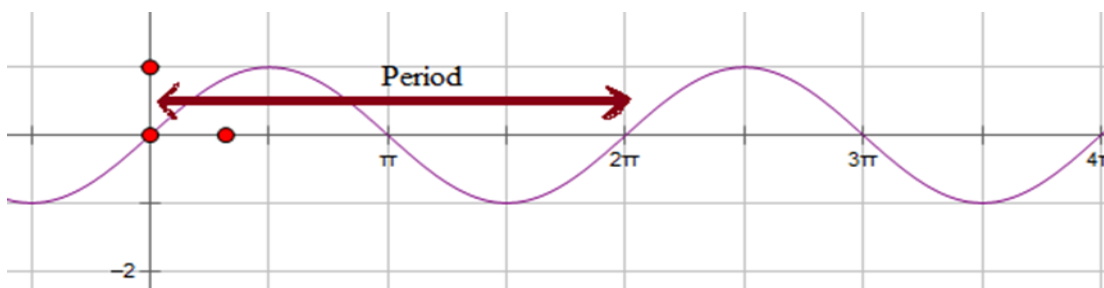
Figure 3. Recognizing the period.

Recognizing period (coefficient B of x)

At the beginning of the interview, I showed Emma Task 1 in which the graph of the function $f(x) = sin(2x)$ was given (see Figure4) and she was asked to identify the sinusoidal function

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represented by the graph. In order to complete Task 1, Emma first focused her attention on the given graph and waited for visual feedback from the graph (her attention was on *holding wholes* according to Mason's classification). Emma stated:

> It is $f(x) = sin(\frac{1}{2}x)$. It is a sine graph because it starts at 0 and it should be $sin(\frac{1}{2}x)$. The sine graph start at 0 and then π and 2π, but in this one is 0, π,$\frac{2\pi}{3}$. This is half of sine graph, because the period here is π, while it is 2π in the original sine curve.

The above statement indicates that the participant recognized incorrectly the function for the given graph, determining it to be $f(x) = sin(\frac{1}{2}x)$. Analyzing the situation using Mason's (2008) framework it can be concluded that Emma *reasoned on the perceived properties* of the sinusoidal functions and from there she *determined (*incorrectly*) relationships* between the visual representation and the symbolic representation. Emma recalled the fact that the period of a canonical sine function is 2π, whereas the period of the curve given in Task 1 was π . She thus concluded that the given curve represented the function $f(x) = sin\left(\frac{1}{2}x\right)$. Emma then connected the period of the sine curve, which was π radians, with the coefficient of x in her suggested sinusoidal function. Her statement illustrates that Emma, in fact, divided the argument x by 2 because the period of the canonical function (2π) was divided by 2 in the given graph (the period was π).
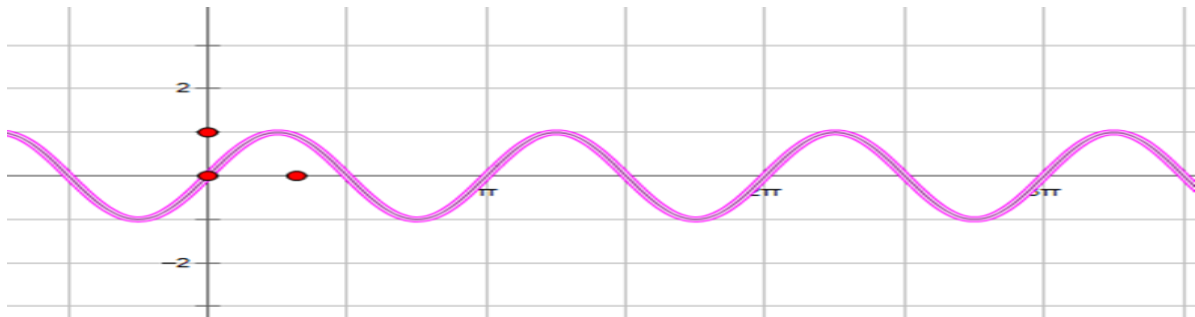
Figure 4: Graph of function $f(x) = sin(2x)$.

Detecting Emma's mistake in recognizing the proper function for the given graph in Task 1, I showed her the graph of $f(x) = sin(\frac{1}{2}x)$. Observing the graph of the function $f(x) = sin(\frac{1}{2}x)$ made the participant realize that the graph of the suggested function did not correspond to the given curve. At this time Emma stared at both graphs #1 and #2 (see Figure 5) for a while and she *held the graphs (#1 and #2) as wholes*. She then began to *describe in detail* the given graph (#2 in Figure 5) in respect to the graph of $f(x) = sin(\frac{1}{2}x)$. Emma stated:

> ….so, if $f(x) = sin(\frac{1}{2}x)$ is like this, so it is going to finish at 4π. So this is going to be the whole graph. So it should not be $\frac{1}{2}x$, it should be 2x. Because when we have $\frac{1}{2}x$ we can see that it ends at 4π. But if I put here 2x, I compressed it and I can…have this curve finishes at π π...The period of sine graph is 2π but this one is compressed, so it is $f(x) = sin(2x)$, but $\frac{1}{2}x$ is expansion in fact.

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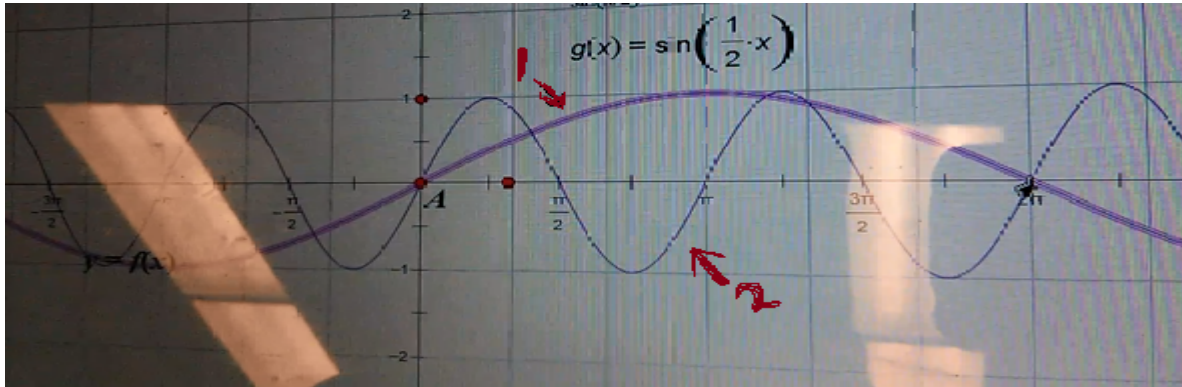
Figure 5: Graphs of $f(x) = sin(\frac{1}{2}x)$ and $f(x) = sin(2x)$.

As it is indicated from the above statement, Emma compared the end point (or the length of a full cycle) of the curve #2 with that of curve #1, considering the origin (0, 0) as a beginning of a cycle ("....$\frac{1}{2}x$ *we can see that it ends at 4π. But…I can…have this curve finishes at π…*"). In other words, by linking the end points of the full cycles (in both curves and comparing them with the graph of the canonical function), Emma was able to *find relationship* between the visual representation and the symbolic representations. She chose the number 2 (which was the reciprocal of the coefficient of $\frac{1}{2}\pi$) as a coefficient for x in the sinusoidal function. As such, she eventually recognized the correct period and thus the proper function for the given curve.

Emma's proper realization in Task 1 directed her to complete successfully similar tasks having a whole number for the coefficient of x. As an example, when approaching Task 4 in which the function was $f(x) = sin(4x)$ and a wavy curve was given, Emma was able to assign correctly the coordinates in the given wavy displace such that it represents the graph of $f(x) = sin(4x)$. After gazing at the given function in Task 4, she expressed:

> …I know that 2π is here [see Figure 6] because 1, 2, 3, and 4 periods is between 0 and 2π and here are 1 and -1…

The above excerpt shows that Emma perceived *properties of sinusoidal functions* ("*…period is between 0, 2π and here are 1 and -1*"). The feedback she received from Task 1 (the fact that there is a direct relationship between the coefficient of x in the sinusoidal functions and the number of repeated full sine cycles between 0 and 2π) allowed Emma to assign coordinates properly in Task 4. In other words, Emma was able to realize period from the given function and therefore assign axes successfully on the sinusoidal curve. Considering Emma's success in Task 1 and Task 4, one might conclude that she was able to recognize period and also sinusoidal functions, from their graphs, and vice-versa, successfully. However, Emma performed differently on the other interview tasks.
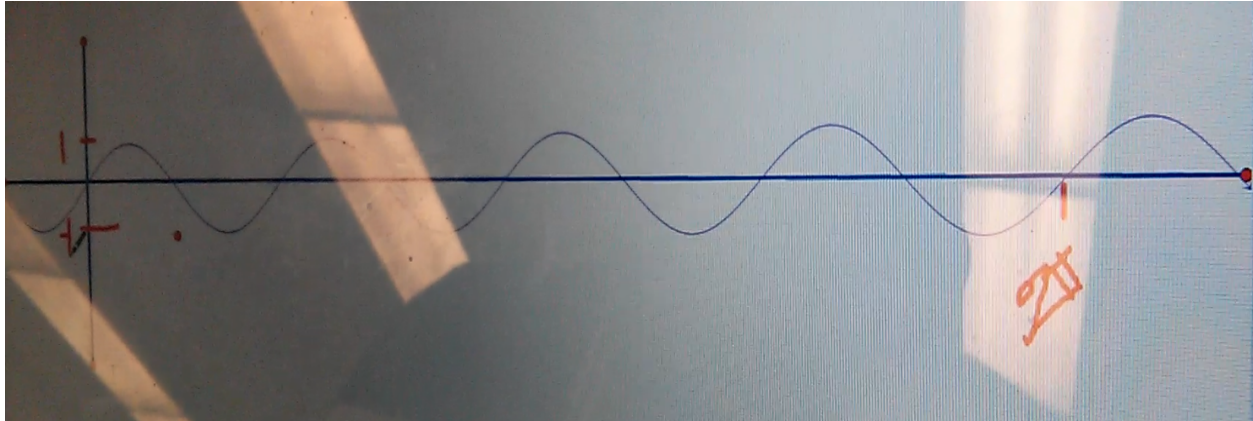
Figure 5. Emma adjusts coordinates for Task 4.

As an example, when completing Task 2, in which the graph of the function $f(x) = sin(\frac{x}{3})$ was given, after *holding the graph as whole* for a long pause, Emma did *discern some details* from the x-axis. She then stated:

> …It is sine of x over something because if it is sine of x it would end here [at 2π]…ok, it is $f(x) = sin(\frac{x}{3})$ because there are one, two and three spaces here between 0 and this point and again one, two, three here… (see Figure 6).

As it appears from the above statement, Emma counted the number of 'blocks' between 0 (the point A in Figure 6) and the point in which the curve intersected the x-axis (point B) and again from point B to another point in which the graph intersected the x-axis (point C). Since the distance between the points A and B, and B and C was 3 blocks, Emma put the fraction $\frac{1}{3}$ for the coefficient of x in the suggested sinusoidal function. It appears that she was eager to find an opposite relationship between 3π and the coefficient of x which was $\frac{1}{3}$. This evidence illustrates that Emma was unable to *recognize appropriately the relationship* between graphical representations of the sinusoidal function and its symbolic representations.
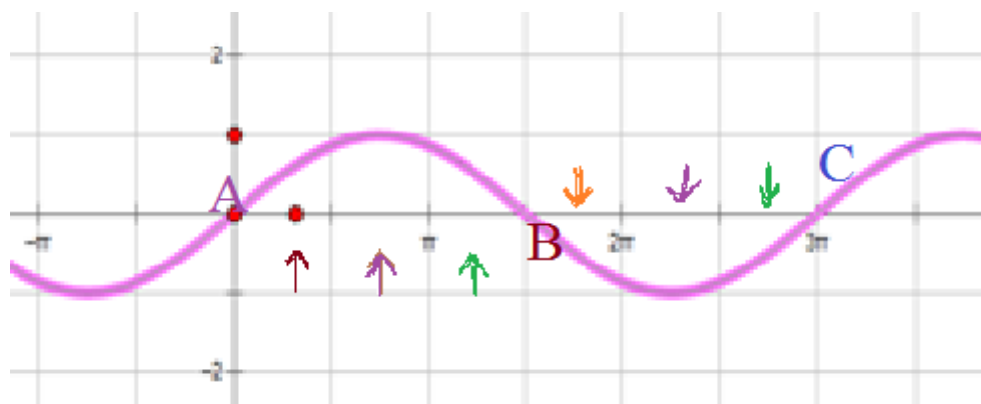
Figure 6. Emma counting the blocks between the points.

Emma's unsuccessful attempt in recognizing period and its relation with the coefficient of x in the sinusoidal function in Task 3 was typical of further errors in the other tasks having fractions for the coefficient of x. In other words, in Tasks 3 and 5, as in Task 2, Emma was unable to recognize

period successfully. As it was mentioned previously, it seems that the fractional coefficient was problematic, because Emma often attempted to reverse the point in which a full curve was finished (which was 3π in the Task 3) in order to find a coefficient for x in the sinusoidal function. Although applying this method directed Emma to determine proper functions when the coefficient of x was a whole number, it did not work for the other tasks. Emma, in fact, should find the relation between the period of a canonical function which is 2π and the point 3π in the given graph ($3\pi = \frac{2\pi}{B}$, so $B = \frac{2}{3}$) in order to identify the coefficient of x in the sinusoidal function.

Discussion

The findings of this study show that the student recognizes period and transformations in different manners when the coefficients of x in the sinusoidal function are whole numbers and when they are fractions. The data from this research demonstrates that Emma is capable in matching the algebraic representations with the graphical representations when the coefficient of x was a whole number. These results are in contrast with the findings of Leinhardt, Zaslavsky and Stein (1993), Yerushalmy and Schwartz (1993), and Knuth's (2000) studies in which a group of undergraduate students were unable to use graphical representations to complete mathematics problems in the symbolic form. The contribution of this research is in connecting together the participant's understanding of transformations, graphs and periodicity, whereas the previous research studies focused distinctly on the concepts of transformations (e.g., Yerushalmy and Schwartz, 1993), graphs (Brow, 2005) and periodicity (van Dormolen and Zaslavsky, 2003).

The findings, however, illustrate that Emma was unable to recognize period correctly, when the factor of x was not a whole number in the sinusoidal functions. That is, she was unable to connect the graphs with the sinusoidal functions when the factor of x was a fraction. As such, further research studies are required to investigate how undergraduate students interconnect the three concepts of transformations, graphs and periodicity when the coefficient of x is a fraction.

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WHAT DOES IT MEAN TO "UNDERSTAND" CONCAVITY AND INFLECTION POINTS?

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The calculus concepts of concavity and inflection points are often given meaning through the shape or curvature of a graph. However, there appear to be deeper core ideas for these two concepts, though the research literature has yet to give explicit attention to what these core ideas might be or what it might mean to "understand" them. In this paper, I propose a framework for the concavity and inflection point concepts, using the construct of covariation, wherein I propose conceptual (as opposed to mathematical) definitions that can be used for both research and instruction. I demonstrate that the proposed conceptual definitions in this framework contain important implications for the teaching and learning of these concepts, and that they provide more powerful insight into student difficulties than more traditional graphical interpretations.

Keywords: Post-Secondary Education, Advanced Mathematical Thinking, Learning Trajectories (or Progressions)

At the level of calculus, and beyond, the twin concepts of concavity and inflection points are essential components of a complete understanding of function behavior. As such, several researchers have begun to examine how students think about these two concepts (e.g., Baker, Cooley, & Trigueros, 2000; Gómez & Carulla, 2001; Tsamir & Ovodenko, 2013). The way these concepts have been studied, as well as how they are often portrayed in textbooks, is usually deeply connected to graphical interpretations and meanings of the concepts (e.g., Baker, Cooley, & Trigueros, 2000; Stewart, 2014). Yet, on occasion, calculus education researchers seemed to have acknowledged other important interpretations or meanings of these concepts. For example, Tsamir and Ovodenko (2013) discussed how students used symbolic representations, including $f'(x)$ and $f''(x)$, to define and reason about the two concepts, and Berry and Nyman (2003) described teaching activities that develop these two concepts through physical movement. Furthermore, Carlson, Jacobs, Coe, Larsen, and Hsu (2002) explained how covariational reasoning is deeply connected to making sense of function behavior, which is closely linked to concavity and inflection points. All of this research has given us important information on how students use concavity and inflection points in activities like graphing, or on general difficulties students may have with them, or on types of reasoning that are required to make sense of them. Yet, these studies have often taken implicit stances on what the concepts concavity and inflection points actually mean. Thus, the question is raised: What are the *core ideas* we might say are contained in the concepts concavity and inflection points, and, consequently, what does it mean for a student to *understand* these ideas? While it is possible to claim that these answers can, or should, reside purely in graphical terms, such as the shape or curvature of a graph, the research literature seems to suggest that there are ideas more fundamental than simply the "shape of a graph" for these concepts. Since research has often not made explicit what these core ideas are, beyond graphical interpretations, it is important that we, as a research community, debate what might make up the core ideas of concavity and inflection points and an "understanding" of them.

Like the research literature, calculus textbooks also tend to focus on graphical interpretations of concavity and inflection points (e.g., Hughes-Hallett et al., 2012; Stewart, 2014; Thomas, Weir, & Hass, 2009). However, despite the heavy graphical treatment of the concepts, they are frequently given definitions through non-graphical language. For example, Thomas et al. (2009) define concavity based on whether "f' is increasing [or decreasing] on I" (p. 203), and in Foerster (2010),

concavity is defined by the sign of the second derivative (p. 373). Even though these books subsequently focus on graphical meanings in their treatment of the concepts, we see hints at other ideas for concavity and inflection points beyond just the shape of a graph. Again, we are left with the question as to what the core ideas are that we want to ascribe to concavity and inflection points and what an "understanding" of them might consequently look like.

In this paper I propose an argument for a particular conceptual framework that could be used as a way to *conceptually* define concavity and inflection points, which then provides one possible answer for what the core ideas are and what it might mean to "understand" them. I articulate my stance through covariational reasoning (Carlson et al., 2002), in that I see covariation as more than just "important" to an overall understanding of these concepts, but I have come to see it as the single core idea that makes up the *essence* of these two concepts. In laying out this conceptual framework, I discuss its connection to the common graphical approach to these concepts, and I discuss some pedagogical implications. I hasten to note that my stance is influenced by my examination of these concepts in real-world contexts (see Gundlach & Jones, 2015), and as such, I fully acknowledge that my view is certainly not the *only* point of departure for a conceptual discussion on concavity and inflection points. The fact that many studies and textbooks discuss concavity in other ways speaks to this. However, I propose my perspective as a way to launch a debate on what a shared idea of the *core ideas* of concavity and inflection points might be and what an *understanding* of them might look like for calculus education.

Arguments For and Against Having a Conceptual Framework

In one sense, one could argue that the various ideas about concavity and inflection points in the research literature and in the textbooks that I outlined in the introduction are all basically different ways to express the same idea, and that consequently there would be no need for a framework such as this. However, contained in this argument is exactly the issue I want to address: if they are all different ways of expressing the same idea, *what*, exactly, is that underlying idea? Similarly, one could argue that we just define an "understanding" of these concepts through the idea of making connections between representations, much as some have couched an understanding of *function* through representational connections. Yet, just as Carlson et al. (2002) have shown that there is a deeper level of function understanding in covariational reasoning, I believe concavity and inflection points also have deeper meanings beyond just connections in the ways we externally represent them to each other through graphs or symbols.

In another argument, one could dismiss the need for a framework by indicating that these different approaches in the literature and textbooks are all equivalent mathematically and that one approach or representation can be translated into another approach or representation. However, despite the *mathematical* consistency across the different approaches and representations, I argue that they are quite *conceptually* different from one another and would therefore each have separate learning implications (as opposed to mathematical implications). To develop a shared notion of what it means to *understand* concavity and inflection points, the conceptual nature, and not purely the mathematical nature, of these two concepts must be attended to. In other words, my argument is similar to—though not congruent to—Tall and Vinner's (1981) distinction between concept definition and concept image. I use the comparison to Tall and Vinner simply to illustrate the difference I see between mathematical consistency versus conceptual consistency. Despite the possible mathematical equivalence of the various definitions and uses of concavity and inflection point, it is important that we distinguish what are the core conceptual ideas that might make up the concepts.

A Proposal of a Conceptual Framework for Concavity and Inflection Points

Covariational Reasoning

Since I am using covariational reasoning as the foundation of my proposed conceptual framework, in this section I briefly describe the covariation construct as laid out in the work of Carlson and colleagues, which is rooted in research on understanding function behavior (Carlson, 1998; Carlson et al., 2002; Oehrtman, Carlson, & Thompson, 2008). Covariational reasoning is defined to be "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson et al., 2002, p. 354). In the framework, five levels of "mental actions" are described, which correspond to increasingly sophisticated cognitive activities. These begin with simply recognizing that the two variables depend on each other, moving to coordinating the "direction" and "amount" of change, followed by a coordination of how the rate of change changes. The five mental actions of the covariational framework are summarized in Table 1.

Table 1: Mental Actions of the Covariation Framework
(Carlson et al., 2002, p. 357; Oehrtman et al., 2008, p. 163)

Mental action	Description of mental actions
1	Coordinating the *dependence* of one variable on another variable
2	Coordinating the *direction* of change of one variable with changes in the other
3	Coordinating the *amount* of change of one variable with changes in the other
4	Coordinating the *average rate-of-change* of the function with uniform increments of change in the input variable
5	Coordinating the *instantaneous rate of change* of the function with continuous changes in the independent variable for the entire domain of the function

Conceptual Objects Produced by the Mental Actions

While Carlson and colleagues obviously bring up the concepts of concavity and inflection points in their work, the framework is focused on an understanding of *function*. Yet, since mental actions 4 and 5 deal quite explicitly with concavity, I propose to define the "concavity concept" in terms of the conceptual *objects* (in the spirit of Sfard, 1991; Sfard & Linchevski, 1994) potentially produced by covariational reasoning at mental actions 4 and 5. In other words, simply put, I define concavity conceptually as "the covariation between the rate of change and the independent variable." Inherent in this definition is the claim that concavity cannot be truly understood in the absence of a mastery of mental actions 4 and 5. I acknowledge that this proposed definition extends the covariation framework beyond its original intent, in that the framework was originally meant to capture cognitive *activities* involved in coordinating change (Carlson et al., 2002, p. 354). By contrast, in this paper I am proposing a definition for a conceptual *object*. Yet, I believe the cognitive activities performed in the mental actions 4 and 5 can produce a conceptual object, which is the very covariation that exists between the rate of change itself and the independent variable. Thus, the concept of covariation dealing explicitly with the rate of change is, then, exactly the concept of concavity I propose.

To define the "inflection point concept," I note that mental actions 4 and 5 can be seen as essentially recycling through mental actions 1–3, with the "dependent variable" quantity being replaced by the "rate of change" quantity. In other words, mental actions 4 and 5 recursively trace through the first three mental actions again, but with the more sophisticated layer of one of the variables being the rate of change (see Table 2). Note that mental action 2 coordinates the *direction* of change, which I interpret to mean *increasing* or *decreasing*. Within mental action 2, one can track places where a *switch* in the *increasing/decreasing* of the rate of change takes place. Thus, within this mental action, if a switch in increase/decrease is identified, a conceptual object can be produced,

which is "change in the direction of covariation," which has much in common with the concepts of maximum or minimum. I then define this "change in covariation" as the "inflection point concept" precisely when one of the covarying quantities is the rate of change itself. Note that in this definition, an inflection point is *not* a point on a graph, but rather a "switch in increase/decrease" of the rate of change as the independent variable changes. Of course, it can be represented graphically as a point, but it is only that—a visual representation of a deeper idea centered on covariation.

Table 2: Mental Actions 4 and 5 Recycling through Mental Actions 1, 2, and 3

Mental Actions: 4. Coordinating *average rate-of-change* 5. Coordinating *instantaneous rate of change*	1. Coordinating *dependence* of rate of change on independent variable 2. Coordinating *direction* (*i.e. increase/decrease*) of change in the rate of change with respect to the independent variable 3. Coordinating *amount* of change in the rate of change with changes in independent variable
Conceptual objects produced	→ Concavity concept = *covariation* between rate of change and variable → Inflection point concept = change in *direction* of covariation

I point out that I am careful to state definitions for the concavity *concept* and the inflection point *concept*, as I am not attempting to define them mathematically. A mathematical definition is a way to take a conceptual idea and create a formal statement for it (see Tall & Vinner, 1981; Zandieh & Rasmussen, 2010). By contrast, this framework is not intended to create a standardized *formal* definition, but rather to put forward an argument for what we, as a field, should consider to be the core conceptual ideas for these two concepts, which may then be represented in various possible ways in textbooks or other mathematical writings.

Comparing the Framework to the Common Graphical Approach

In this section, I address two issues: First, if the framework is simply a repeat of the ways that concavity and inflection points are already typically approached and used, then it is of little use. This framework should offer something beyond what one could already find in other presentations on these two concepts. Second, despite the need for the conceptual framework to offer something new, it must also resonate with the way in which that concept *is* commonly used, defined, and represented in the mathematical community, including in educational research literature and textbooks. Otherwise, it may be that the framework does not even capture what the community considers to be "concavity" and "inflection point."

To address the first of these issues, I describe how concavity and inflection points are often approached and represented in the research literature and in textbooks. Many studies that deal with concavity and inflection points discuss them through the activity of graphing, or through graphical images (e.g., Asiala, Cottrill, Dubinksy, & Schwingendorf, 1997; Baker et al., 2000; Gómez & Carulla, 2001; Tsamir & Ovodenko, 2013). Similarly, many textbooks I have examined primarily define and discuss these concepts through the graphical register (e.g., Finney, Demana, Waits, & Kennedy, 2012; Hughes-Hallett et al., 2012; Smith & Minton, 2008; Stewart, 2014; Thomas et al., 2009; Zill & Wright, 2011). Many textbooks do provide a definition of concavity based on the first derivative increasing or decreasing, yet these books still seem to ascribe concavity as only a feature of a *graph*. For example, Finney, Demana, Waits, and Kennedy (2012) begin their definition with, "The *graph* of a differentiable function $y = f(x)$ is…" (p. 197, emphasis added), and Smith and Minton (2008) start their definition with, "The *graph* of f is…" (p. 238, emphasis added). The book

by Briggs, Cochran, and Gillett (2015) is the only book I have examined that indicates that the *function* itself might be considered concave up or down. Consequently, my conceptual framework offers a new perspective that takes a stance against convention, in that I do not define the concavity and inflection point concepts through graphical terms at all, but as concepts rooted in the ideas of covariation.

Moving to the second issue, given that my stance goes contrary to convention, is my conceptual framework even consistent with what the community thinks of as "concavity" and "inflection points?" In many of the studies and textbooks listed in the preceding paragraph, the discussion regarding the relationship between an increase in the derivative, f', and the shape of the graph focuses on how the slopes of the tangent lines become more steep or less steep. However, one can ask, steeper according to what? Implicit in these kinds of statements is the idea of getting steeper as one moves *left to right*. If one moves *right to left*, the steepness trend would reverse. Thus, even these strictly graphical approaches have covariation inherently embedded in them: the slopes (quantity one) change with respect to the independent variable (quantity two).

Similarly, "inflection point" is often described as a change in one kind of graphical shape to another kind of graphical shape. Inherent in this description is the need to attend to whether the slopes' steepness increases or decreases (i.e. "direction"). These increases or decreases correspond to the mental action 2 with slope or rate of change as one of the quantities. A *switch* in that increase/decrease of slopes or rates of change aligns with the definition of the "inflection point concept" in this framework. Thus, this framework does speak to the same "concepts" discussed in the typical graphical approaches.

Implications for the Teaching and Learning of Concavity and Inflection Points

So far in this paper, I have (a) outlined potential *conceptual* (not mathematical) definitions of concavity and inflection points, (b) shown that this framework differs from conventional approaches, and (c) shown that despite the difference, it still speaks to the same concepts of concavity and inflection points used by the community. However, if this framework yielded no worthwhile implications for how concavity and inflection points are to be taught and learned, then it would still be little more than an academic exercise of small value. I believe that this conceptual framework for concavity and inflection points contains significant implications for the teaching and learning of these concepts. In this section, I first outline what it might look like for a student to "understand" these concepts according to my framework. I then re-examine examples of student difficulties described in the literature regarding the concavity concept and the inflection point concept to show how this framework can further illuminate the nature of some of these difficulties and provide ways to address them.

I was recently involved in a study in which we examined how students made sense of concavity and inflection points in real-world contexts (Gundlach & Jones, 2015) . While the students in the study exhibited a range of interpretations of concavity and inflection points, one student in particular showed an ability to think about, reason about, and make sense of concavity and inflection points in a range of contexts, from intuitive contexts (temperature) to more abstract contexts (the size of the universe). He also demonstrated a facility with the type of graphing problem used in Baker et al. (2000). I believe his proficiency with these concepts stemmed from the fact that he had mastery of the covariation mental actions and seemed to have codified these into some kind of conceptual objects—or at least he had begun to. For example, in one prompt, he was asked to describe how concavity related to a person's height over their lifetime. He first discussed the early period of a person's life, which he stated reflected "concave up."

> *Student*: The way I thought about it is that, over time, it seemed like the height increases. There's a bigger increase of height as you get older, up to a certain point… So from here to here

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[indicates two points in time] it's less increase of height, whereas from here to there, about the same [time] length, it's a greater increase of height, and this would be concave up.

Here the student's comments reflected mental action 4, wherein average rates of change over equal-sized intervals were considered. What is important to note is that he realized that one of the covarying "quantities" in this case is the "increase of height," or the growth rate. Thus, the student has demonstrated a *dependence* of the growth rate on the time variable. As he moved to discuss a concave down period, he exhibited mental action 5 by dropping finite intervals of time from his explanations, and switched instead to continuously changing rates.

> *Student*: They're starting to get close to their full height, whatever that height happens to be. It'll still be increasing, it'll be increasing at a decreasing rate, which means that—that would have to be an inflection point [in order] for that rate, at which their height is changing, to change. And at that point it would begin to be concave down.
> *Interviewer*: …If the curve [i.e. the graph of the height function] doesn't go back down, is it not concave down? What would you say about that?
> *Student*: I'd still say it's concave down… that rate at which they're growing slows down, it still is concave down.

The student used the idea of a continuously changing growth rate, and stated that an increasing growth rate is defined as concave up, while a decreasing growth rate is defined as concave down. Note that he also described the inflection point as being dependent on a "change" in the growth rate from a growth rate that is increasing to a growth rate that is decreasing. Thus, he seemed to have encapsulated the "change in direction" from mental action 2, with one "quantity" being the growth rate. These sophisticated mental actions seemed to have produced understanding of covariation and change that were associated with the terms "concavity" and "inflection point." As such, this student seemed to demonstrate an *understanding* of these concepts according to this framework.

In this illustrative example of understanding, I wish to highlight that this student's discussion of and meaning for concavity and inflection points were *not* dependent on the shape of a graph, even though a graph was drawn during the discussion. Rather, it was his coordination of a changing *rate of change* with the dependent variable that drove his thinking and seemingly produced an understanding of the conceptual objects defined in this paper.

I now switch to focus on examples from the literature regarding student difficulties. First, I discuss a student difficulty described in Baker et al. (2000), in which at least two of the students in the study (Carol on page 567 and Jack on page 568) held the idea that concave up meant the graph was increasing and concave down meant the graph was decreasing. This idea was a source of difficulty for the students when it conflicted with information about the first derivative that seemed to contradict it. The question then becomes, if these two students knew that the *first* derivative dealt with increase/decrease, why did they also apply the idea of increase/decrease to the *second* derivative? Baker et al. proposed the idea that the students were not at an "inter-property" level, meaning they could not coordinate information from two different properties simultaneously. However, it seemed like the students attempted to do so, which is what created the conflict in the first place. To provide an alternate reason, note that this difficulty is consistent with the framework's interpretation of mental actions 4 and 5 being a recycling of mental actions 1–3. That is, making sense of concavity goes through the same cognitive activities as making sense of slope or rate of change. The problem is that the activities required in making sense of concavity take on an additional layer of sophistication since the recycled mental actions 1–3 now deal with one of the quantities being the "rate of change." Since it is likely that current graph-based approaches provide little instructional support to developing these sophisticated, layered mental actions, then it is possible that the recycled layer of mental actions 1–3 collapsed to the original layer of mental actions 1–3. It may

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be that "concavity" was reduced in some sense to the same thing as "rate of change." Thus, the two distinct conceptual objects (concavity versus rate of change) became blurred into a single conceptual object, rate of change. To separate out these two objects, instruction would have to highlight the fact that concavity essentially retraces through the same concept as rate of change, but now *using* rate of change as one of the covarying quantities. I am in no way claiming that this would be an easy feat, but a necessary one if students are to fully understand these concepts.

In a second example, Tsamir and Ovodenko (2013) describe several students identifying inflection points as places where "the graph keeps increasing, but the slope changes dramatically," like a mountain trail changing from a gentle upward slope to a steep, difficult climb (p. 421). The reasons given by Tsamir and Ovodenko for this difficulty was that students look too holistically at graphs, or base their reasoning too much in real-world contexts (p. 421). However, this explanation does not provide much by way of how to address this problem, other than to be "less holistic" or to not use "real-world contexts." By contrast, I believe my framework provides a much deeper reason for this difficulty. In general, an "inflection point" is typically presented as a *change* in something (e.g., Hughes-Hallett et al., 2012; Stewart, 2014; Thomas et al., 2009), and the issue is in *what* changes exactly. There is a significant cognitive demand in recycling through mental actions 1–3 inside of mental actions 4 and 5, replacing one "variable" with the "rate of change" quantity. Specifically, an inflection point arises by noting a change in the "increase/decrease" inside mental action 2, if one quantity is understood to represent the rate of change. The students in Tsamir and Ovodenko's study may have been looking for something that met the usual criterion of a "change in something" when seeking to identify inflection points. Indeed, the idea of the graph going from a slower increase to a sudden, dramatic increase seemed to give the students some kind of "change" happening in the rate of change. Yet, despite the students' possible recognition of the need for a change in the rate of change, they did not have a fully-formed object associated with a switch in the *direction* from mental action 2. Only a switch of *increase-to-decrease* (or vice versa) in the rate of change is appropriate for identifying an inflection point. Thus, this framework would indicate that the students in that study were, in fact, making decisions intelligently, but without a fully formed conceptual object of *what* exactly should change about the rate of change. In other words, instead of looking at students' inability to attend to relevant features of a graph, or of particular shapes, it may be that we should help students see that a change from increasing rates of change to decreasing rates of change (or vice versa) is what constitutes an inflection point.

Conclusion

In this paper, I have shown that despite the heavy graphical emphasis on concavity and inflection points, these concepts may have more important core meanings than as the "shape of a graph." What the core ideas are has important ramifications for the teaching and learning of these concepts, and how we view, understand, and address student difficulties. I have proposed a framework that *conceptually* (not mathematically) defines concavity as the concept of *covariation* in which one quantity is the rate of change, and inflection points as a *change in direction* of this covariation. Thus, this framework provides one possible answer to the question, "What does it mean for a student to understand concavity and inflection points?"

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CHILDREN'S LINGUISTIC AND NUMERIC UNIT COORDINATION

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Detailing is a linguistic tool for mathematical argumentation in which given mathematical information is operationalized through one's warrants to support a claim. Recent literature suggests that students' detailing is related to their early algebraization. This study examined 168 elementary students' use of detailing in two mathematical argumentative tasks in relation to their enacted scheme of multiplicative unit coordination. A convergent mixed methods approach was used to analyze students' argumentative writing qualitatively, and merge these findings with quantitative indicators for students' multiplicative reasoning. Results suggest a statistically significant relationship between students' detailing and their multiplicative unit coordination.

Keywords: Elementary School Education, Number Concepts and Operations, Reasoning and Proof

The development of mathematical argument in the elementary grades is relatively undertheorized. Although various descriptions and models of how children come to engage in more sophisticated mathematical argumentation and/or proof have been proposed (Blanton & Kaput, 2011; Krummheuer, 2007; Morris, 2009; Tall et al., 2011), the vast majority of such descriptions focus only on generalized arguments. This bias towards generalization is understandable, given the necessity of generalization in the desired development of proof processes. However, it has neglected aspects of children's mathematical argumentation that may precede inference and generalization. Following Peirce, Kosko (2016) defined the purpose of mathematical argument as establishing an acceptable mathematical claim of truth. According to Peirce (1903/1998), argument necessarily involves inference for generalization. Yet, the first step of argumentation, colligation, precedes inference. *Colligation* occurs when an individual conveys the collective warrants for their argument as a singular copulative proposition. The copulative proposition can then be used to support a generalized claim. For example, in proving *the sum of two consecutive odd integers is divisible by two*, an individual may use the equation $n + (n + 2) = 2n + 2 = 2(n + 1)$ as part of their proof. The equation includes several propositions that, collectively, support the claim to be proven. Colligation in mathematical argument is facilitated by various linguistic tools (Kosko & Singh, 2016a). However, this particular study focuses specifically on the linguistic tool of detailing, in which the given information from a task is operationalized via a reference chain as a means of providing cohesion for a copulative proposition (Kosko, 2016; Kosko & Singh, 2016a).

Recent study of children's detailing in mathematical argument has identified a potential relationship with abstraction of number (Kosko, 2016; Kosko & Singh, 2016a; Kosko & Singh, 2016b). Specifically, children's abstraction in unit coordination of number has been found to relate to their abstraction of linguistic information units in argumentation. In the present study, we seek to investigate this phenomenon further by studying children's engagement in detailing in relation to their demonstrated unit coordination in multiplicative contexts. Therefore, the purpose of this study is to examine children's detailing enacted in MAW across several tasks in relation to their enacted multiplicative coordination of units.

Theoretical Framework

Detailing as Colligation in Children's Mathematical Argumentative Writing
The present study takes a Peircian semiotic view of mathematical argument as a theoretical lens, while applying Systemic Functional Linguistics (SFL) as an analytic lens. Peirce (1903/1998)

defined *argument* as a sign that synthesizes various propositions to establish a generalizable claim. Further, arguments synthesize other abstracted signs including, but not limited to, copulative signs. *Copulative signs* are the synthesized set of propositions used to foreground the inference towards a generalizable claim. As such, copulative signs are part of arguments, but do not include logical inference or a generalizable claim. The formation of a copulative sign involves the action of colligation, or the collective expression of propositions into a singular copulative proposition. Colligation is facilitated by various linguistic tools. Kosko and colleagues (Kosko, 2016; Kosko & Singh, 2016a; Kosko & Singh, 2016b; Kosko & Zimmerman, 2015) have identified at least two such linguistic tools: nominalization and detailing. Nominalization occurs when two or more mathematical linguistic objects are metaphorically conveyed as one (Halliday & Matthiessen, 2004). For example, $4n+2$ is considered as a singular expression, but includes the discrete nominal objects $4n$ and 2 as being summed. Further, $4n$ could be considered as the product of two discrete nominal objects (4 and n). Thus, nominalization facilitates colligation as a tool for abstraction of multiple nominal objects. Although nominalization is useful, and often essential, for colligation, the present study limits its focus to the linguistic tool of detailing.

Kosko (2016) describes detailing as the operationalization of given information through the construction of a reference chain through the warranted propositions supporting a claim. According to Halliday and Matthiessen's (2004) approach to SFL, reference serves the role of establishing cohesion for a text. In this manner, reference chains can be used to establish new information and re-establish given information for conveying and establishing *information units*. Information units are grammatical units that establish new information based on given information. Detailing goes beyond more common applications of reference chains in that given information is continuously operationalized and built up over multiple warranted propositions (i.e., information units). The following third grade student's mathematical argumentative writing provides an example of detailing The child was asked to respond to the Cuisenaire-rods based task *If a red rod is 5, a yellow rod can't be 9 because…* Elements of the detailed reference chain are in bold, with grammatical clauses separated by "//":

> The yellow rod can't be 9 // because **1 red one is 5** // and **that** does not take it all up // so you put **another 5** // and **it's** still not big enough // but **it [is] basically 10** // and **it's** smaller than the yellow // so the yellow can't be 9.

The child's writing begins and ends with their non-generalized claim that a yellow can't be 9 long. The clauses between serve the role of the copulative proposition, which is cohesively bound by a detailed reference chain. The given, **1 red one is 5**, is operationalized in the information unit **it [is] basically 10**. The operationalized referent, **10**, is exophoric in that it presents new information not in the information unit. Although technically new information, **10** is established in the information unit from the given information in a manner that represents a transformed, or operationalized, version of the given referent. Thus, **10** is part of a detailed reference chain that is used to establish the claim of yellow not being 9 long. Halliday and Matthiessen's (2004) definition identifies both referents as different information units, but detailing allows for the formation of a unique type of reference chain that establishes a copulative proposition. Specifically, **it [is] basically 10** serves as a link in the detailed reference chain connected to the given **1 red one is 5** as well as the referent in the proposition "**it's** smaller than the yellow." Although each proposition provides a different exchange of given and new information, each new information referent is a transformed version of the given information allowing for cohesion. Thus, detailing allows for all three propositions to collectively serve as a copulative proposition to support the child's claim.

As can be gathered from the preceding description, detailing allows for the linguistic coordination and cohesion of different information units into a singular copulative proposition. The primary aim of the present study is to investigate whether children's choice of detailing (i.e.,

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linguistic unit coordination) coincides with their demonstrated ability to coordinate mathematical units (i.e., number). Thus, we briefly discuss multiplicative unit coordination to foreground a description of a conjectured relationship between these two concepts.
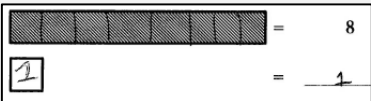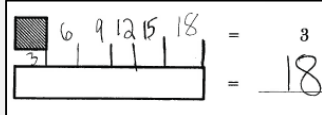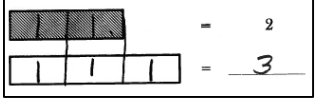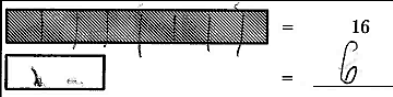
Children's Multiplicative Unit Coordination

This study considers multiplicative reasoning from the perspective of scheme theory, with specific focus on Hackenberg's (2010) multiplicative concepts. According to Steffe (1994) multiplicative schemes require the coordination of at least two levels of units, and to develop multiplicative concepts students require the anticipatory use of such schemes. Specifically, students may construct schemes, or collections of actions, *in activity* (as they are engaged in a task), or they may use them in anticipation of the actions they expect to engage with the task (i.e., *anticipatory schemes*). Hackenberg (2010) suggests three stages of students' multiplicative concepts with the subsequent concept requiring more the use of anticipatory schemes where in activity schemes may have been used previously. The transition from less to more sophisticated multiplicative concepts is gradual, and intermediate levels are developed successively through internalizing in-activity schemes such that they begin to be used as anticipatory schemes (Norton et al., 2015).

Anticipatory and in-activity schemes described above are based on qualitative assessments and assume that a student transitions gradually from enacting schemes in activity, to using schemes in an anticipatory manner. However, examination of whether students enact multiplicative schemes either in activity or in an anticipatory manner incorporates a depth of analysis that requires more time than can be devoted when considering larger sample sizes. Such is the case in the present study ($n = 168$). Therefore, we adopt the approach of Kosko and Singh (in review), in which evidence of schemes *enacted* by students is identified from written work. Such evidence is, by its nature, an artifact of the activity generally observed by qualitative teaching experiments. Therefore, enacted schemes, as defined in the present study, do not distinguish between in-activity and anticipatory schemes. However, enacted schemes are further assessed with enacted reversible schemes to attempt a closer representation of anticipatory schemes than in-activity. Although enacted schemes do not allow for the critical distinction between in-activity and anticipatory schemes, their use does allow for larger scale data collection (Kosko & Singh, in review).

Following Hackenberg's (2010) three multiplicative concepts, but with the caveat of using enacted schemes in place of in-activity and anticipatory, the present study considers students' multiplicative reasoning via four tiers. The tiers align with Hackenberg's (2010) multiplicative concepts. Multiplicative Tier 0 (MT-0) involves pre-multiplicative schemes (i.e., counting by 1s with records of counting). The other three tiers (MT-1, MT-2, and MT-3) are similar to Hackenberg's three stages respectively with the primary difference that the basis of classification in our case is the enacted rather than the anticipatory scheme. In other words a student belonging to MT-1 can be considered to fall in Hackenberg's first multiplicative concept. Thus the first multiplicative tier (MT-1) involves the enacted coordination of two levels of units, and MT-2 involves the enacted coordination of three levels of units. MT-3 involves the enacted coordination of three levels of units, as does MT-2, but shows clear evidence of disembedding with non-1 units, and is considered more likely to correspond to Hackenberg's (2010) definition of the third multiplicative concept.

Table 1: Students' Unit Coordination and Enacted Schemes

Tiers	Students' ways of Unit coordination	Enacted schemes	Examples
MT-0	No unitization or unit coordination	Iterating 1 units *n* times	
MT-1	Students' may coordinate two levels of units	Partitioning into *n* parts to find 1 units.	
		Iterating non-1 units *n* times	
MT-2	Students' may coordinate three levels of units	Partitioning into *n* parts to find non-1 units.	
		Disembedding a unit to iterate *n* times.	
MT-3	Students' can coordinate with three levels of units even with rational numbers.	Decompose partitions into non-1 units (may include coordination of partitions in both length models).	

Coordination of Mathematical Quantitative and Mathematic-Linguistic Information Units

Recent examinations of elementary children's use of detailing have found relationships between whether students engage in detailing and their success on tasks requiring multiplicative reasoning (Kosko, 2016; Kosko & Singh, 2016a), as well as the emergence of nominalization in mathematical argument and the development of relational conceptions of equivalence (Kosko & Singh, 2016b). Such findings follow prior research identifying relationships between students' generalizations and their success with early algebra tasks (e.g., Blanton & Kaput, 2011; Morris, 2009), but examines the interplay between the development of mathematical argumentation and early algebra at a finer gran size of analysis. The present study posits an even more specified view of this interplay. Specifically, children's use of detailing colligates information units to create copulative signs (Kosko, 2016; Kosko & Singh, 2016a; Kosko & Zimmerman, 2015). Children's creation of copulative signs points to an ability to abstract multiple meanings as one. In a very similar fashion, children's coordination of quantitative units in ways mathematics education researchers would consider as multiplicative also points to an ability to abstract multiple meanings as one. Although these two types of coordination point to the same ability to abstract meaning, we conjecture that they are not necessarily the same, but similar enough to co-occur more often than not. However, such a potential interplay is important, given the potential of one type of coordination to influence the development of the other (Kosko, 2016).

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The interplay conjectured here provides a much needed mechanism for explaining observed co-development of early algebra and mathematical argument, with a specific focus on multiplicative reasoning and detailing. Because detailing involves the coordination of multiple information units into what Pierce (1903/1998) refers to as a copulative proposition, and multiplicative reasoning at its initial tiers involves the coordination of multiple non-1 quantities into a singular number, we conjecture that both forms of coordination involve a similar form of abstraction. Thus, they should be more likely to co-occur than not. Our rationale for this conjecture lay in the nature of both kinds of coordination. Specifically, information units are composed of multiple nominal elements such that each proposition must be considered, at least tacitly, as its own entity. Similarly, multiplicative reasoning requires that a non-1 unit be considered at least as a grouping of 1s that can be operated upon. Coordination of propositions (in the form of information units) via detailing requires a level of linguistic coordination beyond simply providing a sequence of propositions. Rather, the propositions must be cohesively joined, and this is an aspect of linguistic coordination that seems to become more prevalent beginning in the second grade (Kosko & Zimmerman, 2015). Likewise, multiplicative reasoning involves the coordination of number beyond counting by 1s; a form of unit coordination that appears to begin emerging more prevalently in second and third grade (Kosko & Singh, in review). Therefore, the level of coordination involved in detailing and multiplication is similar in that both move beyond the basic operations of their domains, but involve at least some movement to coordination units of units. Should our conjecture hold true, we would anticipate seeing a relationship between children's multiplicative reasoning and their detailing across different types of tasks. Thus, we ask the following research question: How does children's multiplicative unit coordination relate to the presence of detailing in their mathematical argumentative writing?

Methods

Sample and Measures
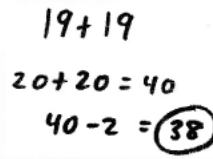
Data were collected in May 2015 from 168 second and third grade students in two suburban school districts in a Midwestern U.S. state. Second grade students were enrolled in four different teachers' classrooms ($n = 76$) and third grade students were enrolled in three different teachers' classrooms ($n = 92$). Participants completed two packets across two weeks. In week one, the packet included a multiplicative reasoning assessment. In week two, the packet included six mathematical argumentative writing tasks, although we limit our discussion in the present paper to only two tasks for sake of space and simplicity (Tasks 3 and 4).

The multiplicative reasoning assessment was developed by Kosko and Singh (in review) and includes 12 items designed to assess students' use of different enacted unit coordination schemes. We used a rubric involving 11 codes to assess students' demonstrated work (Kosko & Singh, in review).The students were assigned to a particular tier if they received correct scores to over half of the items in that tier. As an example we assigned students to MT-2 who got 3 out of 4 items (aligned with MT-2) correct, provided they also meet the criteria for lower tiers. Our analysis placed 67.5% students in MT-0, 18.4 % in MT-1, and 14.1 % in MT-2. None of the students were placed in MT-3. A Cronbach's alpha coefficient of .86 was calculated for the enacted schemes in all 12 items across different tiers, suggesting sufficient reliability of the assessment (for further information on the assessment, see Kosko & Singh, in review).

The mathematical argumentative writing tasks included six tasks with three incorporating length model representations of arithmetic and three incorporating the use of expressions and equations for arithmetic. The two tasks discussed in the present paper are presented in Table 2. Task 3 required students to use Cuisenaire rods (a color-coded length model) and assume that a red rod is 5 long, although it cannot be physically partitioned into 5 sub-units. Kosko (2016) found that use of this task encouraged more detailing than a similar task in which allowed for such physical partitioning.

Similarly, Task 4 requires students to consider three separate expressions/equations in relation to one another. To do so, it was assumed that detailing would be necessarily enacted to communicate the strategy cohesively (i.e., that each expression/equation serves as an information unit that can be colligated via detailing).

Table 2: Mathematical Argumentative Writing Tasks

Task 3*	Task 4
If a red rod is 5 long, a yellow rod can't be 9 because…	Omar was asked to find the answer for **19 + 19**. This is what he wrote on his paper: $19 + 19$ $20 + 20 = 40$ $40 - 2 = \boxed{38}$ Explain how Omar solved the problem and why it does (or doesn't) work.

*Students completed task 3, and other length model tasks, with Cuisenaire rods.

Analysis

The present study incorporates a data-transformation variant of convergent mixed methods design (Creswell & Plano Clark, 2011). Specifically, an SFL approach to examining functional grammar was used to qualitatively analyze second and third grade students' use of reference and detailed reference chains in their mathematical argumentative writing. Findings were organized into classifications that were quantified into variables for Chi-Square analysis. Quantitative data was collected from the multiplicative reasoning assessment and merged with the quantified SFL analysis for the mixing of data. Un-quantified qualitative findings were then used to help interpret quantitative findings from the Chi-Square analysis.

Qualitative analysis of functional grammar. We used SFL to examine the presence and patterns of reference use in children's mathematical argumentative writing. Reference is part of the textual metafunction of grammar, with its primary role to promote cohesion and coherence to an audience (Halliday & Matthiessen, 2004). A child communicating mathematically may use isolated referents, or may use reference chains that link referents in two or more propositions. In the present study, Kosko's (2016) description of detailing is used to distinguish between general reference chains and detailed reference chains. Detailed reference chains were coded for the operationalization of initial given referents in a manner that colligated two or more information units. Information units are clause-level grammatical units that connect given and new referents to convey information (Halliday & Matthiessen, 2004). Detailing, as defined in this paper, creates reference chains with endophoric and exophoric references that allow for two or more information units to be considered holistically as a colligated proposition.

The structure of mathematical writing tasks typically provides given information and an indication for a claim to be established; similar to Herbst and Chazan's (2011) described norms for providing a proof problem in high school Geometry. This structure allows for the construction of hypothetical reference chains that are more likely to be written to form a copulative proposition. For Task 3 (see Table 2), a complete and detailed reference chain should include a reference to the red rod being 5 long, operationalization of this given stating that two reds are 10 long, and an extension of this latter proposition to convey that 10 cannot be less than 9. Presence of variations of these three propositions was coded as detailing, but no other variations of detailing were observed. Both authors examined data for the initial information unit, incomplete detailing (i.e., two but not all three information units present), and complete detailing. Data were quantified, and coding was found to have strong interrater reliability (K=.67). Coding was then dichotomized to compare prevalence of detailing and not detailing in the quantitative analysis (*M*=.22, *SD*=.42).

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Similar to coding of Task 3, Task 4 had one identified and coded detailed reference chain including two information units: a proposition conveying that Omar added 1 to each 19, and a proposition conveying that 2 be subtracted from the total. Incomplete detailing was observed when referents were not operationalized in a manner to colligate the two information units. As with the prior discussed task, data were quantified and coding was found to have strong interrater reliability (K=.90). Data were then dichotomized to compare prevalence of detailing and not detailing (M=.38, SD=.49).

Quantitative analysis of multiplicative reasoning and detailing. Chi-Square statistics were calculated to investigate whether the presence of detailing in students' MAW coincided with their multiplicative tiers. The relationship between detailing and multiplicative tier was found to be independent from chance both for Task 3 ($\chi^2(df=2) = 8.121$, $p = 0.017$) and Task 4 ($\chi^2(df=2) = 18.971$, $p = 0.000$). In order to better understand these findings, a post hoc analysis was used to examine the differences between specific observed and expected frequencies within individual cells of the Chi-Square contingency table. The adjusted standardized residuals for each cell in the 2x3 table (i.e., dichotomous detailing code X three observed multiplicative tiers) were calculated, and represent a statistic similar to a z-score for the difference between observed and expected counts in each cell (Haberman, 1973). Critical values for adjusted standardized residuals were ±2.0. Statistically significant and positive residuals were observed for the presence of detailing for students at MT-2 for Task 3 (2.5) and Task 4 (3.1), as well as for students at MT-1 for Task 4 (2.4). Students at MT-0 were observed to engage in detailing less than expected by chance for both Task 3 (-2.6) and Task 4 (-4.3). Therefore, the multiplicative tier a student was placed was found to coincide with the presence or absence of detailing, with the observed frequencies are outside those expected by chance. However, differences in magnitude of adjusted standardized residuals suggest that the relationship may vary by task.

Discussion and Conclusion

Findings from this paper suggest that elementary children's enactment of detailing in their mathematical argumentative writing on two tasks is not independent from their demonstrated ability to coordinate units multiplicatively. However, it is important to note that both analyzed tasks solicited detailing that directly linked endophoric with exophoric references between propositions. It is likely that some tasks may solicit less (or more) sophisticated reference chains. For example, some mathematical writing tasks may solicit reference chains that link only endophoric references together, which is less linguistically complex than detailing solicited from the tasks in this paper. It is also feasible that a task may solicit the linking of exophoric references between two information units, providing a more linguistically complex example of detailing than presented in this study. Such variations of detailing and reference use in mathematical argument are in need of further study. Although the present study did not explore such variations, there are specific and significant implications of these findings.

Earlier in this paper, we conjectured that linguistic coordination of information units via detailing and unit coordination via demonstrated multiplicative reasoning were similar enough in their observable structure to co-occur among elementary children. The findings of the present study provide additional evidence for this conjecture, as does recent work in this area (Kosko, 2016; Kosko & Singh, 2016b). Although these two types of coordination point to the same ability to abstract meaning, we conjecture that they are not necessarily the same. From a theoretical perspective, this suggests we should not expect a students' demonstrated ability with one coordination type to automatically coincide with the other. Yet from a practical perspective, findings here suggest that the similarity of each coordination type may allow for improved instruction of each via incorporation of the other.

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LAS FUNCIONES SINUSOIDALES Y LOS FENÓMENOS ARMÓNICOS

THE SINUSOIDAL FUNCTIONS AND THE HARMONIC PHENOMENA

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En esta investigación se discute el relacionar la descripción de ciertos fenómenos armónicos (física), con modelos que los representan (matemáticas), para así promover el entendimiento del período, la frecuencia y la amplitud de las funciones trigonométricas, en particular de las funciones sinusoidalesl. Esta es una investigación en curso, donde se emplea tecnología digital, como el sensor de movimiento, para generar gráficas de las funciones sinusoidales. En ellas se identifican rasgos del movimiento que siguen los fenómenos armónicos. Se hace un planteamiento que permite transitar alternativamente del análisis del fenómeno físico al del modelo matemático. Se conjetura que la forma en que se abordan las funciones sinusoidales puede permitir que se asigne significado al modelo $f(x) = A\cos(wx + \varphi)$.

Palabras clave: Educación Post-Secundaria, Modelación, Tecnología

Planteamiento del Problema

Los autores que en matemática educativa han realizado investigaciones sobre las funciones trigonométricas (Blacket y Tall, 1991; Brown, 2006; Buendía y Cordero, 2005; Hertel y Cullen, 2011; Kendal y Stacey, 1996; Moore, 2009; Weber, 2005), entre otros], concuerdan que independientemente de la estrategia que se emplee, las etapas iniciales del aprendizaje de las funciones trigonométricas están llenas de dificultades, lo cual propicia que la mayoría de los alumnos de l4 a 17 años de edad (*high school*), no interpreten adecuadamente los conceptos involucrados con este tipo de funciones, en particular, con la función sinusoidal. A pesar de ello, las investigaciones que sobre la enseñanza de las funciones trigonométricas se han realizado, son escasas (Moore, 2010).

En asignaturas de física e ingeniería es indispensable un buen entendimiento de las funciones trigonométricas*;* sin embargo, existe la problemática de que el objetivo en los cursos de matemáticas es construir significados de los parámetros de las funciones sinusoidales, mientras que, de manera particular, el objetivo en los cursos de física es construir interpretaciones de los fenómenos (armónicos) y profundizar en el entendimiento de éstos con las funciones sinusoidales; pero, el mundo físico y el mundo de los objetos matemáticos están desvinculados, de modo que la frontera entre ellos es inconexa; por ejemplo, muy pocos alumnos manejan la característica fundamental de las funciones trigonométricas: la periodicidad (matemáticas), y son menos los que pueden aplicar este tipo de funciones como modelo de fenómenos periódicos (física). Al respecto, la NCTM (1989) dentro de sus estándares (grado 9-12), para el estudio de la trigonometría (estándar 9), define que uno de los objetivos a cubrir cuando se estudia esta asignatura es que los alumnos exploren fenómenos periódicos de la vida real usando las funciones seno y coseno. Sin embargo, para entender ciertos fenómenos periódicos que ocurren de manera cotidiana o para que los alumnos de bachillerato continúen con estudios superiores de ingeniería, es esencial cruzar la frontera o tender puentes entre la física y la matemática, por lo que en este trabajo se identifican, y analizan qué significados asocian los alumnos a la función sinusoidal, a través del estudio entre la conexión de modelos físicos de fenómenos armónicos, como el péndulo y el sistema masa resorte o el caminar (regular o uniformemente) de los alumnos, con las representaciones matemáticas de éstos, como las gráficas; estas últimas representaciones serán generadas por los fenómenos a analizar, un sensor de movimiento y un programa graficador.

En este reporte de investigación, se comentarán los resultados que hasta el momento se tienen sobre el movimiento corporal de los alumnos y el péndulo. La intención de esto es llegar a definir el objeto matemático que modela a dichos fenómenos, la función sinusoidal, planteándose las siguientes preguntas de investigación: (1) ¿Qué acciones permiten describir a los fenómenos armónicos y qué representaciones surgen de dicha descripción?, (2) ¿A partir de las representaciones de los fenómenos armónicos, cómo se redescribe a estos fenómenos y qué resignificación se genera del modelo matemático que los representa?, (3) ¿El uso de herramientas digitales, de qué manera apoya en la construcción de significados de la función sinusoidal?

Como objetivo*s* se tienen: (1) Analizar, describir y sistematizar los significados que los estudiantes asocian a la relación entre el modelo de la función sinusoidal y los fenómenos armónicos, (2) Describir y analizar los argumentos utilizados por los alumnos, para justificar, explicar y/o probar conjeturas cuando se emplean herramientas digitales en la determinación de la función sinusoidal como modelo de fenómenos armónicos.

Perspectiva Teórica

En la construcción de significados, implícita o explícitamente, los estudiantes ponen en juego concepciones personales o espontáneas; en el caso de la enseñanza de la matemática en el salón de clase, comúnmente se dejan de lado dichas concepciones y se ven a los objetos matemáticos como productos acabados, desvinculados de su origen e independientes del sujeto que los estudia. Difícilmente se toma en cuenta que cualquier acercamiento a un objeto matemático transforma la relación que se tenga con él y por consiguiente la representación del mismo (Moreno, 2014).

Para darle significado a las funciones sinusoidales, en esta investigación, se considera como punto de partida a las representaciones espontáneas o implícitas y a los fenómenos que las funciones sinusoidales modelan, ya que esto puede ayudar a una reinterpretación y resignificación de las mismas, de tal suerte que éstas puedan representarse de manera explícita, **(**Pozo, 2006). De hecho, Hitt (2006, 2013) enfatiza la importancia de las representaciones espontáneas (producto de representaciones funcionales) de los estudiantes, y los procesos de objetivación que permiten realizar una evolución de esas representaciones en la construcción del signo y su significado.

Como la investigación está en curso, hasta el momento, el marco conceptual considerado para el diseño y aplicación de actividades piloto se fundamenta en que al analizar el fenómeno físico, los alumnos logran un entendimiento inicial, el cual se da de manera intuitiva y lleva a hacer una representación implícita (espontánea, primer acercamiento al objeto matemático), para que a partir de esta representación se reinterprete el fenómeno físico; es decir, se redescriba al fenómeno, lo cual implica una resignificación de los símbolos de la representación matemática (modelo matemático) para llegar una representación institucional, general y/o abstracta es decir el objeto formal: la función sinusoidal, (Figura 1).
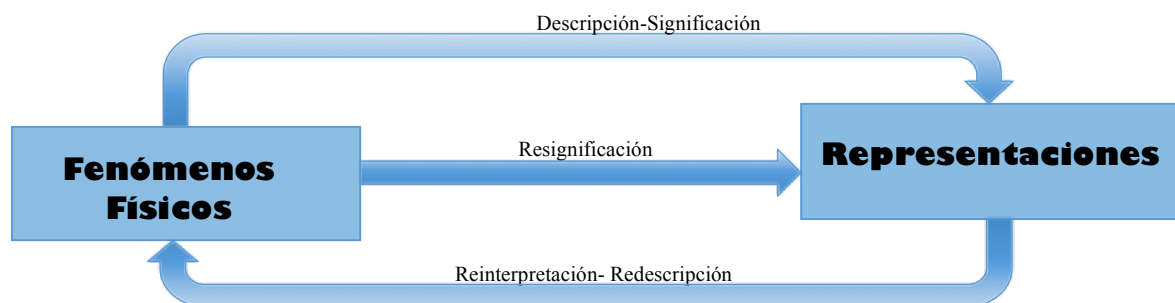
Figura 1: Resignificación de la función sinusoidal.

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Diseño de Investigación

La investigación es de tipo cualitativa. Las actividades piloto que aquí se describen se aplicaron a un grupo de 10 alumnos que se encontraban cursando el primer semestre de bachillerato (15-16 años), lo que supuso no habían tenido ningún acercamiento al tema de las funciones sinusoidales. La descripción del esquema anterior (Figura 1), como tal, determina el diseño de la investigación. El paso (i) requiere una trayectoria hipotética de aprendizaje (Simon, 1995; Simon y Tzur, 2004), un diseño de actividad para los sujetos que participan en la toma de datos. En el paso de (ii) se toman datos por escrito y en vídeo para saber qué reinterpretaciones plantean los alumnos. Finalmente, en el paso (iii), desde el fenómeno, se toman datos para saber qué resignificaciones realizan los sujetos.

Análisis de Datos

Hasta el momento, el experimento se ha realizado en tres etapas: 1) exploración, 2) interpretación-representación y 3) reinterpretación-resignificación. El desarrollo de la primera etapa se dio en 3 sesiones de 2 horas cada una; la segunda y tercera etapas se aplicaron en 2 sesiones de 2 horas. El objetivo de la primera fue identificar las concepciones previas que los alumnos tenían al describir un fenómeno armónico. En la segunda, el objetivo que se siguió fue identificar las representaciones implícitas que, con respecto al fenómeno físico y modelo matemático, lograron describir los participantes. La tercera etapa tuvo por objetivo, que a partir de la representación inicial que los alumnos tuvieron del fenómeno físico (segunda etapa), se redescribiera a éste, para así llegar a una descripción explícita (institucional) del modelo matemático $f(x) = A\cos(\omega t + \varphi),$ el cual describe al fenómeno analizado.

Primera Sesión. Descripción del Fenómeno (Concepciones Preliminares)

Se solicitó a los alumnos: Describir con el mayor detalle posible, cada uno de los fenómenos que observaron (péndulo, sistema masa resorte y movimiento circular de un objeto atado a un cordón, dicho objeto seguía una trayectoria circular vertical y circular horizontal).

Los participantes realizaron la descripción asociándola a aspectos físicos de los modelos presentados, materiales y colores. Dieron el nombre correcto de uno de los dispositivos (el péndulo). Los alumnos mencionaron ideas sobre conceptos como fuerza, movimiento, distancia, velocidad y aceleración; aunque éstas en su mayoría fueron incorrectas o ambiguas. En las respuestas se mostró la recurrencia a concepciones o ideas aprendidas previamente; o bien, a situaciones que, aparentemente, a simple vista se observaron (aun y cuando éstas eran imprecisas). Por ejemplo, el total de alumnos mencionó a la *fuerza* como la causa del movimiento de la lenteja, o que *la fuerza* en un momento determinado *se acababa*, y por consecuencia *el objeto ya no estaría en movimiento*, así como que una vez que un *cuerpo estaba en reposo ya no actuaba sobre él ninguna fuerza*, etc. Las concepciones mostradas por los alumnos en esta actividad se quedaron en la frontera de la física y eran, en gran medida, basadas en representaciones de naturaleza implícita, algunas de ellas similares a como en sus inicios la humanidad intentó conceptualizar el movimiento. Ningún alumno se apoyó en el uso de alguna representación que implicara un concepto formal, una fórmula o el uso, en particular, de algún signo aritmético, algebraico o diagramático. Las concepciones implícitas fueron el punto de partida para que, por parte de los alumnos, surgieran rasgos esenciales de las funciones sinusoidales. Se pretendió un cambio conceptual o representacional de éstas (segundo movimiento del proceso planteado en el marco teórico. Figura 1).

Segunda Sesión. Empleando Herramientas Digitales

Además de que los alumnos se ambientaran con el uso de las herramientas digitales, el objetivo de esta actividad fue que los estudiantes identificaran y asociaran la manera en que se representa el movimiento en una gráfica cartesiana; el análisis se centró en el comportamiento y representación del fenómeno físico, (1er movimiento. Significación. Figura 1); para ello, los alumnos caminaron frente

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a un sensor de movimiento, que conectado a una computadora generaba la gráfica con rasgos particulares del andar de los jóvenes. Por ejemplo:

Alejandro: Cuando los compañeros caminaban de un punto hasta acercarse al sensor, la gráfica marcaba que en la distancia empezaba de 5 hasta disminuir a 0, porqué cada vez se acercaba más y el sensor lo reconocía muy cerca, por eso iba descendiendo y cuando empezaba cerca del sensor y terminaba alejándose, la gráfica lo reconocía desde 0 hasta que se detenía en el segundo 3.1 a 3.5, eso pasaba porque se detenía, y cuando no se detenía la gráfica seguía subiendo.

En el caso de Alejandro, la representación inicial que generó incluyó la toma de datos numéricos, a partir de los cuales regresó a la redescripción del fenómeno (2do. Movimiento. Figura 1).

Posteriormente se les pidió a los alumnos que, con lápiz y papel, bosquejaran la gráfica de la acción de asistir a la escuela si ellos partían de su casa, debiendo regresar a ella después de tomar clases.
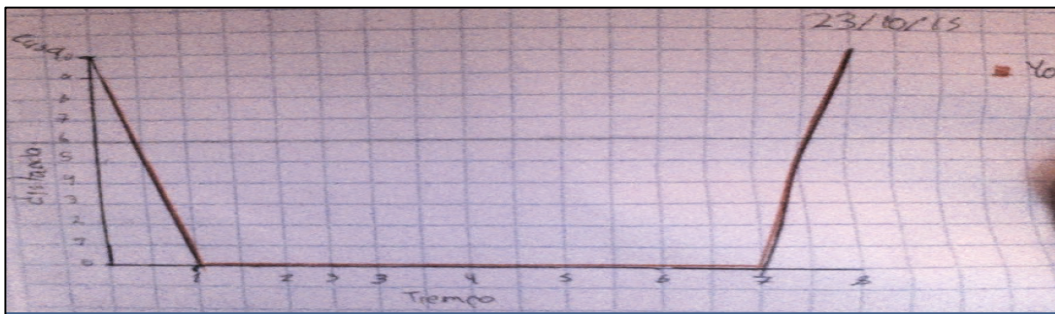
Figura 2. Gráfica generada por Alma sobre un recorrido.

Alma: Es lo mismo, solo que el sensor es la escuela.

Lo expuesto por Alma, representó un ejemplo de que se ha construido una representación explícita del sensor al considerarlo como referente, ya que logró trasladar la situación de las gráficas generadas con el sensor a un nuevo contexto, estableciendo un nuevo punto de referencia, la escuela.

En este actividad los alumnos empezaron a rebasar la frontera de una situación física cotidiana hacia la representación matemática. A continuación se solicitó a los alumnos que bosquejaran, también en lápiz y papel, la acción anteriormente planteada, pero ahora correspondiente a una semana y sin considerar el tiempo que permanecían en la escuela.

En este punto, los alumnos identificaron características del movimiento periódico (al cual ellos se refirieron como repetitivo); de manera básica determinaron el período, en el caso de Alma, ella estableció que tarda 2 horas en ir a la escuela y 2 horas en regresar a su casa. (1er movimiento. Figura 1).
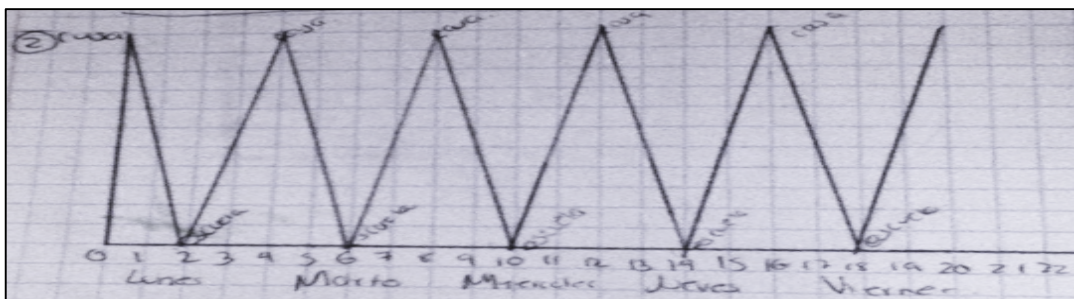
Figura 3. Gráfica generada por Alma sobre 5 recorridos.

La cuestión de la no diferenciabilidad, se trató en una sesión posterior. Los alumnos, después de un largo debate, concluyeron que *es imposible obtener* (caminando), *una gráfica con picos, ninguna persona se mueve tan rápido y logra representar el movimiento de esa manera*. (2do. Movimiento. Figura 1). Posteriormente, una gráfica sinusoidal fue presentada a los alumnos (sin que ellos supieran que se trataba de una gráfica de ese tipo), se les solicitó que caminando frente al sensor hicieran los movimientos necesarios para reproducirla. Los alumnos caminaron de diferente manera; por ejemplo, tratando de seguir la representación icónica de la gráfica, levantando las rodillas o como algunos de ellos dijeron *como si caminara un borracho* o asociaron el tipo de gráfica a la velocidad con la que se caminaba. Después de un número considerable de intentos, a Carlos se le ocurrió que el movimiento debía ser *haciendo el mismo paso, es decir ir y regresar*. El alumno parándose frente al sensor realizó un movimiento que consistió en dar dos pasos hacia el sensor y dos pasos alejándose del mismo.
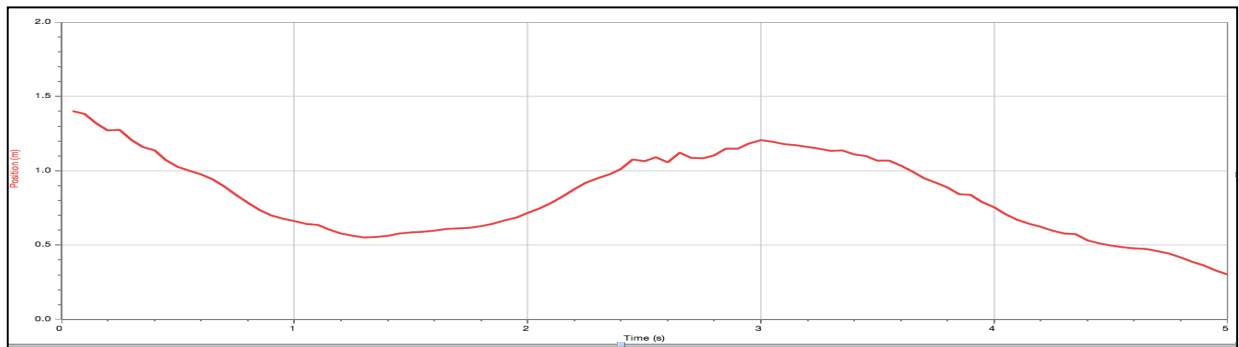
Figura 4. Gráfica generada por Carlos al caminar.

Profesora: ¿Por qué se te ocurrió que el movimiento debía ser así?
Carlos: Porqué si se acercaba uno (hacia el sensor), iba bajando (la representación gráfica); si se aleja uno (hacia el sensor), iba subiendo (la misma representación).

Después de realizar esta acción, Carlos asoció que este tipo de gráficas son generadas por un movimiento periódico, ya que realiza la siguiente pregunta:

Carlos: ¿Se puede poner la mano en el sensor?

Carlos se refirió, a si el movimiento que generaría la gráfica solicitada, era posible hacerlo, en lugar de la manera anterior, acercando la mano hacia el sensor y alejando la mano del mismo, ambos movimientos repetidas veces, Carlos verificó la veracidad de su conjetura frente al sensor.

En una actividad anterior, se analizó el movimiento rectilíneo; éste, además de servir de ambientación en el uso del sensor, se tomó como referencia para identificar la características de la gráfica cuando un alumno se acercaba o se alejaba del sensor, Carlos logró integrar estos tipos de movimiento en uno solo (movimiento de carácter repetitivo a intervalos de tiempo iguales), lo cual le permitió definir las acciones a seguir para generar, con ayuda del sensor, una grafica del tipo sinusoidal. (2do. Movimiento. Figura 1).

En la segunda parte de esta sesión, se empleó un péndulo simple. Las características del movimiento pendular se representaron en el plano cartesiano, primero se hicieron manualmente (lápiz y papel), posteriormente con la mediación del sensor de movimiento. Solamente tres alumnos lograron mostrar un significado explícito al determinar la gráfica *manual* de manera correcta, justificando que recordaron lo que ocurrió cuando se solicitó reproducir la gráfica sinusoidal con movimientos corporales, los alumnos relacionaron el *caminar de manera uniforme* frente al sensor, con el *movimiento repetitivo del péndulo*, (2do. Movimiento. Figura 1). Para verificar o rectificar la gráfica bosquejada por cada alumno, se obtuvo la gráfica del movimiento pendular a través del

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sensor, especificando que la gráfica mostrada correspondía a una función sinusoidal. Acto seguido, se discutieron las características de la gráfica, el principal rasgo a resaltar fue, como los jóvenes lo mencionaron, *la repetición o existencia de un patrón*. (2do. Movimiento. Figura 1).

Posteriormente, se eligió aleatoriamente una gráfica de las obtenidas en la actividad anterior y se solicitó a los alumnos mencionar los elementos que observaban en la gráfica (2do. Movimiento. Figura 1), primero identificaron las *oscilaciones*, los alumnos las refirieron como *montañas* o *máximos*, después de un análisis y discusión, se concluyó que hay *montañas invertidas o mínimos* a considerar, la justificación la dio Carlos argumentando que las *montañas* representaban las idas del péndulo y las *montañas invertidas* los regresos de éste. La profesora estableció que lo mencionado por Carlos se conoce como *oscilación* y que el tiempo que implica una oscilación, se identifica como *período (P)*.

La *amplitud* fue el segundo elemento mencionado, éste fue asociado con la *altura de las montañas*, la profesora retomó la idea de *máximo*, como el punto más alto de la *montaña* y *mínimo* al punto más bajo de la *montaña invertida*, al igual que el *período*, el valor de la *amplitud*, es posible determinarlo directamente con el menú del programa de cómputo empleado en la investigación. Sin embargo, la profesora mencionó un procedimiento analítico para calcular la *amplitud* |A|.

> *Profesora:* Una vez que se localizan las coordenadas del punto máximo y del punto mínimo, se suma los valores absolutos de las ordenadas y el resultado se divide entre 2, el resultado será el valor de la ordenada que junto con cualquier valor de la abscisa (tiempo), será el eje de equilibrio, ya que la amplitud puede definirse como la distancia entre el punto máximo o mínimo y el eje de equilibrio.

Al preguntar a los alumnos qué representa la *amplitud* en el fenómeno físico, fue necesario repetir el experimento del péndulo y después de un largo análisis o debate, Mariano mencionó que *la amplitud era el dibujo de las oscilaciones*, Carlos relacionó que *lo que el péndulo abre* (arco de oscilación) *era lo que en la gráfica se veía como la altura*.

El desplazamiento vertical (V), fue identificado como *la distancia de péndulo* (en la posición de equilibrio) *al sensor*, Samantha mencionó que *a mayor distancia la gráfica salía más arriba*. A los alumnos no les llamó la atención que las gráficas no empezaran en el origen cartesiano, por lo que no identificaron *el desplazamiento horizontal,* ni que el origen de la gráfica estaba determinado por el sensor, por lo que la profesora lo explicó (definiendo que forma parte del modelo matemático).

Cuando la profesora preguntó qué nombre recibían las funciones que se habían estado graficando, ningún alumno lo recordó (No se da el 2do. Movimiento. Figura 1). La profesora conjuntó lo discutido anteriormente y mencionó que la manera de representar matemáticamente este tipo de funciones es: $f(t) = A\cos(Bt + H) + V$, (3er. Movimiento. Figura 1); la representación de los parámetros, fue elegida por los alumnos) A es la *amplitud*; B la velocidad *angular*, está relacionada con el *período* (P), $B = \frac{2\pi}{P}$; H el *desfase o desplazamiento horizontal* y V el *desplazamiento* vertical. En la siguiente actividad, a los alumnus se les solicitó determiner el modelo matemático de la siguiente gráfica (Figura 5.

Figura 5. Gráfica modelada.

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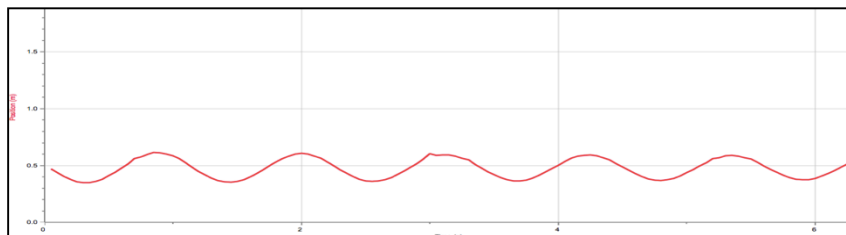
Los valores de los parámetros de la gráfica anterior fueron determinados por 2 de los alumnos (Samantha y Carlos); a excepción del *desfase*, Mariano calculó los valores del resto de los parámetros; los otros alumnos solo calcularon la *amplitud* y el *desplazamiento* vertical,

$$f(t) = 0.12cos(1.12t - 0.46) + 0.47$$

Posteriormente se pidió a los alumnos que comprobaran los valores determinados de la gráfica, Carlos lo hizo ayudándose del menú del programa que se empleó para generar las gráficas.

Carlos: Oscilaciones son 5.513, el 513 porque con la opción *analice* y después *examine* se obtiene el *time* en 6.2 y *position* en 513 y el 5 porque son 5 oscilaciones completas. El período 6.2 entre 5.513=1.12. El desfase, igual, con *analice* y *examine* se ve que es de 0.46 y el desplazamiento vertical de 0.47. La amplitud, en la segunda oscilación, el mínimo es de 0.354 y el máximo de 0.607, 0.354+0.607=0.961 entre 2=0.4805; 0.607-0.4805=0.1265. (3er. Movimiento. Figura 1).

Carlos determina la representación analítica de la gráfica presentada (Figura 5), aunque confundió el período con la velocidad angular; al solicitarle que aplicara la relación $B = \frac{2\pi}{P}$, él cambió el valor de la velocidad: $f(t) = 0.12cos(5.5t - 0.46) + 0.47$. Carlos obtuvo de la gráfica los desplazamientos vertical y horizontal, lo cual indicó que tuvo una representación explícita de los conceptos previamente analizados; sin embargo, *para* calcular la amplitud no la relacionó con la distancia (vertical) del punto máximo de la gráfica (o punto mínimo) con el valor del desplazamiento vertical, ya que nuevamente calculó dicho valor; por lo tanto, de la amplitud (aunque determina el valor correcto), no tiene la representación explícita completa.

Resultados Preliminares

La primera etapa permitió identificar las concepciones que los alumnos tienen al describir un fenómeno armónico. En la segunda, se lograron identificar las representaciones implícitas que lograron describir los alumnos, así como las ventajas y obstáculos de las tecnologías digitales en las que nos apoyamos. A partir de esta etapa, se pudo observar que 2 alumnos (de 10), lograron pasar de una representación implícita o espontánea a una representación explícita, el resto de los alumnos solo asignaron significado a la amplitud y el desplazamiento vertical, al lograr trasladar estos conceptos de la representación gráfica al fenómeno físico analizado y viceversa. Sin embargo, falta experimentar estas dos etapas con los modelos del sistema masa resorte y el movimiento circular y así dar respuesta a la primera pregunta de investigación. A demás falta concluir la tercera etapa de experimentación, en la cual, partiendo de la representación analítica, se reinterpretará al fenómeno físico, cerrando de esta manera un primer ciclo como el mostrado en la figura 1. Se buscará que los alumnos resignifiquen al fenómeno armónico analizado en la primera sesión, lo cual permitirá sortear la frontera que existe entre la física y la matemática, llegando así a una redescripción del fenómeno y a la resignificación del modelo algebraico institucional, enfocándose en consolidar conceptos relativos al período $P = \frac{2\pi}{B}$, la frecuencia y la amplitud.

Hasta el momento se observó que el sensor de movimiento y los modelos físicos de los fenómenos armónicos, así como el diseño y guía de la actividad fueron herramientas esenciales; a través de éstas se eliminaron algunas de las dificultades que las representaciones de las funciones sinusoidales tienen por si mismas; por ejemplo la forma de generar la representación gráfica, permitió a los alumnos identificar y asignar significado a los signos que representan los parámetros de las funciones sinusoidales; cuando fue necesario modificar dichos parámetros, se hizo de una manera relativamente sencilla, solo se tenían que caminar frente al sensor o repetir el experimento, y comprobar si las predicciones o conjeturas eran ciertas o debían replantearse.

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In this research we discuss relating the description of certain harmonic phenomena, with models that represent them, with the goal of promoting the understanding of the period, the frequency, and the amplitude of the trigonometric functions, particularly referring to sinusoidal functions; in this way the frontier between sinusoidal functions and one of its areas of application to physics is preserved. This is an ongoing research study where digital technology like a movement sensor is employed in order to generate graphics related to sinusoidal functions, in which the movement features followed by harmonic phenomena are identified. It is important to mention the use of an approach where it is possible to transit alternatively from the analysis of the phenomena to the mathematical model. It is conjectured that the way it approaches sinusoidal functions could help assign meaning to the model f(x) =Acos(ωx+ϕ).

Keywords: Post-Secondary Education, Modeling, Technology

Problem Statement

The authors who have done research in mathematics education related to trigonometric functions (e.g., Blacket & Tall, 1991; Brown, 2006; Buendía & Cordero, 2005, Hertel & Cullen, 2011; Kendal & Stacey, 1996; Moore, 2009; Weber, 2005), agree that regardless of the strategy employed, the initial stages of learning trigonometric functions are filled with difficulties, which causes most of the students between 14 to 17 years of age (High school), to incorrectly interpret the concepts related to this type of functions, in particular, referring to the sinusoidal function. Nevertheless, research studies associated to the teaching of trigonometric functions have been scarce (Moore, 2010).

In subjects related to physics and engineering a good understanding of the trigonometric functions is essential; while the goal in mathematics courses is to build meaning for the parameters of the sinusoidal functions, the aim in the physics courses is, in particular, to build interpretations of the (harmonic) phenomena and deepen in the understanding of these regarding sinusoidal functions; but, the world of physics and the world of mathematical objects are not linked, and as a result the border between them is disjointed; for example, very few students know of the fundamental feature of trigonometric functions: the periodicity (mathematics), and even fewer are the ones who can apply this type of functions as a model of periodic phenomena (physics). Regarding this situation, the NCTM (1989) within its standards (grade 9-12), for the study of trigonometry (standard 9), defines that one of the objectives to be covered when studying this subject is for students to explore real life periodic phenomena using the functions sine and cosine. However, in order to understand certain periodic phenomena that happen on a daily basis or for high school students to continue with engineering studies at the university level, it is essential to cross the border or build bridges between physics and mathematics, which is why in this paper the meanings students associate to sinusoidal functions are identified and analyzed through the study of the connections between physical models of harmonic phenomena, like the pendulum and the mass spring system or the walking (regularly or uniformly) of students, with their mathematic representations, such as graphics; these representations will be generated by the phenomena to be analyzed, a movement sensor and a graphing program.

In this research report, the current results regarding body movement of the student and the pendulum will be discussed. The intention is to be able to define the mathematical object modeling such phenomena, the sinusoidal function, posing the following research questions: (1) Which actions allow to describe harmonic phenomena and which representations arise from such descriptions?, (2) Based on the representation of harmonic phenomena, how can these phenomena be redescribed and which resignification of the mathematical model that represents them can be generated?, (3) How does the use of digital tools support the building of meanings of the sinusoidal functions?

As objectives, we have: (1) Analyze, describe and systematize the meanings students associate to the relation between the model of the sinusoidal function and harmonic phenomena, (2) Describe and analyze the arguments used by students, in order to justify, explain and/or try conjectures when employing digital tools to determine a sinusoidal function as a model of harmonic phenomena.

Theoretical Perspective

When building meaning, implicitly or explicitly, students put into play personal or spontaneous conceptions; in the case of in-class mathematics teaching, these previously mentioned conceptions are commonly left aside and mathematical objects are seen as finished products, un-linked to their origin and independent from the subject that studies them. It is hardly ever taken into account the fact that any getting close to a mathematical object transforms the relation one has with it and as a result its representation of itself (Moreno, 2014).

In order to give meaning to sinusoidal functions, in this research, spontaneous or implicit representations and phenomena modeled by sinusoidal functions are considered as a starting point, since this could help a reinterpretation and resignification of themselves, so that they can be represented in an explicit manner, (Pozo, 2006). As a matter of fact, Hitt (2006, 2013) makes emphasis on the importance of spontaneous representations (product of functional representations) from students, and the observation processes which allow to perform an evolution of these representations when building sign and meaning.

As the research is ongoing, for now the conceptual framework considered for the design and application of pilot activities is based on the fact that when analyzing a physical phenomena, students achieve an initial understanding, which is achieved intuitively and leads to an implicit representation (spontaneous, first contact with the mathematical object), so that from this representation the physical phenomena can be reinterpreted; that is, the phenomena is redescribed, which implies a resignification of the representation of mathematical symbols (mathematical model), to reach an institutional representation, general and/or abstract which means the formal object: the sinusoidal function, (Figure 1).
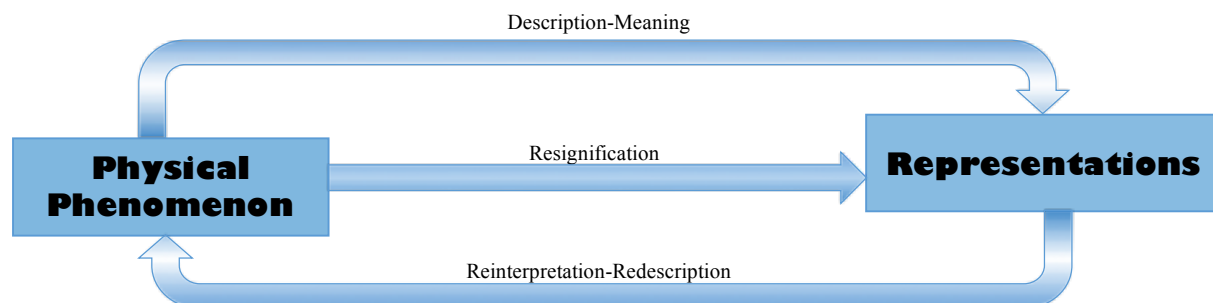
Figure 1. Resignification of sinusoidal function.

Research Design

It is a qualitative research. The pilot activities described in here were applied to a group of ten students during their first semester of high school (15-16 years of age), which meant that they had not had any kind of previous exposure to the subject of sinusoidal functions. The description on the previous scheme (Figure 1), as such, determines the design of the research. The step i) Physical phenomenon to representation requires a hypothetical learning trajectory (Simon, 1995; Simon & Tzur, 2004), a design of activities for the subjects who take part in the study. On step ii) Representation to physical phenomenon, written and video data are collected in order to identify the reinterpretations that students pose. Finally, on step iii), from the Physical phenomenon, data are taken in order to identify the resignifications that the subjects perform.

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Data Analysis

Up until now, the experiment has been performed in three stages: 1) Exploration, 2) interpretation-representation and 3) reinterpretation-resignification. The development of the first stage took place in three two-hour sessions each; the second and third stages were applied in two two-hour sessions each.

The objective of the first stage was to identify the previous conceptions students had when describing harmonic phenomena. At the second stage, the objective was to identify the implicit representations that the participants were able to describe according to the physical phenomena and mathematical model. The main objective for the third stage was that starting from the students' initial representation of the physical phenomena (second stage), it was redescribed, in order to achieve an explicit description (institutional) of the mathematical model $f(x) = A\cos(\omega t + \varphi)$, which describes the analyzed phenomenon.

First Session. Description of the phenomena (Preliminary Conceptions)

Students were asked to: Perform a detailed description of each and every single one of the phenomena they observed (pendulum, mass spring system and circular movement of an object tied to a cord, an object which follows a vertical circular trajectory and horizontal circular trajectory).

The participants performed the description associating it to physical aspects of the presented models, materials and colors. They gave the correct name to one of the devices (pendulum). The students mentioned ideas related to concepts such as force, movement, distance, speed and acceleration; even though they were in their majority incorrect or ambiguous. Their answers showed recurrence to concepts or ideas previously learned; or, situations which, apparently, were observed at first sight (even when they were inaccurate). For example, all students mentioned *force* as the main cause for the lentils movement, or that at any given moment force would be over, and as a result the object would *no longer be moving*, as well as that once a *body was at rest no longer was any force applied*, etc. The conceptions shown by students in this activity were stuck in the physics frontier and were, in their vast majority, based on representations of implicit nature, some of them were similar to how in the beginning mankind tried to conceptualize the concept of movement. No student relied on the use of any representation that would have implied a formal concept, a formula or the use, in particular, of any arithmetic, algebraic or diagrammatic sign. The implicit conceptions were the starting point to understand essential features of sinusoidal functions. A conceptual or representative change of this (second movement of the process described in the theoretical framework. Figure 1) was intended.

Second session. Employing digital tools.

Besides getting the students acclimated to the use of digital tools, the objective of this activity was for students to identify and associate the manner in which movement is represented in a Cartesian graph; the analysis was centered around the behavior and representation of the physical phenomena, (1st movement. Signification. Figure 1); for this, students walked in front of a movement sensor, which plugged to a computer generated the graphic with the particular features of the students walk. For example:

Alejandro: When the fellow students walked from a certain point up until getting close to the sensor, the graphic pointed that the distance started from 5 decreasing up to 0, because each time they got closer and the sensor detected that they were very close, that is why it was decreasing and when they started closer to the sensor and finish further away, the graphic recognized it from 0 until they stopped at the second 3.1 to 3.5, that happened because they stopped, and when they did not stop the graphic continue to increase.

In the case of Alejandro, the initial representation he generated included the compilation of numeric data, from which he came back to the redescription of the phenomena (2nd. Movement. Figure 1).

Afterwards students were asked to use a pencil and paper, in order to sketch the graph of the action of coming to school starting at home and including their having to return home after class.
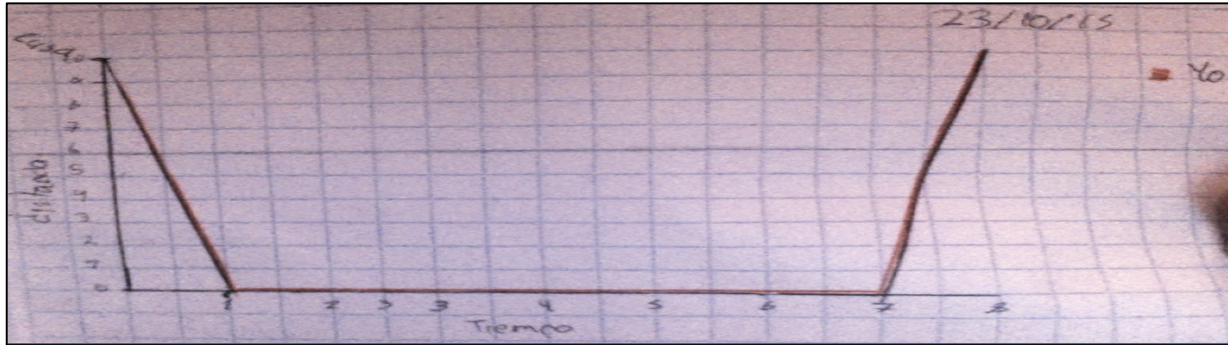
Figure 2. Alma's graph of one round trip.

Alma: It is the same thing only that the sensor is the school.

What was exposed by Alma, represented a clear example that an explicit representation of the sensor has been built by considering it a reference point, given that she was able to translate the situation from the graphics generated by the sensor to a new context, establishing a new reference point, the school.

In this activity students started to leave behind the frontier of a daily physical situation towards the mathematical representation. Next students were asked to sketch, also using a pencil and paper, the action previously posed, but now applied to a week and without considering how much time they spent in school.

At this point, students identified features of periodic movements (to which they referred as repetitive); they determined the period in a basic manner, in the case of Alma, she established that it takes her two hours to get to school and two hours to go back home. (1st movement. Figure 1).
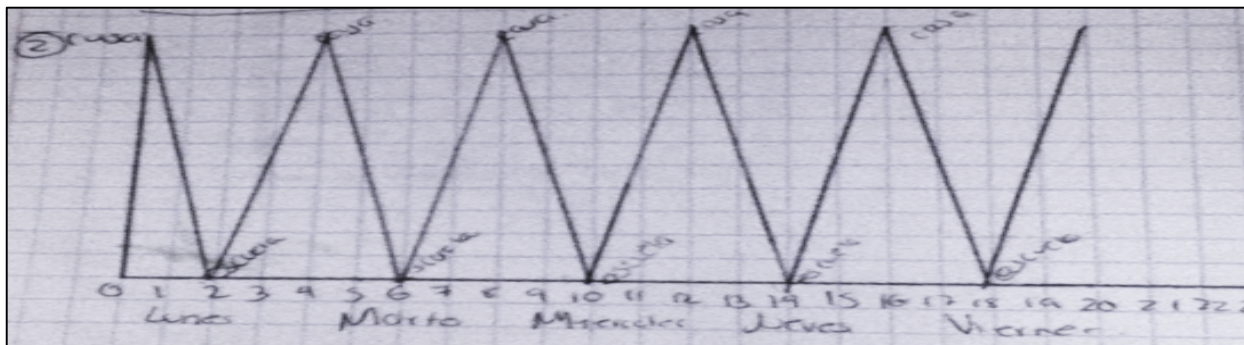
Figura 3. Alma's graph of five round trips.

The question regarding the no differentiability, was approached in a later session. Students, after a long debate, concluded that it is impossible to obtain (walking) *a graphic with peaks, no person is able to move that fast and achieve that kind or representation.* (2nd. Movement. Figure 1).

Afterwards, students were showed a sinusoidal graph (without them knowing they were dealing with that kind of graph), they were asked to perform the necessary movements while walking in front of the sensor in order to reproduce it. Students walked in different ways; for example, trying to follow the iconic representation of the graphic, lifting their knees or as some of them stated *walking*

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like a drunk person or associating the type of graphic to the speed with which they walked. After a considerable amount of attempts, Carlos thought the movement should *be making the same walk, meaning to go and come back*. Standing in front of the sensor the student performed a move consisting of walking two steps towards the sensor and two steps backing away from it.
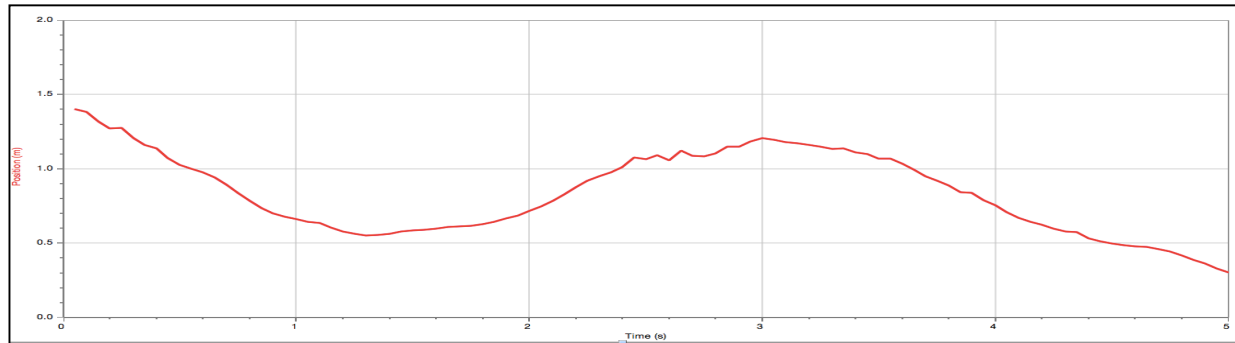
Figure 4. Graphic generated by Carlos at the moment of walking.

Teacher: Why did you think the movement should be like that?
Carlos: Because if one gets closer (to the sensor), it was going down (the graphic representation); if one gets away (from the sensor), it was going up (the same representation).

After performing the action, Carlos associated that this type of graph is generated by a periodic movement, given that he stated the following question:

Carlos: Could I place my hand on the sensor?

Carlos was referring to whether it was possible to perform the movement that would generate the requested graph in a different way, by reaching his hand closer to the sensor and moving the hand away from it, repeating both movements several times; Carlos verified the veracity of his conjecture in front of the sensor.

On a previous activity, rectilinear movement was analyzed; this, aside from working as a setting for the use of the sensor, was taken as a reference to identify the features of the graph at the time when a student was getting closer or further away from the sensor; Carlos was able to integrate this kind of movement in one (movement with a repetitive nature at equal time intervals), which allowed him to define the actions to follow in order to generate, with aid from the sensor, a sinusoidal kind of graphic. (2nd. Movement. Figure 1).

On the second part of this session, a simple pendulum was employed. Pendulum motion features were represented on a Cartesian plane, first manually (pencil and paper), later on with the use of the movement sensor. Only three students were able to show an explicit meaning at the moment of determining the *manual* graph in a correct manner, justifying that they remembered what happened when they were asked to reproduce the sinusoidal graphic with body movements, students related *walking uniformly* in front of the sensor, with the *repetitive movement of the pendulum*, (2nd. Movement. Figure 1). In order to verify or rectify the graph sketched by every student, the pendulum motion graphic was obtained using the sensor, specifying that the shown graph corresponded to a sinusoidal function. Immediately afterwards, the graph features were discussed, the main feature to be pointed out was, in the manner the young people mentioned, *the repetition or existence of a pattern*. (2nd. Movement. Figure 1).

Afterwards, one of the obtained graphs on the previous activity was chosen randomly and students were asked to point out the elements observed in it (2nd. Movement. Figure 1), at first they identified the *oscillations*, students referred to them as *mountains or maximums*, after analyzing and

discussing, it was concluded that there are *inverted mountains or minimums* to be considered, the justification was given by Carlos, arguing that the *mountains* represented the going of the pendulum and the *inverted mountains* its coming back. The teacher established that what Carlos had mentioned is known as *oscillation* and that the time employed by an oscillation, is identified as *period (P)*.

The *amplitude* was the second element mentioned, this was associated with the *height* of the *mountains*, the teacher turned back to the idea of the *maximum*, as the highest point of the mountain and *minimum* to the lowest point of the *inverted mountain*, just like the *period*, it is possible to determine the value of the *amplitude* directly with the menu of the software employed in this research. Nevertheless, the teacher mentioned an analytic procedure to calculate *amplitude* $|A|$.

> *Teacher*: Once the coordinates of the highest and lowest points have been located, the absolute values of the ordinates are added up and the result is divided by two, the result will be the value of the ordinate which together with any value of the abscissa (time), will be the midline (translation). Given that, amplitude can be defined as the distance between the highest or lowest point and the balance axis.

When asking students what the amplitude represents in the physical phenomena, it was necessary to repeat the pendulum experiment and after a long analysis or debate, Mariano mentioned that the *amplitude was the drawing of oscillations*, Carlos related that *how much the pendulum opens* (arc of oscillation) *was what in the graphic is observed as height*.

The vertical displacement (V), was identified as the *pendulum distance* (on a position of equilibrium) *to the sensor*, Samantha mentioned that *at greater distance the graph was higher*. Students did not pay attention to the fact that the graphs did not start on the Cartesian origin, which is why they failed to identify the *horizontal displacement*, nor that the origin of the graph was determined by the sensor, so the teacher explained it (making it clear that it is part of the mathematical model).

When the teacher asked the name of the functions that had been graphed, no student remembered (The 2[nd.] Movement is not given. Figure 1). The teacher put together what had been previously discussed and mentioned that the mathematical way of representing this functions is: $f(t) = Acos(Bt + H) + V$ (3rd. Movement. Figure 1; the representation of the parameters, was chosen by the students), A is the *amplitude*; B is the *angular* speed, it is related to the *period (P)*; $B = \frac{2\pi}{P}$; H is the *phase shift or horizontal displacement* and V the *vertical displacement*.

On the next activity, students were asked to determine the mathematical model of the next graph (Figure 5).
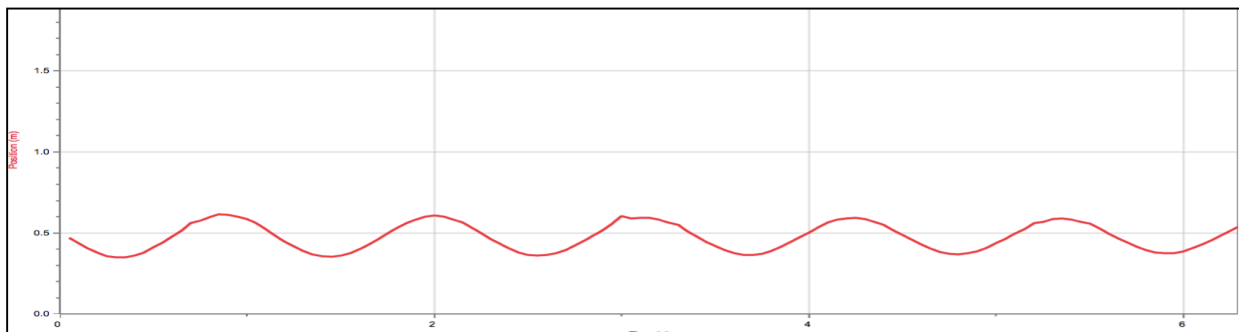
Figura 5. Model of the graph.

The values of the parameters in the previous graph were determined by two of the students (Samantha and Carlos); with exception to the *phase shift or horizontal displacement*, Mariano

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calculated the values of the rest of the parameters; the rest of the students just calculated the *amplitude and the vertical displacement,* $f(t) = 0.12cos(1.12t - 0.46) + 0.47$

Afterwards, students were asked to check specific values of the graph, Carlos did it with the help of the menu of the software used to generate the graphics.

> *Carlos:* Oscillations are 5.513, the 513 because with the option *analyze* and then *examine* the *time* is obtained at 6.2 and *position* at 513 and the 5 because there are 5 complete oscillations. The period 6.2 divided by 5.513=1.12. The phase shift, also, with *analyze and examine* shows that it is 0.46 and the vertical displacement of 0.47. The amplitude, on the second oscillation, the minimum is 0.354 and the maximum is 0.607, 0.354+0.607=0.961 divided by 2= 0.4805; 0.607- 0.4805=0.1265. (3rd. Movement. Figure 1).

Carlos determined the analytic representation of the given graph (Figure 5), even though he confused the period with the angular speed; when asked to apply the relation $B = \frac{2\pi}{P}$, he changed the speed value: $f(t) = 0.12cos(5.5t - 0.46) + 0.47$. Carlos obtained the horizontal and vertical displacements from the graphs, which indicated he had an explicit representation of the previously analyzed concepts; nevertheless, when calculating the amplitude he did not relate it to the distance (vertical) highest point of the graph (or lowest point) with the value of the vertical displacement, given that he calculated such value again, as a result; regarding the amplitude (even though he determines the correct value), he does not have a complete explicit representation.

Preliminary Results

The first stage allowed us to identify the conceptions students have when describing harmonic phenomena. In the second stage it was possible to identify the implicit representations described by the students, as well as, the advantages and obstacles of the digital technology we rely on. From this stage, it was possible to observe that 2 students (out of 10), were able to go from an implicit or spontaneous representation to an explicit representation, the rest of the students were only able to assign meaning to the amplitude and vertical displacement, by getting to translate these concepts from the graphic representation to the physical phenomena and vice versa. Nevertheless, more experimentation is needed in these two stages with the models of the mass spring system and circular movement, to then give an answer to the first research question. Besides, the third experimentation stage is yet to be concluded, in which, starting with the analytic representation, the physical phenomena will be reinterpreted, as a result closing a first cycle as it is shown on Figure 1. It will be sought for students to resignify the harmonic phenomena analyzed on the first session, which will allow to avoid the existing frontier between physics and mathematics, reaching in this way a redescription of the phenomenom and the resignification of the institutional algebraic model, focusing in consolidating concepts related to period $P = \frac{2\pi}{B}$, frequency and amplitude.

Up until now it has been observed that the movement sensor and the physical models of the harmonic phenomena, as well as the design and guidance of the activity were both essential tools; through these, some of the difficulties that the representations of sinusoidal functions representations present were eliminated; for example the way in which to generate the graphic representation, allowed students to identify and assign meaning to the signs which represent sinusoidal functions, when it was necessary to modify these parameters, it was done in a relatively simple way, it was as simple as just walking in front of the sensor or repeat the experiment, and verify if the predictions or conjectures were true or if they had to be reassessed.

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RELACIONES DINÁMICAS ENTRE CONVENCIMIENTO Y COMPRENSIÓN EN LA CONSTRUCCIÓN DE SUSTENTOS

DYNAMIC RELATIONS BETWEEN CONVINCEMENT AND COMPREHENSION IN THE CONSTRUCTION OF GROUNDS

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En trabajos anteriores los autores de este escrito identificaron relaciones puntuales entre comprensión y convencimiento. Con base en la Teoría Fundada, en la investigación actual se analizan los cambios que se dan en dichas relaciones como respuesta a condiciones cambiantes. Para su estudio se analiza una interacción, a distancia, entre un tutor y una estudiante. En un primer nivel se realiza un microanálisis -acudiendo al Modelo de Toulmin- en el cual se describen e interpretan las relaciones entre comprensión y convencimiento que ahí surgieron. Desde una perspectiva más general, en un segundo nivel se adelantan explicaciones sobre esas relaciones y sus cambios. Al final se sugieren propuestas para la práctica educativa.

Palabras clave: Razonamientos y Demostraciones, Trayectorias de Aprendizaje

Antecedentes y Objetivos

En investigaciones diversas se ha destacado el peso que durante los procesos didácticos tienen el convencimiento y la seguridad en los hechos de las matemáticas que vivencian los agentes de clase. Por ejemplo, Krummehuer (1995) resaltó el convencimiento asociado a los soportes de los argumentos, para cuyo análisis utilizó el Modelo de Toulmin; despunta en su aplicación la omisión de los calificadores modales Q, ausencia que señalaron Inglis, Mejia-Ramos, and Simpson (2007). Estos autores sostienen que uno de los objetivos de la instrucción debe ser el desarrollo de habilidades de los estudiantes para igualar "adecuadamente" tipos de garantías con calificadores modales Q (p.3). Acorde con esa perspectiva, en un análisis puntual de los estados de confianza de los estudiantes, Foster (2016) sugiere que un alumno 'bien calibrado' en un tema es aquel que confía en sus respuestas correctas y duda de las que no lo son. A diferencia de esos reportes, el presente documento –que se orienta de acuerdo a la metodología de la teoría fundada y sigue la idea de que los argumentos se desarrollan en el marco de interacciones sociales que se dan en el aula (Krummheuer, 1995)- identifica y adelanta algunas explicaciones sobre fenómenos dinámicos e integrales del salón de clases de matemáticas; específicamente, se busca conocer ¿Cuáles son los posibles cambios que durante el proceso de construcción de argumentos en el aula de matemáticas se dan en las relaciones entre el convencimiento en torno a resultados de esa disciplina, y su comprensión? así como ¿Cuáles son las posibles razones de la presencia de distintas relaciones durante ese proceso?

Marco Teórico

Para analizar las participaciones de los estudiantes se recurrió al Modelo de Toulmin (Toulmin, Rieke, & Janik, 1984). En este modelo, un argumento está compuesto por una afirmación (C), datos (D), garantías (W), un soporte (B), y calificadores (Q). Toulmin, Rieke, y Janik (1984) consideran que Q consiste en "el grado de confianza que puede ser adjudicado a las conclusiones dados los argumentos disponibles para apoyarlas" (p. 85, 1984). En esa interpretación de Q se supone implícitamente un sujeto experto que califica. A diferencia, en el presente escrito se acepta explícitamente que es el sujeto que argumenta el que califica la fuerza de los componentes del argumento, y se considera que ese sujeto (que participa en un foro virtual) vivencia un estado de

convencimiento, o bien de presunción o duda en un enunciado matemático -los que Rigo (2013) denomina "estados epistémicos de convencimiento"-, cuando cumple con alguno(s) de los criterios que aparecen en la Tabla 1 (Martínez & Rigo, 2014).

Tabla 1: Instrumento Teórico-Metodológico para Distinguir Estados Epistémicos

Elementos del habla	La persona recurre a enfatizadores del lenguaje que pueden revelar un mayor grado de compromiso con la verdad de lo que dice, por ejemplo, cuando la persona usa el modo indicativo de los verbos (e.g. "es").
Acción	El sujeto realiza acciones consecuentes con su discurso.
Determinación	La persona manifiesta de manera espontánea y determinada su adhesión a la veracidad de un enunciado matemático
Interés	Las participaciones de una persona que interviene con interés en torno a un hecho matemático específico en un foro virtual son: *sistemáticas* (es decir, el sujeto contesta todas las preguntas dirigidas a él de la manera más detallada posible), *informativas* (sus afirmaciones, procedimientos y/o resultados son suficientemente informativos), *claras y precisas*.
Consistencia	La persona muestra consistencia en sus distintas intervenciones.

El contenido matemático de los fragmentos elegidos para este estudio es el de la resolución de ecuaciones lineales. En este escrito se considera al Modelo 3UV (Ursini, Escareño, Montes & Trigueros, 2005) como el procedimiento paradigmático escolar para encarar ese tipo de tareas. De acuerdo a ese modelo, los aspectos que indican una comprensión de la variable como incógnita específica cuando se resuelven ecuaciones lineales son: interpretar la variable simbólica que aparece en una ecuación como la representación de valores específicos (aspecto I1); determinar la cantidad desconocida que aparece en ecuaciones o problemas, realizando operaciones algebraicas, aritméticas o de ambos tipos (aspecto I4) y sustituir la variable por el valor o valores que hacen de la ecuación un enunciado verdadero (aspecto I3).

Al resolver una ecuación lineal los estudiantes pueden fundar sus argumentos en distintos marcos generales, es decir, en diferentes soportes, los cuales pueden estar conformados por constituyentes diversos. Uno de los constituyentes de los soportes se relaciona con los recursos de sustentación en los que se funda el argumento. A esos recursos Rigo (2013) les llama "esquemas epistémicos" de sustentación. Según la autora, mientras algunos sustentos se vertebran en torno a razones matemáticas, como las instanciaciones de reglas generales, otros se articulan en torno a consideraciones extra-matemáticas, como los esquemas operatorios que se activan cuando se introduce una regla sin justificación, basada posiblemente en la autoridad que se le otorga a las matemáticas. Por tanto, se pueden presentar soportes matemáticos y soportes extra-matemáticos para los argumentos. Otro de los constituyentes hace referencia al carácter aritmético o algebraico del argumento. En este escrito se sugiere (cf. Martínez y Pedemonte, 2014) que la resolución de una ecuación se basa en el álgebra cuando el sistema de referencia en los datos contiene literales, y el "núcleo del argumento" (i.e., sus D y su C) presenta una estructura de tipo deductivo, la que es posible explicitar a través de las garantías, ya que éstas descubren la estructura que articula el argumento. Se dirá que una resolución se soporta en la aritmética cuando el sistema de referencia en los datos se da por ensayo y error numérico, y el núcleo del argumento presenta una estructura inductiva. Otro indicador para determinar si el soporte contiene constituyentes aritméticos o algebraicos está relacionado con los elementos conceptuales que el alumno pone en juego cuando realiza el aspecto I3. I3 presupone el desarrollo, aunque sea sólo de manera intuitiva y tácita, del siguiente argumento: a) Considerar en la ecuación $ax+b=0$ un valor específico para x, i.e., que $x=r$, $r \in \mathbb{R}$; b) Instanciar en la ecuación, i.e., $a(r)+b=0$; c) Realizar operaciones aritméticas; d) Derivar

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(eventualmente) una tautología aritmética: m=m; e) Desprender de d) que a) es una suposición correcta (de otro modo no se derivaría de ella una tautología), y que a(r)+b=0 es una proposición verdadera, esto es, que r hace verdadera a la proposición ax+b=0 (la cual es abierta, ya que carece de un valor de verdad), y que por tanto, r es una solución para dicha ecuación. Cuando el alumno procesa I3 con la conciencia de lo que significa que un valor específico $r \in \mathbb{R}$ "satisface una ecuación y resuelve el problema" (Ursini et al., 2005, p. 27), esto es, cuando tiene algunas intuiciones relacionadas con los pasos a) al e) del argumento antes expuesto, en este documento se considera que I3 coadyuva a su comprensión de la variable y que el soporte de su argumento contiene un constituyente algebraico. Cuando I3 queda sólo como un argumento incomprensible y rutinario para el estudiante que va sólo del paso a) al d) y él lo aplica solamente con el propósito de verificar ("en la aritmética", terreno seguro para el alumno) si los valores obtenidos son correctos, en este documento se considera que ese aspecto I3 coadyuva poco a la comprensión de la variable y que el soporte de su argumento incluye constituyentes aritméticos.

Metodología y Técnicas De Recuperación de la Información

Este escrito forma parte de un trabajo más amplio, inspirado en los procedimientos de la teoría fundada (Corbin & Strauss, 2015), cuyo objetivo es describir y construir explicaciones teóricas de fenómenos relacionados con los estados epistémicos que se dan en el salón de clases de matemáticas. El estudio se llevó a cabo en un diplomado -a distancia- cuyo propósito es fortalecer la formación de asesores que enseñan álgebra a adultos. Los datos que se usaron para el estudio quedaron registrados en la plataforma Moodle para su posterior análisis y forman parte de la interacción que un tutor mantuvo con sus estudiantes (en particular, con Belarmina). El tutor, quien propuso y guió las actividades, es uno de los autores de este trabajo. En trabajos anteriores, los autores identificaron relaciones puntuales entre comprensión y convencimiento. El trabajo actual está orientado por el "principio de cambio" de la teoría fundada, según el cual, los fenómenos se conciben como continuamente cambiantes. De modo que, a diferencia de otros estudios, las relaciones entre comprensión y convencimiento se consideran como parte de un proceso dinámico en el que dichas relaciones pueden modificarse en respuesta a condiciones cambiantes. Para analizar dichos cambios en las relaciones, se eligió una pieza de una interacción entre el tutor y Belarmina. El análisis se divide en dos apartados según el nivel de profundidad. Considerando los contextos de interacción, un primer nivel está dirigido a un micro análisis de las relaciones que se dieron entre los estados epistémicos del sujeto (Q) y su nivel de comprensión, al amparo de un soporte (B) específico. Para este análisis, la interacción se separó en fragmentos (distinguidos con un numeral) y se organizó en argumentos conforme al Modelo de Toulmin. En un segundo nivel, las relaciones entre comprensión y convencimiento se concentraron en tablas de análisis con el fin de obtener una perspectiva general del cambio de dichas relaciones a lo largo de la interacción. Para ofrecer explicaciones de las complejas relaciones entre comprensión y convencimiento, se tomó en cuenta el contexto en el que éstas surgieron y el principio de la teoría fundada, según el cual, los actores son capaces de tomar decisiones precisas de acuerdo a las opciones de que disponen.

Análisis de Resultados. Primer Nivel de Análisis: Descripción e Interpretación

1a Participación de Belarmina: Expresión de una Tendencia Algebraica y Operatoria

A manera de diagnóstico, el tutor propuso resolver a los estudiantes: Rosa tiene una balanza en equilibrio, de un lado una pesa de 5 kg y del otro una pesa de 2kg y un bulto de fierro. ¿Cómo puede hacer para saber el peso del fierro? En la Figura 1 se muestra la respuesta de Belarmina.

1.1	5=2+x donde x es el bulto de fierro	**D1.** a: 5=2+x donde x es el bulto de fierro; b: x=5-2	→	**C1.** 3kg
1.2	entonces x=5-2=3kg.	**W1.** a: La solución de la ecuación esta al lado derecho del signo igual (I1); b: Trasposición de términos (I4)		
		B1. a: Esquemas operatorios y álgebra (I1); b: Esquemas operatorios y álgebra (I4)		

Figura 1. Análisis de la 1ª participación de Belarmina. Argumento 1.

En su primera intervención, Belarmina experimentó seguridad en la aplicación de I1 e I4, la cual se deja ver en el uso del enfatizador "es" en 1.1, al actuar con base en las expresiones que derivó y mostrar determinación por publicar su respuesta. La aplicación de I1 e I4 la hizo conforme a esquemas operatorios (que se revelan por el carácter implícito de las reglas que enunció) y a una perspectiva algebraica, que se refleja a través de la estructura deductiva y el sistema de referencia algebraico en los datos.

Intervención del Tutor: Cuestionamiento del Soporte

| 2.2 | Una vez planteada la ecuación acostumbramos usar "trasposición de términos", pero ¿por qué funciona? Para averiguarlo realicemos la siguiente actividad. |
| 2.3 | Da clic en el interactivo, arma la ecuación en la balanza y llega a la solución. Describe paso por paso cómo llegaste a la solución. Por ejemplo: -2x-4=4x-4; Para dejar sola a la x realizo lo siguiente: 1.-Sumo a ambos miembros 4. La ecuación nos queda: -2x=4x; 2.- Sumo a los dos miembros 2x. La ecuación nos queda: 0=6x; 3.- Divido a los dos miembros entre 6. La ecuación nos queda: 0=x. La solución es 0. |

Como respuesta a Belarmina, el tutor cuestionó (v. 2.2) el constituyente operatorio sobre el cual la estudiante apoyó I4 (v. B1b). En la Figura 2 aparece lo que la alumna respondió.

2a Participación de Belarmina: Seguridad en el Sustento Algebraico y Operatorio

3.1	Para dejar sola a la x en: -4x-4=8x-4	**D2.** a: -4x-4=8x-4; b: sumo a ambos miembros 4; c: -4x=8x; d: sumo a los dos miembros 4x; e: 0=12x; f: divido a los dos miembros entre 12; g: 0=x	→	**C2.** La solución es x!!!
3.2-3.3	Sumo a ambos miembros 4. La ecuación nos queda: -4x=8x			
3.4-3.5	Sumo a los dos miembros 4x. La ecuación nos queda: 0=12x	**W2.** b-f: Determinar el valor de la literal con las propiedades de la igualdad (I4); g: La solución de la ecuación se encuentra a la derecha del signo igual (I1)		
3.6-3.7	Divido a los dos miembros entre 12. La ecuación nos queda: 0=x	**B2.** b-f: Álgebra y razones matemáticas (I4); g: Álgebra y esquemas operatorios (I1)		
3.8	La solución es x!!!			

Figura 2. Análisis de la segunda participación de Belarmina. Argumento 2.

Belarmina desarrolló I4 con base en las reglas promovidas por el tutor (v. W2b-f), y apoyada en un soporte algebraico y matemático (v. B2b-f), extendiendo su comprensión en este aspecto. Pero nuevamente, la alumna también afianzó su argumento en esquemas operatorios (B2g) cuando en el paso de D2g a C2 dio una interpretación incorrecta del signo igual (v. W2g) que la llevó a contravenir I1. Sobre la aplicación de W2g (relacionada con I1) e I4, Belarmina experimentó seguridad que mostró con el uso de enfatizadores (!!!), al actuar siguiendo las reglas que enunció y al mostrar determinación e interés por publicarlas. Como respuesta, el tutor cuestionó el uso implícito de la garantía W2g relacionada con I1: 1. ¿Qué entiendes por la solución de una ecuación? 2. ¿La solución de una ecuación puede expresarse con literales? ¿Por qué? En la Figura 3 se analizan las respuestas dadas por Belarmina.

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3a Participación de Belarmina: Duda Asociada a la Aparición de Razones Matemáticas

4.1	1.- [La solución es] Encontrar el valor que al sustituirlo en la ecuación por la incógnita permita llegar a una igualdad	**D3.** La solución es encontrar el valor que al sustituirlo en la ecuación por la incógnita permita llegar a una igualdad (I3)	**C3.** La solución de una ecuación no se puede expresar con literales (I1)
4.2	2.- Tutor, tengo duda en esta pregunta pero checando la pregunta de arriba entonces no se puede expresar con literales porque vamos a encontrar su valor. Corríjeme.		
		W3. Si la solución de una ecuación es el valor de la literal entonces la solución no se puede expresar con literales (I1)	
		B3. Razones matemáticas y álgebra	

Figura 3. Análisis de la tercera participación de Belarmina. Argumento 3.

En su participación, Belarmina parafraseó con seguridad I3 (ver en 4.1 el uso indicativo de los verbos y el empleo de I3 para derivar otra regla) en su versión aritmética, lo cual hizo conforme a esquemas operatorios y aritméticos. De esta versión escolar de I3, la estudiante dedujo con duda (ver 4.2) una conclusión C3 acorde con I1, conforme a una garantía soportada en razones matemáticas y algebraicas, ayudando a su comprensión de I1. Con el fin de que la estudiante aplicara las proposiciones que enunció, el tutor preguntó: 1.- ¿Cuál es el valor de la incógnita?; 2.- ¿Cómo comprobamos que ese valor es solución de la ecuación? En la Figura 4 se analizan las respuestas de Belarmina.

4a Participación de Belarmina: Duda al Aplicar una Nueva Garantía en los Datos

5.1	1.- [El valor de la incógnita] sería 0	**D4.** a: D2g y C3; b: Sustituimos el valor en la ecuación; c: -4x-4=8x-4; d: -4(0)-4=8(0)-4; e: -0-4=0-4; f: -4=-4; g: Existe igualdad	**C4.** Sería 0
5.2	2.- Espero y estar bien, si no, me corrigen. [Para comprobar] lo sustituimos en la ecuación		
5.3	-4x-4=8x-4; -4(0)-4=8(0)-4; -0-4=0-4; -4=-4	**W4.** a: W3 (I1); b-g: Versión aritmética de I3 (I3)	
5.4	Existe una igualdad en ambos lados	**B4.** a: Razones matemáticas y álgebra (I1); b-g: Razones operatorias y aritmética (I3)	

Figura 4. Análisis de la cuarta participación de Belarmina. Argumento 4.

En 5.1, Belarmina aplicó C3, relacionado con I1, con cierta inseguridad (ver el uso del mitigador "sería") que en su participación anterior soportó en razones matemáticas y bajo una perspectiva algebraica. En 5.3 la estudiante aplicó D3, relacionado con I3, bajo esquemas operatorios y aritméticos y lo hizo con duda (ver 5.2). A continuación, el tutor solicitó resolver: Bety tuvo que cobrar $178 de un billete de $200. Ella le preguntó al cliente si traía cambio y él le dijo que traía $3. Ella aceptó. ¿Cuánto tiene que regresar? Esta tarea, similar a la que Belarmina enfrentó en su primera participación, la planteó el tutor para identificar cambios que se dieron en su resolución después de la interacción. En la Figura 5 se analiza la respuesta de Belarmina.

5a Participación de Belarmina: Seguridad en un Soporte Algebraico

4.1	procedemos a despejar la incógnita;	**D5.** a: 200+3=178+x; b: 203=178+x; c: 203-178=178-178+x	**C5.** x=25
4.2	200+3=178+x;		
4.3	203=178+x; 203 -178=178-178+x;	**W5.** a: Interpretar la variable como un valor específico (I1); b-c: Determinar la literal con las propiedades de la igualdad (I4)	
4.4	x=25 que es el cambio que tiene que regresar Bety	**B5.** a: Razones matemáticas y álgebra (I1); b-c: Razones matemáticas y álgebra (I4)	

Figura 5. Análisis de la quinta participación de Belarmina. Argumento 5.

Belarmina aplicó I1 (aún cuando D5c pudo activar W2g) e I4 (esta vez con las propiedades de la igualdad), aspectos que previamente re-construyó con el tutor bajo un soporte matemático. Como en los casos precedentes, ella los administró con seguridad, estado que dejó ver mediante el uso de enfatizadores ("procedemos", "es"), de acciones congruentes con lo que enunció y la determinación e interés que exhibió al publicar su respuesta. Sin embargo, la estudiante dejó de aplicar I3, la cual construyó y empleó en sus contribuciones precedentes (v. D3 y D4). Así que el tutor cuestionó su solución C5: ¿Cómo podemos comprobar que el valor que obtuviste para la incógnita es solución de la ecuación? La respuesta a esta pregunta se expone en la Figura 6.

### 6a Participación: Seguridad al Aplicar una Nueva Regla

5.1	Sustituyendo lo que vale x, que en este caso es 25, en la ecuación 200+3=178+x	**D6.** a: 200+3=178+x; 200+3=178+25; 203=203 → **C6.** x=25
5.2	200+3=178+25; 203=203	**W6.** a: Si al sustituir un valor en una ecuación existe una igualdad, entonces ese valor es solución de la ecuación
5.3	de esta manera podemos comprobar que es correcto porque en ambos lados es la misma cantidad.	**B6.** a: Esquemas operatorios y Aritmética

Figura 6. Análisis de la sexta participación de Belarmina. Argumento 6.

En esta participación, Belarmina parafraseó tácitamente la versión aritmética de I3 y la aplicó en D6 bajo esquemas operatorios y aritméticos. En esta ocasión, ella mostró seguridad en esa versión de I3, al utilizar enfatizadores (e.g. "es") cuando la enunció, actuar conforme a ella (en 5.1 y 5.2) y mostrar determinación e interés por explicitarla. Enseguida, el tutor le pidió resolver:-4x-16=9x+1. En la Figura 7 aparece lo que la alumna respondió.

7a Participación: Omisión de un Procedimiento Aritmético

6.1	Tutor, esta es mi respuesta	**D7.** a: -4x-16=9x+1; b: -4x-16+16=9x+16+1; c: -4x=9x+17; d: -4x-9x=9x-9x+17; e: 13x=17 → **C7.** X=17/13
6.2	Ecuación: -4x-16=9x+1	
6.3	-4x-16+16=9x+16+1 (propiedad usada suma); -4x=9x+17; -4x-9x=9x-9x+17 (propiedad usada resta); 13x=17 (propiedad usada división)	**W7.** a: Interpretar la variable como un valor específico (I1); b-e: Determinar la literal con las propiedades de la igualdad (I4)
6.4	X=17/13	**B7.** a: Razones matemáticas y Álgebra (I1) b-e: Razones matemáticas y Álgebra (I4)

Figura 7. Análisis de la séptima participación de Belarmina. Argumento 7.

Para resolver la ecuación, Belarmina activó los aspectos I1 e I4 siguiendo con puntualidad el procedimiento sustentado algebraicamente que construyó con el tutor. Como en las ocasiones previas, asociado a este esquema la estudiante pareció experimentar convencimiento; se ve al usar el indicativo de los verbos cuando presentó su respuesta ("esta es"), actuar conforme a las reglas que enunció y mostrar determinación e interés por publicar y explicar su respuesta. Pero nuevamente, ella dejó de aplicar I3, por las (posibles) razones que se exponen en lo que sigue.

Análisis de resultados. Segundo nivel: explicaciones plausibles

Se exponen algunas explicaciones viables sobre las complejas relaciones entre los estados epistémicos y la comprensión que pueden surgir en los procesos de desarrollo de argumentos y en los de cambios y conformación de sus respectivos soportes (v. trayectorias Tabla 2, 3, 4).

Tabla 2: Dinámica de Relaciones Entre Estados Epistémicos, Comprensión y Soporte de I4

Participación / Categoría	Primera	Segunda	Tercera	Cuarta	Quinta	Sexta	Séptima
Calificador	Seguridad	Seguridad	No aplica	No aplica	Seguridad	No aplica	Seguridad
Comprensión	Acorde	Acorde	No aplica	No aplica	Acorde	No aplica	Acorde
Soporte	Operatorio	Matemática	No aplica	No aplica	Matemática	No aplica	Matemática
	Álgebra	Álgebra	No aplica	No aplica	Álgebra	No aplica	Álgebra

Tabla 3: Dinámica de Relaciones Entre Estados Epistémicos, Comprensión y Soporte de I1

Participación / Categoría	Primera	Segunda	Tercera	Cuarta	Quinta	Sexta	Séptima
Calificador	Seguridad	Seguridad	Duda	Duda	Seguridad	No aplica	Seguridad
Comprensión	Acorde	Discorde	Acorde	Acorde	Acorde	No aplica	Acorde
Soporte	Operatorio	Operatorio	Matemática	Matemática	Matemática	No aplica	Matemática
	Álgebra	Álgebra	Álgebra	Álgebra	Álgebra	No aplica	Álgebra

Tabla 4: Dinámica de Relaciones Entre Estados Epistémicos, Comprensión y Soporte de I3

Participación / Categoría	Primera	Segunda	Tercera	Cuarta	Quinta	Sexta	Séptima
Calificador	No aparece	No aparece	Seguridad	Duda	No aparece	Seguridad	No aparece
Comprensión	No aparece	No aparece	Acorde	Acorde	No aparece	Acorde	No aparece
Soporte	No aparece	No aparece	Operatorio	Operatorio	No aparece	Operatorio	No aparece
	No aparece	No aparece	Aritmética	Aritmética	No aparece	Aritmética	No aparece

En la Tabla 2, se observa que el estado de seguridad vivenciado por Belarmina cuando aplicó reglas relacionadas con I4 permaneció sin alteraciones, aún cuando en la segunda intervención la estudiante cambió el constituyente operatorio del soporte a uno matemático al introducir nuevas reglas matemáticas (W2b-f). Esto revela la prontitud con la que la estudiante asimiló garantías relativas a I4 y que un aumento de comprensión puede ir acompañado de seguridad. Pero como se verá en lo siguiente esta asociación no siempre es inmediata.

Con respecto a I1 (Tabla 3), en su primera participación Belarmina experimentó seguridad cuando aplicó una regla según la cual la solución está a la derecha del signo igual (W1a) apoyada en un esquema operatorio que, en conjunción con la trasposición de términos, la llevó a obtener una respuesta acorde con I1. Con las propiedades de la igualdad que ella utilizó en su segunda intervención (W2b-f), dejó de trasponer términos, pero mantuvo su seguridad en torno a la regla W1a y la aplicó, lo que la condujo a trasgredir I1; esto puede explicar cómo es que la estudiante asoció seguridad a una proposición incorrecta. Los cuestionamientos del tutor a W1a, condujeron a la estudiante en su tercera participación a explicitar reglas (W3) acordes con I1 bajo esquemas matemáticos, ayudando a su comprensión. Sin embargo, aquí Belarmina dudó. Lo anterior desvela que un incremento de la comprensión no va necesariamente aparejado de un fomento en la seguridad, porque lo primero supone reacomodos cognitivos que suelen propiciar estados de inseguridad. Dicha inseguridad continuó en su cuarta intervención; no obstante, en la quinta y séptima, Belarmina prefirió esa garantía W3 (por sobre W1a) y le asoció seguridad.

En cuanto a I3, Belarmina parafraseó la versión operatoria y aritmética en su tercera participación pero sólo la aplicó cuando se lo solicitó el tutor, primero con duda y luego con seguridad. Se esperaría que, como ocurrió con I1 e I4, ella incluyera este aspecto I3 en su séptima participación; pero nuevamente, ella lo dejó de aplicar, aún cuando antes lo había manejado con seguridad, y aún cuando le hubiese permitido detectar el error en el que ahí incurrió. La trayectoria de resoluciones de Belarmina permite suponer fundadamente que esa omisión puede obedecer a que la versión aritmética y operatoria de I3 la llevaría al terreno de la aritmética dejando relativamente a lado el del álgebra, constituyente en el que ella sustentó sus resoluciones desde el inicio. Esta consideración es consistente con la respuesta que dio Belarmina cuando el tutor preguntó por su seguridad en torno a la solución que dio a las ecuaciones: "Yo me sentí segura de mis respuestas. En este caso seguimos

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los pasos para solucionar una ecuación donde buscamos la incógnita y utilizamos las propiedades de suma, resta, multiplicación y división." A diferencia de Belarmina, otras estudiantes como Jeymi asociaron su seguridad a la comprobación (I3, versión aritmética); ella así se expresó: "Muy segura porque en la comprobación me resulta la igualdad". Basar la seguridad en un proceso aritmético, en lugar de asociarla a procesos algebraicos, se puede explicar porque, como apuntan Martínez y Pedemonte (2014), los estudiantes utilizan la aritmética para entender el álgebra.

Consideraciones finales: sugerencias para la práctica didáctica

Con base en el marco teórico-interpretativo, siguiendo los principios metodológicos de la teoría fundada y mediante un estudio de caso, en los dos niveles de análisis aquí expuestos se describen e interpretan pero también se explican distintos procesos que puede desarrollar un profesor para que el alumno, durante la dinámica de ese proceso, llegue a adquirir seguridad de un conocimiento bien fundamentado (v. la idea de estado 'bien calibrado' de Foster, 2016). En esos procesos puede aparejarse, por ejemplo, un aumento de comprensión con duda, porque lo primero significa reacomodos cognitivos ante cuestionamientos o contradicciones que suelen propiciar estados de inseguridad. Asimismo, una disminución de la comprensión puede asociarse a seguridad porque las relaciones entre la seguridad y razones extra-matemáticas pueden mantenerse a falta de razones matemáticas. La diversidad de relaciones que surgen entre comprensión y convencimiento son parte de la construcción natural del conocimiento en condiciones ordinarias, y son relevantes porque pueden orientar las intervenciones del profesor: la seguridad de un estudiante coligada a incomprensión, por ejemplo, puede requerir el cuestionamiento del soporte, la construcción de nuevas garantías y su aplicación; la inseguridad aunada a la comprensión de una garantía, por otro lado, puede requerir de su aplicación en diferentes contextos. Adicionalmente, sería deseable que el profesor considerara que los procesos de conformación de un soporte y la seguridad hermanada parece ser un fenómeno local: respecto a un tema, respecto a un constituyente del soporte y también, respecto a la formulación y aplicación de una determinada garantía. Esto es significativo porque alerta sobre el hecho de que los cambios en un aspecto puntual pueden no extenderse de manera automática a otros.

In previous publications the authors of this paper identified specific relations between comprehension and convincement. On the basis of Grounded Theory, this research analyzes the changes that arise in said relations as a response to changing conditions. For their study, the authors analyze an interaction, at a distance, between a tutor and a student. At an initial level, a micro-analysis is undertaken -resorting to the Toulmin Model- which describes and interprets the relations that arise there between comprehension and convincement. From a more general standpoint, at a second level the authors contribute explanations concerning those relations and their changes. Finally, the authors suggest proposals for educational practice.

Keywords: Reasoning and Proof, Learning Trajectories (or Progressions)

Background and Objectives

Sundry research papers have underscored the importance of convincement and confidence of the mathematics facts within the didactic processes that are experienced by classroom agents. For instance, Krummehuer (1995) highlighted the convincement associated with the backings of arguments, and for his analysis he used the Toulmin Model. Of note in his application was his omission of the Q mode qualifiers, and this absence was noted by Inglis, Mejia-Ramos, and Simpson (2007). The latter authors maintain that one of the objectives of learning should be development of students' abilities to "adequately" equate types of warrants with Q mode qualifiers (pg. 3). According

to that standpoint, in an analysis of the states of student confidence, Foster (2016) suggests that a student that is "well-calibrated" in a topic is a student that has confidence in his/her correct answers and has doubts concerning incorrect answers. Unlike those reports, this paper -aligned with the Grounded Theory methodology and following the idea that arguments are developed within the framework of social interactions that arise in the classroom (Krummheuer, 1995)- identifies and puts forward several explanations concerning dynamic and integral mathematics classroom phenomena. Specifically, the paper seeks to know what the possible changes are that arise in the relations between convincement of mathematics results and understanding thereof during the process of building arguments in the mathematics classroom. It further seeks to know what the possible reasons are for the presence of different relations during said process.

Theoretical Framework

The authors used the Toulmin Model (Toulmin, Rieke, & Janik, 1984) in order to analyze student participations. In that model, arguments are made up of a claim (C), data (D), warrants (W), a backing (B) and qualifiers (Q). Toulmin, Rieke, and Janik (1984) consider that Q consists of "the degree of confidence that can be assigned to the conclusions given the arguments available to support them" (p. 85, 1984). That interpretation of Q implicitly assumes the existence of an expert subject that qualifies. Unlike the foregoing, this paper explicitly accepts that it is the subject that expresses the argument that qualifies the strength of the argument's components, and the authors contend that said subject (who participates in a virtual forum) experiences a state of convincement or of presumption or of doubt concerning a mathematics statement -which Rigo (2013) calls "epistemic states of convincement"-, when it meets some of the criteria that are included in Table 1 (Martínez & Rigo, 2014).

Table 1: Theoretico-Methodological Instrument for Differentiating Epistemic States

Elements of Speech	The person resorts to language emphasizers that can reveal a degree of commitment to the truth with what s/he states. For example, when the person uses the indicative mode of verbs (e.g. "is").
Action	The subject carries out actions that are consistent with his/her discourse.
Determination	The person spontaneously and determinedly expresses her/his adherence to the veracity of a mathematics statement.
Interest	The participations of a person who takes part with interest in a specific mathematics fact in a virtual forum are *systematic* (that is, the subject answers all of the questions posed of him in the most detailed manner possible), *informative* (his claims, procedures and/or results are sufficiently informative), *clear* and *precise*.
Consistency	The person's different interventions are consistent.

The mathematics content of the fragments chosen for this study is the solution of linear equations. Model 3UV (Ursini, Escareño, Montes & Trigueros, 2005) is used in this paper as the school paradigmatic procedure to face this type of tasks. According to said model, the following are the aspects that are indicative of an understanding of the variable as a specific unknown when linear equations are solved: interpreting the symbolic variable that appears in an equation as the representation of specific values (aspect I1); determining the unknown amount that appears in equations or problems, by performing algebraic, arithmetic or both types of operations (aspect I4); and substituting the variable for the value or values that make the equation a true statement (aspect I3).

Upon solving a linear equation, students can base their arguments on varying general frameworks. That is, they can base them on different backings that can be made up of different constituents. One of the constituents of the backings is related to the sustentation resources upon which the argument is founded. Rigo (2013) calls those resources "epistemic schemes" of sustentation. According to the author, while some grounds are supported by mathematical reasons, such as instantiations of general rules, others are articulated with respect to extra-mathematical considerations, such as the operational schemes that are activated when an unjustified rule is introduced, possibly based on the authority afforded to mathematics. Consequently, mathematical and extra-mathematical backings can be presented for arguments. Another of the constituents refers to the arithmetic or algebraic nature of the argument. In this text, the authors suggest (cf. Martínez & Pedemonte, 2014) that the resolution of an equation is algebra-based when the system of reference of the data contains literal numbers, and the "core of the argument" (i.e., its D and C) presents a deductive-type structure, which may be made explicit by way of warrants because said warrants uncover the structure that articulates the argument. A resolution will be said to be backed by arithmetics when the system of reference in the data arises by numerical trial and error, and the core of the argument presents an inductive structure. Another indicator for determining whether the backing contains arithmetic or algebraic constituents is related to the conceptual elements that the student brings into play when she performs aspect I3. I3 assumes development, even if only intuitive and tacit, of the following argument: a) In equation $ax+b=0$, considering a specific value for x, i.e. that $x=r, r \in \mathbb{R}$; b) Instantiating in the equation, i.e. $a(r)+b=0$; c) Carrying out arithmetic operations; d) (Eventually) Deriving an arithmetic tautology: $m=m$; e) Inferring from d) that a) is a correct assumption (otherwise a tautology could not be inferred from it), and that $a(r)+b=0$ is a true proposition, that is that r makes the proposition $ax+b=0$ (which is open, since it lacks a value of truth) true, and that r is consequently a solution for said equation. When the student processes I3 with the awareness of what it means for a specific value of r to $\in \mathbb{R}$ "satisfies an equation and solves the problem" (Ursini et al., 2005, p. 27), that is, when the student has a few intuitions related to steps a) to e) of the afore-mentioned argument, the authors of this paper deem that I3 aids the student's comprehension of the variable and that the backing of his/her argument contains an algebraic constituent. When I3 is left simply as an incomprehensible and routine argument for the student going from step a) to d) and the student only applies it in order to verify ("in arithmetic", safe ground for the student) whether the values obtained are correct, in this paper the authors deem that that aspect of I3 is of little help to comprehension of the variable and that the backing of the argument includes arithmetic constituents.

Methodology and Information Recovery Techniques

This paper is part of a broader work, inspired by the procedures of Grounded Theory (Corbin & Strauss, 2015), which objective is to describe and construct theoretical explanations for phenomena related to the epistemic states that arise in the mathematics classroom. The study was carried out during a distance learning, certificate program whose purpose is to strengthen the training of consultants who teach algebra to adults. The data used for the study were registered on the Moodle platform for subsequent analysis, and they are part of the interaction of one tutor with his students (in particular with Belarmina). The instructor, who proposed and guided the activities, is one of the authors of this paper. In previous works, specific relations between comprehension and convincement were identified. This paper is guided by the "principle of change" of Grounded Theory, according to which the phenomena are understood to be continuously changing. Hence, unlike other studies, the relations between comprehension and convincement are deemed to be part of a dynamic process in which said relations can shift as a response to changing conditions. A segment of one interaction between the tutor and Belarmina was chosen in order to analyze those relational changes. The analysis is divided into two sections according to the level of depth. Given the

interaction contexts, a first level aims at a microanalysis of the relations that arose among the epistemic states of the subject (Q) and her level of comprehension, under the cover of a specific backing (B). For this analysis, the interaction was separated into fragments (identified using number) and it was organized into arguments pursuant to the Toulmin Model. At a second level, the relations between comprehension and convincement were concentrated into tables of analysis so as to obtain a general perspective of the change in those relations throughout the interaction. In order to put forward explanations of the complex relations between comprehension and convincement, the authors took into account the context in which they arose and the Grounded Theory principle, according to which the actors are able to make precise decisions in line with the options available to them.

Analysis of Findings. First Level of Analysis: Description and Interpretation.

Belarmina's First Participation: Expression of an Algebraic and Operational Tendency

For purposes of diagnosis, the instructor proposed that the students resolve the following: Rosa has a balanced scale; on one side is a 5 kilo weight, and on the other there is a 2 kilo weight and a bundle of iron. What can she do to find out how much the iron weighs? Figure 1 depicts Belarmina's answer.

1.1	5=2+x where x is the bundle of iron	**D1.** a: 5=2+x where x is the bundle of iron; b: x=5-2	→	**C1.** 3kg
1.2	so x=5-2=3kg.	**W1.** a: The solution to the equation is on the right hand side of the equal sign (I1); b: Transposition of terms (I4)		
		B1. a: Operational schemes and algebra (I1); b: Operational schemes and algebra (I4)		

Figure 1. Analysis of Belarmina's first participation. Argument 1.

In her first intervention, Belarmina experienced security in her application of I1 and I4, which can be seen in her use of the emphasizer "is" in 1.1, by acting on the basis of the expressions that she derived and showing determination to announce her answer. She applied I1 and I4 according to operational schemes (revealed by the implicit nature of the rules that she enunciated) and to an algebraic perspective, reflected by way of the deductive structure and the algebraic reference system in the data.

Instructor Intervention: Questioning the Backing

2.2	Once the equation has been raised, we generally use "transposition of terms". But why does it work? To find out, let's do the following activity.
2.3	Click interactive, compose the equation on the scale and arrive at the solution. Give a step-by-step description of how you reached the solution. For example: -2x-4=4x-4; I do the following to have the x stand alone: 1. I add 4 on each side. The equation is then: -2x=4x; 2. I add 2x to both sides. The equation is then: 0=6x; 3. I divide both sides by 6. The equation is then: 0=x. The solution is 0.

To answer Belarmina, the tutor questioned (see 2.2) the operational constituent that the student used to support I4 (see B1b). Figure 2 contains the student's answers.

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Belarmina's Second Participation: Confidence in the Algebraic and Operational Grounds

3.1	To have the x standing alone in -4x-4=8x-4	**D.2** a: -4x-4=8x-4; b: I add 4 to each side; c: -4x=8x; d. I add 4x to each side; e: 0=12x; f: I divide both sides by 12; g: 0=x	**C2.** The solution is x!!!
3.2-3.3	I add 4 on each side. The equation is then: -4x=8x	**W2.** b-f: Determine the value of the literal number with the properties of the equality (I4); g: The solution to the equation is on the right hand side of the equal sign (I1)	
3.4-3.5	I add 4x to both sides. The equation is then: 0=12x		
3.6-3.7	I divide both sides by 12. The equation is then: 0=x	**B2.** b-f: Mathematics reasons and algebra (I4); g: Operational schemes and algebra (I1)	
3.8	The solution is x!!!		

Figure 2. Analysis of Belarmina's second participation. Argument 2.

Belarmina developed I4 based on the rules that the tutor promoted (see W2b-f), and supported by an algebraic and mathematics backing (see B2b-f), extending her comprehension in said aspect. But yet again the student secured her argument on the basis of operational schemes (B2g), when in the step from D2g to C2 she provided an incorrect interpretation of the equal sign (see W2g) which led her to contravene I1. With respect to W2g (related to I1) and I4, Belarmina experienced security which she demonstrated by using emphasizers (!!!), by acting in adherence to the rules that she expressed and by showing determination to and interest in announcing them. In answer, the tutor questioned the implicit use of the W2g warrant related to I1, as follows: 1. What do you understand to be the solution of an equation? 2. Can the solution of an equation be expressed in literal numbers? Why? Figure 3 provides an analysis of Belarmina's answers.

Belarmina's Third Participation: Doubt Associated with the Appearance of Mathematical Reasons

4.1	1. [The solution is] To find the value that, after using it to substitute the unknown in the equation, makes it possible to arrive at an equality.	**D3.** The solution is to find the value that, after using it to substitute the unknown in the equation, makes it possible to arrive at an equality (I3).	**C3.** The solution of an equation cannot be expressed in literal numbers (I1).
4.2	2. Instructor, I am doubtful about this question, but after checking the above question then it cannot be expressed using literal numbers because we are going to find its value. Correct me.	**W3.** If the solution of an equation is the value of the literal number, then the solution cannot be expressed in literal numbers (I1).	
		B3. Mathematics reasons and algebra	

Figure 3. Analysis of Belarmina's third participation. Argument 3.

In her participation, Belarmina paraphrased I3 with security (see in 4.1 the indicative use of verbs and use of I3 to derive another rule) in her arithmetic version, which she did in keeping with operational and mathematics schemes. From this school version of I3, the student deduced with doubt (see 4.2) a C3 conclusion in line with I1, in keeping with a warrant backed by mathematics and algebraic reasons, helping her comprehension of I1. The tutor asked the following so as to have the student apply the propositions enunciated: 1.- What is the value of the unknown?; 2.- How can we prove that that value is the solution of the equation? Figure 4 provides an analysis of Belarmina's answers.

Belarmina's Fourth Participation: She Doubts when Applying a New Warrant to the Data.

5.1	1.- [The value of the unknown] would be 0
5.2	2.- I hope to be right. If not, correct me. [To prove] we substitute it in the equation.
5.3	-4x-4=8x-4; -4(0)-4=8(0)-4; -0-4=0-4; -4=-4
5.4	Both sides are equal.

D4. a: D2g and C3; b: We substitute the value in the equation; c: -4x-4=8x-4; d: -4(0)-4=8(0)-4; e: -0-4=0-4; f: -4=-4; g: Equality exists → **C4.** It would be 0

W4. a: W3 (I1); b-g: Arithmetic version of I3 (I3)

B4. a: Mathematics reasons and algebra (I1); b-g: Operational reasons and arithmetic (I3)

Figure 4. Analysis of Belarmina's fourth participation. Argument 4.

In 5.1, Belarmina applied C3 related to I1 with a certain some insecurity (see the use of mitigator "would") which in her previous participation she backed using mathematics reasons and a certain algebraic perspective. At 5.3, the student applied D3 related to I3, under operational and arithmetic schemes and she did so with doubt (see 5.2). After that, the instructor asked that the students solve the following: Bety had to charge $178 from a $200 bill. She asked the customer if he had any change, and he told that her that he had $3. She accepted the coins. How much does she have to give him back? This task, similar to the task Belarmina faced in her first participation, was raised by the instructor in order to identify changes that had arisen in her resolution subsequent to the interaction. Figure 5 contains an analysis of Belarmina's answer.

Belarmina's Fifth Participation: Security in an Algebraic Backing

4.1	Let's isolate the unknown;
4.2	200+3=178+x; b:
4.3	203=178+x; 203 -178=178-178+x;
4.4	x=25 which is the change that Bety has to give back

D5. a: 200+3=178+x; b: 203=178+x; c: 203-178=178-178+x → **C5.** x=25

W5. a: Interpret the variable as a specific value (I1); b-c: Determine the literal number with the properties of the equality (I4)

B5. a: Mathematics reasons and algebra (I1); b-c: Mathematics reasons and algebra (I4)

Figure 5. Analysis of Belarmina's fifth participation. Argument 5.

Belarmina applied I1 (even though D5c could have activated W2g) and I4 (this time with the properties of equality), aspects that she had previously re-constructed with the instructor with a mathematics backing. Like in the preceding cases, she managed them with confidence, a state that she revealed by way of her use of emphasizers "let's", "is"), of actions that were consistent with her statement and the determination and interest that she exhibited when announcing her answer. However, the student stopped applying I3, which she had built and used in her preceding contributions (see D3 and D4). So the instructor questioned her solution at C5, as follows: How can we verify that the value that you obtained for the unknown is the solution to the equation? The answer to that questions can be found in Figure 6.

Sixth Participation: Security when Applying a New Rule

5.1	By substituting the value of x, which is 25 in this case, in the equation 200+3=178+x
5.2	200+3=178+25; 203=203
5.3	so in this way we can prove that it is correct because it is the same amount on both sides.

D6. a: 200+3=178+x; 200+3=178+25; 203=203 → **C6.** x=25

W6. a: If there is equality when a value is substituted in an equation, then that value is the solution to the equation.

B6. a: Operational schemes and Arithmetic

Figure 6. Analysis of Belarmina's sixth participation. Argument 6.

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In this participation, Belarmina tacitly paraphrased the arithmetic version of I3 and she applied it in D6 under operational and arithmetic schemes. This time, she showed confidence in the version of I3, by using emphasizers (e.g. "is") when enunciating, by acting in conformity with the version (at 5.1 and 5.2) and by showing determination to and interest in explaining it. The tutor then asked her to resolve -4x-16=9x+1. Figure 7 contains the student's answer.

Seventh Participation: Omission of an Arithmetic Procedure

6.1	Tutor, this is my answer:	**D7.** a: -4x-16=9x+1; b: -4x-16+16=9x+16+1; c: -4x=9x+17; d: -4x-9x=9x-9x+17; e: 13x=17 → **C7.** X=17/13
6.2	Equation: -4x-16=9x+1	
6.3	-4x-16+16=9x+16+1 (property used: addition); -4x=9x+17; -4x-9x=9x-9x+17 (property used: subtraction); 13x=17 (property used: division)	**W7.** a: Interpret the variable as a specific value (I1); b-e: Determine the literal number with the properties of equality (I4)
6.4	X=17/13	**B7**. a: Mathematics reasons and algebra (I1) b-e: Mathematics reasons and algebra (I4)

Figure 7. Analysis of Belarmina's seventh participation. Argument 7.

To resolve the equation, Belarmina activated aspects I1 and I4, specifically following the algebraically-supported procedure that she constructed with the instructor. As on previous occasions, associated with this scheme the student appeared to experience convincement. This can be seen by her use of indicative verb forms when she presented her answer ("this is"), by acting in conformity with the rules that she enunciated and by showing determination to and interest in announcing and explaining her answer. But once again, she did not apply I3 due to the possible reasons put forth below.

Analysis of Findings. Second Level: Plausible Explanations

Some viable explanations are presented concerning the complex relations between the epistemic states and comprehension that can arise in argument development and change processes and in conformation of their respective backings (see trajectories Tables 2, 3, 4).

Table 2: Dynamics of Relations Among Epistemic States, Comprehension and Backing of I4

Participation Category	First	Second	Third	Fourth	Fifth	Sixth	Seventh
Qualifier	Confidence	Confidence	Does not apply	Does not apply	Confidence	Does not apply	Confidence
Comprehension	Agreement	Agreement	Does not apply	Does not apply	Agreement	Does not apply	Agreement
Backing	Operational Algebra	Mathematics Algebra	Does not apply	Does not apply	Mathematics Algebra	Does not apply	Mathematics Algebra

Table 3: Dynamics of Relations Among Epistemic States, Comprehension and Backing of I1

Participation Category	First	Second	Third	Fourth	Fifth	Sixth	Seventh
Qualifier	Confidence	Confidence	Doubt	Doubt	Confidence	Does not apply	Confidence
Comprehension	Agreement	Disagreement	Agreement	Agreement	Agreement	Does not apply	Agreement
Backing	Operational Algebra	Operational Algebra	Mathematics Algebra	Mathematics Algebra	Mathematics Algebra	Does not apply	Mathematics Algebra

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Table 4: Dynamics of Relations Among Epistemic States, Comprehension and Backing of I3

Participation Category	First	Second	Third	Fourth	Fifth	Sixth	Seventh
Qualifier	Does not appear	Does not appear	Confidence	Doubt	Does not appear	Confidence	Does not appear
Comprehension	Does not appear	Does not appear	Agreement	Agreement	Does not appear	Agreement	Does not appear
Backing	Does not appear	Does not appear	Operational Arithmetic	Operational Arithmetic	Does not appear	Operational Arithmetic	Does not appear

In Table 2 one can see that the state of confidence experienced by Belarmina when she applied rules related to I4 remained unchanged, even when in her second intervention she changed the operational constituent of the backing to a mathematics constituent by introducing new mathematics rules (W2b-f). The foregoing speaks to the promptness with which the student assimilated warrants relative to I4 and that an increase in comprehension can be accompanied by confidence. However, as shall be seen below, said association is not always immediate.

With respect to I1 (Table 3), during her first participation Belarmina experienced confidence when she applied a rule according to which the solution is on the right hand side of the equal sign (W1a), supported by an operational scheme that, together with the transposition of terms, led her to obtain an answer in conformity with I1. With the properties of equality that she used in her second intervention (W2b-f) she no longer transposed terms but maintained her confidence with respect to rule W1a and applied it, which led her to infringe upon I1; this can explain the reason the student associated confidence with an incorrect proposition. The tutor's questioning at W1a led the student in her third participation to explain rules (W3) compliant with I1 under mathematics schemes, helping her comprehension. However, this is where Belarmina had doubts. The foregoing reveals that an increase in comprehension does not necessarily go hand in hand with developing confidence, because the former assumes cognitive readjustments that usually foster states of doubt. That doubt continued through her fourth intervention, albeit in the fifth and seventh Belarmina preferred the W3 warrant (over W1a) and associated confidence with it.

As regards I3, Belarmina paraphrased the operational and arithmetic version in her third participation, and only applied it when prompted by her instructor, first with doubt and then with confidence. One would expect that, as happened with I1 and I4, she would include that I3 aspect in her seventh participation. But once again, she stopped applying it even though she had handled it with confidence before and even though it could had allowed her to identify the mistake that she had made at that point. Belarmina's trajectory of answers enables one to have grounds to assume that said omission may be due to the fact that the I3 arithmetic and operational version would lead her to the land of arithmetic, leaving the land of algebra relatively to the side, a constituent upon which she had based her resolutions from the very beginning. The foregoing consideration is consistent with the answer that Belarmina provided when the instructor asked about her confidence concerning the solution that she had given to the equations, that is: "I felt sure of my answers. In this case we followed the steps to solve an equation in which we sought the unknown and we used the properties of addition, subtraction, multiplication and division. "Unlike Belarmina, other students such as Jeymi associated their confidence with the verification (I3, arithmetic version); the latter expressed herself by saying: "Very sure because in the verification I get the equality." Basing confidence on an arithmetic process instead of associating it with algebraic processes can be explained because, as pointed out by Martínez and Pedemonte (2014), students use arithmetic to understand algebra.

Final Considerations: Suggestions for Didactic Practice

On the basis of the interpretive theoretical framework, following methodological principles of Grounded Theory and by way of a case study, in the two levels of analysis presented here the authors

describe, interpret and explain different processes that teachers can develop during the dynamics of said process so that students can acquire confidence in well-founded knowledge (see Foster's (2016) idea of the "well-calibrated" state). In those processes, for instance, an increase in comprehension can be paired with doubt, because the former means cognitive readjustments when faced with questioning or contradictions that generally foster states of doubt. Moreover a reduction of comprehension can be associated with confidence because the relations between confidence and extra-mathematical reasons may be maintained in the absence of mathematics reasons. The diversity of relations that arise between comprehension and convincement are part of the natural construction of knowledge under ordinary conditions, and they are relevant because they can guide the teacher's interventions. A student's confidence jointly tied to incomprehension, for example, may require questioning the backing, construction of new warrants and their application. Doubt tied to comprehension of a warrant, on the other hand, may require that it be applied in different contexts. Additionally, it would be desirable for teachers to consider that the processes of conforming a backing and the matched confidence seem to be a local phenomenon. That is to say they concern a topic, they concern a constituent of the backing, and they also concern formulation and application of a given warrant. This is significant because it raises a flag regarding the fact that changes relative to a specific aspect may not automatically extend to others.

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THE FERRIS WHEEL AND JUSTIFICATIONS OF CURVATURE

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This report discusses the results of semi-structured clinical interviews with ten prospective secondary mathematics teachers who were provided with dynamic images of Ferris wheels. We asked the students to graph the relationship between the distance a rider traveled around the Ferris wheel and the height of the rider from the ground. We focus on the different quantitative and non-quantitative ways of thinking in which students engaged to justify the curvature of their drawn graphs. We also discuss how these ways of thinking relate to reasoning covariationally about directional change and amounts of change.

Keywords: Modeling, Cognition, Problem Solving

Introduction

Authors of the Common Core State Standards for Mathematics (CCSSM) (National Governors Association Center for Best Practices, 2010) recommended that educators provide students with repeated and sustained opportunities to model situations via constructing and comparing relationships including those involving constant and changing rates of change. Researchers have also shown that quantitative reasoning (i.e., the analysis of a situation into a quantitative structure (Thompson, 2011)) and covariational reasoning (i.e., students conceiving situations as composed of measurable attributes that vary in tandem (Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E., 2002)) are critical for numerous K-16 topics (Ellis, 2007; Johnson, 2015; Moore & Carlson, 2012). These researchers have argued for more research to understand how students reason when constructing and comparing relations between quantities.

Given that modeling and reasoning about rates of change are important components of mathematical reasoning, this study aims to answer researchers' calls to explore and to gain a better understanding of students' reasoning about quantities given modeling situations. Namely, we draw on the framework proposed by Carlson et al. (2002) with a particular focus on students' mental actions of directional change and amounts of change. We analyze students' reasoning against the backdrop of these covariation mental actions in the context of their justifying the curvature of graphs they constructed based on a given dynamic situation. Prior research on univariational reasoning and shape thinking also inform our analysis of students' reasoning.

Literature Review

Carlson et al. (2002, pp. 357-358) described a framework explicating students' covariational reasoning that includes five mental actions. This study focuses specifically on Mental Action 2 (MA2) and Mental Action 3 (MA3) of Carlson and colleagues' framework. MA2 is "coordinating the direction of change of one variable with changes in the other variable" (Carlson et al., 2002, p.357). Behaviors indicative of MA2 include "constructing an increasing straight line" and verbalizing "the direction of change of the output *while* [emphasis added] considering changes in the input" (Carlson et al., 2002, p. 357). MA3 involves considering (and possibly comparing) amounts of change in one quantity for amounts of change in another quantity. Some researchers (Castillo-Garsow, Johnson, & Moore, 2013; Johnson, 2012) have clarified the ways in which students reason about amounts of change. For example, Johnson (2012) provided insights into particular ways students reason about the intensity of varying quantities by "systematically varying one quantity and simultaneously attending to variation in the intensity of change in a quantity indicating a relationship between covarying quantities" (p. 327). Castillo-Garsow et al. (2013) further distinguished students imagining

the intensity of change in terms of successive chunks or in terms of a smooth, dynamic image of covariation.

Moore and Thompson (2015) clarified students' covariational reasoning in the particular context of graphs. They termed students' *emergent shape thinking* to be a student conceptualizing a graph (in ways potentially compatible with those described in the previous paragraph) as an in-progress trace of two united varying quantities. A student thinking emergently justifies the curvature of a drawn graph in terms of covarying quantities or magnitudes represented along the relevant axes. Highlighting that educators cannot take students' covariational reasoning as a given, Moore and Thompson (2015) identified other ways of thinking that we show students might use when justifying the curvature of a graph. The authors termed students' *static shape thinking* – "operating on a graph as an object in and of itself" (p. 784). Students' static shape thinking includes both iconic translations and thematic associations. Iconic translations involve associations between the visual features of the situation and graph (Monk, 1992), and thematic associations are between the graph and the event phenomena (Thompson, 2015). Lastly, one example of a quantitative justification is univariational reasoning, in which a student considers how one quantity's value changes without any systematic coordination with another quantity's value (Leinhardt, Zaslavsky, & Stein, 1990).

Theoretical Framework

We approach knowledge as actively constructed in ways idiosyncratic to the knower; an individual's knowledge is fundamentally unknowable to another (von Glasersfeld, 1995). When referring to a quantity, we do not imply that a quantity exists independent of a person conceiving it. Quantities are personally constructed measurable attributes (Steffe, 1991; Thompson, 2011). Similarly, relationships between quantities are constructed by the individual; the relationship hinges of the individual's understanding of each quantity and their image of the situation, which may not be compatible with a researcher's intentions (Moore & Carlson, 2012). Although an individual's knowledge is taken to be fundamentally unknowable to another person, it is possible for a person to make inferences about another individual's thinking. These inferences are based on the person's interpretations of the words and actions of the individual. These inferences are the basis of the person's constructed model of an individual's mathematics, referred to as *the mathematics of students* (c.f., *students' mathematics*) (Steffe & Thompson, 2000).

Methodology

This study focuses on student work from one problem during semi-structured clinical interviews (Clement, 2000; Goldin, 2000). We conducted the study with ten students (eight female, two male) who were enrolled in a secondary mathematics teacher education program at a large university in the southeast United States. The problem consisted of two tasks given during the first of three interviews with each student. At the time, the students were either enrolled in or had completed their first content course in the secondary mathematics education program, and all the students had completed a calculus sequence and at least two additional mathematics courses (e.g., linear algebra, differential equations, etc.) with at least C letter grades. Some of the students had completed additional courses in both mathematics and mathematics education, but these courses did not focus on ideas directly related to the interview tasks. Two members of the research team were present at each interview, and each interview was videotaped and digitized for analysis. We used an open (generative) and axial (convergent) approach based on grounded theory (Corbin & Strauss, 2008) and in combination with conceptual analysis (Thompson, 2008; von Glasersfeld, 1995) to analyze the results. That is, we used an iterative approach to construct viable models of the subjects' thinking and their ways of reasoning.
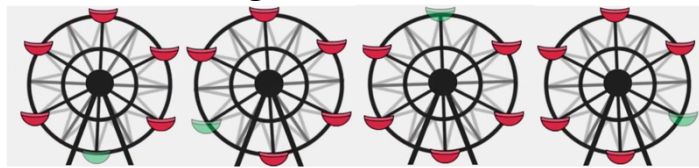
Task Design – The Ferris Wheel

Figure 1. Animation Snapshots of the Ferris Wheel Task I and Task II.

This study focuses on the Ferris wheel problem, which is split into two tasks. First, students view an animation of a Ferris wheel rotating clockwise (Desmos, 2014) continuously; the Ferris wheel has a green cart that begins at the bottom to represent the location of the rider (Figure 1). He or she then responds to Task I: "Graph the relationship between the total distance the rider has traveled *around* the Ferris wheel and the rider's distance from the ground." Once completed, the student views an animation of a Ferris wheel that moves at a slower speed and pauses three times at the locations seen in the latter three snapshots (Figure 1). We then provide Task II: "Watch the video, which dictates a second rider's trip around the Ferris wheel. Is the previous graph relevant? Explain." The goal of this problem is to (1) provide a student with two situations with perceivable differences, yet the same underlying relationship (ignoring explicit parameterizations) between two distances, and (2) afford MA3; for successive equal amounts of distance traveled around the Ferris wheel for the first quarter of a trip, the rider's height above the ground increases and the amount by which the height increases is also increasing.

Results

We organize students' justifications for the curvature of their graph into two categories: non-quantitative justifications and quantitative justifications. A non-quantitative justification involves reasoning in which some non-quantitative aspect of the given situation implies or necessitates a (possibly perceptual) feature of the drawn graph. A quantitative justification includes some consideration of the quantities and changes in at least one of the conceived quantities. We are not implying that a student providing non-quantitative justifications did not reason covariationally or quantitatively, only that some aspect of their justification was non-quantitative. For instance, students commonly justified their graphs via directional change (MA2), but provided non-quantitative justifications when prompted to explain the curvature of their drawn graphs further.

Non-Quantitative Justification of Curvature

The first example of a non-quantitative justification for the curvature of the graph is consistent with students' static shape thinking and iconic translation. In other words, the student relies on shapes in the situation to justify his or her drawn graph. In this task, students who exhibited static shape thinking justified the curvature of the graph based on the shape of the wheel (i.e., circular wheel implies circular graph). To illustrate, Rory created her graph (Task I) drawing a curved shape, but only described directional change (MA2) when justifying her graph:

> *R*: It starts total distance and then it gets higher about half way, so say this [marks point on the top center of her graph] is our total distance…It peaks about half way, so, however many seconds, um, and then it just goes up, I would say, uh, at a quadratic rate [plots points on the left side of her graph and connects them with a curved line, completes right side of her graph]…it goes up and down in sort of a parabola shape [Figure 2a].
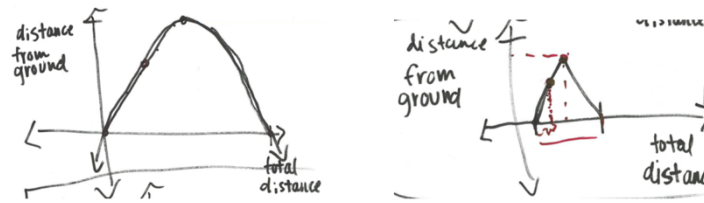
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Figure 2. (a, left) Rory's First Graph and (b, right) Second Graph.

When asked to explain the "quadratic rate" further, Rory explained her "parabola shape":

R: …As we go up higher and higher, the total distance increases…but when we come down about that half-way point where the [cart] is, um, exactly 90 degrees in the opposite direction, so where it just passed [*at the top of the Ferris wheel*], um, that's where it peaks [*pointing to the top of her graph*] and that's where it's the highest, but it's only about half the total distance that it's gone. So, it could also reflect, I guess, an absolute value graph [*draws Figure 2b excluding axes and labels*], but a downward facing upside down absolute value graph…I feel like because it's more circular in nature, and not like necessarily linear, um, to where it, I guess it would be like a hexagon or something with lines connecting, I would feel more parabola shaped just because it's curved, rather than an absolute value. But I could see how it would be reflecting an absolute value graph as well, so. But, I would still go with my parabola or quadratic shape as I was saying…

I: …the parabola shape because you said it's, what's curved? Something's curved?

R: The, the Ferris wheel in and of itself is a circle.

We infer that Rory understands that both a linear (i.e., "absolute value") and a curved graph (i.e. "quadratic") capture the appropriate directional relationship (MA2). However, her explanation for why she chose to draw a curved graph is based on the shape of the Ferris wheel. Thus, she engaged in both covariational reasoning and an iconic translation, with the iconic translation implying that the graph be "parabola" and not "linear" in shape.

A second example of a non-quantitative justification is thematic shape thinking. Students' thinking of graphs thematically entail their making associates between some perceived phenomenon of a situation and features of a graph (Thompson, 2015); in such a case, relationships between quantities are *implications* of assimilating some perceived phenomenon (Moore & Thompson, 2015). In the case of Task II, a student thinking thematically could involve his or her associating pauses that occur in a trip with some perceptual feature in the graph (e.g., horizontal lines, bolded points, or breaks in the graph) that "shows" the pauses.

Moreover, students might note the constant rotational speed of the Ferris wheel and make associations with the graph on the basis of constant speed (i.e., students conclude that constant speed unquestionably implies a graph composed of linear segments, and varying speeds unquestionably imply a graph composed of non-linear curves). For example, after Rory explicitly stated "relationships don't necessarily mean anything to do with the, the actual physical being being circular" after the previous excerpt, she abandoned her graph (Figure 2a) and watched the Ferris wheel again. Then, she made the following observation about the Ferris wheel:

R: It looks more at a constant rate now that I look at it. So it would probably be more of this absolute value shape… [*completes Figure 2b*]… And I don't, again, if it speed up, sped up like at some point to where the higher it went, it would represent more of the quadratic shape [*pointing to Figure 2a graph*]. So I would go with this graph [*Figure 2b graph*].

Rory perceived the Ferris wheel rotating at a constant speed and concluded that her graph should now have an "absolute-value shape" (i.e., linear). Notice also that her language in describing her

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graph is still based on shapes (i.e., the absolute value *shape* and the quadratic *shape*). Because Rory's justification relied on a perceived phenomenon (i.e., constant speed) implying a shift in a perceptual feature in her (distance-distance) graph (i.e., *quadratic* to a *absolute-value* shape), we infer her actions to indicate thematic shape thinking.

Quantitative Justification of Curvature

Univariational Reasoning. When a student engages in univariational reasoning, he or she alludes to and possibly compares changes in some quantity to make a claim about curvature. This comparison involves looking at different regions of the graph, although not systematically with respect to changes in another quantity, as Johnson (2012) described. For instance, in the Ferris wheel problem, students imagined that the rider's distance from the ground appears to "stay longer" at the top of the Ferris wheel than in the middle, concluding that there should be a difference in how these two regions appear on their graphs. Although their justifications considered the magnitude of the height and how this magnitude changed in some way in certain regions of the graph, the students did not explicitly or systematically hold in mind variations of the total distance traveled around the Ferris wheel; the students often made these comparisons for implicit total distance intervals of unequal sizes or interval sizes that are too large (in the sense that the relationship between the two quantities changes within the chuck under consideration).

For instance, consider Louise who initially drew her graph (Figure 3a without numerical values) considering directional changes in both quantities (i.e., "distance around [*the Ferris wheel*] is always increasing" and "distance from the ground grows and then stops and then goes back down" for one rotation around the wheel). When we asked her to justify the curvature of her graph, she considered a graph with a "peak" similar to Figure 2b. At one moment, she mentioned that a curved graph "looks better" but that maybe she is "just thinking that 'cause it's a curved circle [*referring to the shape of the Ferris wheel*]." Unsatisfied with this conclusion, she considered how the height changes in a way that we consider indicative of univariational reasoning. The following transcript includes her final justification for the curvature of her graph:

L: It's not like curved curved, but it's sorta curved. It's a small curve. Just because, it's. Because the distance around if you look at the top [*of the wheel*]…this whole area is kind of the peak [*points to range at the top of the wheel*], a little bit. Like it kind of like travels at the top, just a little bit. Um, so I think it kinda travels at the top here a little bit.

I: What about like through here [*pointing to the line segment between 0 and 5, Figure 3a*]?

L: Well, I think, 'cause it's continually growing. I mean like, coming from the ground it's not really staying anywhere. Like, it's not stopping there [*pointing at location along left side of the wheel*] and being like, "Oh, I'm gonna chill here". You know. It's constantly going, but at the top it kind of has to like go around the curve, so it's at the peak for a little longer than on here [*motioning along the right side of the Ferris wheel*]. And the same thing down here [*motioning along the bottom of the Ferris wheel*]. That's kinda why it's curved down here as well [*pointing to the bottom of Figure 3b*]. Because it, you know, stays at the bottom for just a, just like a tad bit longer than it would on the sides.
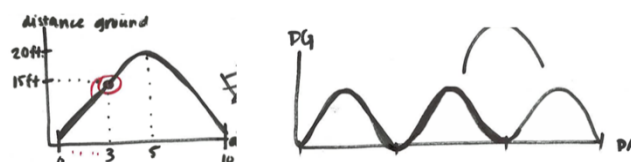

Figure 3. (a) Louise's Initial Graph and (b) Graph for Three Revolutions.

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Louise split the Ferris wheel into four sections to justify the curvature of her graph: the top and bottom sections and the two sides. Within each section, she imagined the height with respect to the experience of the ride to draw conclusions about the intensity of height changes. Because the height seemed to "stay" near the top and bottom sections "longer", she concluded that these sections of her graph are curved. For the sides of the Ferris wheel, she discussed the height as "constantly" changing; she represented those sections on her graph with (nearly) linear segments. Overall, her description included the intensity of one quantity's changes in one of her four sections (of unspecified size) without referencing how the second quantity (i.e., distance traveled around the wheel) related to these changes in height. This reasoning contrasts with the covariational reasoning Johnson (2012) described, which involves reasoning about the intensity of a quantity because Louise is not systemically comparing the intensity of height changes to another quantity (e.g., intensity of changes in distance traveled around the Ferris wheel).

Conflated Amounts of Change. When a student conflates amounts of change, he or she compares the changes in one quantity with respect to another quantity by considering how equal successive changes in one quantity relate to variations in the other quantity. However, he or she may be unsure how to imagine or represent the given quantities and their changes in the situation or the graph. The former leads to considering changes of a quantity in the situation that we, as observers, consider different than requested. For the Ferris wheel problem, a student might depict equal changes of "total distance" along the horizontal diameter of the wheel instead of equal changes in "total distance" around the wheel. The latter leads to students (possibly unknowingly) conflating which axes they use to represent particular quantities.
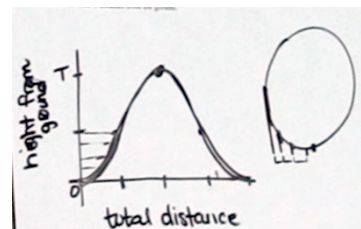

Figure 4. Kim's Graph and Diagram.

Kim began Task I by plotting points for each quarter turn and then each remaining one-eighth turn of the Ferris wheel. She connected the points using curved lines (Figure 4). She indicated reasoning directionally (MA2) when she described "he comes back down" and "he's back at 0" while tracing along the arc of the right side of the Ferris wheel.

When asked to justify connecting her plotted points, she immediately began drawing a diagram of the situation in order to relate the situation to her graph (Figure 4). In her explanation, she made two conflations. First, turning to her diagram, Kim marked off equal changes *along the ground* (as indicated by the horizontal segments in her diagram in Figure 4) and compared changes in height. This reasoning led her to conclude that her graph needed to show "height was increasing at an increasing rate". Second, when she turned to the corresponding section in her graph, she marked equal changes along the vertical axis (i.e., equal changes in the axes labeled *height from the ground*) and appropriately concluded that she "drew it like it's increasing at a decreasing rate, though." Thus, the equal changes she was comparing in both situations did not represent the same quantity (from our perspective). However, in both her diagram and her graph, she made appropriate conclusions for the relationships between the quantities she was comparing (e.g., with respect to the drawn graph, abscissa values do increase at a decreasing rate as ordinate values increase from 0). Although Kim was able to reason with the given quantities in order to plot points on her graph and reason using

MA2, she conflated the quantities when attempting to reason about amounts of change to justify the curvature of her graph.

Amounts of Change. We argue that a student reasoning emergently about the curvature of their graph with respect to the situation they are modeling occurs when he or she is able to exhibit MA3 on the given situation (or a student's diagram of the situation) and then create/modify/justify his or her graph in terms of such a relationship. For example, April justified her graph as she drew it. She marked equal changes of "arc length or total distance around" the Ferris wheel, identified the change in height for each interval using the situation, made conclusions about rates of change for given intervals around the wheel, and finally indicated how these changes are represented on her graph (Figure 5).
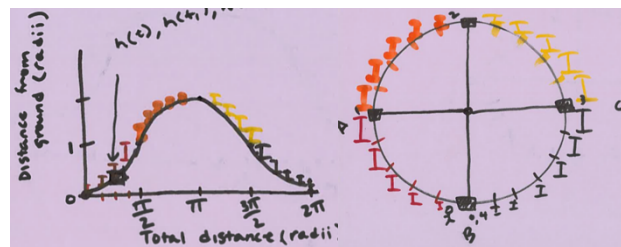

Figure 5. April's Justification of Curvature Using Amounts of Change.

Discussion

This paper focuses on types of reasoning students engage in when modeling a dynamic situation. The results corroborate several of Carlson et al.'s (2002) and Saldanha and Thompson's (1998) claims. Overall, many students (8/10) reasoned directionally (MA2) from the researcher's perspective, but they often constructed graphs without considering amounts of change *prior* to their drawing a graph; several students (8/10) had difficulty (quantitatively) justifying the curvature of their drawn graphs. In our study, we identified specific ways of thinking students exhibited when attempting to justify the curvature of their graphs. Students who reasoned non-quantitatively exhibited forms of static shape thinking when asked to justify the curvature (or lack thereof) of their graphs. We also noticed students who took a quantitative approach. However, before students could reason effectively about amounts of change (MA3), they needed to construct not only the change in one quantity, but also the change in that quantity with respect to a change in another given quantity. Students who were only reasoning about the change in one quantity without systematically considering changes in another quantity reasoned univariationally. Furthermore, students who realized that they needed to compare the change in one given quantity with respect to a change in another given quantity were not always sure how to represent these quantities on either their graph or the situation, often conflating quantities from our perspective. Piaget (1974) described that the path to consciousness is a dynamic, active system, which is consistent with the idea that quantities are not intrinsic attributes of graphs and situations. Furthermore, he claimed, "knowledge does not proceed from the subject or from the object, but from interaction between the two" (p. 335). Providing students with tasks that afford opportunities to reason quantitatively and covariationally encourages students constructing situations constituted by quantities, and perhaps helps to promote the construction of quantitative MA3 reasoning.

Acknowledgments

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CONSTRUCTING COMMUNAL CRITERIA FOR PROOF THROUGH CRITIQUING THE REASONING OF OTHERS

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Fifty-seven students in mathematical content and secondary mathematics methods courses from four U.S. universities participated in an instructional sequence to generate communal criteria defining mathematical proof within their respective classrooms. Participants completed a proof-related task before class, worked together in small groups to evaluate instructor-selected arguments, communally agreed upon criteria for evaluating a proof based on their evaluations, and then revised their original argument to meet the communal criteria after class. Similar criteria were constructed across the four classrooms. Moreover, the four authors coded and compared students' initial and revised arguments with respect to proof schemes to identify specific shifts in students' work after the instructional sequence. Results indicated a majority of students' proof schemes changed in their revised argument with specific trends aligning with their class-based criteria for proof.

Keywords: Reasoning and Proof, Post-Secondary Education, Classroom Discourse, Teacher Education-Preservice

Current reforms have called for a stronger emphasis on mathematical reasoning and proving in school mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA & CCSSO], 2010; National Council of Teachers of Mathematics [NCTM], 2000). The challenge of meeting this recommendation is that students typically observe their teacher presenting a polished and completed proof (Stylianou, Blanton, & Knuth, 2009). Using this method of proof instruction suggests to students that their teacher is the primary authority to judge the validity of their proof (Harel & Sowder, 1998). Students then believe that the goal of learning proofs involves strategies that replicate similar problems (Bleiler, Ko, Yee, & Boyle, 2015). Such instructional methods complicate students' ability to judge the validity of mathematical arguments because students are not given the opportunity to "construct viable arguments and critique the reasoning of others" (NGA & CCSSO, 2010, p. 6), a necessary standard for mathematical practice. Therefore, it is not surprising that research has shown that students at all grade levels have considerable difficulty in judging the validity of a proof (Healy & Hoyles, 2000; Ko & Knuth, 2013; Segal, 1999; Weber, 2010).

Although students' difficulties with proof are well documented, to date, little research has explicitly articulated a classroom community's criteria for mathematical proof, including social dimensions necessary for proof validation. Bleiler, Thompson, and Krajčevski (2014) explain that "many researchers have called for the explicit instruction to proof validation to ameliorate common difficulties related to accepting inductive arguments or focusing on local rather than global elements of an argument" (p. 109). Concomitantly, Hanna and Jahnke (1993) remind us that "the social process of verification through which proof becomes accepted in the mathematical community should somehow be imitated in the school" (p. 433). To address this research gap, we designed an intervention where students: (1) constructed an initial argument justifying their generalization to the Sticky Gum Problem (SGP) depicted in Figure 1 before class; (2) critiqued instructor-selected

arguments in small groups and developed a list of five communal criteria necessary for writing proofs during class; and (3) revised their initial argument to the SGP to meet the classroom-based criteria for proof after class described in Table 1. In this paper, we investigated how students' written arguments differed before and after our instructional sequence. To answer this research question, we used Harel and Sowder's (2007) proof schemes to identify how the students' arguments shifted given their engagement in the instructional sequence.

The Sticky Gum Problem

Ms. Hernandez came across a gumball machine one day when she was out with her twins. Of course, the twins each wanted a gumball. What's more, they insisted on being given gumballs of the same color. The gumballs were a penny each, and there would be no way to tell which color would come out next. Ms. Hernandez decides that she will keep putting in pennies until she gets two gumballs that are the same color. She can see that there are only red and white gumballs in the machine.

1) Why is three cents the most she will have to spend to satisfy her twins?
2) The next day, Ms. Hernandez passes a gumball machine with red, white, and blue gumballs. How could Ms. Hernandez satisfy her twins with their need for the same color this time? That is, what is the most Ms. Hernandez might have to spend that day?
3) Here comes Mr. Hodges with his triplets past the gumball machine in question 2. Of course, all three of his children want to have the same color gumball. What is the most he might have to spend?
4) Generalize this problem as much as you can. Vary the number of colors. What about different size families? Prove your generalization to show that it always works for any number of children and any number of gumball colors.

Figure 1. The Sticky Gum Problem originally created by Fendel, Resek, Alper, and Fraser (1996).

Table 1: Instructional Sequence to Create Communal Criteria to Understand Proof

Before-Class Activity	• Each student solves the Sticky Gum Problem (SGP) and submits the SGP electronically prior to the before-class activity. • The instructor chooses five distinct arguments submitted by their students.
During-Class Activity	• Students break into small groups of 2-4 students. Each group (1) looks at the same five instructor-selected classmate's arguments; (2) discusses and decides which of the five selected arguments are mathematical proofs; (3) determines how they decided whether each argument is a mathematical proof; and (4) creates a list of five proof criteria through small and whole group discussions. • All small groups rejoin the entire class. The entire class has discussions, compares the small group's criteria, and determines a class-wide communal list of five criteria for proof.
After-Class Activity	• Each student revises their original argument for SGP to satisfy the communal class-based criteria and submits their revised argument.

Theoretical Framework

To investigate how developing communal criteria for proof affected students' written arguments, this study builds on the frameworks about proof in the mathematics classroom (Stylianides, 2007), the process of proving (i.e. ascertaining and persuading, Harel & Sowder, 1998, 2007), and proof schemes (Harel & Sowder, 1998, 2007).

Proof in the Mathematics Classroom

A traditional perspective of proof is "a formal and logical line of reasoning that begins with a set of axioms and moves through logical steps to a conclusion" (Griffiths, 2000, p. 2). This definition describes a formal individual process of writing a proof, but does not acknowledge the negotiation aspects of constructing and validating proofs within a mathematics community. Stylianides (2007) suggested that proof is a connected sequence of assertions because it

(1) uses statements accepted by the classroom community (set of accepted statements) that are true and available without further justification; (2) employs forms of reasoning (modes of argumentation) that are valid and known to, or within the conceptual reach of, the classroom

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community; and (3) is communicated with forms of expression (modes of argument representation) that are appropriate and known to, or within the conceptual reach of, the classroom community. (p. 291)

Stylianides created this definition emphasizing proof that balances mathematics as a discipline and a learning tool. We use Stylianides' definition of proof as a frame of reference to interpret students' characteristics of proof and their classroom-based criteria for proof.

Ascertaining and Persuading

To understand how a mathematics community can support students' understanding of what counts as proof, it is also necessary to consider individuals' ideas of what constitutes a proof during the proving process. To incorporate both the individual and the communal aspects of proving, we use Harel and Sowder's (2007) action of proving as "the process employed by an individual (or a community) to remove doubts about the truth of an assertion" (p. 808). Furthermore, Harel and Sowder (1998, 2007) indicated proving as ascertaining (convincing oneself) and persuading (convincing one's community). Weber's (2010) research found that undergraduate mathematics majors would judge a deductive argument as proof, even if they were not convinced by its argument or the argument was logically flawed. Both Weber (2010) and Segal (1999) emphasize the need to push individual's convictions into public discussion, which was an aim of our instructional sequence. When students have to judge the validity of an argument, they are forced to negotiate between ascertaining and persuading because the judgement is based on how the argument would convince themselves and others. To this end, we designed our instructional sequence around students judging the validity of instructor-selected arguments, which allowed them to use their understanding of what counts as proof to create a discursive space and reach consensus on their judgements.

Proof Schemes

As the purpose of this study is to examine the effect that our instructional sequence would have on students' arguments, we use Harel and Sowder's (2007) proof schemes to analyze students' work because a proof scheme consists of "what constitutes ascertaining and persuading for that person (or community)" (p. 809). Harel and Sowder divided proof schemes into three main categories, which are external conviction proof schemes, empirical proof schemes, and deductive proof schemes. External conviction proof schemes mean that conviction is made by the form of an argument, such as a teacher, a textbook, or symbolic representations. In the classroom external convictions often take the form of an authoritarian proof scheme (Harel & Sowder, 2007). For empirical proof schemes, the validity of a conjecture is made by specific cases or individuals' mental images. Regarding deductive proof schemes, a conjecture's veracity is determined by logical deduction. Harel and Sowder (1998) make a strong point "that these schemes are not mutually exclusive; people can simultaneously hold more than one kind of scheme" (p. 244). For that reason, it is important for us to look for overlaps within the three proof schemes when analyzing our data.

Harel and Sowder's (1998) original proof-scheme taxonomy aligns with the focus of this study:

Our definitions of the process and proving and proof scheme are deliberatively psychological and student-centered... Thus, this classification is not of proof content or proof method…In contrast, it is the individual's scheme of doubts, truths, and convictions, in a given social context, that underlies our characterization of proof schemes. (p. 244)

In the same ilk, the focus of this research emphasizes the "social context" of the classroom. Hence, we use proof scheme categorizations as comparative tools to look for changes in students' initial and revised arguments. We are not trying to evaluate the correctness of students' arguments but rather assign a qualitative descriptor of students' justification. This allows us to gain evidence of students'

relative convictions (Weber & Mejia-Ramos, 2015) of what constitutes mathematical proof before and after the instructional sequence.

Method

Participants

This study was conducted across four separate courses across four institutions where the authors were the instructors. Two of the courses (Authors 1 and 4) were mathematical content courses, and the other two (Authors 2 and 3) were secondary mathematics methods courses. Concomitantly, three authors were within mathematics departments, and one author was in an education department. There were a total of 57 undergraduate and graduate mathematics or secondary mathematics education majors participating in the study. All of the participants either had completed calculus courses and an introductory proof course or were presently in an introductory proof course.

Design and Implementation of the Instructional Sequence

The instructional design was organized around the SGP (see Figure 1) into a before-class activity, during-class activity, after-class activity (see Table 1). The participants submitted their solutions to the SGP prior to the during-class activity. The four instructors reviewed the arguments submitted by the students and selected five arguments from each class to have the students evaluate with the during-class activity. Arguments were chosen for their diversity in mode of argumentation, mode of argument representation, and sets of accepted statements (Stylianides, 2007). Simultaneously, the four authors selected arguments that would illustrate differing modes of argumentation to promote discourse around what counts as proof.

Students individually evaluated the five selected arguments in class and then worked in small groups to negotiate and come to a consensus on which arguments they believed to be valid proofs. This negotiation provoked important conversation about what defines a proof because negotiations highlighted if the argument persuaded the students, aligned with how the students had ascertained justification, or both. The students were then instructed to articulate three to seven criteria for determining arguments to be proofs. After each small group created their list, the whole class discussed and agreed upon five appropriate criteria for proof in their classroom. Each of the four classes had some differing criteria, but their criteria all showed that a proof needs to be generalizable and have logical structure (see Boyle, Bleiler, Yee, & Ko, 2015 for a more detailed discussion of the communal criteria). An important note is that these two categories, generalizability and logical structure, are similar to what Harel and Sowder (2007) labeled as necessary characteristics of the transformational deductive proof scheme, generality and logical inference.

Data Collection and Analysis

All four authors collected their students' work and class artifacts throughout the instructional sequence. Each student's initial argument and revised argument were coded according to Harel and Sowder's (1998, 2007) proof schemes to identify their changes. As discussed in the theoretical framework section, Harel and Sowder suggested that overlaps of the categories of external conviction, empirical, and deductive proof schemes were to be expected. Thus, three more categories, "empirical/external conviction, external conviction/deductive, and empirical/deductive," were developed and included in our data analysis. We provide Gabriel's work depicted in Figure 2 as an example of the need for the overlap in coding.

Figure 2 is Gabriel's initial argument coded as Empirical/Deductive. It could be easy to classify Gabriel's argument with an empirical proof scheme because of the justification using one example with three children and three colors. However, there are elements of the deductive proof scheme within his work as well. Gabriel generalized the argument with statements, such as "worst case

scenario" and the concluding remark, "if all but one child have all the colors of gum, any color will ensure they all have the same." These statements reference the restrictive characteristics of the transformational proof scheme (Harel & Sowder, 1998), a subset of the deductive proof scheme. Gabriel presumed to understand the restrictions that transform the situation by assuming the worst case scenario and discussing what transformation would be necessary to satisfy the last child receiving the same color. Thus, Gabriel's argument was coded as Empirical/Deductive. It is important that the proof schemes are not evaluating the correctness of students' arguments but rather than describing what characteristics students communicate with their justifications.

Figure 2. Gabriel's original argument classified via empirical and deductive proof schemes.

All four authors met and agreed upon the structure and purpose of the six categorizations, and evaluated 15 randomly-selected arguments to make sure that there was strong interrater reliability. Authors then all double-coded the remaining arguments. There were a total of 54 original and 46 revised arguments, and each of them was coded by at least two authors. Any varying proof schemes were discussed amongst the authors who coded the proof scheme to find consensus. Once all arguments had been coded and consensus had been reached amongst the codes, the authors compared original and revised proof schemes amongst all students.

Results

Table 2 illustrates the coding of each student's before-class argument (rows) and after-class argument (columns). Colors have been added to illustrate blends. For example, empirical is red and external conviction is yellow, so empirical/external conviction is orange. Each cell in Table 2 illustrates the number of students that transitioned from one proof scheme to another. A row or column identified as "None" means that the students offered no argument.

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

Table 2: Proof Schemes of all original and revised arguments

Brefore-Class Proof Schemes (rows) Revised Proof Schemes (columns) TOTAL	Revised None	Revised Empirical	Revised Empirical/External Conviction	Revised External Conviction	Revised External Conviction/Deductive	Revised Deductive	Revised Empirical/Deductive	Original Arguemnt Total
Original None	0	2	0	0	0	1	0	3
Original Empirical	5	5	3	6	0	6	2	27
Original Empirical/External Conviction	1	0	0	0	0	3	0	4
Original External Conviction	2	1	1	3	0	1	0	8
Original External Conviction/Deductive	0	0	0	0	0	2	0	2
Original Deductive	1	0	0	1	0	4	0	6
Original Empirical/Deductive	2	0	0	0	1	3	1	7
Revised Argument Total	**11**	**8**	**4**	**10**	**1**	**20**	**3**	**57**

The main diagonal of Table 2 shows that five empirical proof schemes remained empirical (row2, column2), three external proof schemes remained external convictions (row4, column4), four deductive proof schemes remained deductive (row6, column6), and one empirical/deductive remained empirical/deductive. Altogether, 13 of the 57 participants kept with the same proof scheme after the instructional sequence, demonstrating that the majority of students constructed arguments that provided evidence of altered proof schemes after the instructional sequence. Looking along the diagonal also reveals that no arguments that was coded initially with an overlapping code (e.g. external/deductive), kept the same proof scheme after being revised except for one student whose original argument and revised argument were coded as empirical/deductive.

Table 2 also shows that the number of strictly empirical proof schemes declined from 27 to 8 while the number of strictly deductive proof schemes grew from 6 to 20 after our instructional sequence. For instance, Kimmie's original and revised argument depicted in Figure 3 is an example of this transition. Kimmie's original argument bases its justification around the given information and justifies the general rule using examples. For her revised argument, Kimmie uses generality and logical inference, which are two aspects of Harel and Sowder's (2007) deductive proof schemes. Kimmie's work illustrates how her argument shifted from an empirical proof scheme to a deductive proof scheme after participating in the instructional sequence.

Finally, the external conviction proof schemes increased slightly from 8 to 10 after the instructional sequence. When looking closely at arguments demonstrating external conviction, the students seemed to tend to replicate a method that had been used in one of the five instructor-selected arguments (e.g., mathematical induction on both the number of children and the colors of bubble gum) to construct their revised arguments. More specifically, those students might mimic one of the instructor-selected arguments without being able to fully justify why their generalization to the SGP is always true.
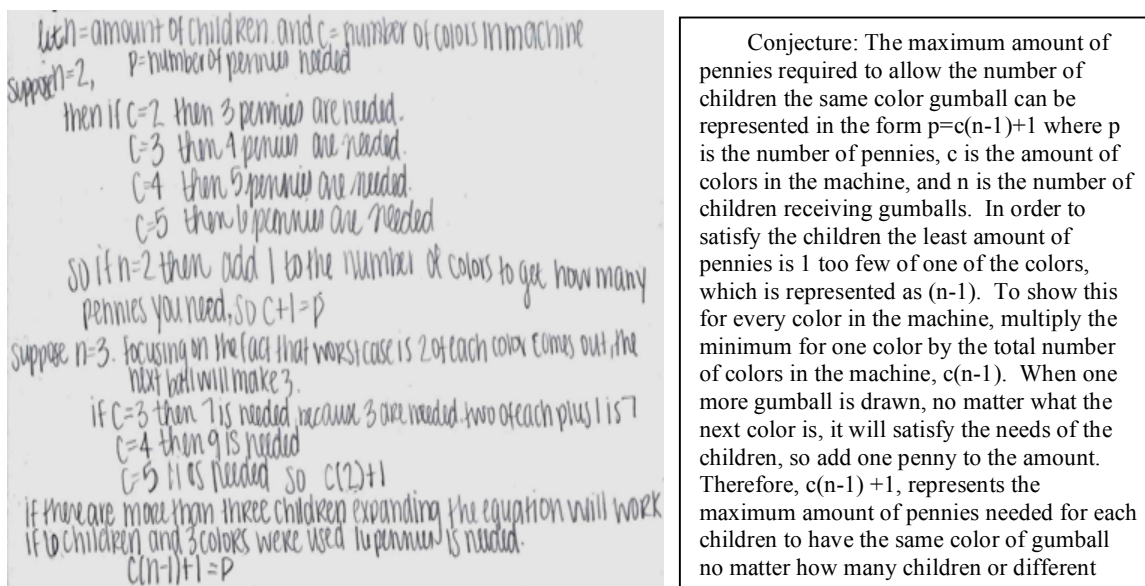
Conjecture: The maximum amount of pennies required to allow the number of children the same color gumball can be represented in the form p=c(n-1)+1 where p is the number of pennies, c is the amount of colors in the machine, and n is the number of children receiving gumballs. In order to satisfy the children the least amount of pennies is 1 too few of one of the colors, which is represented as (n-1). To show this for every color in the machine, multiply the minimum for one color by the total number of colors in the machine, c(n-1). When one more gumball is drawn, no matter what the next color is, it will satisfy the needs of the children, so add one penny to the amount. Therefore, c(n-1) +1, represents the maximum amount of pennies needed for each children to have the same color of gumball no matter how many children or different

Figure 3: Kimmie's initial (hand written) and revised (typed) arguments coded as empirical and deductive, respectively.

Discussion

This study challenges the current borders surrounding students' classroom engagement with constructing and critiquing arguments. Our instructional sequence gives students opportunities to develop criteria for proof through evaluating, discussing, and negotiating what counts as proof. In addition, these results illustrate that the instructional sequence allowed students to judge the validity of arguments, empowering students to generate criteria from those judgements to influence what constitutes as ascertaining and persuading (i.e., Harel & Sowder's proof scheme, 2007) in their classroom community. Thus, this instructional sequence can aid teachers in developing what constitutes a proof through evaluating arguments, as well as discussing and negotiating their classroom-based criteria for proof.

By using Harel and Sowder's (1998) lens of proof schemes, our results show that a majority of students changed their argument after the instructional sequence. Of those changed, the number of empirical proof schemes decreased while the number of deductive proof schemes increased. Also, the number of external conviction proof schemes remained about the same. The decrease in empirical proof schemes and increase in deductive proof schemes might appear to align with the communal criteria for proof students developed, because the two of the criteria shared by all four classes—logical structure and generalizability—are the characteristics of Harel and Sowder's (2007) transformational deductive proof schemes, logical inference and generality.

When evaluating the validity of arguments, Weber and Mejia-Ramos (2015) indicate that students' convictions are often not absolute but relative, meaning that writing an argument is more than justifying one's beliefs. The research described in this paper did not measure the mathematical validity of students' initial and revised arguments. However, we identified characteristics of student justifications with respect to Harel and Sowder's (2007) proof schemes, because our study desired to determine how engaging in our instructional sequence would affect students' revised arguments. When implementing our instructional sequence, one should not see the communal criteria as an absolute final product, but they serve as a means to improve students' understanding of proof and their abilities to write proofs. Thus, we see the generation of classroom-based criteria is not limited to assessment of learning or assessment for learning, but it also serves an assessment *as* learning (Bleiler, Ko, Yee, & Boyle, 2015). A valuable future research study could include multiple iterations

of this instructional sequence with various the proof-related problems to lead to a better understanding of what students have learned about proof and proving over a full semester and across different mathematical content areas.

In sum, this study offers an alternative method of proof instruction to help students take ownership of their proof construction instead of viewing their teacher as the primary authority in the classroom. Moreover, our results show a means in which to have students develop a common understanding of what counts as proof through critiquing the reasoning of others. Thus, this instructional sequence offers a social space in which teachers and students can discuss and negotiate the meaning of what constitutes a proof.

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EXPLORING STUDENTS' UNDERSTANDING OF MEDIAN

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This paper explores middle-grade students' conceptions of median. Describes, where and why they struggle and provides learning trajectory to improve their understanding.

Keywords: Data Analysis and Statistics, Middle School Education

Background and Overview

Measures of Central Tendency (MOCT), including mean, mode and median, are important mathematical concepts typically included in middle grades standards and curriculum (National Council of Teachers of Mathematics [NCTM], 2000). However, the vast majority of research in MOCT has generally focused on the teaching and learning of arithmetic mean, with little study of median or mode (Groth & Bergner, 2006; Mokros & Russell, 1995). The few studies focusing on students' conceptions of median have found that high school and college-level students tend to have more difficulty calculating median than arithmetic mean (Barr, 1980; Zawojewski & Shaughnessy, 2000). Barr (1980) studied high school students' knowledge of median and found that less than half of participating students calculated median correctly and tended to provide incorrect explanations for their answers. Specifically, many high school students did not grasp the concept of median as the middle value in a distribution of data. Similarly, Zawojewski & Shaughnessy (2000) observed that although median is a simpler arithmetic calculation as compared to mean, most undergraduate students in their sample had more difficulty calculating median than statistical mean. Although the literature on students' understanding of median suggests that students have more difficulty with this concept than arithmetic mean, such research does not focus on the nature of such conceptions, or how and why they develop in students, indicating a significant gap in the field's models of students' learning and conceptions of median. Therefore, the purpose of the present study is to explore and model two middle grade students' conceptions of median. The study focused on a small sample of middle grades students both to allow for more in depth analysis of how conceptions are conveyed through student actions and because the middle grades are typically the initial point in K-12 curricula where these concepts are formally studied (NCTM, 2000).

Method

We used a classroom based teaching experiment to build our models of middle grade students' conceptions of median and mode. A *classroom based teaching experiment* involves the teacher-researcher (the first author) in the posing of tasks to construct models of students' conceptions, but also to provide tasks that could scaffold students' learning. As with individual teaching experiments, the approach requires the teacher-researcher to collaborate with an observer (Behr, Wachsmuth, Post, & Lesh, 1984). The approach allows the teacher-researcher to interview students individually or as a group (Behr et al., 1984), and extends the individual-based constructivist teaching experiment to the social context.

The study was conducted in an urban, middle-class school district with 6 sixth-grade students (5 males and 1 female students). Students participated in 14 teaching experiment episodes across 14 weeks in Spring 2015. All episodes were video and audio recorded, with the observer-researcher taking field notes. Each episode took place in a designated room, isolated from external distractions and noise, and all participants sat with the teacher-researcher at a round table. This environment provided the necessary proximity to establish teacher-student relationships and observe students' actions more easily. Episodes involved exchanges in which the teacher-researcher presented a task

focusing on MOCT (mean, median, and/or mode). Participants would work on the task either individually or with peers, and the teacher-researcher would ask probing questions to solicit descriptions of their mathematical thinking. However, each student was asked to explain their answers separately. After each student explained their answers, their written work was collected for further data analysis.

Each episode lasted approximately 45-50 minutes. Tasks for initial episodes focusing on a topic (i.e., median) were prepared ahead of time. However, tasks for all other episode were prepared based on the analysis of the prior episode. Specifically, the teacher-researcher and observer-researcher met and discussed current observations and developed hypotheses following each episode. Based on the hypotheses, a new protocol, including tasks, was developed for the next episode to test the hypothesis(es). For sake of space and simplicity, the present study presents data from episodes 6 and 8 to illustrate preliminary findings for learning trajectories of the participating students Alice and Bob.

Analysis and Findings

Initial conceptions

In episode 6, both Alice and Bob described the median as the middle value of a data set, and generally were able to identify the median with data sets with an odd number of elements. Figure 1 illustrates Alice's written strategy to find median from a data set with an odd number of elements. When asked to explain how she found the median, Alice replied, "Oh, it was easy…you know…you write the number small to big and cross out one number from left and one number from right until you find a number in the middle." This statement conveys a potential interpretation on Alice's part of the median as the middle value in a data set. However, when Alice was asked to find the median for data with an even number of elements, she claimed that there was no median. Figure 2 illustrates Alice's written strategy to find median from a data set with an even number of elements. Noticeably, 17 and 20 are left as is with no continuation additional work by Alice to find the median. As Alice explained, "there is no median because 2 numbers are in the middle…not one. Only one number can be in the middle, not two". Bob's work on both tasks was nearly identical to Alice's with corresponding explanations. Bob explained his position and said, "Median lives in the middle of all numbers…when there is 7 numbers then we have a middle number, but when we have 10 numbers then we do not have a middle number…no median".

Figure 1. Alice's response to data with an odd number of elements for median**.**

Figure 2. Alice's response to data with an even number of elements for median.

Although a seemingly simple 'mis'conception of the definition of the median, we conjecture that both students' conception of median is more nuanced than a simple misinterpretation. Specifically, Alice's description of how to find the median focused on marking off the discrete elements in the data set (once they were sorted). At no point did Alice or Bob data set, but instead referred to the elements within the set. We conjecture that these references, both in the students' spoken descriptions and in their written work, indicate that Alice and Bob were considering the elements as

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discrete objects of the data set instead of seeing the elements as part of a unitized data set. Similar ways of operating mathematically have been described regarding children's counting schemes. Discussing Steffe's schemes for children's number sequences, Oliver (2000) described the difference between pre-numerical counting schemes in which the end number is not a unitized collection of 1s (i.e., a child may count 1,2,…5 but not view the 5 as five 1s), and initial number sequences where they are able to consider a number as a collection of 1s in activity. In the present study, Alice and Bob demonstrated a similar focus on operating on discrete elements in the data set to consider the median as an end result of those discrete elements, instead of as a property of the unitized data set. We refer to this initial scheme, demonstrated in episode 6 by both Alice and Bob as the middle-value scheme. *Middle-value* is the number that is in the middle of a data set with an odd number of elements.

After the learning trajectory

During the teaching experiment, we sought to promote perturbation in students' initial conceptions of median. Therefore, we asked students to begin plotting the data on a number-line. The number-line was chosen due to its potential for presenting the distribution of elements in a data set. Thus, we anticipated that the number-line might facilitate students' operating on data sets and not only data elements. So, students began using the number-lines to model their process for finding the median during episode 8. At one point in the episode, students were provided a data set with odd number of elements and asked to find the median. Alice and Bob initially wrote down each number on the number-line and tried to find which number lay in the center/middle (see Figure 3). Once they found the center/middle number they marked the number and claimed that number as median.
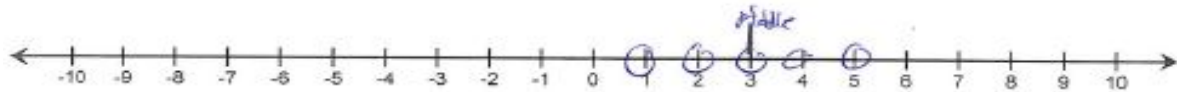

Figure 3. Alice's response to data with an odd number of elements for median.

After finding the median from a data set containing odd numbers of element, Alice and Bob were asked to find the median from a data set containing an even number of elements. Alice began to model the data on the number-line, as in the prior task, and used the number-line to sort the elements of the data on the number-line. Once all data were sorted, Alice identified the space between 2 and 3 (see Figure 4). Alice said, "this set has no middle value…but, we can find it…the middle number has to be in the middle of 2 and 3…we can add 2 and 3 and divide them in half…yes, then 2.5 is the median…(giggling)". Alice provided the response of 2.5 as a middle value of the given discrete elements. We interpret Alice' actions as an application of partitioning the data set when it was considered as a unitzed entity, or a *whole*. This is a distinctly different scheme than the middle-value scheme demonstrated in episode 6. We refer to this scheme as the center-value scheme because it involves operating on a unitized data set. *Center-value* is the calculated middle number from a data set with an even number of elements.

Figure 4. Alice's response to data with an even number of elements for median.

Similar to Alice, Bob said, "I see now…the middle number has to be between 2 and 3. I think, it is 2.5…it is the middle of 2 and 3…I am sure. Although it is tempting to consider the number-line as an intervention that allowed the student to clarify their definition of median, in episodes that

followed, both Alice and Bob continued to use a number-line representation when asked to find median in future tasks. Specifically, when asked to find the median, students would sketch the portion of a number line representing the middle values in a sorted data set (often excluding all other values). This suggests a particular conception of median as the middle that has yet to be fully interiorized, but suggests a new scheme for operating on data to find median. Specifically, the student continued to determine median in activity and used the number line representation to scaffold such engagement. In episodes that followed, Alice and Bob were able to find the median. We conjecture this is due to the construction of a new scheme that allowed them to consider the data set as a whole (instead of only as discrete data elements). Therefore, they were more easily able to identify the middle position of the data set (using a segment of a number line) and determine the middle of the sorted data set as the median value.

Conclusions

Findings of the present study indicate that middle grade students' conceptions of median may include middle-value and center-value schemes. Further, when presented a more visual representation of the data set, Alice and Bob both demonstrated construction of center-value schemes to consider the data set as a unitized whole, instead of as discrete data elements. Alice continued using a portion of a number line to find median, suggesting that the center-value scheme continued to be applied. These findings have several implications. A primary implication for research is that students' conception of the median is more complex than previously recognized. In regards to practice, and given the observed construction of the center-value scheme, these findings suggest that students should be afforded opportunities to manipulate data prior to being introduced to algorithms and purely symbolized definitions of these concepts.

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MAKING THE IMPLICIT EXPLICIT: A CALL FOR EXPLORING IMPLICIT RACIAL ATTITUDES IN MATHEMATICS EDUCATION

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While disparities in the quality of mathematics instruction and learning opportunities are well documented in the literature for African American students, mechanisms that produce these disparities are harder to detail. We argue that work on implicit racial attitudes can contribute significant insights into the ways in which instruction and learning opportunities are differentiated. We use four literatures to support our argument: the quality of mathematics instruction for African American students, teacher-student relationships, racial comparisons of teacher expectations, and racial microaggressions in mathematics. Integrating implicit racial attitudes in our work can achieve more equitable mathematics learning for African Americans.

Keywords: Equity and Diversity, Teacher Beliefs, Instructional Activities and Practices

Disparities in the quality of instruction and mathematics achievement for African Americans are well documented (Ladson-Billings, 1997; Lubienski, 2002). However, the mechanisms that produce such disparities are less clear (Battey, 2013; Lubienski, 2002). In light of the conference theme on questioning borders, we draw on work both inside and outside mathematics education to argue that research needs to explore implicit racial attitudes to better understand dynamics that produce lower-quality instruction and learning opportunities for African American students. We use four literatures to support our argument: the quality of mathematics instruction African American students receive, relationships developed with teachers, racial comparisons of teacher expectations, and racial microaggressions in mathematics. However, before delving into this research, we briefly define and examine the literature on implicit racial attitudes.

Implicit Racial Attitudes

Researchers have found that although explicit racist attitudes have diminished, implicit racist attitudes have not (Dovidio & Gaertner, 2008). Implicit racial attitudes are characterized by unconscious feelings and beliefs, which can be orthogonal to publicly professed attitudes (Greenwald & Banaji, 1995). Implicit attitudes result from exposure to stereotypes and are characterized as evaluations and beliefs automatically activated by the presence of a particular stimulus, in this case, racial interactions (Dovidio, 2001). Contemporary research has found that whites assert that prejudice is wrong and that these values about equality inhibit the direct expression of implicit racial attitudes (Dovidio & Gaertner, 2008). Implicit attitudes are not endorsements for prejudiced views; rather, they represent exposure to broad deficit ideologies.

What keep these beliefs implicit are proxies used for discussing race. Labels that reference culture, family values, or testing categories serve as proxies for race since, socially, we have a difficult time discussing these complex issues (DiME, 2007). This is even more problematic given that whites, who would not discriminate explicitly, tend to discriminate against African Americans when bias can be attributed to a factor other than race (Dovidio & Gaertner, 2004). Racial proxies cloud instances when teachers access stereotypes, which can unconsciously affect their behavior in classrooms including limiting expectations and lowering the quality of instruction offered to African American students.

Quality of Mathematics Instruction

Research shows that mathematics teachers in urban schools are more likely to teach mathematics vocabulary out of context, disconnect taught procedures from students' thinking, use unexplained procedures, assess students based on following steps rather than student thinking or even correct/incorrect answers, and use fewer resources such as manipulatives even when available (Ladson-Billings, 1997; Lubienski, 2002). However, we still do not understand the mechanisms that deliver lower-quality mathematics instruction to African American students. For example, Hill demonstrated that lower teacher knowledge is directly related to student minority status (Hill, 2007). Shechtman and colleagues (2010) found an indirect relationship between teacher knowledge, instruction, and student learning in mathematics because while knowledge was correlated with student learning, it was not correlated with instructional quality. This signifies that these variables may be mediated by factors that the field is not considering, especially for African American students. Factors such as implicit racial attitudes or negative relationships could mediate the effects of teacher knowledge such that many African American students do not receive high-quality mathematics instruction.

Teacher-Student Relationships

Teacher-student relationships have been found to be a critical component of both psychosocial and academic development. Research has found that teachers rate relationships with African American students as more conflictual than their relationships with white students and that conflict predicted lowered standardized test performance in mathematics (Jerome, Hamre & Pianta, 2009; Pianta & Stuhlman, 2004). Teacher-student relationships relate to student learning, regardless of the quality of instruction. Our own work has shown that the quality of relationships is unrelated to the quality of instruction for African American students (Battey & Neal, under review). For example, handling of student thinking and framing student ability were not linked to the quality of problem solving and mathematical discussions. In work from the African American students' perspective, they often attribute stronger teacher-student relationships to placement in higher-tracked classes (Leyva, under review). With students' awareness of their underrepresentation in higher mathematics classes, they come to see themselves at a relational disadvantage with their mathematics teachers and can be dissuaded from seeking support. African American students had to exert extra effort in order to manage teachers' possibly implicit racially biased attitudes. When teachers bring implicit racial attitudes with them unchecked into mathematics classrooms, the burden is placed on students to navigate these spaces. Implicit attitudes hold potential for expanding our understanding of this relational phenomenon, especially in classrooms where high-quality relationships are not developed. Similarly, it can affect teachers' expectations of student behavior and engagement.

Racial Comparisons of Teacher Expectations for Behavior and Engagement

The broader literature in education has found that African American students receive more negative consequences for their behavior. Ferguson (2000) found that the behavior of black[1] boys was rated as more intentional, aggressive, and that punishments were more severe than for white boys performing the same behavior. While punishment of students can remove them from mathematics instruction, it can also impact teachers' perceptions of their cognitive abilities. For instance, Neal and colleagues (2003) found that teachers' perceived African American movement styles as lower in achievement, more aggressive, and more in need of special education services. These studies reveal that racial perceptions of behavior can impact teacher expectations of students' cognitive aptitude and possibly lower the quality of instruction.

Gershenson and colleagues (2015) examined expectations of the same black student by a black and a non-black teacher. Black teachers held significantly higher expectations for students than non-black teachers and mathematics teachers largely drove those expectations. Since the teachers in the

study rated the same students, the authors argue that this is evidence that teachers were unconsciously racially biased. Dovidio (2001) furthered this line of work in mathematics through testing white students divided into 3 groups: explicitly biased whites, implicitly biased whites, and non-prejudiced whites. After solving a logic problem together with black students, both implicitly biased and non-prejudiced whites perceived interactions as positive, while blacks only perceived interactions with non-prejudiced whites as positive. In addition, non-prejudiced whites grouped with blacks solved the problem significantly quicker and groups containing implicitly biased whites solved the problems significantly slower. The implications for mathematics education are that unconscious racial interactions can impact students' problem solving and implicit racial interactions, unconscious for whites, are perceived by black students.

Racial Microaggressions in Mathematics Classrooms

When African American students are asked about their racial and mathematics identities, they often discuss microaggressions. Microaggressions are automatic insults or slights, sometimes unconscious, directed toward oppressed groups (Sue, 2008). Mathematics education research has found a number of interactions considered microaggressions by African Americans. In McGee and Martin's (2011) research, they note instances of framing students as not smart, not belonging in mathematics classrooms, or not being capable of completing work. The interactions drove students to prove stereotypes wrong and note the hyper-visibility they felt in mathematics classrooms. Martin (2006) also noted an instance when a teacher, rather than offering support or advice, told an African American student that they had gone as far as they could mathematically. Finally, Berry (2008) studied how teachers and administrators discriminated against African American males in three ways: resisting to place African Americans in gifted programs, focusing on behavior rather than cognitive abilities, and pre-diagnosing students as having ADHD (Berry, 2008). If mathematics education researchers are noting African American students as experiencing microaggressions in classrooms, then someone must be enacting them. All of these instances are consistent with implicit racial attitudes. Understanding more about this phenomenon can only increase our understanding of transforming mathematics instruction.

Discussion

Research is clear that African American students generally receive low-quality instruction and experience microaggressions in mathematics spaces. Outside work has found that teachers perceive their relationships with African American students as conflictual and that teachers have negative perceptions of student behavior and cognitive abilities. All of these findings are consistent with the enactment of implicitly racially biased attitudes. Combining this work with research on mathematics opportunities to learn has the potential for shifting our understanding of mechanisms that impact mathematics access and opportunity for African American students.

We are not arguing that teachers are aware of ways in which racial attitudes affect their expectations and relationships. Actually, they are likely unaware of the subtle ways in which race impacts their interactions. And yet, high levels of exposure to negative interactions cause students of color to experience frustration and anxiety, leading to disengagement, dropping out, and lower grades (Feagin, Vera, & Imani, 2001; Solórzano, Allen, & Carroll, 2002). Whether teachers are aware or not, their interactions with African American students have significant repercussions. In looking for implicit bias, we might uncover the unconscious ways in which internalized racial narratives are shaping teachers' interactions in undesired ways, with the ultimate goal of supporting teachers to resist these deficit narratives.

Endnote

[1]We use the term black to remain consistent with the author's language and do this throughout the paper.

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AN EXPLORATION OF COLLEGE ALGEBRA STUDENTS' UNDERSTANDING OF HIGHER ORDER POLYNOMIAL FUNCTIONS

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This paper reports on a study of College Algebra students that examined their understanding of polynomial functions of degree three or more. The study included interviews with students as they completed mathematical problem tasks, enabling a focus on the students' on-going cognitive actions. The results summarize how the students' existing conceptual structures guide their actions to produce and make connections among polynomial function graphs.

Keywords: Algebra and Algebraic Thinking, Cognition, Problem Solving

The mathematics community agrees that the study of non-linear functions must have a prominent place in the school mathematics curriculum (National Council of Teachers of Mathematics [NCTM], 2000). This paper is the first in a series of studies we will conduct with College Algebra students, addressing their understanding of non-linear functions. The current study examines students' understanding of polynomial functions, $f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x^1 + a_0, n \geq 3$, with particular emphasis on how students' conceptions guide their actions to produce and make connections among graphs of polynomial functions that are of degree three or more.

While studies of non-linear functions have mostly focused on quadratic functions (Ellis & Grinstead, 2008), there are few studies of higher order polynomial functions (Curran, 1995; Teachey, 2003). Our research questions were: 1) What is the nature of students' understanding of graphs of higher order polynomial functions? 2) How do students express their understandings in mathematical problem solving situations involving higher order polynomial functions?

Theoretical Framework and Related Literature

Though studies of learners' general knowledge of non-linear functions (Vinner & Dreyfus, 1989; Moschkovich, Schoenfeld & Arcavi, 1993) proved useful to our analyses, we noted some limitations. For example, the majority of studies incorporate a multiple representation view of functions, i.e., that the learners' function knowledge can be viewed as consisting of different representations such as tables, graphs and formal algebraic rules. We agree with Thompson's (1994) view that the multiple representations view may not be the best way to characterize the learners' knowledge of functions; rather, such knowledge may be more effectively characterized as a single conceptual structure that can be expressed in a variety of ways (such as tables, graphs and algebraic rules) in specific situations:

> The core concept of "function" is not *represented* by any of what are commonly called the multiple representations of function, but instead our making connections among representational activities produces a subjective sense of invariance … We should instead focus on them as representations of something that, from the students' perspective, is *representable*, such as aspects of a specific situation. (Thompson, 1994, p. 24)

In order to examine the students' conceptual structures of polynomial functions in terms of the 'subjective sense of invariance,' alluded to by Thompson, we investigated the different ways that students made connections among graphs of higher order polynomials in specific problem solving situations. We also analyzed, from the students' perspective, what the graphs represented to them. For example, we note that while quadratic functions have graphs that have a conceptual 'sameness' to them, higher order polynomial functions have a greater set of conditions for the learner to

consider. Therefore, we hypothesized that in order to produce a subjective sense of invariance for higher order polynomial functions, the learner must have a certain conceptual structure of the properties that enables him or her to see how the class of functions of a particular degree can vary in some ways yet still be similar in terms of the degree of the polynomial. Furthermore, we believed that their prior experiences with quadratics might limit their developing a sense of invariance when examining higher order polynomials. Finally, in our examination of students' understanding of what the graphs represented to them, we noted Moore and Thompson's (2015) *shape thinking* construct as a useful way to characterize and classify ways that students conceive of graphical relationships. Specifically, *static shape thinking* "entails assimilations and actions based on perceptual cues and the perceptual shape of the graph" (Moore & Thompson, 2015, p. 784). In contrast, *emergent shape thinking* involves understanding a graph *simultaneously* as what is made (a trace) and how it is made (covariation).

Methodology

A total of twenty College Algebra students participated in the current study. Observing college students solving mathematics problems has proven to be an effective way of modeling the processes of problem solving (Cifarelli & Sevim, 2014; Carlson, 1997; Schoenfeld, 1992) and thus giving insights into their conceptual understanding. Each student was interviewed twice, for forty-five to sixty minutes. All interviews were videotaped.

Table 1: Selected Tasks Used in the Study

Task 1: Without using the graphing functions of your calculator, sketch the graph of the function.	$f(x) = x^3 - 4x^2 - 7x + 10$
Task 2: Which of the following polynomial functions might have the graph shown here: (a) $y = x^4(x-a)^2(x-b)^3$ (b) $y = -x^3(x-a)^3(x-b)^3$ (c) $y = x^2(x-a)^2(x-b)^2$ (d) $y = x(x-a)(x-b)$ (e) $y = -x^2(x-a)(x-b)^4$	

We include the tasks given in Table 1 for the following reasons. Task 1 is a typical textbook problem that requires the students to sketch a graph from a given polynomial equation in expanded form. This task requires that students use their knowledge of end-behavior of the graph to determine a rough initial sketch, find and classify the zeros of the function, and plot additional points as necessary to refine the sketch. In contrast, Task 2 is non-traditional, focusing on the more difficult problem of determining a functional relationship from observation of an actual graph. Finally, we believe that these tasks, especially Task 2, would provide us with data that would both help us better understand the students' conceptual structures and also to make some specific comparisons to the work of Moore & Thompson (2015), particularly to determine if their findings extend to higher order polynomial functions of degree three and higher.

Results and Discussions

The results seem to support our hypothesis regarding how students' prior experiences with quadratics might limit their developing a sense of invariance when acting on polynomial functions of degree three or higher. Specifically, most students related the graph of the given polynomial

functions to parabolas in some way; and all demonstrated some degree of *static shape thinking* (Moore & Thompson, 2015). We will illustrate some of these findings with results from two students, with pseudonyms BD and SE.

In solving Task 1, BD used synthetic division to find a zero, x=1, found other zeros using the quadratic formula and then sketched the graph, stating that "even's touch and odd's cross." In sketching the graph, BD first made a general sketch based just on the location of the zeros, which he then modified by locating the y-intercept and one point on the graph (3, -20), and making estimates of the locations of the relative maximum and minimum points on the graph. When asked to further explain the graphs of these higher order polynomials, BD stated that: "it's all described by either a cubic function or a quadratic function, it's parabolic with an unknown middle or cubic with unknown middle." We inferred that BD could see some differences between quadratics and polynomials of degree 3 in terms of the zeros, and that he could sketch the graph by finding the zeros and some additional points, but we did not observe explicit treatment of the mathematical attributes of the graph as properties of covariation, *emergent shape thinking* (Moore & Thompson, 2015). On the other hand, BD did appear to have a coordinated understanding of the properties of cubic functions in general, particularly the classification of the zeros of the function as well as their end-behaviors.

In contrast to BD's inferred understanding of connections between the function rule and its graph, SE did not appear to make such conceptual connections. In solving Task 2, SE interpreted the given graph as a series of parabolas and recalled properties of quadratic functions in a process of elimination to make her selection.

SE: These are the ones I usually have trouble with. Where it touches the x-axis at 3 points. Usually these I'll see which one makes most logical sense to me.
Interviewer: What's hard about this problem?
SE: I think it's one of those that look very difficult but actually you can dissect it and see which one, see how it works. Yes. I'll just start from left to right and see which one. Obviously the first curve right here (*she points at choice A in Task 2*), so my choices are x^4. It's positive because it's going upwards so it can't be B or E. So it leaves me with x^4, x^2 and x.

In her analysis of the portion of the graph that includes the point (0, 0), SE ruled out the correct answer, E, reasoning that the graph "opens up" and thus the formula cannot have a term with a negative sign such as $-x^2$.

SE: B, in that problem it was a negative and it's also a parabola which means something in there had to be squared. And then there's another parabola here.
Interviewer: So you're thinking of breaking that picture up into parabolas?
SE: Yes. Looking at this parabola, B is going down making it negative. These are what go through my head. We have to pay attention to points a and b because that's what are given. I'd rule out D because this is a parabola right here. There's got to be a reason it goes down and comes up right here (*she points at the portion of the graph from x=a to x=b*). It can't be D because there's no squared. I'd rule out D so would be left with A and C.

We inferred that SE's conception of graphs is consistent with Moore and Thompson's (2015) static shape thinking in that she viewed polynomial graphs as collections of parabolas, thus focusing on graphs-as-wires with certain properties, and seeking to match the graph with the given quadratic expressions based on those properties. She had difficulty in self-initiating her solution activity and demonstrated little computational competence beyond carrying out synthetic division, a skill she could carry out only when the possible roots were provided to her. Although she could verify a particular x-value to be a zero for a given function, she was

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not able to use the result to find other zeros nor was she able to sketch the graph without assistance from the interviewer.

Episodes from BD's interview suggest that he had a more unified view of polynomial functions and their graphs than did SE. In terms of their conceptual structures, we inferred that BD had a well-developed conceptual structure that allowed him to demonstrate flexibility, coherence and reversibility in his actions. More than SE, he could self-initiate problem solving activity and demonstrated that he could connect and coordinate the various properties of polynomial functions. In addition, he demonstrated a sense of reversibility in his actions to the extent that he could reflect on the problem and state specifically what must be true regarding the various properties (zeros, end-behavior).

Implications

We believe the experiences of students BD and SE are somewhat typical of College Algebra students in how they solve polynomial function problems. However, we were struck by the general finding that, for many students the ways that they examined higher order polynomial functions was dominated by their knowledge of quadratic functions, applying properties of quadratic function graphs when analyzing the higher degree polynomials. This finding suggests that it may be important for mathematics educators to have the instructional goal of having students first develop a sense of invariance in working with various polynomial functions before offering them graphs as objects in-themselves with various properties.

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USING PHOTOGRAPHS TO AID IN CHILDREN'S DEVELOPING UNDERSTANDING OF MULTIPLICATION

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In this study, two students were provided with photographs of realistic arrays and given the opportunity to contextualize their thinking and to share their multiplication strategies over 6-7 sessions. This data was then analyzed using a nested chain of semiotics. This study challenges boundaries for what is considered 'traditional' mathematics instruction and instructional strategies that develop in tandem with a child's understanding of their mathematical world. Students were able to demonstrate a progression of understanding, and improve their structuring and enumerative qualities by referencing real-world photographs of arrays.

Keywords: Classroom Discourse, Elementary School Education, Modeling

Typical US textbooks tend to contain illustrations that do not support a student's meaningful construction of mathematical content (Sutherland, Winter & Harries, 2001). Yet, as the external world influences students' mathematical learning (Freudenthal, 1968; Sutherland et al., 2001) photographs could be an excellent resource for instruction. In this study, photographs of "naturally occurring arrays" (i.e. the arrangement of windows or floor tiles) were used to elicit students' multiplicative thinking structures. By analyzing students' abilities to mathematize these structures (Freudenthal, 1968), I explore the following question: To what extent can photographs be used to support a meaningful construction of multiplication?

Theoretical Framework

Students must be given opportunities to interact with the real world while actively engaging their developing mathematical structures through conversation (Sfard, 1998). Unfortunately, there is a tendency to teach mathematics as a set of rules and processes to be memorized, as opposed to developing conceptual skills through constructing relationships (Hiebert, 2013). This study challenges the boundaries between traditional mathematics instruction and instructional strategies that focus on students' methods for multiplicative reasoning when dealing with the structuring and enumerative qualities using photographs of arrays.

Spatial Structures and Arrays

Due to the use of arrays in classroom instruction, elements of *spatial structuring,* defined as "the mental operation of constructing an organization or form for an object or set of objects," are foundational to the explanation of students' representations (Battista, Clements, Arnoff, Battista, & Borrow, 1998, p. 503). When students view the array structures within photographs, they are identifying the shape as it exists in reality, reflecting, and then re-presenting it as their own structure, which is part of the spatial structuring and multiplicative developmental process.

Semiotics

A child's meaning for an object, such as an array, is conveyed to the world through their signs, or words (Seeger, 2005). This is the process of semiotics, and it links previous meaning with new. Using Presmeg's (2006) framework of nested semiotic chains allows for the analysis of children's constructed mathematical knowledge by revealing their process of connecting previously made relations while formulating new ones (Clark & Kamii, 2000). The previous relations are "nested" in the new; the knowledge that comes before is necessary to develop understanding at more

sophisticated levels. It is this nested function that gives the semiotic chain its power for mathematics education (Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997; Presmeg 1997, 2006).

Methods

This study took place in an urban public charter school. One 3rd and one 4th grader will be discussed for this analysis. Over 6-7 individual sessions, photographs of multiplicative arrays were presented in a progressive sequence to students to explore their abilities to abstract, reflect, and re-present what they see using their own knowledge of mathematical structures. At the beginning of each session, they were shown a photograph of an array and told, "Describe the picture using math." Additional questions were asked in order to clarify students' comments or drawings. The images (see Figure 1) were used as models, not as instructional guides.

Analysis

Students' responses were analyzed using Presmeg's (2006) nested semiotic chain. Any words, drawings, or other representations were recorded, collected, and later analyzed as signs, using the semiotic framework. The pictures of arrays provided the student opportunities to share their strategies. By recording strategies or algorithms, the student was afforded opportunities to contextualize thinking while negotiating meaning within the structures of the photos.

Results

I give two examples from each student in an effort to demonstrate the multiplicative progression each student undertakes: one from early in the study and one from later.

Nyla

Nyla was nine years old and in third grade. She spoke fast, counted quickly and always by one. She often "drew" a shape in the air before she explained the mathematics to me and seemed confident in both the shape and the number of "chocolates" in the photograph (Figure 1a):
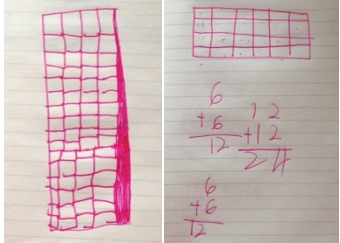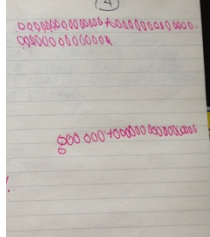

| (a) Week 1, *Chocolate Bar* | (b) Week 4, *Marbles* | (c) Initial (left) and Final (right) Chocolate Bar | (d) Unitized addition |

Figure 1. Photograph models and Nyla's re-presentations for photographs of arrays by Phillips (2005). *Mathematical world exploring mathematics through photographic images.* Emeryville, CA: Key Curriculum Press.

> *Interviewer:* How many do you think are here?
> *Nyla: (referred to image, counted up rows of four counting by ones silently, her lips moving; drew 6 columns with four in each in the air)* Twenty-three…Because look, they all have four in them. See…one two three four.
> *Interviewer:* How do you know there's one there? *(pointing to the covered portion of the bar)*
> *Nyla:* Because I can see a piece of chocolate…look, there's ten, twenty, wait. 1 2 3 4 5 6 7…15… It's going to be more than 23. I'm telling you that. It's going to be fine.

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Unlike her "drawing in the air", this shape was 4 columns, but lacked the six composite units necessary to reproduce her count (Figure 1c). She drew individual squares within the shape, four in each row, until she filled up the space. This indicates Level 2 of Battista's (2007) spatial structuring framework. She had wanted 23 or 24, but could not coordinate her model with her mathematical estimate.

Table 1: Nyla's semiotic chain for the initial *Chocolate Bar* (Figure 1c)

Object	Representamen	Interpretant
Photo of chocolate bar	Draws rectangle and fills in squares one at a time	Her drawing
Her drawing (Figure 2a)	Counts by one	=68

Her iteration of one, and knowing there were groupings of four, were not difficult for her to represent, but she became lost and could not quantify the amount she wanted. I noted that she counted and tagged by one. Nyla subitized the small arrays, though all the mathematics she performed were from signs re-presenting her count, starting with 'one' every time (Figure 2b).

She can partition space, but cannot name it in a multiplicative way. Lacking the numerical understanding necessary to iterate a composite unit, she struggled to quantify the space that she created. This was seen again when she counted the 24 squares in her *final* drawing of the chocolate bar (Figure 1c), four weeks later.

Table 2: Nyla's semiotic chain for counting the squares in the last *Chocolate Bar* (Figure 1c)

Object	Representamen	Interpretant
Chocolate Bar	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24	24
24	1,2,3,4,5,6. 1,2,3,4,5,6. 1,2,3,4,5,6. 1,2,3,4,5,6.	6 + 6 and 6 + 6
6 + 6	1,2,3,4,5,6,7,8,9,10,11,12	24 is the same as 12 + 12

Jessica

Jessica was ten years old and in fourth grade. A verbal-processor, she often repeated, not only what I said to her, but her own thinking as she worked through a problem. She did not rush, but demonstrated a thoughtful and careful process of mathematizing. In Week 1, Jessica coordinated her spatial thinking with discrete objects, not as a collection, but as individual units, in that she counted and tagged each visible element of the chocolate bar array (Figure 1a).

During Week 2, Jessica demonstrated her developing multiplicative thought process. She rationalized as she described the lower array of stars (Figure 1b): "This one…one two three going across, one two three four going down. Three times four is twelve, so there should be twelve so…1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12. Twelve, hmm, my estimate was correct."

Table 3: Jessica's semiotic chain of meaning in referencing *Cityscape*

Object	Representamen	Interpretant
Cityscape	1, 2, 3; 1, 2, 3, 4	3 across; 4 down
3 across; 4 down	3 x 4 is 12, so there should be 12	3 x 4 = 12?
3 x 4 = 12?	1,2,3,4,5,6,7,8,9,10,11,12	Yes, 3 x 4 = 12

She did not rest with the multiplication fact, which was not a fact to her since she stated it *should be twelve,* and instead relied on counting by ones to verify her thinking.

By Week 4, Jessica's ability to both conceptualize and represent number developed. Her semiotic chain indicated a deepened relationship between spatial structuring and enumeration skills (Battista et al., 1998). She no longer counted by one, but felt confident drawing a sign in the shape of a 3x3

array. She had developed in her multiplicative thinking by including multiple levels of abstraction when describing mathematical situations (Clark & Kamii, 1996). She structured her array with discrete items (three columns of three) and extended it to a continuous model using the discrete objects as individual units. The use visual imagery allows students to coordinate and structure arrays (Battista et al., 1998). Her semiotic chain for *Marbles* (Figure 1c) is as follows:

Table 4: Jessica's semiotic chain for *Marbles*

Object	Representamen	Interpretant
Marbles	3, 3, 3	They are all equal
Equal rows of three	Three rows of three	3 x 3 array

Conclusion

Nyla demonstrated spatial structuring skills using a countable unit of one with which to structure space. She recognized the repeatedness of the arrangement of the arrays that it was the same throughout. Nyla improved her spatial structures by modeling the context within the photographs, creating effective tools for her mathematization processes. With further interactions, she may have developed composite counting strategies, as research suggests (Battista et al., 1998).

Conversely, Jessica developed from representing an array additively to re-presenting multiplication *as an array*, through the process of modeling and constructive reflection. In our remaining sessions, she used the spatial representation of an array to represent her multiplicative thinking. This demonstrated an essential understanding of structure-related enumeration from Level 2, to Level 7 (Battista, 2007).

Photographs give students a realistic mental model with which to mathematize their worlds and challenge the boundaries of traditional illustrations currently used in the United States

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LEARNING ABOUT MODELING IN TEACHER PREPARATION PROGRAMS

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This study explores opportunities that secondary mathematics teacher preparation programs provide to learn about modeling in algebra. Forty-eight course instructors and ten focus groups at five universities were interviewed to answer questions related to modeling. With the analysis of the interview transcripts and related course materials, we found few opportunities for PSTs to engage with the full modeling cycle. Examples of opportunities to learn about algebraic modeling and the participants' perspectives on the opportunities can contribute to the study of modeling and algebra in teacher education.

Keywords: Modeling, Teacher Education-Preservice, Algebra and Algebraic Thinking

Mathematical modeling is a critical component in school mathematics as it supports students in developing a way of thinking and interacting with the world that will be necessary later in life. Through modeling, students develop skills necessary to successfully make sense of and interact with complex mathematical systems (Lesh & Doerr, 2003). To teach mathematical modeling, teachers must learn new ways of thinking about and interacting with mathematics. Anhalt and Cortez (2015) argued, "Mathematical modeling problems present unique challenges for teachers, who are not typically required to take courses on modeling as part of their preparation" (p. 2).

In response to new guidance from *Common Core State Standards for Mathematics* (*CCSSM*) related to modeling (National Governor's Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010), there is a need for research exploring whether and how teacher preparation programs have made changes in their curricula to prepare secondary pre-service teachers (PSTs) to learn about modeling based on *CCSSM*. *CCSSM* included mathematical modeling as a mathematical practice across preK-12 grades and as a mathematical content area for High School (NGA & CCSSO, 2010). The *CCSSM* modeling cycle is described as a complex, iterative process in which assumptions are made, tested mathematically, interpreted and validated, and then potentially revised multiple times. Beginning teachers should be familiar with all steps of the mathematical modeling process as described in the *CCSSM*, should have had opportunities to develop their own expertise, and should be aware of the importance of meta-cognitive reflection throughout the modeling process. We report findings from research focused around modeling, conducted as part of a collaborative NSF research project, *Preparing to Teach Algebra*, which investigated opportunities secondary mathematics teacher preparation programs provided to learn about algebra. In this paper, we aim to answer a question, "What opportunities do secondary mathematics teacher preparation programs provide to learn about algebra related to the modeling standards described in *CCSSM*?"

Methods

To investigate PSTs' opportunities to learn algebra and modeling, we conducted interviews with secondary PSTs and instructors of required courses, and collected instructional materials (e.g., syllabi, project descriptions) used in the courses at five universities. We call these universities: Great Lakes University (GLU), Midwestern Research University (MRU), Midwestern Urban University (MUU), Southeastern Research University (SRU), and Western Urban University (WUU). Different

from other universities, WUU only admits PSTs who have already completed a Bachelor's degree in mathematics.

Among all the courses required in the secondary mathematics preparation program at the universities, we selected mathematics courses that include algebra content (e.g., Linear Algebra, Geometry, Statistics and Probability). If the teacher education program required mathematics for teachers courses (e.g., Algebra for Teachers, Geometry for Teachers), we interviewed an instructor from each course. We also interviewed an instructor from each of the mathematics education courses (e.g., Secondary Mathematics Methods) and general education courses (e.g., Teaching in a Diverse Society) that may include opportunity to learn about equity in algebra. Table 1 summarizes the course types and the number of instructors interviewed at each university.

Table 1: Number of instructor interviews by course type

Course Type	GLU	MRU	MUU	SRU	WUU	Total
Mathematics (M)	5	4	6	5	0	20
Mathematics for Teachers (MfT)	1	2	0	3	0	6
Mathematics Education (ME)	3	5	3	4	1	16
General Education (GE)	1	1	1	1	2	6

Additionally, two groups of 3-4 PSTs from each university were interviewed. We designed interview protocols in parallel for instructors and focus groups. We asked instructors and students about modeling generally first. Then we asked specifically about modeling process as described in *CCSSM*. In addition to instructor interviews, we collected corresponding instructional materials.

To analyze the interview transcripts and course materials, we developed six codes, named for each element of modeling process described in *CCSSM*: a) identifying and selecting variables, b) formulating a model, c) analyzing and performing operations, d) interpreting the results, e) validating the conclusions, and f) reporting on the conclusions. Two pairs of us independently coded each transcript based on these nodes using NVivo 10 and we then compared our coding and resolved any discrepancies.

Findings

We address our research question by describing how each group of participants responded to the interview questions related to the opportunities for PSTs to learn about modeling in algebra. The number in Table 2 represents the number of opportunities (e.g., tasks, discussions, lecture, problem) that instructors or PSTs reported related to each element of modeling process at the five institutions. Note that no General Education instructors at any university described opportunities to learn modeling, so we do not include them below.

Mathematics Instructors. Mathematics instructors reported several opportunities for PSTs to learn about "formulating a model by creating and selecting appropriate representations" and "validating the conclusions." One example of "formulating a model" is presented by Probability and Statistics instructor from GLU: "like my take-home final, they were doing a lot of this [formulating a model] because they had data, and they had to analyze it." In terms of "validating the conclusions," Linear Algebra instructor at GLU reported that he provided a unique solution to a traffic flow problem that led PSTs to validate conclusions. Other instructors described how PSTs identified variables, performed operations, and interpreted the results, but no one reported an example of "reporting on the conclusions."

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Table 2: Opportunities to learn about modeling in different types of courses

Elements of Modeling Process	M (20)	MfT (6)	ME (16)	GE (6)	FG (10)	TOTAL
a)Identifying and selecting variables	2	4	3	0	2	11
b)Formulating a model	4	5	6	0	5	20
c)Analyzing and performing operations	1	2	1	0	3	7
d)Interpreting the results	1	2	3	0	7	13
e)Validating the conclusions,	3	2	5	0	5	15
f)Reporting on the conclusions	0	3	3	0	2	8
TOTAL	11	18	21	0	24	74

Note. Number of instructors (or number of focus groups) interviewed in parentheses.

Mathematics for Teachers Instructors. Mathematics for Teachers instructors reported a number of opportunities even though they were few (6 of 48 instructors). One example illustrates how they asked PSTs to engage with modeling and to use the activity for teaching their own students: PSTs identified "functions to fit the data given movie money data" and considered "how to use this activity in future teaching."

These instructors also provided more opportunities related to the modeling process "identify and select variable" compared to the other participants. A following task introduced by Secondary Math Connection 1 instructor at SRU presented such modeling process:

> Ana is sitting in the bucket of a Ferris wheel. She is exactly 46.7 feet from the center and is at the 3 o'clock position as the Ferris wheel starts turning. Sketch graphs that represents Ana's different location on the Ferris wheel.

We see in this task that variables were not given in the problem statement. When PSTs sketched graphs, they decided which quantity would be labeled as independent or dependent variables.

One modeling process reported by not Mathematics instructors, but Mathematics for Teachers instructors instead, was "reporting on the conclusions." A Secondary Mathematics Connections 1 instructor at SRU said, "validating conclusions, reporting on conclusions, if the mindset's done, then this all gets wrapped up into one. You're always interpreting, validating and reporting within that process." This instructor connected several modeling process: interpreting, validating, and reporting the results.

Mathematics Education Instructors. As a group, Mathematics Education instructors provided more opportunities related to "validating the conclusions" than the other instructors. An activity provided by the Secondary Mathematics Methods instructor from WUU shows that PSTs had the opportunity to compare their answers with others during the modeling process. Mathematics Education instructors also provided several opportunities for PSTs to formulate a model. A Secondary Math Methods instructor at MUU described PSTs opportunity to discuss the meaning of variables in context and generate a model using representations when solving for the number of border tiles in a pool. The discussion led by this instructor might help PSTs make connections between the real-world situation (e.g., border tiles for a pool) and mathematics.

Focus Groups. While instructors most often reported opportunities to "formulating a model by creating and selecting appropriate representations," PSTs most often described the element, "interpreting the results." Specifically, they described a process of interpreting the results in the given context: "it's a lot of word problems to where when you're given an equation, it's not just okay x=3 it's like, in the problem we wrote this equation to model this situation. What does that variable mean?" [Focus Group from SRU]. As described, PSTs considered the meaning of each variable in

the equation that represents the problem context. PSTs also mentioned that it would be helpful to have more time validating and refining their process of solving a problem. A PST from MRU said,

> I feel like there is more of a "This is what you need, get really good at this skill" but there's not like a "Explore it and refine what you've seen."

This example shows that PSTs considered the value of exploring the problem on their own and refine the solutions, but they thought such opportunity was not provided much in their program.

Discussion and Conclusions

Findings show that both instructors and PSTs reported examples of opportunities that addressed few elements of modeling process. More opportunities to learn about modeling might be revealed from teacher education programs if our study is extended to mathematical modeling, rather than algebraic modeling. Also, if we had observed classroom practice, we may have been able to find more opportunities addressing the full modeling cycle. Nonetheless, the findings from this study show several opportunities for PSTs to learn about algebraic modeling, but also show that not all opportunities involved the full modeling cycle recommended by *CCSSM*. This result shows that PSTs need more opportunities to explore the complete modeling process (Anhalt & Cortez, 2015; Pollack & Garfunkel, 2013).

Certain modeling elements were emphasized by PSTs and instructors. PSTs reported the value of validating and refining the process of solving a modeling problem. Some PSTs remembered how a mathematics education instructor emphasized reporting and justifying answers throughout the entire modeling process. A mathematics for teachers course instructor also provided an example of reporting conclusions, saying that this process can be connected with interpreting and validating the results. Despite interviewing more mathematics instructors (20 instructors) than mathematics education instructors (16 instructors), more opportunities to learn about modeling described in *CCSSM* were reported by mathematics education instructors. Mathematics instructors, who did not provide any example related to the last element of modeling process (reporting on the conclusions), need to consider specific ways of helping PSTs report conclusions throughout modeling process.

Overall, this study incorporates the large data including interview transcripts of 48 course instructors and 10 focus groups from five universities, and corresponding course materials, which enabled us to present the overview of learning opportunities reported by instructors and future teachers. We also presented the extent and element of modeling process addressed in different courses and found that few opportunities involving the entire modeling cycle. This result supports the need for future teachers to learn about more modeling activities that address all the modeling processes.

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PRODUCTIVITY AND THE DISTRIBUTION OF AUTHORITY DURING COLLABORATIVE MATHEMATICS PROBLEM-SOLVING

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This research brief describes an analytic approach to examining the relationship between productivity and the distribution of student authority during collaborative mathematics problem-solving among elementary learners. Our approach focuses on distributions of both intellectual and social authority – shared, concentrated, and contested – and its relations to how students engage with the collaborative task and one another. We then discuss the utility for such an approach to examine how students work together on collaborative mathematics tasks.

Keywords: Classroom Discourse, Elementary School Education, Equity and Diversity, Problem Solving

Introduction

The new Common Core State Standards in Mathematics expect students in today's schools to more centrally engage in mathematical argumentation with their peers, including constructing mathematical conjectures and critiquing the reasoning of others. These expectations shift authority to students who now have a central role in authoring and evaluating mathematical ideas and managing collaborative tasks. Dynamics around authority, inherent in peer-led mathematical argumentation, affect who gets to speak, as well as whose ideas are judged as worthwhile and ultimately influential.

Group dynamics during collaborative mathematics problem-solving have typically been analyzed in relation to shared metacognition and other regulatory processes (Barron 2003; Goos, & Galbraith,1996; Goos, Galbraith, & Renshaw, 2002; Iiskala, Vauras, Lehtinen, & Salonen, 2011). Less often considered are the relationships of authority among peers and how these relationships shape the ways in which students engage with the collaborative task and one another's ideas. In this research brief, we frame the collaborative mathematics problem-solving process to include two key dimensions: (1) the productivity of the collaboration and (2) the distribution of intellectual and social authority of the collaborating students.

We build on Engle and Conant's (2002) notion of productivity and situate our study in classrooms that embody all four of the principles to fostering productive disciplinary engagement. With respect to productive engagement, Engle and Conant (2002) state:

> …students' engagement is productive to the extent that they make intellectual progress, or, in more colloquial language, 'get somewhere'… such productivity might involve things like recognizing a confusion, making a new connection among ideas, or designing something to satisfy a goal. (p. 403)

Here we frame productivity in terms of students' engagement with one another and the task as they move the collective work forward. These forms of engagement might include: offering an idea, expressing agreement or disagreement, engaging in parallel independent work, building or connecting on peers' ideas, and so on. Of interest is how these forms of engagement might relate to the relations of authority students construct during collaborative work. We draw on Engle, Langer-Osuna, and McKinney de Royston's (2014) and Langer-Osuna's (2011, 2016) constructs of intellectual and social authority. By intellectual authority, we mean interactional moments wherein students are positioned as a credible source of mathematical information. This might include moments where a particular student's help is sought, or moments where a particular student is driving the intellectual work of the group. By social authority, we mean interactional moments

wherein students are positioned as having the right to issue directives to their peers or otherwise manage the group (Langer-Osuna, 2011, 2016; Wood & Kalinec, 2012). While both forms of authority can co-occur, we make an analytic distinction between management and the intellectual work in order to capture their interrelations.

We view this work as crossing critical intellectual borders between the problem-solving literature and research on relations of authority, which is fundamental to addressing issues of access and equity in mathematics classrooms. Our field knows relatively little about how students construct relations of authority in collaborative mathematics classrooms and its effects on the collaborative problem-solving process.

Study Context and Data Sources

This study is based on a broader teacher-initiated partnership between the research team and an elementary instructional team of five teachers (grades K, 1, 2, and 4) in an urban area of Northern California. The teachers sought support in implementing the Contexts for Learning Mathematics (CFL) instructional units (Fosnot 2007) to support student-led collaborative mathematical work. These units draw on relevant story-based contexts to engage students in inquiry-based activity and rich mathematical discussions. The research team sought to capture how students made sense of and negotiated collaborative mathematical work and to provide an ongoing external lens to partner teachers on their implementation of the unit of their choice.

Each teacher chose one CFL unit to implement in her class in collaboration with the research team. In the Kindergarten classroom the teacher taught a unit "Bunk Beds and Apple Boxes: Early Number Sense." In the first grade and two second grade classes the teachers taught the unit, "The Double Decker Bus: Early Addition and Subtraction." In the fourth grade class the teacher taught the unit, "The T-Shirt Factory: Place Value, Addition and Subtraction." Data was collected in each of the five classrooms before, during and after the teacher-chosen unit. Units lasted between three and five weeks, differing by teacher and content.

Participating teachers all worked at the same elementary school, which serves predominantly bilingual Latino and Pacific Islander students from lower-income neighborhoods. Two-thirds of students are designated as ELLs and 93% participate in the free or reduced price lunch programs. We worked with teachers over the summer to support their capacity to notice practices related to the following core ideas: (1) eliciting and responding to student mathematical thinking, (2) student group dynamics and their relation to privileged and marginalized student engagement in collaborative peer work, and (3) bilingual students' forms of mathematical communication (Moschkovich, 1999). During the academic year, participating teachers and the research team met bi-monthly to collectively watch video clips from the teachers' classrooms and engage in discussion focused on how students negotiated their collaborative work.

We collected multiple forms of data. During the academic year, we videotaped whole class discussions and student-led table work throughout the duration of the teacher-chosen unit in each of the five classrooms, capturing between one and four instructional days per week. We used one video camera to capture whole class discussions, and two additional video cameras to capture two focal tables during student-led group work. We additionally collected pre- and post-assessments of individual mathematics problem solving related to the concepts within the instructional unit and their feelings about learning mathematics. We also collected student work throughout the unit. Here, we focus exclusively on videos of student-led table work.

Analytic Process

We offer a novel analytic approach that can capture these interrelations and be usefully applied across multiple video records. We recognize the dearth of publications focused on describing the analytic process of classroom-based research and hope to contribute to greater transparency. The

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purpose of our analytic process is two-fold: (1) to code the collaborative problem-solving process in terms of both authority; and (2) to characterize their relations.

We focus on dyadic (and, at times, group) student problem solving sessions in classrooms (grades K, 1, 2, and 4) across the course of a typically one-month long unit. Specifically, we focus on a total of 49 videos of table work gathered during these units of study: 8 videos of Kindergarten, 8 videos of 1st grade, 14 videos of 2nd grade, and 19 videos of 4th grade.

We repeatedly viewed the videos of table work in order to develop a coding scheme that captured the ways in which students engaged with one another and the task. We were initially guided by Engle and Conant (2002) to look for evidence of "moving the work forward" and student moves characterized in the literature, including "authoring ideas", "expressing agreement/disagreement", "building on the ideas of others", and so on. We added codes until multiple videos from a range of grade levels could be adequately captured with a complete, minimal set of codes. A sample of coded behaviors and moves include (a) building on or connecting to one another's ideas, (b) explaining or thinking aloud, (c) verifying the work of self or others, (d) disagreement or critique, (e) requesting help, and (f) offering ideas.

We then turned to coding interactional dynamics that capture different distributions of both social and intellectual authority: (a) shared authority, (b) concentrated authority, and (c) contested authority. We drew on Engle et al. (2014) and Langer-Osuna (2016)'s analysis of intellectual and social authority. Their work coded bids for authority and their uptake (accepted or rejected) at the interactional level. Because of the number of videos focused on here, coding each interaction in relation to authority bids and their uptake would be too onerous. We code instead interactional events we call authority distributions: shared, concentrated, or contested.

Interactional dynamics representing shared intellectual authority include events where bids for intellectual authority are made by and taken up for more than one student; shared social authority include events where bids for the management of the task or peers are likewise made by and taken up for more than one student. Examples of shared intellectual authority include events where multiple students author ideas for moving the task forward that are considered by peers; examples of shared social authority include events wherein multiple students take part in the management of the work, including the distribution of tasks and the management of off- and on-task behavior. Interactional dynamics representing concentrated authority include events where bids for intellectual/social authority are taken up for only one student; bids by other students are rejected or ignored. Examples of concentrated intellectual authority include dynamics where one student controls the intellectual work of the group. Concentrated social authority dynamics include interactions where one student controls the management of the group. Interactional dynamics representing contested authority include events where multiple bids are ignored or rejected, with no uptake for any student. Contested intellectual and social authority events include dynamics where the right to take control or contribute is explicitly contested across a series of interactions.

Using these codes, we then generate profiles of productivity based on patterns of authority distribution. Intellectual and social authority distributions were organized as a 3 (intellectual authority distributions) X3 (social authority distributions) possible combinations (e.g., instances of shared intellectual/shared social authority, shared intellectual/concentrated social authority, etc). Patterns of student engagement in the collaborative work (productivity) are then compared across the 9 authority distributions across grade levels. From these profiles, researchers are set to investigate numerous questions pertaining to how students navigate the forms of authority afforded to them in student-centered collaborative classrooms. Some of these questions might include: how do students manage collaborative work and the co-construction of mathematical ideas as they engage in complex, open-ended tasks together? What kinds of interactions tend to lead to productive collaborations, and what kinds of interactions tend to make productive collaborations fall apart? What kinds of classroom norms or teacher moves support or hinder how students negotiate their collaborative

work? These questions are central to understanding effective, inclusive collaborative mathematics classrooms.

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DORI AND THE HIDDEN DOUBLE NEGATIVE

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This case study elucidates the difficulty that university students' may have in unpacking an informally worded theorem statement into its formal equivalent in order to understand its logical structure and facilitate constructing a proof. This situation is illustrated with the case of Dori who encountered just such a difficulty with a hidden double negative. She was taking a transition-to-proof course that began by having students first prove formally worded "if-then" theorem statements that enabled them to construct proof frameworks, and thereby make initial progress on constructing proofs. However, later, students, such as Dori, were presented with more informally worded theorem statements to prove. We discuss what additional linguistic difficulties students might have when interpreting informally worded theorem statements and structuring their proofs.

Keywords: Post-Secondary Education, Advanced Mathematical Thinking, Reasoning and Proof

This paper sits at the border between linguistics and mathematics education. It considers linguistic obstacles that university students often have when unpacking informally worded mathematical statements into their formal equivalents. This can become especially apparent when students are attempting to prove such statements. We illustrate this with an example from Dori, who was taking a transition-to-proof course that began by having students construct proofs for formally worded "if, then" theorem statements. Early on, she was introduced to the idea of constructing proof frameworks (Selden & Selden, 1995, 2015) and was successful. Later, she encountered difficulty when attempting to interpret and prove an informally worded statement with a hidden double negative.

Theoretical Perspective

We adopt the theoretical perspective of Selden and Selden (2015) and consider a proof construction to be a sequence of mental or physical actions, some of which do not appear in the final written proof text. Each action is driven by a situation in the partly completed proof construction and its interpretation. For example, suppose that in a partly completed proof, there is an "or" in the hypothesis of a statement yet to be proved: *If A or B, then C*. Here, the situation is having to prove this statement. The interpretation is realizing that *C* can be proved by cases. The action is constructing two independent sub-proofs; one in which one supposes *A* and proves *C*, the other in which one supposes *B* and proves *C*.

A proof can also be divided into a formal-rhetorical part and a problem-centered part. The *formal-rhetorical* part is the part of a proof that depends only on unpacking and using the logical structure of the statement of the theorem, associated definitions, and earlier results. In general, this part does not depend on a deep understanding of, or intuition about, the concepts involved or on genuine problem solving in the sense of Schoenfeld (1985, p. 74). The remaining part of a proof has been called the *problem-centered* part. It is the part that *does* depend on genuine problem solving, intuition, heuristics, and a deeper understanding of the concepts involved (Selden & Selden, 2013).

Proof Frameworks

A major feature that can help a student write the formal-rhetorical part of a proof is a *proof framework*, of which there are several kinds, and in most cases, both a first-level and a second-level framework. For example, given a theorem of the form "For all real numbers x, if $P(x)$ then $Q(x)$", a first-level proof framework would be "Let x be a real number. Suppose $P(x)$. … Therefore $Q(x)$,"

with the remainder of the proof ultimately replacing the ellipsis. A second-level framework can often be obtained by "unpacking" the meaning of $Q(x)$ and putting its (second-level) framework between the lines already written for the first-level framework. Thus, the proof would "grow" from both ends toward the middle, instead of being written from the top down. In case there are subproofs, these can be handled in a similar way. A more detailed explanation with examples can be found in Selden and Selden (2015). A proof need not show evidence of a proof framework to be correct. However, we have observed that use of proof frameworks tends to help novice university mathematics students write correct, well-organized, and easy-to-read proofs.

The Formal-Informal Distinction and Linguistic Obstacles

An *informal statement* is one that departs from the most common natural language version of predicate and propositional calculus or fails to name variables. For example, the statement, "differentiable functions are continuous," is informal because a universal quantifier is understood by convention, but is not explicitly indicated; because the variables are not named; and because it departs from the familiar "if-then" expression of the conditional. Such statements are commonplace in everyday mathematical conversations, lectures, and books. They are not ambiguous or ill-formed because widely understood, but rarely articulated, conventions permit their precise interpretation by mathematicians, and less reliably, by students. In our experience, mathematicians, including those with no formal training in symbolic logic, move easily between informal statements and their equivalent more formal versions.

We suggest that an informal version of a theorem will often be more memorable, that is, be more easily brought to mind and used. However, it may also be more difficult to prove, and given a proof, be more difficult to validate, than a formal version. This suggests the question: Can undergraduates who have taken a transition-to-proof course reliably unpack an informally stated theorem into its formal version? Our earlier paper (Selden & Selden, 1995) indicates that the answer to this question is often no. Because the inability to unpack an informally written theorem statement into a formal version can often prevent a student from constructing a proof, we think that the informal way that a theorem is stated can be a *linguistic obstacle*. Such an obstacle need not be a mistake or misconception (i.e., believing something that is false). Indeed, the obstacles mentioned in the earlier paper (Selden & Selden, 1995) are related to difficulties with unpacking the logic of informally worded mathematical statements.

The Case of Dori and the Hidden Double Negative

What happens when a student is confronted with a hidden double negative in a theorem statement and wants to construct a proof? Using our field notes and photographs, we report on an beginning mathematics masters student, Dori, who in a tutoring session at the end of an inquiry-based transition-to-proof course, was confronted with the task of proving: *A group has no proper left ideals*. (This was in a semigroup setting, in which a nonempty set L is a left ideal of a semigroup S if $L \subseteq S$ and $SL \subseteq L$.) Dori had already experienced proving theorems on sets, functions, real analysis, and abstract algebra (semigroups). She had available to her the course notes, with all previous definitions and theorems. In the same tutoring session, she had just taken 40 minutes to prove, with some difficulty and backtracking, that group inverses are unique. Specifically, she had just proved: *Let G be a group with identity* 1. *If g,* g,' $g'' \in G$ *with $gg' = g'g = 1$ and $gg'' = g''g = 1$, then $g' = g''$.*

Dori, who was working at three seminar room blackboards, next began to prove the theorem about left (semigroup) ideals by writing the theorem statement on the middle board. She wrote: *A group has no proper left ideals*. Dori then looked up various definitions, such as that of left ideal and proper, in the course notes. We then talked with her about what "not proper" means, after which she wrote $GI \neq I$ and $GI \subset I$ on the right board and suggested doing a proof by contradiction. We were surprised at this suggestion, and now speculate this might have been because of the word "no" in the

theorem statement. At the time, however, realizing that this would not be a productive approach, we suggested that Dori write a proof framework as she had been accustomed to doing in the past. She continued writing, below the theorem statement on the middle board:

Suppose G is a group and I is a left ideal of G.

...
Then G=I.
Therefore, I is NOT a proper left ideal of G.

We then suggested that Dori write in her scratch work the properties of a group and of a left ideal of a semigroup. She wrote these additional observations correctly on the right board. These included noting that G has an identity and inverses, that I being a left ideal means that $GI=\{gi \mid g \in G, i \in I\}$, $GI \subseteq I$, and $I \neq \emptyset$. Dori also noted the existence of the identity element, $1 \in G$ and that there is an $i \in I$ and so $i \in G$. In addition, Dori drew an appropriate diagram of the situation.

The emphasis, in Dori's scratch work, on what it means for I to be a proper ideal may not have been helpful, as, according to her proof framework, she was trying to show that I was *not* proper, namely, the negation. It is often difficult for university students to form proper mathematical negations; instead, they often formulate the opposite, as they would in everyday life (Antonini, 2001). Somehow, Dori did *not* note, at this point, that in order to show that $G=I$ (the penultimate line of her proof framework), all she needed to show was $G \subseteq I$.

Difficulties Inherent in Converting the Theorem Statement to its Formal Version

As Dori was working diligently on her scratch work, it appeared to one of us that the informal wording of the theorem statement might be causing Dori difficulty. So, while Dori continued her scratch work, this one of us decided to try to translate the theorem into "if-then" format, judging that it might be easier to comprehend. It became clear that there were two negations involved in the phrase "no proper". The first was contained in the word "no". The second was hidden within the word "proper", which means that the ideal, I, is a proper subset of G, namely, that I$\neq$G. Thus, there is a double negation in the statement of the theorem. Having noted this, we went on to use this observation to write the theorem statement in a positive "if-then" way as, *If I is a left ideal of G, then I=G,* on the left blackboard. We went over this version of the theorem statement with Dori. The positive "if-then" formulation of the theorem has the following apparent advantages: (1) The notation has been introduced. (2) It is in the formal "if-then" form, from which a proof framework can be written in a straightforward way. (3) It does not have a hidden double negation, but rather is entirely positive and straightforward.

Dori had had no trouble introducing the notation. Perhaps this was because of the theorem on inverses that she had just proved earlier; it already contained the notation G for a group and 1 for the identity element. With encouragement from us, towards the beginning of her proof attempt, Dori had initially written a proof framework, introducing the letters G for the group and I for a left ideal of G, and scrolling to the bottom, had written $G=I$ in the penultimate line and had concluded in the final line that I is not a proper left ideal of G, as well as having produced some scratch work. After discussing with her the positively worded version of the statement, namely, *If I is a left ideal of G, then I=G*, we suggested that she "Suppose $1 \in I$" to see what happens. Dori wrote "Let $1 \in I$" and also, "Let $g \in G$, $i \in I$, so $gi \in I$. Let $i=1$, so $g \cdot 1=g \in I$." This essentially completes the argument that $G \subseteq I$, and hence, proves the theorem. From start to finish, this entire proving episode took 45 minutes. Dori, because she was proving and thinking aloud in front of us, gave us unique insight into her proving process—something not possible with written work alone or even classroom proof presentations.

The Hidden Double Negative

Did the presence of a hidden double negation in the informal version of the theorem statement cause Dori difficulty? We cannot say for sure. However, it seems quite clear that the informal version

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of the statement, like many such informal versions, while definitely memorable, is difficult for students to unpack into its formal (positive) version. It is well-known to cognitive psychologists that negations are hard to decode and understand. Pinker (2014) reasoned as follows:

> The cognitive difference between believing that a proposition is true (which no work beyond understanding it) and believing that it is false (which requires adding and remembering a mental tag), has enormous implications for a writer [and a reader]. The most obvious is that a negative statement like The king is not dead is harder on the reader than an affirmative one like The king is alive. Every negation requires mental homework, and when a sentence contains many of them the reader can be overwhelmed. Even worse, a sentence can have more negations than you think it does. Not all negation words begin with n; many have the concept of negation tucked inside them, such as a few, little, least, seldom, though, rarely, instead, doubt, deny, refute, avoid, and ignore. The use of multiple negations in a sentence…is arduous at best and bewildering at worst... (pp. 172-173)

The word "proper" in the above informally worded theorem statement has a double negation "tucked" inside it, and according to Pinker, would be arduous and bewildering.

In future research, it would be interesting to investigate the sorts of linguistic difficulties students have when unpacking informally worded theorem statements, other than (a) introducing suitable notation; (b) structuring a proof when a theorem is not in "if-then" format; and (c) dealing with hidden double negatives.

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AN EXPLORATION OF FIRST GRADE STUDENTS' ENGAGEMENT IN MATHEMATICAL PROCESSES DURING WHOLE GROUP DISCUSSIONS

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This paper describes the manifestation of mathematical processes within two whole group interactions in a first grade classroom. A case study research design was employed, in which an in-depth look at the whole group mathematical discourse within the classroom occurred. Preliminary results indicate students' engagement in emergent forms of the mathematical processes of making sense of problems, creating a mathematical model, and argumentation, which might serve as a basis for a framework describing young children's mathematical thinking.

Keywords: Classroom Discourse, Elementary School Education

Introduction

The role of communication in young students' learning of mathematics has been the focus of study in current and past research. Observations of young children have shown that their conceptions of the world are variable, and bound to the context with which the child associates that knowledge (Sophian, 1999). Based on this notion of variability in children's cognitive development, an implication from this research calls for opportunities in which students communicate their solution methods, through which their thinking transitions from being intuitive in nature to a more explicit state (Sophian, 1999).

Emphasis on communication and other mathematical processes in the learning of mathematics is also described in the Common Core State Standards for Mathematical Practice (SMP) (NGA & CCSSO, 2010) as behaviors, dispositions and thought processes integral to doing mathematics. These standards describe processes and habits of mind grounded in mathematics as a discipline, and do not prescribe a particular teaching method for students' development of them. However, the nature of the standards implicitly calls for an instructional approach in which students communicate their mathematical thinking and provide justification based on reasoning as a means for learning new mathematical ideas and concepts.

Although the processes exist in standards documents, there is much work to be done if the SMP are to be a part of children's formal mathematical experiences. Instructional approaches, which regularly engage students in the SMP, are challenging to implement, particularly with 6-year old children for whom actively participating in mathematics-focused discussions is a new experience. Furthermore, mathematics teachers continue to make sense of the SMP (Olson, T., Olson, M., & Capen, 2013), and more research on their enactment in the classroom is needed (Heck, Weiss, & Pasley, 2011).

The aim of this qualitative study is to describe how the mathematical processes of communicating, reasoning, and forming an argument manifest in students' whole group discourse in the classroom, through which their thinking is made explicit. Possible implications from this research include the formulation of a framework that describes the development of mathematical processes in the thinking of early elementary students.

Methods

A case study design was employed, in which classroom observations of the mathematical discourse of 20 first grade students and their teacher occurred. Observations were video- recorded one–two times a week for a period of four–six weeks, in November, February, and May. Researcher field notes were taken during observations, with semi-structured teacher interviews audio-recorded

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once a week. Video and audio recordings were transcribed and segments of interactions that exemplified any one of the SMP's characteristics were coded and analyzed.

Selection of Subjects

The first grade mathematics classroom for this case study was purposefully selected to be one in which developing mathematical discourse was consistent with the teacher's instructional goals. The teacher had recently completed a school-based professional development project, in which developing mathematical discourse through a problem-based approach had been a focus. Furthermore, she regularly engaged her students in solving open-ended problems and tasks. The tasks primarily came from the newly state-adopted mathematics program, *Stepping Stones* (Burnett, Irons, DePaul, Stowasser, & Turton, 2013), published by Origo Education.

Findings and Implications

The findings for the study reveal the degree to which students were provided with opportunities to engage in the SMP, and whether or not those opportunities were facilitated and capitalized upon by the teacher and students. Furthermore, the findings indicate emergent versions of the SMP, as they relate to students' development in their ability to communicate mathematically and engage in whole group discourse. Students were introduced to the cultural practice of presenting and explaining their strategies and solutions using mathematical language and visuals to the whole class at the beginning of the year. The interactions presented here occurred in February, after students had been introduced to the practice of focusing on their mathematical thinking during whole group discussions.

Making Sense of and Modeling Mathematical Situations

Students were regularly provided with opportunities to collaboratively work through open-ended problems without being shown a pre-determined solution path, and they regularly engaged in making sense of problems for themselves. They were also encouraged to formulate mathematical models for problem situations. The following example was one in which these mathematical processes became salient. The class had worked on a problem in which they needed to determine the number of eggs eaten on Saturday and Sunday knowing that the total was 12. The only constraint was that more eggs had been eaten on Saturday than on Sunday. The discussion was coming to a close, when the teacher quickly put Lane's paper on display:

> *Teacher:* Ok, so look, I wanted to share, this was actually Lane and Greg. Look at their equation. Why did they have 12 and with a minus and question mark, minus a question mark?
> *Lane:* Because we don't see the numbers, we can't see the numbers that we need to make twelve.
> *Teacher:* Ah, so Lane said, it doesn't say about the other numbers. But Lane knew that you had to make....
> *Lane:* Twelve.
> *Teacher:* Twelve. They had to equal up to twelve. So that's how she was thinking about this equation, that something and something has to make.....

This interaction highlights how Lane attempted to make sense of the problem situation, and wrote an equation to model how she thought about the problem. Although Lane was not able to come to a complete solution, through the discourse and presentation, her mathematical thinking is made explicit.

Forming an Argument

As students engage in discourse focused on their solution methods, the definitions and assumptions underlying their assertions, questions, and refutations become the content of the lesson (Lampert, 1990). This view supports students' engagement in mathematical conflict, in which there is a back-and-forth discussion between at least two parties, whereby substantiation of mathematical thinking occurs. In the following excerpt, there is evidence of an emerging argument presented by Ned, who is the only student who sees that blocks might be added to both sides of a balance scale in making the position of the balance true (see Figure 1). The directions for the task were to draw more blocks to make the picture true, then to write the corresponding numbers below the balance.
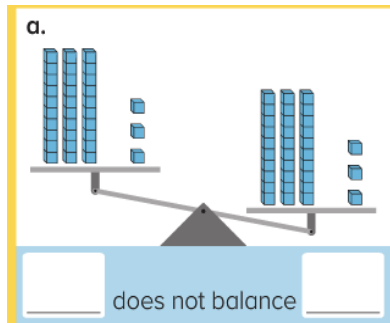
Figure 1. Picture of Balance Scale Problem.

In the following interaction, Peter has just presented his solution to the class, in which he drew ten more blocks on the right pan of the balance in Figure 1, making the position of the balance true. After further questioning by the teacher, Ned argues that although Peter has added ten to the right side, one or two more blocks could still be added to left side and the picture would continue to hold true:

> *Teacher*: Ok, you said yes. You think he could have drawn it over here? (points to left pan)
> *Ned:* Because um, he could make, um, this side that I'm pointing to right now, (points to right pan) go down, he could pretend it's going down, but he could add one more to that side (points to left pan) or that side (points to right pan).
> *Teacher:* Ok, so this is what I think you're saying. You're saying he could have drawn onto this side (points to left pan).
> *Peter:* Only if it was going down (refers to left pan).
> *Teacher:* But–if it was going down (to Peter)? Or do you mean right now (to Ned)? So how many could he draw on this side? (points to left pan)
> *Ned:* He could–like one or two?
> *Teacher:* Ok, (draws in one on left side) he could put one more over there.
> *Jackie:* But the directions say you could only put it on one side.
> *Ned:* It doesn't say you have to go on this side. It doesn't say you have to.

With the teacher's facilitation in making Ned's thinking explicit, the students were able to engage in an emergent debate about the blocks on the balance. Ned was the only student in the class who saw that blocks could be added to both sides of the balance and presented his argument to the rest of the group. During the course of the year, this was the only observed interaction in which the students engaged in conversation involving differing views. For six year-olds just learning to participate in academic discussions, the act of disagreeing in a respectful way that focuses on a person's mathematical thinking, and not on the person's self can be challenging. In this first grade class, although students were beginning to question each other's mathematical thinking, the interactions did not often sustain beyond the asking of an initial question or making a statement. However, it is

evident that as students gain facility with explaining their thinking and asking each other questions, their ability to form an argument will develop.

The findings from this research show that early elementary students are capable of engaging in mathematical processes if they are provided the opportunities to do so through mathematical discourse. For first grade students, the teacher's skill in facilitating discussions in which students have opportunities to explain their thinking for themselves and consider each other's ideas, while at the same time are assisted with making their thinking explicit, is paramount. As more teachers become comfortable with teaching with these mathematical processes in mind, continued study into other classrooms in which mathematical discourse as an approach to teaching and learning mathematics might allow for the development of a framework delineating levels of development for early elementary students.

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EFFECTS OF A VIABLE ARGUMENT (PROOF) ON STUDENT ACHIEVEMENT

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In the United States, Common Core State Standards for School Mathematics call for students to develop viable argumentation practices connected to content learning. The article reports on a research study that explores the efficacy of an intervention for improving Grade 8 students' viable argumentation practices as they learn mathematics content. Treatment students' scores on an argument and reasoning assessment and state standardized assessments are compared to comparison students' scores on these assessments. Findings demonstrate that treatment students outperformed comparison students on both measures.

Keywords: Learning Trajectories (or Progressions), Middle School Education, Reasoning and Proof

Introduction

In the United States, Common Core State Standards for School Mathematics, CCSSM, (National Governors Association Center for Best Practices & Council of Chief State School Officers [NGACBP & CCSSO], 2010) calls for developing students' mathematical practice with *constructing viable arguments and critiquing the arguments of others,* which should be connected to students' content learning. However, interventions that develop students' viable argumentation practices alongside content learning are scarce, if not absent, from the mathematics education literature. At present, the plausibility of learning viable argumentation (proving) practices alongside content has not been empirically supported. The present study explores the efficacy of an intervention for improving Grade 8 students' viable argumentation practices as they learn mathematics content.

Theoretical Framework

Viable Argumentation, Proof, and Proving

We adopt the term "viable argument" to be consistent with the terminology in the CCSSM. The term is not explicitly defined in that document, but it can be assumed the term is used instead of "proof" to note that there are proof-like, but informal, arguments that can be made in K-12 classrooms. Some of the arguments, such as *generic example arguments*, that we will label as "viable" are considered proofs by some authors (e.g., Balacheff 1988; Russell, Schifter, & Bastable 2011; Stylianides 2007). We reserve the term *proof* to describe what others might call *formal proof* or *mathematical proof*: a sequence of statements that demonstrate the truth of a mathematical statement using prior results and logic.

To us, an argument is *viable* if it has a clear, explicit, and appropriately-worded claim and support for that claim that involves acceptable data and warrants. We use broad criteria for acceptable data, including examples, diagrams, prior results and definitions, narrative descriptions, etc. We place more burdens on the warrant by noting the types of warrants that are dubbed acceptable and the types that are not. Using logical necessity, referencing prior results, and describing through narrative the meaning of a definition can be acceptable warrants.

Methods

The LAMP Intervention

The LAMP intervention is the product of the National Science Foundation (NSF) funded Learning Algebra and Methods for Proving (LAMP) project (DRL#1317034). The LAMP intervention is the practice of teaching and learning with and through viable argumentation and a learning trajectory for viable argumentation knowledge and practices. LAMP teachers are encouraged to included viable argumentation in every lesson they teach and on assessments. This includes asking students to develop explicit claims using the language of mathematics (e.g., "for all" and "there exists") and to support claims with viable referents and warrants.

The LAMP learning progression is a sequence of 24 project-developed lessons that target learning a collection of argumentation practices and conceptions as students learned content. Lessons progress in sophistication to build a coherent understanding of viable argumentation. The lessons' argumentation goals were developed from existing proof, proving, and argumentation literature; which is available from the first author upon request. The following list illustrates our assumptions about what Grade 8 students need to know and understand about viably argumentation in the context of Grade 8 content: 1) *Students need a framework to guide their argument constructions and critiques.* 2) *Claim construction is integrally linked to viable argumentation and explicit training on the language of mathematics is needed.* 3) *Students should construct and explore generalizations but should be skeptical of empirical evidence.* 4) *Students should construct referents to support their generalizations and that these referents should express a conceptual insight that structurally links the conditions of the generalization to its conclusion.* 5) *Students need explicit instruction on valid reasoning schemes to align their "every day" reasoning with valid reasoning approaches.* 6) *Students need a variety of modes of argumentation and meta-theories justifying them to construct and critique arguments viably.*

Research Questions

1. To what extent does the LAMP intervention improve students' abilities to write viable arguments and critique the arguments of others?
2. To what extent does the LAMP intervention impact student achievement?

Sample

Data were collected from 210 students (treatment N=117, comparison N=93) taught by seven teachers (4 treatment, 3 comparison) in six public or charter schools during the 2014-15 academic year. Three schools included only students in the treatment condition, two schools included only students in the comparison condition, and one school included students in both treatment and comparison classrooms (the sub-group). Treatment teachers were recruited to the project and were asked to identify a comparison teacher/classroom whose students were similar to their own. In the treatment condition, teachers employed the LAMP intervention and the LAMP teaching practice of including viable argumentation as a regular feature of instruction in Grade 8 algebra and/or traditional mathematics classes. Comparison teachers continued with their "business as usual" lessons/instruction.

All schools enrolled fewer than 600 students. Student populations were majority white, fairly evenly distributed between males and females, had small percentages of English Language Learners, had under 15% of students qualified for Special Education/IEPS, and primarily rural (except the one large school that included treatment and comparison conditions). These similarities enhance the comparability of the samples because differences in outcome scores are less likely to be due to variations in contexts.

Data

Smarter Balance Assessment Consortium (2014–2015). Adopted by 18 states including Idaho, the Smarter Balanced (SBAC, 2012) program develops accountability assessments consisting of two components: performance tasks and an end-of-year computer-adaptive assessment. The end-of-year math assessment was used in this study to measure student achievement. Scale scores were used to determine the extent to which students' overall achievement changed. Categorical mathematics levels based on scale scores were used to measure mathematics achievement categories (below basic, basic, proficient, or advanced).

The Argumentation Assessment contains two types of open-ended items: Four items assess students' ability to construct viable arguments, and one item assesses students' abilities to critique the argument of others. Items were developed by reviewing prior research on proof and proving (e.g., Healy & Hoyles, 2000), reviewing state assessments (e.g., SBAC release items), and with feedback from the external advisory board in Year 1 of the project. The scoring rubric assesses students' abilities to construct viable arguments. Each item is rated on a scale from 0 (*no elements of a viable argument)* to 3 (*elements of a viable argument)*. The rubric was developed through iterative phases involving review and rubric development by the research and evaluation team, a calibration meeting to enhance scoring accuracy among three expert raters, and a pilot sample with 25 randomly selected student assessments. Scores for all problems were summed to create a final score from 0 to 15. Pre and post Argumentation Assessment results from the treatment and the comparison students were analyzed to examine the change in students' ability to construct viable arguments and critique the arguments of others. The sample size varied for each outcome measure as follows: Argumentation Assessment (t- N=85; c- N=17; Smarter Balance Assessment Consortium (SBAC) Scale Score (t-N=117; c-N=93); SBAC Math Level (t-N=106; c-N=93); and SBAC Math Claim 3: Communicating Reasoning (t-N=115; c-N=74).

Data Analysis. Independent samples t-tests were used to assess group differences between treatment and comparison on the SBAC achievement scale scores. Chi-square tests were used to identify differences between treatment and comparison groups on mathematics levels. Paired sample t-tests for the treatment and the comparison groups were used to determine if there were statistically significant changes in overall argumentation and each of the five problems on the assessments prior to and following participation in the LAMP intervention. Effect sizes utilizing Cohen's *d* were reported for statistically significant findings for the achievement outcomes.

Results

Students in the treatment condition outscored students in the comparison condition on the SBAC (2012) state test scale scores (t = -4.98, $p < .001$), with a medium sized effect (Cohen, 1992), Cohen's d = .689. Similarly, within the subgroup, students receiving LAMP intervention scored more highly than the comparison students (t = -3.34, $p < .001$), Cohen's d = .639.

A greater proportion of treatment students were in higher mathematics levels relative to the comparison group ($\chi^2 (3) = 31.40$, $p < .001$). There were differences in the distribution of student sub-scores for the mathematics claim 3: communicating reasoning ($\chi^2 (2) = 12.26$, $p < .01$) suggesting treatment students may score significantly higher on communicating reasoning scores. In the subgroup, a greater proportion of treatment students scored proficient or advanced ($\chi^2(3) = 10.34$, $p < .05$). There were also trend differences in the distribution of sub-scores for the mathematics claim 3 ($\chi^2(2) = 5.48$, $p = .065$), indicating treatment students score higher on communicating reasoning beyond effects of school context.

Paired sample *t*-tests demonstrate that treatment students made significant pre-post gains ($t = 2.85$, $p < .01$) on the Argumentation Assessment and the comparison group did not ($t = 1.62$, $p = 1.24$). Item level analysis showed that treatment students made significant pre-post gains on Problem 1 ($t = 3.60$, $p < .01$), Problem 2 ($t = 3.64$, $p < .01$), and Problem 4 ($t = 3.55$, $p < .01$). The

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comparison students showed significant gains only on Problem 4 ($t = 3.10$, $p < .01$). Within the subgroup, treatment students demonstrated significant gains ($t = 5.76$, $p < .01$), while the comparison students' total scores decreased significantly ($t = -5.19$, $p < .01$). These subgroup results should be interpreted with caution due to the small sample size (t- N=34, c-N=9).

Discussion

This study did not control for baseline differences between the treatment and comparison students on mathematics achievement. In 2013, Idaho adopted SBAC but did not report the 2013-2014 SBAC scores, so 2012 SBAC baseline data was unavailable. This study cannot rule out selection bias. The research team had no control over which students received LAMP intervention. Another limitation is the low completion rate of the post assessment by comparison students (N=17). At the end of the school year, comparison teachers reported feeling pressed for time. Despite these limitations, the findings support an assertion that the LAMP intervention improves students' ability to construct critique arguments viably. The findings also support that the LAMP intervention did not hinder student content learning, and perhaps enhanced it.

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JUMPING THE ACHIEVEMENT STEREOTYPE THROUGH MODEL-ELICITING ACTIVITY: A COMPARISON AMONG STUDENTS AT DIFFERENT PERFORMANCE LEVELS

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Keywords: Modeling, Problem Solving

Introduction, Theoretical Framework, and Methodology

This study investigated students' thinking, understanding, and mathematical development when solving mathematical classroom open-ended, real-life context tasks in which students were required to model a solution. Specifically, Thought-Revealing activities known as Model-Eliciting Activities (MEAs) (Lesh et. al 2001) were used to compare and contrast the reasoning and solutions of 11th grade students at different levels of performance in Monterrey, Mexico. Two dimensions of contrast and comparison were considered: The quality of the students' solutions and the sophistication of the mathematical construct. MEAs are mathematical open-ended, client-driven, learning tasks that usually encourage students to mathematize by "quantifying, coordinatizing, categorizing, algebratizing, and systematizing" (Lesh and Doerr, 2003, p. 2) real-life situations to produce a model. According to Lesh and Doerr (2003), models are conceptual systems composed of elements (e.g. variables, tables, symbols, procedures, and procedures), which interact among themself, and where relations, operations, and rules dictate how these elements interact. Six principles (Lesh et al. 2001) guide the design of MEAs: 1. reality, 2. model construction, 3. self-evaluation, 4. model externalization, 5. simple prototype, and 6. model Generalization. In this research study, two MEAs were implemented during the summer and fall 2015. Sixty-five students of different level of achievement (e.g., low, average, and high performance) as measure by tests and classroom activities– participated in the study. A design-based research methodology was used to implement the two case studies of MEAs adapted to the context and reality of the target populations. For the analysis of the quality of the students' solutions, I considered all the artifacts and written material students invented and created during the solving process. For the sophistication of the solution, I considered the elements – i.e., variables, graphs, symbols, notations, tables, list, and procedures – used during the mathematical-construction process, or model development. Some of the question addressed in this research included: To what extent do the solutions to MEAs provided by students at different levels of performance differ? What are the mathematical elements that students used during the problem-solving process, and how do these differ among students labeled as low, average, and high performance? Preliminary findings include the ability of the low-performance students to propose and develop more creative solutions. Furthermore, solutions provided by the average students were not as different as the one provided by the low-performance. Finally, although the high-performance students used more sophisticated elements in their procedures, their solutions were incomplete and not well explained.

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CROSSING BORDERS AND BUILDING BRIDGES WITH PROBLEMS

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"Connected Mathematical Thinking"

This is what I call thinking that seeks ways to bridge or unify a variety of different things. Its value is that the knowledge one then holds, instead of being fragmented and disconnected, becomes networked and more coherent.

But connected mathematical thinking does not happen automatically; it requires a special kind of skill and disposition, and these can be developed through experience. So if one wants students to engage in connected mathematical thinking, they should have opportunities to learn it. How can one design instruction that presents students with situations that are rich in connection making possibilities, and then gives dedicated attention to such connection making?

This study presents two approaches to this, both conceived around the relationships between mathematics problems and mathematics itself. One approach uses what I call, "cross-domain problems." The other uses what I call, "common structure problem sets." Each of these problem designs calls for significant mathematics connection making, but of very different kinds in the two cases.

Many Mathematics in the Same Problem

A *cross-domain problem* is defined to be one whose solution draws resources from two or more domains of mathematics. Such problems can present obstacles to students who tend to "type cast" a problem as belonging to a single domain, perhaps as a result of the compartmentalization of the curriculum. The student will mobilize a tool kit from that domain, but then get stuck when the solution process requires crossing a domain boundary. Here are some examples (see Table 1).

Table1: Sample Cross-Domain Problems

Cross-Domain Problem	Comments
Let N be a positive integer. Call a factor d of N isolated if d and N/d are relatively prime. Show that the number of isolated factors of N is 2^r, where r is the number of prime divisors of N.	Starts in multiplicative arithmetic, and crosses into combinatorics.
Show that any product of d consecutive integers is always divisible by d!	Multiplicative arithmetic application of combinatorics (formula for n-Choose-d).
What is the base-1,000 representation of $N = 48,574,623,791,105$?	Often unnoticed connection of our language and notation for large numbers with the concepts of place value.
Find all (real valued) functions f(x) of a real variable x that satisfy the condition: (*) $\|f(x) - f(y)\| = \|x - y\|$ for all real numbers x and y.	A geometry problem (isometries of the line) that appears to be an algebra or pre-calculus problem.

The Same Mathematics in Many Problems

A *common structure problem set* is defined to be a (diverse) set of mathematics problems that can demonstrably be shown to have a substantial mathematical structure in common. Finding, and articulating, that common structure is a kind of synthesizing mathematical action that resembles theory building practices in disciplinary mathematics. An illustrative set of thirteen problems will be presented, together with a design for its instructional use will be presented.

RELACIONES DE EQUIVALENCIA PARA LA DEMOSTRACIÓN EN ÁLGEBRA

EQUIVALENCE RELATIONS FOR PROOF IN ALGEBRA

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Palabras clave: Álgebra y Pensamiento Algebraico, Razonamientos y Demostraciones

Las actividades de demostración en álgebra en el nivel superior requieren, entre otras cosas, de un manejo adecuado de la operatividad, así como de contar con un buen número de expresiones equivalentes, por ello es importante desarrollar el significado del signo "=", que en la escuela básica se asocia sólo a una actividad operativa, mientras que en álgebra se ve asociado también a la equivalencia, proceso que debe ser desarrollado tanto en sus aspectos relacionales como en los operacionales, Kieran (1981).

Algunos de los recursos algebraicos que son usados con frecuencia en la demostración algebraica pasan desapercibidos, tal es el caso de procedimientos como: 1. Agrupar y desagrupar la expresión. 2. Realizar una operación descrita. 3. Sumar cero, que es la base de indicaciones como completar productos notables. 4. Multiplicar por uno de manera adecuada como en el caso de "multiplicar por el conjugado". 5. Factorizar y expandir.

A pesar de ser recursos válidos, aunque se debe considerar los dominios de definición de las expresiones, para el estudiante éstos son sólo trucos usados arbitrariamente, por lo que se debe hacer un esfuerzo para incorporarlos como algebraicamente aceptables y en particular sustentados en la idea de equivalencia. Es aquí en donde consideramos que los problemas planteados deben incorporar aspectos tanto operacionales como relacionales con el objeto de formar una base para el tratamiento de la demostración en álgebra.

Por ejemplo:

1. Para mostrar que $x \cdot 0 = 0$, primero se suma un cero $x \cdot 0 = x \cdot (0 + 0)$, después se distribuye $x \cdot 0 = x \cdot (0 + 0) = x \cdot 0 + x \cdot 0$ y sumando $-(x \cdot 0)$ a los extremos se concluye que $x \cdot 0 = 0$.
2. Para mostrar que si $a \cdot x = a$, donde $a \neq 0$, entonces $x = 1$; primero tenemos que $ax = a$ y como $a \neq 0$ existe $a^{-1} \neq 0$, multiplicando entonces a^{-1} por la izquierda obtenemos que $a^{-1}ax = a^{-1}a$, haciendo la operación nos resulta que $1x = 1$, con lo que se concluye que $x = 1$.

Los estudiantes cambian su actitud respecto a la equivalencia, luego de que se hace mención de que existen recursos disponibles para modificar las expresiones algebraicas que persiguen un objetivo específico como el de la demostración.

Keywords: Algebra and Algebraic Thinking, Reasoning and Proof

The activities of proof in algebra at the university level require, among other things, an adequate handling of operability. These activities also include equivalent expressions. Therefore it is important develop the meaning of the "=" sign, which is associated to a operability activity in elementary school, while in algebra it is also associated with equivalence; in this level both relational and operational aspects need to be developed (Kieran, 1981).

Some of the algebraic resources that are often used in the algebraic proof pass unnoticed, such is the case with the following processes: 1. Grouping and ungrouping the expression, 2. Executing a described operation. 3. Adding zero, which it is the basis of the instructions to get notable products, 4. Multiplying by an expression equivalent to one, as in the "multiply by the conjugate" case, 5. Factoring and expanding expressions.

Despite being valid resources, which should be considered domains of definition of the expressions, for the student these are just tricks that are arbitrarily used. Efforts should be made to incorporate them as algebraically acceptable and in particular based in the idea of equivalence. It is here where we consider that the teaching must incorporate both relational and operational aspects in order to form a basis for the treatment of proof in algebra. For example:

1. To show that $x \cdot 0 = 0$, first we have to add a zero $x \cdot 0 = x \cdot (0 + 0)$, later we distribute $x \cdot 0 = x \cdot (0 + 0) = x \cdot 0 + x \cdot 0$ and adding $-(x \cdot 0)$ to the extremes we conclude that $x \cdot 0 = 0$.
2. To show that if $a \cdot x = a$, where $a \neq 0$, then $x = 1$; first we have $ax = a$ and how $a \neq 0$ exists $a^{-1} \neq 0$, multiplying a^{-1} to the left we obtain that $a^{-1}ax = a^{-1}a$, by doing the operation we find that $1x = 1$, thus we conclude that $x = 1$.

The students change their attitude towards equivalence, after mentioning that there are resources available to modify the algebraic expressions that follow a specific objective like proof.

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MANIPULATIVES IN MATHEMATICS LEARNING: DOES IT MATTER IF CHILDREN VIEW THEM AS TOYS OR TOOLS?

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Manipulatives—concrete objects that represent abstract concepts—are commonly used in mathematics instruction, but they are not always effective at promoting learning (Moyer, 2001; Sowell, 1989). One possible source of this difficulty is the need to consider an object simultaneously both as an object and as a symbol (Uttal, Scudder & DeLoache, 1997). From this perspective, one possible explanation for why students differ in their ability to learn from manipulatives is that they differ in how they view the manipulatives. Some children may attend to their function as representations of mathematical ideas, and some may attend to their function as physical objects.

Allowing children to play with mathematical manipulatives, before learning with the manipulatives, may decrease their ability to view the manipulatives as representing mathematical concepts. When allowed to play with manipulatives, children may have difficulty viewing them as symbols, rather than as toys. Further, without highlighting an object's status as a tool used for learning math, children may not make the connection from the manipulative to the concept to be learned.

We examined whether children's construal of a mathematical manipulative—either as a toy or as a tool for doing math—influences their learning from a lesson with that manipulative. Children (7-9 years old) were randomly assigned to one of four conditions for learning about mathematical equivalence. We varied the way in which children were introduced to buckets and beanbags using a 2 (toy representation) x 2 (mathematical representation) experimental design. The session included a problem-solving pretest, a lesson with the manipulative, a problem-solving posttest and transfer test, and a brief assessment of conceptual knowledge. Our preliminary findings suggest that children who were introduced to the manipulatives as tools for doing math displayed greater success at posttest and transfer test, and greater conceptual understanding of equivalence after the lesson.

Children who are encouraged to view mathematical manipulatives as tools for doing math may have greater success connecting the objects to mathematical concepts. This suggests that children may benefit from being encouraged to view manipulatives as mathematical representations, and not as objects, leading them to utilize the manipulatives more effectively, and more effectively link them to mathematical concepts. Teachers might consider providing explicit instructions that enhance the status of the manipulatives as symbols for mathematical ideas.

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SUPPORTING DISCURSIVE SHIFTS TOWARD MATHEMATICAL DEFINING

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Keywords: Instructional Activities and Practices, Reasoning and Proof

The practice of defining differs across academic and non-academic discursive genres. As a result, students often have trouble formulating, interpreting, and using definitions in mathematical proof and problem solving. This design-based research project evaluated the conjecture that under appropriate conditions students could be steered to leverage naturalistic discursive practices in generating mathematical definitions. Specifically, I examined for learning effects of a dyadic game-based activity requiring the resolution of ambiguity in reference.

Theoretical Framework

Mathematical definitions, I submit, are not inherent truisms but rather meaning-relations built to fulfill particular purposes, often via an iterative process of formulating and evaluating a definition with respect to its utility, conformity to intuition, or other criteria. A definition, I further submit, could be viewed as an *epistemic form* (a target structure guiding inquiry), the end product of an *epistemic game* (Collins & Ferguson, 1993). The Specifications Game (SG) was designed to prompt students to reinvent one such epistemic game. The outcomes of SG were interpreted through the lens of commognition as a theoretical framework (Sfard, 2007).

Methods and Data Sources

Four dyads (ages 11, 15, & 22) participated in a pilot implementation of SG. One student attempted to specify a missing shape (e.g., triangle), while the other provide counter-examples (Fig. 1). I then prompted them to examine and modify their list of requisite properties into a conjectured definition and compare and evaluate alternative definitions for the object. Specifications given (e.g., "tall," "straight sides") were analyzed for discursive shifts in the course of game play. All data was examined for participants' meta-discursive rules of defining.

Figure 1. A game board (left) and its pool of game pieces (right). Define the missing shape!

Conclusions

As they apprentice into the field of mathematics, students gradually move away from a view of mathematical definitions as arbitrary systems of constraints couched in non-normative language, towards a view of definitions as purposeful epistemic forms satisfying specific properties. By drawing on familiar cultural practices, setting up mathematical language as a potential solution to ambiguity, distributing the process over two participants, and encouraging reflection, SG supports students in enacting an epistemic game of formulating definitions. In this process, students become sensitized to properties of definitions such as specificity and minimality, resulting in a discursive shift away from mundane practices of defining.

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RELATIONSHIPS AMONG MATHEMATICS AND SCIENCE REASONING PRACTICES

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Previous studies discussed that students in student-centered science classrooms showed higher growth in mathematics than their peers in traditional learning environments (e.g., Adey & Shayer). It is reasonable to assume that practices in science classrooms had significant roles in increasing mathematics achievement (Choi, Seo, & Hand, 2015). To understand how the growth in mathematics occur, we investigated students' *intellectual and cognitive resources* (Bailin, 2002) students used when learning. We hypothesized that reasoning used in science practices influences students' learning of mathematics. Specifically, we want to empirically investigate if reasoning developed in learning science affect fourth-grade students' mathematical reasoning.

We adopted categories of intellectual resources from the TIMSS 2011 assessment framework (Mullis, Martin, Ruddock, O'Sullivan, & Preuschoff, 2009) for the reasoning and applying domains in mathematics and science. We employed the Generalized DINA (de la Torre, 2011) to analyze 2,287 fourth-grade students' responses in subcategories of reasoning and applying domains and conducted the path analysis to elaborate the relationships between reasoning domains in mathematics (MR) and science (SR). Then we compared the direct and indirect relationships between SR and MR with and without mathematics applying (MA) as a mediator.

Without MA, the standardized direct effect of SR on MR was .977 ($p < .01$; CFI=.919, TLI=.869). However, this direct relationship was dramatically changed when MA was added. The standardized direct effects of SR on MA and MR is .587 and .065 respectively ($p < .01$ for both coefficients; CFI=.938, TLI=.913). The direct effects of SR on MR was changed from .997 to .065 by considering MA between SR and MR. The indirect effect of SR on MR was .561, which indicated that 95.57% of the total effects of SR on MR are indirect through MA.

A possible reason for this indirect relationship is that SR directly influences MA through students' solving science-context problems while indirectly affects MR focusing on unique aspects of domain-specific reasoning skills. The findings indicate that students could experience the common practices in the two subject areas disjointedly or teachers do not have good understanding of those practices either. The findings suggest that educators should consider the shared cognitive skills in the two disciplines into classroom-level and curriculum-level practices when developing and employing curricula.

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MIDDLE SCHOOL STUDENTS' CONCEPTIONS ON PROPORTIONAL REASONING

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Proportions are an important mathematics concept taught during middle school. In fact, proportional reasoning is "a milestone in student's cognitive development" (Lobato & Ellis, 2010, p. 48) and plays a critical role in developing algebraic thinking and function sense (National Council of Teachers of Mathematics, 2013; National Mathematics Advisory Panel, 2008). However, ratios and proportions are traditionally difficult concepts as Lamon (2007) stated: "the most difficult to teach, the most mathematically complex, the most cognitively challenging" (p. 629).

In this study, students (n = 59) in grades 6-8 in the Midwest of the United States solved a set of eight tasks involving ratios and proportions in an open-ended manner. Contextual tasks and non-contextual tasks were mixed in the assessment set. From initial coding, we constructed the following codes to describe what methods or misconceptions made by the students: additive reasoning (A), cross multiplication (CM), equal amount (E), equivalent ratio/fraction (Erf), and confused about "whole" (W). Then, the students' written responses were coded, using the method of content analysis. Using inter-rater reliability, we have 95% agreement on one task among two authors.

We would like to share the result of one of the tasks, which involves the context of proportional division because the students in all grades struggled the most to solve this task. Approximately 10% of them were able to find a correct answer. Other students made multiple types of errors, such as using additive reasoning instead of multiplicative reasoning (36%), incorrectly applying an algorithm of equivalent ratios (19%) or cross multiplication (5%), and confusing with the whole quantity (16%). This reveals that knowing algorithms does not guarantee students can solve a proportional task as Tjoe (2014) argued. This implies that these students have not developed proportional reasoning although they have procedural knowledge of proportions.

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WORD PROBLEM SOLVING: PARSED MODELS

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The Common Core State Standards (CCSSI, 2010) specially emphasizes the importance of word problem solving. However, since several millions of students in the U.S. have low reading comprehension (Heller et al., 2007, Biancarosa et al., 2006), we can expect for many students, challenging word problems are unapproachable. To help students comprehend word problems, a *word problem parsed modeling* technique is developed. This technique is similar to *bottom-up parsing* and concentrates on developing a thorough understanding of a word problem's details.

It is known for a long time, two strategies—writing meaningful chunks of sentences one per line and breaking sentences into smaller, simple sentences—improve reading comprehension of low-level readers (Mason et al., 1978). This study developed *word problem parsed modeling* to combine two methods—chunking text into separated short sentences and writing these sentences, one per line. The method also involves abbreviations and pronoun decoding.

This study is based on active production—students' success with chunking, pronoun decoding, elimination of unnecessary data, and solving the problem—signaling their mathematical comprehension. Observations of students' work and informal interviews lead to designing a unit teaching word problem parsing. Next, this unit was used with multiple groups of students. The participants were 4th–8th graders, 20–30 students per grade annually, attending a Midwestern U.S. suburban learning center from 2008 to 2016.

Students' parsing models demonstrated students' misconceptions connected with decoding pronouns in complex in compound sentences and some simple sentences as well. For example, the sentences "A is greater than B and greater than C" and "A is greater than B, which is greater than C" were frequently decoded the same way. Additionally, students could not parse sentences involving clauses such as *that of*, *as much as*, and *in turn*. After remediation instruction, 8–15 classes in parsing, **all participants** were able to create effective parsed models. Such high performance was a result of an *individual pace* approach used in the learning center, where each student spends as much time as needed on a topic for understanding on the level of application.

A combination of an active production method with parsing helped create a set of short parsing exercises and challenging word problems, which significantly improve students' comprehension of word problems' text.

Besides word problem solving, the approach can be used to improve comprehension of a broad-range of informative texts. A further study could test the parsing technique with a larger variety of students.

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THE INTERPLAY BETWEEN STUDENTS' CONCEPTIONS OF PROOF AND PROOF-RELATED CLASSROOM NORMS AND PRACTICES

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While there is a wealth of research on students' conceptions of proof, less is known about how students' conceptions of proof relate to their experiences in classrooms. Although, through task selection, instructional emphases and practices, teachers are influential on what meanings and understandings of proof and proving students construct, very few studies (e.g., Harel & Rabin, 2010; Martin, McCrone, Bower, & Dindyal, 2005) have investigated the relationship between instructional practices and students' understanding of proof. Therefore, this study aims to investigate the relationship between students' conceptions of proof and instructional practices, by taking into account teachers' conceptions of proof and classroom norms as well.

To better understand the intricate relationships between teachers' conceptions of proof, their instructional practices, and their students' conceptions of proof, a single case study method (Yin, 2003), which allows an in-depth examination, was employed. Thus, a secondary (ages 15-16) mathematics class was observed and videotaped for 18 lessons; classroom artifacts such as lessons plans, homework, and student work were collected. Additionally, semi-structured, task-based interviews were conducted with the teacher and the students. The teacher was interviewed at the beginning and at the end of the study. 18 students (out of 31 students) were interviewed throughout the observation phase, and 7 focus students were interviewed again two months after the observations were complete. The semi-structured interviews included questions asking participants to evaluate hypothetical student proofs, to describe what proof and proving mean to them and how they view the role of proof, as well as to prove a given mathematical statement.

The data analysis process includes two foci. While the first analysis approach focuses on the teacher's and the students' conceptions of proof, the second approach focuses on the classroom norms and instructional practices related to proof. The first analysis examines the participants' discourse when they talk about their views about the nature of proof, the roles and purposes of proof, what constitutes proof, and what they understand about proof. The second analysis identifies the instructional practices and norms related to proving. Then, these analyses are incorporated by examining how the teacher's conceptions of proof are manifested in her instructional practices and, in turn, how those practices contribute to the students' developing conceptions of proof.

The poster will present in what ways the identified classroom norms and practices relate to the students' conceptions of proof. Implications of the identified links between instructional practices and students' proof conceptions will be discussed.

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COLLEGE STUDENTS' PERCEPTIONS OF PROOF METHODS AND THEIR LECTURE NOTES

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Research on proof at the tertiary level has largely focused on students' difficulties with understanding, constructing, and evaluating proofs and on teaching strategies. This study examined what perceptions of five proof methods—direct proof, proof by contrapositive, proof by contradiction, proof by cases, and proof by induction—college students developed during a transition-to-proof course, and what relations there exist between students' perceptions about the proof methods and their class notes; such perceptions might relate to their proof construction and proof validation. Social constructivism perspectives, along with the theoretical work of Wood, Cobb, and Yackel (1995), were used to ground the analysis of this study. Such perspectives consider knowledge as socially and culturally constructed (Ernest, 1990), thus helping in the understanding of students' learning in the context of a mathematics classroom. For this smaller study, I used open coding (Strauss & Corbin, 1990) after identifying and separating the parts in which three participants provided their thoughts about the five proof methods over the course of four interviews conducted while enrolled in a transition-to-proof course. I analyzed and summarized my participants' perceptions of the proof methods as well as their class notes, focusing on what they wrote down during the lectures on the five proof methods.

The three participants had similar perceptions about the proof methods even though their preferences for methods were somewhat different. All of the participants viewed proof by contrapositive as a kind of direct proof. Similarly, two of the participants considered proof by cases as a type of direct proof. Most participants liked direct proof, proof by contrapositive, and proof by cases, but proof by contradiction was the least liked method. Because of difficulty with algebraic manipulations in the inductive step, two of the participants disliked proof by induction compared to the other methods. An analysis of their lecture notes found that these notes could provide evidence of why they had similar views about proof methods. The lecture notes also showed that their instruction presented the five proof methods in the following order: direct proof, proof by contrapositive, proof by cases, proof by contradiction, and proof by induction. Presenting them in such an order seemed to impact the level to which the participants were familiar with each proof method; it also coincided with the order of their proof preferences. The preliminary results show that we can foresee what perceptions students might construct about proof methods based on the content of their lecture notes. Further study is needed to examine comments made when presenting proof methods in lectures and the order of presentation when considering reasons for students' preferences for proof methods.

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THE ROLE OF LOGIC IN STUDENTS' UNDERSTANDING OF PARAMETERS IN ALGEBRA

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The purpose of this study is to analyze the role of logic in college students' understanding of parameters in algebra, as they cross the border from elementary to advanced mathematics. The research on students' understanding and difficulties with parameters in school algebra has been scarce (Postelnicu & Postelnicu, 2015). Using a theoretical framework proposed by Postelnicu & Postelnicu (2015), prior research reveals that students not yet exposed to formal logic have difficulties *identifying parameters*, and with *parameters in action*, i.e., working with mathematical objects identified with the help of parameters. We propose to address those difficulties by employing logical quantifiers. Epp (2003) argues about the benefecial role of logic in teaching advanced mathematics. Logical quantifiers may also play a beneficial role in students' understanding of parameters in algebra, and help the students *identify parameters* (discriminate between parameters and other variables), and using *parameters in action* (unpacking the mathematical object and working with its corresponding mathematical process, and reifying a mathematical object again, according to the specified restrictions of the task, if any). For example, we can define a family of lines this way - for every real number k, let L be the line with the equation $4x + ky = 5k$, where x and y are real numbers. This way the parameter k can be discriminated from the other variables x, y, the domain of k is clearly stated, and the mathematical object defined with the help of parameters, the family of lines L, may be easier to reify by students. We can ask the students to graphically represent the family L, or to prove that for every real number k, L is not parallel to x-axis.

A teaching experiment (Steffe & Thompson, 2000) was conducted by the first author with 26 college students enrolled in a transition to proofs course at a university in United States. The teaching of the logic of quantified statements was augmented by one extra class (1.5 hours) with tasks that required the students to: i) *identify the parameters* and justify their choice; and ii) use the *parameters in action* which implied finding the subdomains of the parameters such that certain conditions to be fulfilled, and graphically representing the mathematical object defined with the help of parameters, in our case families of functions and curves. At the end of the teaching experiment, the students could *identify the parameters* in statements with logical quantifiers, but still had difficulties with *parameters in action*. The difficulties increased when the mathematical objects defined with the help of parameters or the contexts were less familiar to students, or the restriction imposed by the task was formulated with a negation.

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THE EXISTANCE, PERSISTANCE, AND SIGNIFANCE OF COMMON ALGEBRA ERRORS ON STUDENT SUCCESS IN UNIVERSITY MATHEMATICS

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De Morgan (1910) wrote about the difficulties students face in learning mathematics noting common errors related to arithmetic and rational number computation. Since that time, other researchers have catalogued common errors in computation and algebra. In Booth, Eyer, & Paré-Blagoev's (2014) study focused on Algebra I students, student errors were identified and classified. Although, research on students' difficulties with algebra at school has been well documented, methodical studies on the presence of these difficulties and their impact at university level are scarce. For the purpose of this paper, only a portion of the research findings from a larger study will be presented. Specifically, the existence, persistence, and significance of fraction errors found in the work on limits of Calculus I students.

This study was conducted in a mathematics department at a large research university in Southwest of the United States. Data were collected from more than 3000 students' midterm and final examinations from students enrolled in College Algebra, Pre-Calculus and Trigonometry, Pre-Calculus for Business, Life and Social Sciences, and Calculus and Analytical Geometry I. The data generated from the final exams for this course were analyzed using Booth et. al.'s (2014) categories of conceptual errors and the researchers coded the data individually before coming to a common consensus.

We found that many students struggled with the algebra and calculations needed to solve problems with limits. Research on students' understanding of limits and continuity reveal that many students find these concepts difficult to understand. Unfortunately, the results of study indicate that in addition to students' challenges with the concept of limits, their work is replete with common *fraction* errors and errors with *operations*.

The findings of this study reveal that not only the common errors and misconceptions students develop in middle school and high school mathematics exist, but they also persist into their university level mathematics coursework. Furthermore, these common errors and misconceptions cause significant challenges for students and their instructors. It is difficult for students and faculty alike to discern, based on summative test scores, what students know and understand about the new concepts they are being taught in Calculus courses. It may be possible to identify these common errors and misconceptions and work to alleviate them early in a student's university mathematics career and doing so may hold tremendous promise for students' success in their university mathematics coursework and ultimately in their ability to pursue STEM careers. It is the aspiration of this project to identify these common errors in order to create a model of intervention.

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SIXTH GRADE STUDENTS' UNDERSTANDING OF PROBLEM SOLVING TASKS VIA MODEL DIAGRAMS

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The importance of school children learning how to problem solve has been a steadfast goal within the mathematics education community especially since the publication of Polya's (1945) four problem solving heuristics. We note that problem solving is a common theme for K – 12 students in the Common Core State Standards – Mathematics (CCSS).

There are many research studies on problem solving. However, for the purpose of this research study, we narrowed the focus to examine studies that investigated heuristics or addressed specific processes that students use in their completion of problem solving tasks. Developing students' abilities to solve problems is not only a fundamental part of mathematics learning across content areas but also an integral part of mathematics learning across grade levels. As Zhang and Manouchehri (2014) noted the issue of just how to develop problem solving skills among learners continues to be a major dilemma in research on problem solving.   In their research they noted that greater attention should be given to the processes that children use when engaged in problem solving. Their findings are supported by a research study by Pajares and Kranzler (2002) of 329 high school students. They found that students' ability and self-efficacy had strong direct effects on performance on 18 problems that ranged from algebra, arithmetic, to geometry.

The goal of this research study was to get a snapshot of students' ability to complete six word problems and also to gain insight into students' methods of solving the problems (i.e., whether students use model drawing to complete the word problems). Students were given a pre-test in mid-August and a post-test in early April. Students were prompted to use model drawing or tape diagram models to work the problems if possible.   Problems ranged from addition and subtraction word problems to multiplication, fraction, ratio, and percent word problems. The results of the 62 students' responses from the pre- and post-test show that the mean difference was 1.79 items. The p-value was less than 0.0001 showing that the results are highly significant and that the results are likely attributed to the change in the intervention.   Given that none of the students tried the model drawing in the pre-test in August and then overwhelmingly students did significantly better in the post-test by means of completing the model drawing suggests that students are capable of learning a heuristic to solve mathematical word problems.

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MIDDLE GRADES STUDENTS' INTERPRETATIONS OF ADDITION OF FRACTIONS USING AN INTERACTIVE NUMBER LINE

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Keywords: Middle School Education, Rational Numbers, Technology

Addition and Subtraction of Fractions

Students often have trouble with adding and subtracting fractions, especially ones with different denominators (Cramer, Post, & del Mas, 2002). Saxe et al. (2007) state that this is because students are using whole number reasoning. Further, when teachers use fraction representations such as pie charts it may not be clear to students what the fixed unit is. What does addition of fractions mean to students and how does it relate to the number line as emphasized in CCSS-M? This study investigated how middle grade students solve problems that involve the addition and subtraction of fractions using an interactive number line and coordinate the context and the features of the digital tool in doing so. Results reported here come from the study of five middle grades students. Students were given a brief demo of the features of the digital number line and then interviewed separately using a pre-designed set of tasks presented in digital form on a tablet. Each task had a blank interactive number line embedded in it, and the students were allowed to write on blank paper in addition to the digital number line to solve the problem. The interviews were conducted as clinical interviews.

Results

Analyses of interviews showed that students associated the 'equipartition' feature of the number line with finding a common unit. The 'scale' feature was primarily used to construct or extend the number line to account for larger fractions. Although the 'scale' and 'replicate' features are inherently different, students generally used them as if they were the same. This may be due to a lack of distinction between additive and multiplicative structures for the students. The 'measure' feature was used both to perform operations and justify solutions. There were multiple instances of students using the 'measure' feature to operate on two fractions. Additionally, it was common that students used the measure tool to justify a sum or difference they had already computed on paper.

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CHANGE IN TEACHERS' CONCEPTIONS OF ARGUMENTATION

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Argumentation, a core mathematical practice (NGA & CCSO, 2010), has received increased attention in recent years. Although research programs highlight the importance of teachers' mathematical knowledge for their teaching (e.g., Hill, Sleep, Lewis & Ball, 2007), few studies have elicited teachers' thinking about mathematical argumentation and what an argument is.

This study is situated in the context of the Bridging Practices Across Connecticut Math Educators (BPCME) Project, a math-science partnership grant project that aimed to advance participants' knowledge about mathematical argumentation and pedagogy of argumentation. Forty inservice math teachers and coaches (grades 3-12) participated. We collected a range of data on participants' initial conceptions of argumentation and how they changed over time. This analysis draws on data from 37 pre and post surveys where teachers responded to the prompt, "What is a mathematical argument?" Standard qualitative methods (Creswell, 2007) were employed. Open coding revealed two main foci of teachers' responses regarding a mathematical argument: (a) the components of an argument (what must be included) and (b) the purposes of a mathematical argument. We developed a set of codes for each focal area.

Teachers' conceptions of mathematical argumentation varied widely, and teachers changed their conceptions of a mathematical argument during the project period. A prominent trend was that teachers shifted from including non-essential components of arguments, such as multiple representations, "all steps," and "good math vocabulary," to a focus on essential components, such as evidence and reasoned chains of statements. We also found teachers honed their understanding of the purpose of a mathematical argument. Initial responses included an emphasis on explaining one's problem solving process; sharing one's ideas with others; and making conjectures. Although mathematicians engage argumentation for a variety of purposes (de Villiers, 1990), the core function a mathematical argument must accomplish is demonstrating the truth or falsehood of a well-formed claim (Staples, Bartlo, & Thanheiser, 2012) – an idea all but absent from pre survey data, but well represented in the post survey data.

With so few teachers expressing a robust and disciplinary-aligned view of mathematical argumentation at the beginning of our project, these findings urge teacher educators to be aware of teachers' detailed understandings of core constructs now emphasized by new standards.

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CONCEPTIONS OF PROOF, ARGUMENTATION, AND JUSTIFICATION

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Keywords: Reasoning and Proof, Algebra and Algebraic Thinking

Policy recommendations and researchers call for instruction designed to promote students' engagement with proof, argumentation, and justification. However, definitions and descriptions of these practices may vary, and even contradict one another (Cirillo, Kosko, Newton, Staples & Weber, 2015). In an effort to address this issue, a working group was established (Cirillo et al., 2015), advocating that educators clarify and refine their conceptions of these constructs.

In the working group, participants were invited to construct and analyze concept maps. In this poster, we present an analysis of the working group's concept maps, which summarized ways in which proof, argumentation, and justification are interrelated and understood by the participants. In total, we analyzed 44 maps from the perspective of researchers focusing on justification (n=11), argumentation (n=9), proof (n=19), and "other" (n=5). In analyzing the maps, we included open coding and formal techniques, such as Systemic Functional Linguistics (Halliday & Matthiessen, 2004) to identify themes.

Overall, a majority of the researchers used Venn diagrams in their maps. Some participants structured their maps to show hierarchy—most with proof at the top, which given the proportion of participants focused on proof is not surprising. The concepts were represented with both dynamic and static perspectives. The dynamic perspective illustrated how proof, argumentation, and justification are related through interactions; whereas, the static perspective defined how each concept related to the other concepts.

Many participants focused on mathematical argumentation communicated proof as a product or goal, but the role of justification was not as consistently conveyed. Furthermore, several participants who focused on justification treated the concepts of justification and proof as closely related but argumentation as separate. Even among researchers with a common focus, definitions of these constructs differed. Therefore, those conducting research in mathematical argumentation, justification or proof should be mindful of how they position these terms in their research. We recognize this analysis as a step towards developing a framework to describe how the field considers these constructs and the interrelationships among them.

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Chapter 8

Preservice Teacher Education

Research Reports

Brief Research Reports

PRE-SERVICE AND FIRST-YEAR TEACHERS' VIEWS ON THE INFLUENCE OF MATHEMATICS METHODS TEXTBOOKS

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We investigated how pre-service and first-year teachers engaged with the pedagogy textbooks from their mathematics methods courses. Sixteen participants – seven pre-service teachers and nine first-year teachers – were interviewed about their interactions with mathematics methods texts and their perceptions about the texts. This paper focuses on participants' views regarding the messages sent by their mathematics methods textbooks about how to teach. Findings describe the tensions that exist between pedagogy textbooks and participants' experiences in the classroom. Implications for teacher educators are also discussed.

Keywords: Teacher Education-Preservice, Affect, Emotion, Beliefs, and Attitudes, Teacher Beliefs

How do prospective teachers interact with mathematics methods textbooks and how do these interactions shape prospective teachers' views of teaching? As Mesa and Griffiths (2012) noted, "textbooks remain a ubiquitous course component with various implications for the teaching and learning of mathematics at a tertiary level" (p. 86). We suggest that implications also exist for the use of textbooks in the teaching and learning of mathematics education in higher education settings. Yet, there is a lack of research that examines ways in which prospective teachers interact with mathematics pedagogy textbooks. Although there are numerous studies that have looked at the relationships between teachers, learners, and mathematics textbooks in K-12 education (Freeman & Porter, 1989; Haggarty & Pepin, 2002; Herbel-Eisenmann, 2007; Herbel-Eisenmann, 2009; Mesa, 2004; Remillard, 2000), few studies have examined these relationships in higher education settings. Moreover, the focus of studies conducted at the tertiary level have centered on textbook use in content courses, not pedagogy courses (Lithner, 2003; McCrory & Stylianides, 2014; Mesa & Griffiths, 2012; Weinberg & Wiesner, 2011; Weinberg, Wiesner, Benesh, & Boester, 2012). We posit that textbooks are an important component in mathematics methods courses for prospective teachers and the ways in which prospective teachers engage with these texts should be explored. In this paper, we discuss findings from interviews with pre-service and first-year teachers regarding the messages sent by their mathematics methods textbooks about how to teach.

Background and Theoretical Framework

While textbooks play a role in many mathematics methods courses, there are some issues of power that need to be addressed. Several textbook studies address the authoritative nature of textbooks and of the ways in which instructors mediate textbook use (Haggarty & Pepin, 2002; Herbel-Eisenmann, 2007; Herbel-Eisenmann, 2009; Mesa & Griffiths, 2012; Smitherman, 2006; Weinberg & Wiesner, 2011). The authoritative nature of textbooks can create tensions between the messages conveyed in texts and the ideologies of the mathematics education community. Herbel-Eisenmann (2007) investigated the "voice" of a Grade 7 textbook to see if it reflected the ideology emphasized by the *Principles and Standards of School Mathematics* (National Council of Teachers of Mathematics [NCTM], 2000). By using a discourse analytic framework to examine different linguistic features of the textbook, Herbel-Eisenmann found that the human agency in mathematics was often masked through the absence of different pronouns and the presence of modal verbs (i.e., could, might, would). Yet, many mathematics educators and the *Principles and Standards of School Mathematics* (NCTM, 2000) advocate for human agency through "doing mathematics" as one learns.

Tensions also existed with how "the text represented a strongly certain viewpoint of mathematics. This viewpoint is absolutist or Platonist in nature rather than fallible" (Herbel-Eisenmann, 2007, p. 362). This absolutist view can also be seen in pedagogy textbooks. In Smitherman's (2006) analysis of ten different mathematics education pedagogy textbooks, all but one of these textbooks "exhibit modern, rationalist, ideas in mathematics education" (p. 61). For Smitherman, this meant the pre-service teachers' expectations of what it meant to teach were shaped by notions of mathematical proof which were evident in the textbooks: replication is possible; mathematics education is comprised of a universal language; and it is based on a predictable cause-effect relationship. While aspects of these notions are indeed present with mathematics education, they also do not fully take human presence into account. For example, if predictable cause-effect relationships were possible in the classroom, then one could assume that several ideas from students "doing mathematics" would be dismissed as not fitting the prescribed relationship that is predicted and replicated.

Furthermore, Smitherman's (2006) analysis found that "the conversations in these texts [were] one-sided and unilateral" (pp. 63-64). For example, when these textbooks defined mathematics, the opinions and ideas held by the prospective teachers were never acknowledged in any of the textbooks. This emphasized the authoritative nature of the textbooks, in which "the authors of these texts [were] trying to create a particular way of conceiving math" (p. 65). We posit that this also creates an ideological tension with the mathematics learning emphasized in methods courses.

Weinberg and Wiesner (2011) also described conflict created by a textbook's authoritative nature as they applied reader-orientated theory to calculus textbooks. There are three types of readers central to reader-oriented theory: the intended reader (the image of the reader the author has), the implied reader (the qualities needed to read the text in the way the author intended), and the empirical reader (the actual reader). "The authority inherent in school textbooks adds to the tensions between the three readers" (Weinberg & Wiesner, 2011, p. 57). As explained by Weinberg and Wiesner, textbook authors typically envisage readers that employ active reading strategies (implied readers) and readers that are part of the mathematics community (intended readers). This creates a tension with the authoritative nature of textbooks that conveys a specific "truth" of mathematics to empirical readers. Weinberg and Wiesner also posited that if the intended, implied, and empirical "readers are not in alignment, the student will not be able to generate the appropriate mathematical meaning through reading the textbook" (p. 57).

Yet, the relationship between the text and the reader is often mediated through course instructors. However, if there are tensions with the authoritative nature of textbooks, then there are also tensions that exist with the authoritative nature of instructors using those textbooks. Mesa and Griffiths (2012) saw a difference in how instructors used textbooks with students depending on if instructors classified students as "undergrad students" (e.g., students in their first years of tertiary education taking courses to fulfill requirements or students taking remedial courses) or as "math students" (e.g., honors students or those students in upper division or graduate mathematics courses). Mesa and Griffiths posited that "although instructors may want students to read the textbook, or do something more with it than doing the homework or reading the examples, they seem to describe this expectation as reasonable for students in upper division courses, but not for students in lower division courses" (p. 97). This stance is confronting when combined with the role of teachers as mediators of the text as discussed by Haggarty and Pepin (2002). Textbooks have a large influence in the classroom, and teachers control the ways in which the textbooks are used. As Haggarty and Pepin suggested:

> [teachers] decide which textbook to use; when and where the textbook is to be used; which sections of the textbook to use; the sequencing of topics in the textbook; the ways in which pupils

engage with the text; the level and type of teacher intervention between pupil and text; and so on. (p. 572)

Thus, instructors can choose to use a text differently depending on how they classify students, and as Mesa and Griffiths (2012) contended, a "direct consequence of the different schemes that instructors have for using the textbook…results in different opportunities for these students to learn" (p.101).

Methodology

In order to garner multiple perspectives, we sought out participants who would be or were teaching various grade levels. Sixteen participants were chosen through a convenience sampling process (Creswell, 2012), where we invited former students to be interviewed or contacted colleagues who recommended their former students. Amber interviewed three pre-service elementary teachers and five first-year elementary teachers from a large Southwestern city in the United States. Shelly interviewed four pre-service secondary teachers and four first-year secondary teachers from a large Midwestern city in the United States.

Semi-structured, retrospective interviews were conducted in one-on-one settings. The use of retrospective interviews can generate some criticism, because interviewers try to get participants to recall and then reconstruct past experiences from memory. Although these interviews are the least likely type of interview, "… to provide accurate, reliable data for the researcher" (Fraenkel & Wallen, 2000, p. 510), the nature of our research question required us to use retrospective interviews. However, we trusted that the participants' memories were accurate based on *their* perceptions, which were their realities.

The interview protocol included demographic data questions, 11 open-ended questions about textbook use, and a final question to capture anything else the participant wished to share. Here, we focused on participants' responses to two questions within the interview protocol that related to messages sent by the pedagogy textbook: How did the texts you used in methods courses make you think about how to teach? and Considering your experience in the classroom(s) during and after your methods course(s) do you feel that texts can tell you how to teach? Why or why not?

Each interview lasted about one-half of an hour, and each interview was audio recorded. These recordings were later transcribed. After we transcribed the interviews, we also created a document that grouped participants' responses to each question. This helped us look at each participant's responses to all of the questions and to also look at all participants' responses to each of the questions. To analyze the data by utilizing a grounded theory constructivist design, no *a priori* categories for participants' responses, we focused on the meanings ascribed by the participants in the study (Creswell, 2012) and captured their experiences with texts in mathematics methods courses. During Skype™ researcher conversations we first used *in vivo codes*, labels for categories that were phrased in the exact words of the participants, and then created themes that emerged from the *in vivo codes*. As with most qualitative research, we acknowledge that our conclusions are suggestive rather than definitive.

Findings

When asked about how the texts used in mathematics methods courses made participants think about how to teach, approximately 70% of participants (n=11) mentioned the different strategies and ideas the text provided for teaching different concepts. While participants mentioned other aspects of the texts in response to this question, no other response had the same collective agreement by participants (see Table 1). There were only three other responses to this question that were given by at least three participants: an emphasis by the text on student explanations to strategies and solutions, the text leading to a realization of different ways students think about mathematics, and the text as a source of activities for the classroom.

Table 1: Responses to "How did the texts you used in methods courses make you think about 'how to teach?'"

Common Response	Number of Participants
By providing different strategies/ideas for teaching concepts	11
By emphasizing student explanations	4
By helping me see that students think in different ways/that there are different learning styles	4
By being a source for different activities	3

Mathematics methods courses often focus on pedagogical skills and knowledge, pedagogical content knowledge, and how students learn in order to help pre-service teachers develop teaching strategies aligned with these areas. Both the frameworks surrounding methods courses and the texts selected by course instructors work towards building pre-service teachers' knowledge in these areas. Yet, when we asked participants if they felt that texts could tell them "how to teach," about half of the participants (n=7) did not think the text could do this (see Table 2). Six participants gave mixed responses (yes and no), and three participants reported that pedagogy texts could tell them how to teach. Throughout these responses, four themes emerged that described participants' views related to if texts could tell them "how to teach."

Table 2: Responses to "Considering your experience in the classroom(s) during and after your methods course(s) do you feel that texts can tell you 'how to teach?'"

Response	Number of Participants
Yes	3
Yes and No	6
No	7

Textbooks as a Platform for Reflection

One theme focused on using texts as a platform for reflection on teaching. One participant said:

Texts can kind of guide you in the right direction and sort of get you to reflect on your own teaching practices. It's very just reflective…like what would it look like in my classroom? Can I get my students to the same endpoints using those methods? (participant 2, first-year teacher).

Yet, another participant viewed this question in a more personal way and stated:

I think it's beneficial to have texts to give you a solid, to give you a concrete example of what someone else thinks and I think those are beneficial because thinking about other people's thinking is as important as thinking about your own thinking…How can you teach people to teach when teaching is such a personal thing…I think that you have to find your own connections with everything. (participant 5, pre-service teacher)

Both of these quotes from participants emphasize the connections prospective teachers make between the information in the textbook and their own teaching practices. Yet, when asked in the preceding interview question about how the texts they used in their mathematics methods courses helped them think about how to teach, only one participant's response mentioned the textbook providing a reflective component.

Experience is the Best Teacher

Another theme that emerged was that texts cannot tell you how to teach, because experience is the best teacher. As one participant described:

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

> But, when you're in the thick of it, that's when you really learn, I think, you know experimenting and using the ideas that you have, bringing them into the classroom is the most effective. So, kind of a combination of both, but I don't think the book alone can tell you how to teach. (participant 9, first-year teacher)

This feeling was echoed by other participants, including one who explained:

> I feel like [the texts] can't [tell you 'how to teach']. I definitely think they help…But actually being up in a classroom and teaching and getting the feedback from your students, I think that's going to teach you the most. (participant 13, pre-service teacher)

Perhaps, being in the classroom is when it becomes "real" to prospective teachers. In fact, when asked about how the texts used in the mathematics methods courses helped him or her think about how to teach, one participant shared, "I didn't think about how to teach. That's why I think it was such a shock for me going into my 5th grade [placement] classroom" (participant 14, pre-service teacher).

All Students are Different

The next theme that developed from participants' responses was that textbooks did not tell them "how to teach," because all students are different. Interestingly, most of the participants expressing this view were first-year teachers. The response of only one pre-service teacher fell into this theme. That participant said:

> I think the text tries to tell you how to teach and it gives you strategies but a lot of teaching is based on your students and every student learns differently…I think it gives you a general overview and best practices but every teacher is gonna [sic] have to change their teaching for the students that they have in their classroom. (participant 1, pre-service teacher)

One of the first-year teachers whose response fit this theme expanded upon the notion of knowing your students to knowing the community of students:

> There are so many different parts to teaching. There's knowing your kids…So, in the most straightforward sense, no, I don't think [the text] can teach you how to teach…This textbook is for, you know, just a general population of elementary and middle school teachers but we all go off into our own types of communities. (participant 11, first-year teacher)

We also saw the realization of the different ways that students think about mathematics emerge as a common response to the preceding interview question about how the texts participants used in their mathematics methods courses helped them think about how to teach.

Textbooks do not Reflect Reality

The final theme that emerged through participants' responses to these two interview questions concerned the differences between the text and the reality of teaching. Participants often commented that texts represented a "perfect world" of teaching. As one participant explained, "I feel like textbooks can be very idealistic or optimistic and everything's going to go well and the students are going to get it" (participant 12, first-year teacher). This sentiment was echoed in several responses from participants, particularly those participants who were first-year teachers. According to one of the first-year teachers, "How did [the text] make me think about how to teach? I think the way that it made me think isn't like reality once you start doing it" (participant 16, first-year teacher). This is troubling, particularly if it impacts how teachers view their preparation. "You're not going to get a textbook situation in your teaching environment…it's a completely different world when you step into the classroom versus reading a book…I felt almost unprepared for the real world" (participant 4, first-year teacher).

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

Discussion and Implications

The results of our interviews emphasize the different ways pedagogy textbooks are viewed by pre-service and first-year teachers concerning ideas on how to teach. Surprisingly, approximately 81% of participants (n=13) either did not think pedagogy textbooks could tell them how to teach or were mixed in their responses. Only three participants indicated that methods texts could tell them how to teach. While the authoritative nature of texts is often considered when studying textbooks (Haggarty & Pepin, 2002; Herbel-Eisenmann, 2007; Herbel-Eisenmann, 2009; Smitherman, 2006; Weinberg & Wiesner, 2011), participants in this study did not seem to heed the assumed authority of the text on how to teach.

Based on participants' responses and the themes that emerged through them, we suggest that prospective teachers view teaching in a more personal manner than portrayed through pedagogy texts. Many viewed teaching as complex and multi-faceted, and the authority of the textbook conflicted with this view. While the textbook was influential in providing some participants with ideas about the different ways students learn and think about mathematics, several participants also felt tensions between the portrayal of classrooms within the textbooks and their own classroom experiences. We hypothesize that this tension stems from the Platonist view that textbooks present about math and mathematics education (Herbel-Eisenmann, 2007; Smitherman, 2006). If, as Smitherman posited, the notions that shape prospective teachers' expectations about what it means to teach mathematics include that replication is possible and predictable cause-effect relationships, then they may very well think that the textbook represents a "perfect world" of teaching.

The tension between the presentation of classrooms in the textbook and the participants' classroom experiences may also be the result of a misalignment between the intended reader, the implied reader, and the empirical reader (Weinberg & Wiesner, 2011). The participants being the actual readers of the texts (the empirical readers) may not match the profile of the readers the authors envision (the intended reader) and/or may not possess the qualities needed to read the text in the ways the authors intended (implied reader). What skills do prospective teachers need to read the textbook in a way that lessens this tension? Do they need additional classroom experience? Do they need more confidence in their perceptions of mathematics education? Or, do they need a stronger avenue to voice those perceptions? Perhaps, there are other factors that need further research in order for the intended, implied, and empirical readers to align when facilitating the use of pedagogy textbooks in our mathematics methods courses.

That being said, we posit that these tensions signify positive aspects in our participants as readers of the texts. The results of our interviews also suggest that participants leaned towards a critical stance when they indicated that textbooks could not tell them how to teach. This questioning of the texts is something that we would like to promote in our methods courses. Like Draper and Siebert (2004), we agree that for pre-service teachers to "acquire mathematical knowledge and participate meaningfully in mathematical activity, [they] must become adept at creating, negotiating, and consuming texts" (p. 945).

We want to position the textbook as a source of information, not *the* source for information. Helping the students in our methods courses to question and negotiate the information presented in the various texts helps create a "privileging the student" position (Herbel-Eisenmann, 2009) between the students in our methods course, us (as instructors), and the textbook. Furthermore, if we allow these tensions and the classroom experience to be exposed and discussed, we help prospective teachers create their own texts. The results of our interviews suggest that participants viewed experience as the best teacher of how to teach. Several of our participants also recognized that all students are different and responding to the needs of students in the class is important as teachers plan. We look for ways that these ideas can be recognized and built upon in our methods courses to foster the confidence of prospective teachers and position them in ways that value their contributions along with the information presented in the textbook.

Textbooks influence both prospective teachers and instructors of mathematics methods courses. We do not want to downplay their use or their importance in our courses; there is valuable information that comes from textbooks. Instead, we want to position the textbook in a way that mirrors this statement by Herbel-Eisenmann (2009), "When teachers, textbooks, and students come into contact with one another, there is the potential for each of these 'participants' in the classroom to take on responsibility for the introduction and development of mathematical knowledge" (p. 147). Indeed, we want to leverage textbook use in ways that prospective teachers will find meaningful in both their teaching preparation and in their teaching careers.

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PRE-SERVICE SECONDARY MATHEMATICS TEACHERS' PERCEPTIONS OF ABILITY, ENGAGEMENT, AND MOTIVATION DURING FIELD EXPERIENCES

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We conducted a self-study to learn more about how to support the first author's pre-service teachers (PSTs) during their field experiences in an introductory secondary mathematics methods course. The findings highlight how the PSTs' perceptions of the secondary students' mathematics ability was closely related to how they viewed the students' levels of engagement and interest. The data also indicated that PSTs often diminished the role that the cooperating teachers played in co-constructing student ability, motivation and engagement, particularly in situations where the students displayed unfavorable behaviors. The findings inform curricular development for methods courses and field experiences, both for the authors and for the mathematics educators who support PSTs in diverse school settings.

Keywords: Teacher Education-Preservice, Equity and Diversity, Teacher Beliefs

Background and Purpose of the Study

Preparing mathematics pre-service teachers (PSTs) includes supporting them as they develop dispositions toward equity (Hand, 2012), particularly in schools that are labeled as high needs. *High-needs school* is a marker used to describe schools with one or more of the following characteristics: a student population comprised primarily of students of color, a high concentration of English language learners, or a high concentration of students from families with lower socioeconomic status. At times, this label perpetuates deficit-oriented thinking about the students who attend these schools. PSTs often enter mathematics education courses well versed in popular deficit-laden discourse about high-needs schools and the challenges of working in them, deemed as "lay culture norms" by Tatto (1996). This line of thinking often leads to PSTs framing their students' experiences and ways of knowing as unfamiliar (and possibly subordinate) to their own, which negatively impacts PSTs' mathematics teaching practices. In turn, this limits students' academic trajectories and opportunities to learn. Mathematics educators must support PSTs in recognizing and challenging this deficit-oriented thinking.

I, the first author of this work, designed an introductory secondary mathematics methods course with the goal of not only promoting rigorous and standards-based mathematics teaching methods, but also in the hopes of supporting students in unpacking issues of privilege, equity, and ability in the context of diverse mathematics classrooms. One important component of this course is the field experience. Upon analyzing my program's survey data as part of a larger study of programmatic change in our Secondary Education program (Samaras, Frank, Williams, Christopher, & Rodick, in press), I learned that former students found their field experiences unbeneficial. Thus, my graduate assistant and second author, Monique, and I designed a field experience that we hoped students would find relevant and that would highlight aspects of equitable teaching in action.

This self-study (Samaras, 2011) examined the challenges the first author faced as a new mathematics teacher educator in designing a meaningful field experience opportunity for PSTs who were observing mathematics teaching and learning in a high-needs high school. The following question guided our research: What can we learn from the PSTs' field experiences to provide more meaningful fieldwork in secondary mathematics methods courses? The findings gleaned from the PSTs' reflections and observations revealed the tacit labels assigned to students in high-needs schools rooted in PSTs lay culture norms about ability, motivation, and engagement. Our findings

challenge us to find new ways to disrupt these lay culture norms and to help PSTs acknowledge the co-constructed nature of mathematics ability and participation between students and teachers in mathematics classrooms (Hand, 2010).

Theoretical Perspective

Integrated Perspective

This study is grounded in sociocultural and sociopolitical perspectives about mathematics teaching and learning, acknowledging that mathematics classrooms are cultural and political spaces. A sociocultural perspective highlights the role of participation, social interaction, and negotiation in teaching and learning, and it highlights the norms and discourse practices of educational settings and how they influence learning (Cobb & Yackel, 1996). Context is central to learning from a sociocultural perspective, meaning one cannot separate what one learns from the context in which it happens and the people with whom the learning takes place.

We also draw on Nasir and McKinney de Royston's (2013) notion of sociopolitical perspectives in mathematics education. These authors define a sociopolitical perspective as distinct from a sociocultural one in that it accounts for "how…power operate[s] in learning settings, especially as [it] may relate to privilege and marginalization" (p. 266). Sociopolitical attention to mathematics teaching and learning addresses not just how activity is organized, but also how issues of power permeate learning contexts. In this study, we see the political nature of mathematics permeating mathematics learning at three levels – at the school level as well as at the state- and national-levels where policies such as accountability mandates inform instructional choices.

We adopted an integrated theoretical lens to understand the PSTs interpretations of their field experience observations. This means that while the cultural and contextual features of the classrooms our PSTs observed were important, we also accounted for the political nature of mathematics education, which had significant influence over teachers' instructional choices.

Teacher Noticing

As a complement to our integrated perspective, we also draw on contemporary research about teacher noticing, which encompasses "not only the attention that [PSTs] give to classroom actions and interactions, but also their reflections, reasoning, and decisions based on this noticing" (McDuffie, et al., 2014). Mason (2008) noted two facets of teacher noticing which are central to our study, (a) attention, or what teachers notice as they observe practice, and (b) awareness, or how teachers make sense of or interpret what they observe. Given that PSTs' attention and awareness are culturally situated, informed by participation and negotiation in methods class and the classrooms they observed, and influenced by context, we see noticing as complementary to our theoretical perspective.

Methodology

The purpose of self-study methodology is for educators to explore issues directly relevant to the context to which it is applied (Samaras, 2011). The theoretical underpinnings of self-study acknowledge that knowledge production and development is context and cultural sensitive (LaBoskey 2004). Thus, self-study honors teacher voice, in that it allows teachers to respond "to the needs and concerns of *their* students in *their* contexts" (Kosnik, Beck, Freese, & Samaras, 2006, p. x). The five tenants of self-study research include: being self-initiated and focused, aimed at improvement, being interactive, utilizing multiple, primarily qualitative methods, and using validation procedures that are exemplar-based (Samaras, 2011). In our efforts to maintain fidelity to the characteristics of self-study, we initiated this self-study project to improve the field experiences component of the methods course. This study was highly interactive, as we served as critical friends

who provided insider (first author) and outsider (second author) perspectives during data collection and analysis.

Participants

This study examined the perceptions of four PSTs from the first author's introductory methods course who were assigned to a racially, linguistically, and socioeconomically diverse school. The group ranged in age from early 20s to late 50s. Two male and one female PST identified as White and one as an Asian immigrant woman. Three PSTs decided to teach mathematics after having careers in mathematics-related fields. One had recently earned an undergraduate mathematics degree. In their mathematics mathematics autobiographies, each of them expressed comfort and ease with mathematics throughout their K-12 schooling

It is important to note that while we are not direct participants in this study, our roles as instructor and graduate assistant are central. We entered this study aware of the power dynamics between researchers and educators when conducting research (Kvale, 2006), especially when one researcher is also the instructor of the participants. For this reason, the second author led the focus groups, though the first author probed for deeper understanding at times. Since this study, the second author has served as a teaching assistant for a subsequent introductory mathematics methods course and will be the instructor of record for future methods courses. Thus, we both have personal interests in improving this major component of the course.

Context

The context for this study was an introductory secondary mathematics methods course. The goals of the course were to introduce students to reform-minded mathematics teaching and learning via the National Council for Teachers of Mathematics (NCTM) *Principles and Standards for School Mathematics* (NCTM, 2000) and state standards, which encourage student-centered, conceptual approaches to teaching. PSTs had to complete 15 hours of observations in secondary mathematics classrooms. The first author only required PSTs to observe in this introductory course, as they are required to take an active and participatory role in a secondary mathematics classroom during their second methods course. They were encouraged to visit as many classrooms as they could to get a sense of how mathematics instruction varied both across a schools' mathematics department and within a teacher's practice as she worked with different groups of students.

The university's clinical experiences office organized the field experience placements. The PSTs completed their fieldwork at an ethnically, linguistically, and socioeconomically diverse high school in a nearby local district, which according to state-level data, is the lowest performing high school in the district. In fact, at the time of the study, the school faced sanctions for not meeting accountability standards.

Data Sources

Multiple sources of data in this study serve as a means of data triangulation to provide corroborating evidence to support our findings (Miles & Huberman, 1994). Data sources included two focus groups at the beginning and end of the field experience that ranged from 45 minutes to an hour. We also collected each PST's autobiographies where PSTs reflected on their personal experiences as mathematics learners. We also collected their final reflection papers that provided detailed accounts of their field experiences using writing prompts that encouraged PSTs to consider school and community contexts and classroom interactions among students and between students and teachers.

Given the focus on the researcher in self-study, I as the first author and instructor kept detailed personal memos. These memos were written following both focus groups. They focused on the big ideas that I heard and wonderings that I had as I listened and probed PSTs during the focus groups.

As I read my PSTs' reflection papers, I also kept memos of the themes that I noticed as I read. I shared my memos with the second author as we analyzed the data.

Data Analysis Procedures

We employed qualitative methods for this study, including document analysis and focus groups. Document analysis is a systematic procedure for evaluating printed and electronic documents that "requires that data be examined and interpreted in order to elicit meaning, gain understanding, and develop empirical knowledge" (Bowen, 2009, p. 27).

Data analysis began with pre-coding the data (Saldaña, 2009), meaning that we read the focus group transcripts and reflections and flagged, highlighted, or underlined portions that we believed would be important to this study and in need of attention during the coding process. Upon several reads and annotations of the transcripts, these themes eventually became codes and sub-codes about the nature of the PSTs observations. Once we established our initial codes via axial coding (Miles & Huberman, 1994), we used Dedoose software to code the corpus of data. With the help of the interactive data visualization tool in Dedoose, we were able to observe patterns and themes. We each coded the data sets on our own, then came together to share the patterns that emerged as well as our interpretations. We strived to validate our findings by check-coding our ongoing analysis and interpretations of the data and by meeting frequently to share our coding and to resolve discrepancies.

Approaching this work from the theoretical lens described above, we analyzed the PSTs' participation in focus groups as well as what they shared in their written reflections to explore how they drew upon artifacts, tools, fellow PSTs, and us, as faculty and graduate assistant, to make sense of their experiences and to develop new understandings (Putnam & Borko, 2000).

Findings

PSTs' Perceptions of Student Ability, Motivation and Engagement

Ability and reform-based teaching practices. An emergent theme from the data was that the PSTs' assumption that reform-minded mathematics was mainly suitable for advanced students. PSTs wondered if reform-based instructional approaches were suitable for all types of learners, or whether these practices only worked with students who were viewed as "motivated," "smart," or "high flyers" as some PSTs described them. For example, when discussing his frustration with the lack of reform-minded practices in a low-tracked Algebra class he observed, one PST shared, "I would like to see classrooms where [reform-based practices] actually work. I think that would be really interesting. What (sic) are those students? What are the demographics?" Implicit in this excerpt is the PSTs' understanding that the kind of student-centered teaching emphasized in the methods class is suited for particular learners, namely those who are considered to be advanced or "smart."

Ability related to motivation and engagement. In a similar vein, the PSTs' perceptions of student motivation and engagement were often predicated on the academic level of the class. We found numerous instances of this in the corpus of data. PSTs frequently adopted a deficit-oriented perspective when describing unfavorable student behavior or learning outcomes. Peppered throughout the focus group conversations regarding student motivation and ability were phrases such as "Some of these kids don't care," while other PSTs expressed sentiments that students who appeared unmotivated "don't want to learn."

When reflecting on observations in an advanced (International Baccalaureate [IB]) class, one PST shared, "Every single student was paying attention and doing their work. And it could be the fact that it is an IB class, which I would probably think most of the reason is." Given that this PST's reasoning for students being engaged and paying attention in class was because the class was designated for advanced students is another example of how perceptions of student ability influenced

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

the PSTs' perceptions of motivation and engagement. The PSTs often referred to the students in lower-tracked classes as unmotivated and disengaged, while the students placed in the higher-level courses were frequently posited as focused and on-task. These claims were made based on behaviors that the PSTs' observed, not on the students' work or grades in the course. Even when advanced students displayed similar behaviors as those in lower-tracked classrooms, PSTs interpreted the behavior differently. One PST recounted the following:

> Until [lower-tracked students] were called on, they were just sitting there [absorbing] whatever information made it through their skin, and the instructor said, "Look you know you guys have to care"... The [higher tracked] Probability and Statistics course, they were all very focused. So [the instructor] could lecture to the students who wanted to pay attention. There was a girl right in front of me flipping through her magazine and two students on my right on their phones. [The teacher] says, "They are good students, and they will take care of their homework after class."

This excerpt is especially salient. PSTs noticed behaviors that they believed indicated that the lower-tracked students were disinterested and not concerned with learning mathematics (i.e., "waiting for information to seep into their skin"). In the excerpt above, higher-tracked students exhibited similar behaviors, yet because of their status as higher-tracked and "good" the teacher ignored their disinterest during class.

The Impact of Accountability Mandates

PSTs who observed at this particular high school noted the disengagement of the students in lower-tracked classrooms as described above. Upon further exploration, we found that most of the classes where the PSTs noted disengagement were typically remedial or for standardized testing preparation. Cooperating teachers in these classes opted to drill subject matter, while others struggled with pacing and covering all of the necessary material for testing. Ultimately, all of these choices resulted in teacher-centered instruction and selection of materials that emphasized procedures rather conceptual understanding. When looking across all 4 PSTs' reflection papers as well as the focus group data, all of the PSTs mentioned how the pressures of standardized testing influenced their observed teachers' instructional choices; However, most missed the opportunity to connect these instructional choices to student disengagement. This finding also points to a third theme that emerged from the data, the limited acknowledgement of the cooperating teachers' role in co-constructing ability, engagement, and motivation, which we discuss in the next section.

Limited Unpacking of the Cooperating Teacher's Role

We surmise that the PSTs did not see how their cooperating teachers, through discourse, interactions with their students, and curricular choices *positioned* (Herbel-Eisenmann, 2015) students to take up or resist particular opportunities to learn in their classroom interactions. When we questioned the PSTs about how the cooperating teachers' instructional decisions contributed to their students' disengagement and perceived low mathematics ability, most PSTs admitted that they had not considered the role of each teacher in co-constructing these negative interactions that they had mostly attributed to students. This is reminiscent of the teachers in Hand's (2010) study of how secondary teachers were often unaware of how their behaviors toward their students aided in co-constructing opposition in their low-tracked mathematics classrooms.

As we noted this pattern in the data, we began to ponder why the PSTs consistently missed the opportunities to notice their cooperating teachers' roles in co-constructing behaviors they saw as unfavorable. Upon a closer look at the data, we found that PSTs expressed apprehension in framing their cooperating teachers as being ineffective. In fact, during the last focus group, several of them noted that they felt guilty about reporting what they observed. They were afraid of being "teacher bashers" as one PST explained. It is important to note that in our analysis, we found that the PSTs

gave far less attention to how they framed students in their discussions, particularly when describing unfavorable student behavior or learning outcomes. Unknowingly, the PSTs rested the responsibility for learning and engagement squarely on the students' shoulders.

When PSTs had the opportunity to consider the role of their cooperating teacher in co-constructing negative interactions, they gained deeper understanding of the teacher's role in impacting student motivation and engagement in tasks. One PST noted how even something that seemed minimally important on the surface, the way a teacher sets up her classroom, could foster unproductive participation and possibly encourage student disengagement. He shared, "All the desks were in groups of four, which meant that about half the students for the entire class period had their backs to the teacher...and those students were facing the opposite wall and were some of the ones that were the most disengaged. This no doubt encouraged the talkativeness of his class." Another PST admitted that he had not considered the role that teachers play in co-constructing student disengagement or lack of understanding of the content. When reflecting on his experiences in the two classes observed, the PST noted in his final reflection:

> "Good" teachers often create a desire in students to learn. I really enjoyed Mr. O's class. He employed the use of real world problems and all of his students were actively learning...I did not get this sense from Mrs. V's class. Hardly any of the students paid attention, which she did nothing about. I couldn't blame the students because she made Algebra I seem boring."

These quotes show that some of the PSTs had an emerging understanding of how teachers' decisions either afford or constrain opportunities for engagement and motivation.

Implications

As this is a self-study, the findings and analysis inform our practice as instructor and graduate assistant of this course. However, we believe that the implications for this work are meaningful for other mathematics teacher educators. From the findings emerged important factors to consider when designing field experiences for future methods courses. We are also grappling with how we as university-based teacher educators affect fieldwork that often takes place in our absence and under the purview of cooperating teachers.

Within our methods courses, It is essential that we provide opportunities for PSTs to make sense of the contexts of their field experience location sites, especially when placed in high-needs schools. The PSTs needed more time to work through the deficit-laden discourse that influences what they notice and attend to when observing practice. Past research has shown that PSTs who only have one-time or minimal field experiences in high-needs schools tend to deepen their deficit perspectives due to the lack of exposure to unfamiliar communities (Sleeter, 2008). Rushton (2000) argued that new teachers could begin to shift their thinking about students in these settings if they work in these communities over a sustained period of time. Further, shifts in PSTs' deficit perspectives also have implications for what they notice as they observe instruction. McDuffie et al. (2014) note, "Teachers need support in learning to attend to, or notice, students' mathematical thinking and important classroom events and interactions—in other words, noticing is a practice that needs to be developed" (p. 246).

Additionally, unpacking the accountability milieus in which teachers have to make instructional decisions must become a more explicit component of PSTs reflecting upon their field experiences. Since data collection and analysis, we co-taught another introductory methods course. We tried to make noticing with respect to issues of, ability, engagement, and motivation more explicit with respect to our reading selections and class discussions about equity, race and ethnicity, and language. Only one PST explicitly discussed race in his reflection and two directly addressed issues of language, yet most of them noted that the drastic demographics between remedial classes that were primarily Black and/or Latin@ and advanced classes that were predominately White. We also spend

time discussing how mathematics educators can shift from discussions of mathematics ability to discussions of student status (Horn, 2007). Additionally, in the subsequent advanced methods course, the first author has revised the curricular component on assessment to address the political nature of teaching mathematics in addition to more traditional mathematics assessment issues. We discuss how policies regarding mathematics assessment have very distinct outcomes for non-dominant groups of students.

Closely related to the need for PSTs to make sense of school contexts, where accountability mandates permeate mathematics instruction, we are now making space, in class and in the field, for students to think about how the political nature of teaching mathematics impacts teacher agency and instructional choices. Noting how the PSTs were reticent about sharing their observations of less desirable teaching for fear of teaching bashing pushes us to think of how we help students productively critique the cooperating teachers' practices. We ponder whether their reticence could be attributed to the PSTs' status as students who are learning to teach. Their positioning as novices could have led to them deferring to their cooperating teachers' pedagogical and relational choices. While we want our PSTs to implement more reform-minded strategies, we also want them to understand the complex nature of teaching mathematics and all of the factors that impact it. We are also pushing this work outside of the university-based methods course and working toward adopting a model of field experience where we, as instructors, visit classrooms with our PSTs and engage in conversations with cooperating teachers about their pedagogical and relational choices with respect to accountability.

With the attention to classroom context and the larger political forces that shape classroom activity, we believe that we are building a foundation for PSTs to move beyond static, one-sided characterizations of students as not caring about their mathematics courses. We hope that through our curricular changes based on this research, our PSTs come to see mathematics classrooms as dynamic spaces where teachers are co-constructing student engagement, behavior, and their students' mathematics identities. We strive to help them understand that learning mathematics cannot be viewed as a one-sided endeavor where students are framed as not interested in learning and, as a result, teachers are excused from providing effective instruction.

Conclusion

Helping secondary mathematics PSTs shift their dispositions toward teaching and learning mathematics in high-needs schools is complicated and complex work. It is essential as dispositions influence the moment-to-moment decisions mathematics teachers make in their classrooms (Hand, 2012). Supporting PSTs in this way requires attention to content, beliefs, and affect, and it requires PSTs to challenge and confront their own assumptions. Field experiences are viable spaces for doing some of this work. They give students authentic spaces to wrestle with the challenges that practicing teachers face daily. Finding new ways within methods courses to support this type of work continues to be a challenge we pursue.

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A REVIEW OF 25 YEARS OF RESEARCH: ELEMENTARY PROSPECTIVE TEACHERS IN UNIVERSITY MATHEMATICS CONTENT COURSES

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The purpose of this review was to examine the 25-year period of research on elementary prospective teachers (EPTs) in mathematics content courses since publication of the Curriculum and Evaluation Standards for School Mathematics (NCTM, 1989). Analysis included an extensive electronic search and a manual search of 11 well-respected journals. Twenty-four studies met the inclusion criteria. Nineteen of the 24 studies occurred in the context of a reform pedagogy. Results showed positive changes in EPTs' affect were possible, but these shifts were sometimes difficult to come by and encountered resistance from the EPTs. Some studies showed an increase in EPTs' content knowledge, while others did not achieve the desired effects. Results show further study of EPTs' content knowledge is warranted. Implications for course learning experiences and suggestions for future research are offered.

Keywords: Mathematical Knowledge for Teaching, Teacher Education-Preservice, Teacher Beliefs, Teacher Knowledge

Introduction

Similar to many countries, in the U.S. most elementary teachers are prepared as generalists during initial teacher preparation, ultimately assuming positions in schools requiring the teaching of all subjects. Such all-purpose preparation has led to a corpus of elementary teachers needing improved knowledge for effectively teaching mathematics with understanding and proficiency at the level of rigor and depth depicted by the *Common Core State Standards for Mathematics* (CCSI, 2010). The significance of mathematical knowledge should not be underestimated, since teacher knowledge has been linked to teaching effectiveness and ultimately student learning (Hill, 2010). Wu's (2009) assertion rings true: "The fact that many elementary teachers lack the knowledge to teach mathematics with coherence, precision, and reasoning is a systemic problem with grave consequences" (p. 14). Accordingly, considerable resources and efforts have been devoted to both understanding this knowledge and determining efficacious ways of building it (e.g., Ball, Hill, & Bass, 2005). Many institutions of higher education have added specialized mathematics content courses for elementary prospective teachers (EPTs) that according to the Conference Board of the Mathematical Sciences (CBMS, 2012) should be grounded in the perspective that teachers should study the mathematics they teach in depth and from the viewpoint of the teacher.

Such mathematics courses hold challenges specific to elementary teachers, who often have negative affect toward mathematics, including dislike and avoidance (Bekdemir, 2010), and a propensity to espouse traditional, procedural views on what it means to know and do mathematics and on how it is learned (CBMS, 2012). These difficulties are compounded by the perspective of some EPTs who think they do not need to learn more mathematics, as their experiences thus far have provided them sufficient content knowledge needed for teaching in the elementary classroom. Given this increasing awareness of the need for improved mathematical knowledge of elementary teachers via specialized mathematics courses, coupled with constraints particular to this population, research on EPTs in university mathematics content course experiences is critical.

Purpose of the Review

The purpose of this review is to examine the research on EPTs in mathematics content courses since publication of the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989). Specifically, the review examines empirically based articles published from 1990 to 2014 (25 years) with EPTs as participants, who were completing university courses identified as mathematics content courses. We review past research in order to present the state of knowledge on the mathematical preparation of EPTs and also to highlight areas research has left unresolved. We provide summary findings and contributions of the body of work, implications for elementary teacher preparation, and suggestions for future research.

Method

The pool of articles in this review was determined through several cycles of appraisal and scrutiny using a comprehensive search strategy aimed at identifying all available studies. Criteria for initial inclusion were that the study must have: (a) been published between 1990 and 2014, (b) included EPTs in an initial certification program as participants, and (c) occurred in a university mathematics content course/university math for teachers content course/or university course that simultaneously focused on mathematics content and teaching methods as the context. A university course solely focusing on mathematics teaching methods was excluded, as were studies in yearbooks and conference proceedings.

The five-member research team engaged in several cycles of examination to arrive at the final collection of articles. The first round was an electronic search that cast a broad net for evidence of the initial inclusion criteria. The electronic databases explored were ERIC, PSYCH Info, Academic Search Complete, Professional Development Collection, and Psychology and Behavioural Sciences Collection using combinations of the following key words: elementary education, elementary teachers, mathematics, undergraduate, mathematics courses, pre-service teachers, prospective teachers, content courses, and early childhood education. Review of abstracts from this search yielded a total of 54 publications. Simultaneously, a targeted, manual review of abstracts in 11 journals considered to have high scholarly regard in the fields of mathematics education and teacher education were examined. These included *American Educational Research Journal, Cognition and Instruction, Educational Studies in Mathematics, Elementary School Journal, Journal of Mathematical Behavior, Journal of Mathematics Teacher Education, Journal of Research in Mathematics Education, Journal of Teacher Education, School Science and Mathematics, Teachers College Record,* and *Teacher and Teaching Education*, which gleaned a total of 40 articles. After eliminating duplicate articles between the electronic database and targeted journal searches the pool for the next round was 89 articles.

Because the original search cast a wide net that focused only on the content of abstracts, the second round of examination focused on the entirety of the article and involved in-depth scrutiny of each manuscript. All 89 manuscripts were carefully reviewed by at least two researchers using a rubric that applied the initial inclusion criteria. In addition to the criteria listed above, to be included the work had to: (a) be research-based, i.e., the report presented research question(s) or purpose(s); (b) provide a review of relevant research and/or theoretical frame; (c) include a description of methods, including participants, context, data collection, and data analysis; (d) present results; and (e) give some interpretation or discussion. These research-based elements were created drawing from *Standards for Reporting on Empirical Social Science Research in AERA Publications* (AERA, 2006). In addition, we noted in the review if a pedagogical approach or method was indicated. In order to establish agreement on the meaning of the components of the rubric, each member of the research team independently read and examined two pre-determined articles (one qualitative and one quantitative) before convening as a group to consider and discuss consistency of interpretation and application of the rubric across the studies. Three typical reasons for exclusion from the final pool

were studies (a) that failed to meet all the research elements criteria, (b) that appeared to have content course contexts but were in fact methods courses, and (c) that involved participants other than EPTs. Of the 89 articles in this round, 24 qualified for the final pool. Journals represented and number of studies in each included *American Educational Research Journal* (1), *Educational Research for Policy and Practice* (1), *Educational Studies in Mathematics* (4)*, International Journal of Mathematical Education in Science and Technology* (1), *Issues in Teacher Education* (1), *Journal for Research in Mathematics Education* (1), *Journal of Mathematical Behavior* (5), *Journal of Teacher Education* (1), *Mathematics Education Research Journal* (1)*, Mathematics Teacher Education and Development* (1), *School Science and Mathematics* (7).

Summary Findings

We sorted the studies using four distinctive categories from the review rubric: research method, context, purpose/focus, and pedagogical approach of the instruction. A brief overview of each follows.

When considering ***research method***, five studies involved quantitative methods only, nine studies included qualitative methods only, and ten studies involved some combination of qualitative and quantitative methods. Clustering by years is shown in Table 1.

Table 1: Clustering by year of the study and research method

	1990-1994	1995-1999	2000-2004	2005-2009	2009-2014
Quantitative				3	2
Qualitative	2	1	2		4
Mixed		2	1	2	5

With respect to ***context***, 18 of the studies data collection occurred within one mathematics content course (which may have been collected across multiple sections of the same course). In the six remaining studies, context for data collection occurred in a variety of settings including a comparison of EPTs who completed a content course and those who did not, a study of four integrated content/methods courses, and studies that looked at more than one content course.

When classified by the ***purpose/focus*** of the study, twelve studies examined affective factors of the EPTs such as beliefs, attitudes, motivation, and identity. Ten studies investigated various forms and concepts of mathematical knowledge of the EPTs. One study explored both affective factors and mathematical knowledge. The last study examined the development of classroom norms and mathematical justification. A classification by years and focus is shown in Table 2.

Table 2: Clustering by year of the study and focus

	1990-1994	1995-1999	2000-2004	2005-2009	2009-2014
Affective	1	2	2	2	5
Cognitive	1		1	3	5
Other		1		1	

Finally we clustered by ***pedagogical approach*** used in the courses. We termed the pedagogy as reform when it aligned with pedagogy described by NCTM in the *Principles and Standards for School Mathematics* (2000) and/or the *Principles to Action* (2014). Any study that generally described EPTs learning through means such as peer or small group interactions, classroom discourse, modeling and representations, and/or problem solving tasks involving reasoning and sense making, among other elements, was considered to involve reform. Nineteen of the 24 studies were

identified as reform. In the remaining five studies, the pedagogical approach was either not described or was not clear.

Extended Analysis of Courses Involving Reform Pedagogy

Within the nineteen studies identified as using reform pedagogy in the courses, we found, among others, descriptors such as: the instructor acts as a facilitator; students work in small groups; students present their ideas to the class; manipulatives are used; and problems are used to build conceptual understanding, reasoning, and sense-making. Some studies included significant details about the instruction, while others were less forthcoming. In either case, the salience of this instructional focus is evident based on the large portion of the twenty-four studies that involved reform pedagogy (79%). Within these nineteen studies, ten focused on affective factors. Five of the ten examined change that occurred over a course or courses and four were descriptive. Seven of the nineteen studies focused on content knowledge, specifically its development. One of the nineteen examined affective factors and content knowledge, one compared two groups (one having a reform course and one that did not), while the remaining study focused on constructing classrooms norms and mathematical justification within a reform approach.

Studies Not Categorized as Using Reform Pedagogy

Three of the remaining five studies addressed specific areas of mathematics content (i.e., the associative property, order of operations, and polygons), but not within a reform course. All three of these studies were descriptive, relating EPTs' understandings. The two remaining studies explored affective factors. One studied the relationship between motivation to learn mathematics and attitudes toward mathematics and the second conducted a collective case study using data from two previous studies in which the researchers contrasted the EPTs and instructors' perspectives, looking for common themes and relationships. Table 3 displays reform and other instructional approach clustered by year.

Table 3: Clustering by year of the study and pedagogical approach

	1990-1994	1995-1999	2000-2004	2005-2009	2009-2014
Reform	2	3	2	4	8
Other			1	1	3

As evidenced in Tables 1-3, research on the mathematical preparation of EPTs has increased over the last 10 years, has predominantly employed a reform approach to instruction, and as noted earlier, was primarily done within a single course or sections of a single course. Quantitative only methods were employed least.

Space limitations prevent a detailed accounting of each of the 24 studies[1] in this paper. However, some additional summary findings are worthy of mention.

- Several studies across the time frame (e.g., 1996, 2004, 2007, 2014) using both qualitative and quantitative methods and occurring within a single course employing a reform pedagogical environment revealed EPTs generally developing positive affective stances.
- While positive impact of reform pedagogies was reported, there was also evidence of mixed results where EPTs showed some reform perspectives, but continued with some traditional perspectives (e.g., 1993, 2004, 2011, 2014).
- Some studies that employed reform approaches showed little to no change in the overall content knowledge of the EPTs (e.g., 1994, 2007, 2008).

Discussion

It is noteworthy that only 24 studies across the past 25 years were identified as meeting the inclusion criteria. Eleven of these studies (46%) were published during the last five years of the period of review, showing a recent increase in attention on the mathematical preparation of EPTs. This upturn is heartening, but given the importance of elementary teachers having a deep and flexible knowledge of elementary mathematics and that courses taken during teacher preparation offer the best opportunity for development of this knowledge, more inquiry is warranted.

Affective Factors

Of the 24 identified studies, 19 were in the context of courses involving reform pedagogy and more than half (10) of the 19 looked at affective factors. Overall, this review found that a reform instructional approach had positive effects and provided opportunities for constructive shifts in affective factors such as beliefs about mathematics, confidence, motivation, and attitudes. Specifically, six studies showed EPTs developed a positive perspective toward a reform approach to instruction, while three revealed a moderate effect and one showed almost no effect. Interestingly, this last study has the oldest publication date (1993), a time when focus on beliefs and affect was in its infancy.

The significant attention on teacher affect in the studies parallels the emphasis in the field of mathematics education over the time period and also provides credence to the assertion this teacher development construct is highly relevant to EPTs. That is, elementary teachers too often have negative affect toward mathematics, including a fear, dislike, and avoidance of the subject, and this propensity needs to be considered, addressed, and examined during university courses with the goal of affecting change. When considering learning experiences prompting change in teacher affect, the studies showed the use of problem-based instruction affording EPTs opportunities to invent their own solution strategies and engage in productive struggle, in addition to an emphasis on peer interactions, were efficacious.

Though course experiences promoted the desired changes in teacher affect to a degree, some studies noted that EPTs enter courses with such deeply rooted, negative affect that change was difficult and too often persisted at some level throughout and after the courses. Letting go of traditional perspectives on mathematics pedagogy learned over many years as students in classrooms was an arduous process and fraught with resistance. Results from two of the descriptive studies on teacher affect, one from the beginning of the time period (Civil, 1993) and one at the end of it (Chamberlin, 2013) support this assertion. We found that while the more recent group showed some change, both groups of EPTs, separated by a generation, exited their mathematics course still holding *some* traditional perspectives on what it means to teach and learn mathematics.

Content Knowledge

Further consideration of the focus of the identified studies showed an increased emphasis on research on content knowledge over the time period. Of the seven studies focusing on specific content knowledge in this group, four studies revealed improved content knowledge in courses involving reform pedagogy, while the remaining three did not achieve the desired effect. A few studies (some within a reform pedagogy and some that were not) focused on examining specific areas of content knowledge (e.g., fractions, nominal categorical data, ratio, polygons, order of operations, associative property of multiplication) that revealed mixed findings of EPTs learning. Research on more areas of elementary mathematics is clearly needed. Several studies lent credence to the importance of EPTs' understanding the relevance and usefulness of the mathematics learned in the courses, including explicit connections to the elementary classroom. However, one study found instructors and EPTs had conflicting viewpoints on such an emphasis (Hart, Oesterle, & Swars, 2013).

The overall shift in attention from affective factors (e.g., teacher affect, etc.) toward content knowledge is depicted in Figure 1. Over the time period 1989-2014, the percentage of studies focusing on content knowledge increased, while studies of affective factors remained relatively constant (Three studies are not included in this total.)
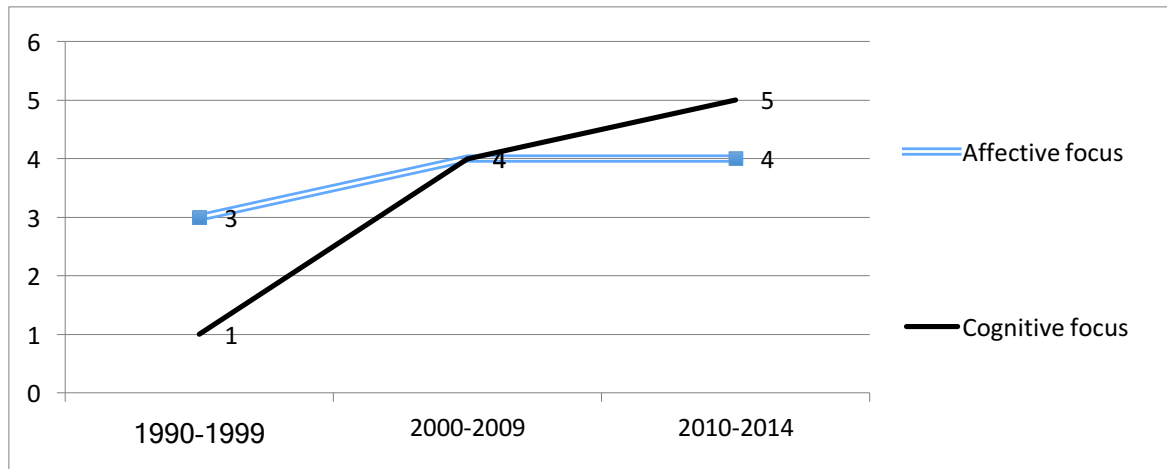
Figure 1: Changes in focus related to the EPTs.

This general trend aligns with the recognition of the need for elementary teachers to have well-developed mathematical knowledge, coupled with the efforts to define the mathematical knowledge needed for teaching (MKT).

Potential Implications

As more and more institutions of higher education require specialized mathematics content courses for EPTs, this analysis revealed certain elements of reform pedagogy generated positive changes. EPTs must have time and opportunities to think about, discuss, and explain mathematical ideas, coupled with a focus on mathematics as a sense making activity. Further, it is imperative EPTs perceive the relevance of the mathematics they are learning to their future profession. Emphasizing problem solving and other mathematical processes (Lubinski & Otto, 2004) and also studying children's thinking (Philipp et al., 2007) are effective mechanisms for promoting learning and change.

Different institutions of higher education include varying mathematics topics in their content courses for EPTs. Regardless of the specific content, an emphasis on problem solving, reasoning, and justification promotes EPTs' learning and change. For example, Liljedahl (2005) suggests that through posing problems, allowing substantial time for working on problems immediately after being assigned, providing time to revisit already assigned problems, working with peers in small groups, and engaging in a reflection about problems can positively transform EPTs' mathematical affect.

Further, while the ostensible purpose of a mathematics content course for EPTs is to learn mathematics, there are strong reasons to incorporate study of children's mathematical thinking into coursework (Philipp et al., 2007). Instead of trying to interest EPTs in mathematics for the sake of mathematics itself, providing connections to something to which they are fundamentally concerned, children, prompts motivation, learning, and change. Studying children's thinking challenges EPTs' beliefs about mathematics and leads to the recognition that their own mathematical understandings are insufficient for teaching elementary mathematics. Such an emphasis also helps EPTs appreciate how important it is for them to know the content for their future roles as teachers. The relevance of the mathematics they are learning to their chosen career path is evident.

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Suggestions for Future Inquiry

Considerable challenges exist for reaching the desired goals for EPTs in mathematics content courses. In general, the findings of this analysis showed positive changes in teacher affect and content knowledge are possible within a reform approach to instruction. But, these changes were sometimes difficult to come by and often encountered resistance from the EPTs. Given the paucity of studies, more inquiry is needed, particularly on the aforementioned means of prompting learning and change (e.g., a focus on problem solving, studying children's thinking, etc.). In addition, longitudinal study over several courses or an entire teacher preparation program is warranted. Further, the sustainability of changes in EPTs as they graduate and become responsible for their own classrooms, and how or if such changes translate into classroom practices, are worthy of study. We note that some longitudinal studies may exist but were not the focus of this review. Likewise, EPTs' individual characteristics or capabilities should be examined in relation to change, given the findings of one study revealing mainly high mathematical achievers in the course significantly changed their beliefs (Emenaker, 1996), in contrast with the results of Philippou and Christou (1998), who found change was not-correlated with any of the individual characteristics tested, including mathematical performance.

As a final note, one concern from this review is the significant number of the original 89 studies that failed to incorporate all the elements of quality research and therefore were not considered to meet the criteria for this project. It is imperative that future research on the mathematics education of both elementary prospective and practicing teachers attains a level of rigor that will inform the field of mathematics education in a trustworthy and legitimate way. After 25 years of research in this area, we are still limited in definitive evidence of how to make a difference with this population of learners.

Endnotes

[1]Full citations for the 24 papers mentioned in this paper will be available at the presentation and can be obtained by emailing the first author.

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FACTORS RELATING TO THE DEVELOPMENT OF MATHEMATICS FOR TEACHING

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This study investigated the effect of different aspects of mathematics knowledge for teaching on performance in upper elementary mathematics methods courses. In the Ontario (Canada) context, prospective teachers have been, until the 2015-2016 school year, able to obtain a Bachelor of Education (BEd) degree in as little as eight months after a different undergraduate degree, with most students taking no university mathematics courses whatsoever prior to the eight month BEd. We examined the effect of performance in a new course on mathematics for teaching as well as an exam in mathematics content, on performance in the methods course. Mathematics knowledge for teaching, as measured by the final grade in the mathematics for teaching course, was found to be a significant predictor of first semester methods course performance, with general content knowledge being less important.

Keywords: Mathematical Knowledge for Teaching, Teacher Education-Preservice, Teacher Knowledge

Introduction

At the time of this study, prospective teachers at our Ontario university would enroll in a one-year Bachelor of Education (BEd) program in order to be certified to teach. This program would occur after the candidates had completed another unrelated degree, such as a BA in History or English. The majority of the prospective teachers in our program have not taken any mathematics courses at the university level prior to entering the BEd program. Within our program, prospective teachers would graduate with a BEd after taking only one thirty-six hour mathematics Methods course. Since specialised mathematical knowledge of teachers has been directly linked to student achievement (Baumert et al., 2010), developing prospective teachers' mathematical understanding, in particular the understanding needed for teaching, is an important aspect of the education program we offer. With the limited number of hours we have to build these understandings, we have been concerned with what program developments we could put in place to increase the opportunity for knowledge development. One of the changes has been a Competency Exam taken by all prospective teachers in September at the beginning of their BEd year. This exam tests only content knowledge found in the elementary curriculum (Ontario Ministry of Education, 2005). More recently, a second program change was undertaken, which supports some of the candidates and is the focus of this research.

At our institution, there is one other way that prospective teachers can achieve their degree: through the Concurrent Education program. This means that they are accepted into the BEd program and their other degree at the same time. This allows for some education courses to be taken earlier, however the majority still occur in the one-year BEd program taken after their first degree is completed. The extra time, however, has allowed for us to institute one thirty-six hour Mathematics for Teaching course prior to the Methods course in the BEd program. The effect of this addition is that these students now have double the hours in mathematics courses particularly aimed at mathematics knowledge needed for teaching. Although this alternate route is an option for our prospective teachers, the majority of prospective teachers, however, do still enter the eight month BEd program after completing their first degree and thus miss the first Mathematics for Teaching course.

We were interested in the effects of these options on performance in the Methods course, hence,

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this study examines the effects different types of mathematics knowledge (as developed through these various components of our program) have on the prospective teachers' understandings of mathematics as needed for teaching. Specifically, we examined the influence that the performance in the Mathematics for Teaching course, and the scores on the Mathematics Competency Exam, have on performance in the Methods course.

Framework

Since mathematics knowledge for teaching is not "just" a knowledge of subject matter (Baumert et al., 2010; Kajander, 2010; Silverman & Thompson, 2008), it is important that teacher knowledge includes a knowledge of students and of teaching mathematics. Simply stating a procedure or procedural steps would not be enough to show mathematical reasoning and support a claim when teaching (Ball & Bass, 2000); indeed mathematics for teaching is often described as including 'more' than the ability to perform standard procedures (Silverman & Thompson, 2008). In many descriptions of teachers' mathematical knowledge, the content a teacher needs may be something that goes beyond, or is somehow distinct from, what any non-teacher studying mathematics would need (Baumert et al., 2010; Ma, 1999). It has been further argued that such knowledge differs from the knowledge incoming prospective teachers typically possess (Chamberlin, Farmer, & Novak, 2008; Davis & Simmt, 2006; Kajander, 2010).

In our recent work we have sought to determine, describe, and unpack specific content pieces that we feel are critical to teacher mathematical capacity. In particular, we have found the models and modelling approach described in Lesh and Doerr (2003) fundamentally helpful in this regard. It is this approach that frames the content areas and approaches chosen for inclusion in the course textbook which was written especially for our Mathematics for Teaching course (see Kajander & Boland, 2014), and which guides the course activities. Developing a deep conceptual sense of elementary mathematics content based on models, relationships, and connections, and all interconnected with reasoning which might be applicable to classroom discourse, form some of the goals of our elementary mathematics courses for teachers. In particular, we strive to focus on the development of "how to gradually decompose and unpack the mathematical rules and operations through the use of representations, and knowledge of how to use representations to develop generalizations" (Mitchell, Charalambous, & Hill, 2014, p. 55).

The Mathematics for Teaching (MKT) model proposed by Ball and her colleagues (Ball, Thames, & Phelps, 2008) makes a distinction among different kinds of mathematical understandings held by teachers, such as the descriptions of 'pedagogical' and 'specialised' knowledge. While at times we find these distinctions have been blurred in our own work, they can be helpful in providing broad ways to describe different components of teacher knowledge. For simplicity in describing our current data, we make use of the terms from three of the categories of the MKT model, specifically, common content knowledge, specialised content knowledge, and the over-arching category of pedagogical content knowledge.

Ball, Thames, and Phelps (2008) begin their discussion of the different types of mathematics knowledge with an examination of common content knowledge (CCK). Based on their research, this knowledge is described as the general mathematics knowledge that any person studying mathematics would know. The Mathematics Competency Exam in our program is designed to test basic elementary school mathematics performance to the eighth grade level. However, the exam does not include items related to the use of models, multiple methods, or connections among ideas, all of which we see as crucial to teachers' knowledge. Rather, the test draws directly from our provincial grade eight mathematics curriculum document (Ontario Ministry of Education, 2005). Hence, for the purposes of the current analysis, we use the common content knowledge acronym, CCK, to refer to the grade in the September writing of this test (the 'Competency Exam'), as we feel it represents knowledge that all school students might be expected to develop. A sample item from the

Mathematics Competency Exam is provided in Figure 1.

A small 1 m^2 garden is to be planted with herbs. One-fourth of the garden will be chives and one-fourth parsley. Of the remaining portion of the garden, one-third will be basil, one-third will be thyme, and the rest will be sage. What fraction of the total garden is sage?

Figure 1. Sample Question from Mathematics Competency Exam.

The next piece of the MKT model that is pertinent to our study is that of specialised content knowledge (SCK; Ball, Thames, & Phelps, 2008). As described in the MKT model, this knowledge is special knowledge required of teachers, and goes beyond what an individual studying mathematics would require. This knowledge, however, does not depend on knowledge of students or knowledge of teaching; it is indeed 'mathematical'. Final course grades in the Mathematics for Teaching course were used as a measure of SCK. While a thorough analysis of every examination item from this course with the MKT framework has not been conducted, the overall descriptions from the literature of important elements of SCK do align well with our course objectives and assessment items, hence a global alignment is arguable. Our course focuses, for example, on models and reasoning about problems that would be important to teachers while in the field, however does not focus on pedagogy or knowledge of students. A sample item from the final course exam is shown in Figure 2.

The following pattern is made of toothpicks:

n: 1   2   3
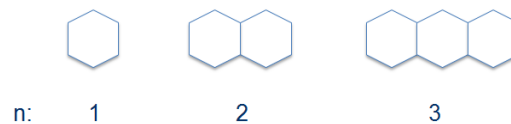
- State a formal or function rule using "n" as the picture number, which determines the number of toothpicks needed for each picture. Redraw the third picture in the above pattern and use it to show how your rule relates to the pattern.

- State a different formal or function rule which relies on a different geometric way of grouping and counting the same pattern of toothpicks. Again redraw the third picture in the above pattern and use it to show how the new rule relates to the pattern.

Figure 2. Sample Item for Mathematics for Teaching Course.

Lastly, we look at one final aspect of the MKT model: pedagogical content knowledge (PCK). This aspect of the model includes sub parts of knowledge of content and teaching, and knowledge of content and students (Ball, Thames, & Phelps, 2008). Based on the MKT model, PCK includes knowledge of models and reasoning in relation to student understanding, including knowledge of typical student errors and so on. To measure this attribute, we used the grade in the Methods course from the fall semester course exam. Items on this exam include extensive use of models and reasoning (as do the Mathematics for Teaching course items), but also items related to typical student errors, misconceptions, and other topics drawn from the MKT model categories of PCK. Thus we use the over-arching name of PCK to refer to the aggregate of our items of this type, as found in the fall Methods course exam. Figure 3 provides a sample item from the Methods course exam.

Consider this problem for the questions below: 1 ⅓ ÷ ¼
- Student A believes the answer is 5 ⅓.
- Student B believes the answer is 5 $\frac{1}{12}$
Is either student correct? If so, which student is correct?
Using reasoning and models, show the thinking of each student that may have resulted in their solution for the question.

Figure 3. Fall Methods Course Exam Item.

Methods

Our study sought to further enhance our understandings of what and how previous experiences in mathematics for teaching, as grounded in the MKT (Ball, Thames, & Phelps, 2008) model, might support further development at the methods course level. We used a quantitative method design to search for relationships between the different factors in our program. We were interested in whether the kinds of specialised content knowledge taught in our Mathematics for Teaching courses were helpful in supporting the development of pedagogical content knowledge in our Methods course. We also examined if there were effects from the scores on the Competency Exam on the development of pedagogical content knowledge. For ease and familiarity, we have named the three variables using the MKT descriptors, as just explained. However, agreement with our naming is not required to examine the results – the names of the variables we term CCK, SCK and PCK could simply be replaced by the course components they respectively represent, which are the Competency Exam grade, the Mathematics for Teaching course final grade, and the Methods course first term exam grade.

Participants

All participants were prospective teachers in the thirty-six hour upper elementary mathematics Methods course. All participants in the Methods courses were asked to participate in the study, and most did (N=71 each year for each of two years). In both years, data were collected from all participants who volunteered during the Methods course. In particular, we were interested in whether each participant had previously taken the new Mathematics for Teaching course (which could have been taken in any previous year since 2010-2011), and if so, how well they did in it.

Descriptive statistics were first used to examine the variables PCK and CCK by separating the groups based on whether or not participants had enrolled in the Mathematics for Teaching course (the final grade in which was recorded as the SCK score). As mentioned, the grade on the Mathematics Competency Exam was used as a measure of CCK, and grades on the Professional year Methods course fall exam were used to measure PCK. Further quantitative analysis was conducted to explore the relationships of the two independent variables SCK and CCK, to see the impact on PCK. Relationships among these three variables were explored using independent t-tests as well as regression analysis. We used independent sample t-tests to compare the PCK as well as the CCK to determine differences between the two subgroups (those with SCK and those without). A regression analysis was also conducted on those who had taken the new Mathematics for Teaching course (SCK) to determine the effect of SCK and CCK on the fall methods exam (PCK).

Since all sections of the Methods course during the three year period were taught by the same instructor (first author) in a similar manner, and all of the previously taken Mathematics for Teaching courses had also been taught by a consistent instructor (second author), we chose to combine the quantitative data sets for both years. In total then, we collected performance data from a total of 142 prospective teacher participants, drawn from two cohorts of 71 participants. Of these 142 prospective

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teachers, only 22 of the prospective teachers had enrolled in the Concurrent Education program and therefore only these 22 had taken the Mathematics for Teaching course.

Results

We began by examining the data collected from the 142 participants in our pool of prospective teachers. As mentioned, 22 of these had taken the Mathematics for Teaching course, and 120 had not, meaning they were registered in the stand-alone eight month BEd program. Thus the subgroup with an SCK score had 22 participants. Of the 142 examined in total, 10 (only 1 from the SCK sub-group) chose not to take the September write of the Competency Exam so did not have a CCK score. Of the 132 participants who took the Competency Exam (21 with a SCK score), 66 of them did not reach the 75% required "pass" score. Of this 66, 10 were in the SCK subgroup. All of the participants had a PCK score since all were enrolled in the Methods course.

Table 1: Percentage Means by Subgroup

Subgroup	CCK Mean score (%) (Competency Exam)	PCK Mean score (%) (Methods course exam)
SCK	74.867 (N=21)	76.500 (N=22)
No-SCK	68.622 (N=111)	67.700 (N=120)

Initial observations showed that the participants with SCK, meaning they had taken the Mathematics for Teaching course, did better in both CCK and PCK (see Table 1). Since 10 of the SCK group did "fail" the CCK ("Competency Exam") portion, further analysis was performed on examining the participants in the SCK sub-group (see Figure 4) to explore this situation further. Only 21 participants are included in the graph since 1 did not have a CCK score. Prospective teachers are given the option of challenging the Competency Exam in September of their BEd year. They are encouraged to take the exam at this time in order to set goals for areas of content that they need to work on prior to taking the exam in March where a score of less than 75% would mean they do not get their BEd degree. Most prospective teachers do choose to the take exam in September for this reason.
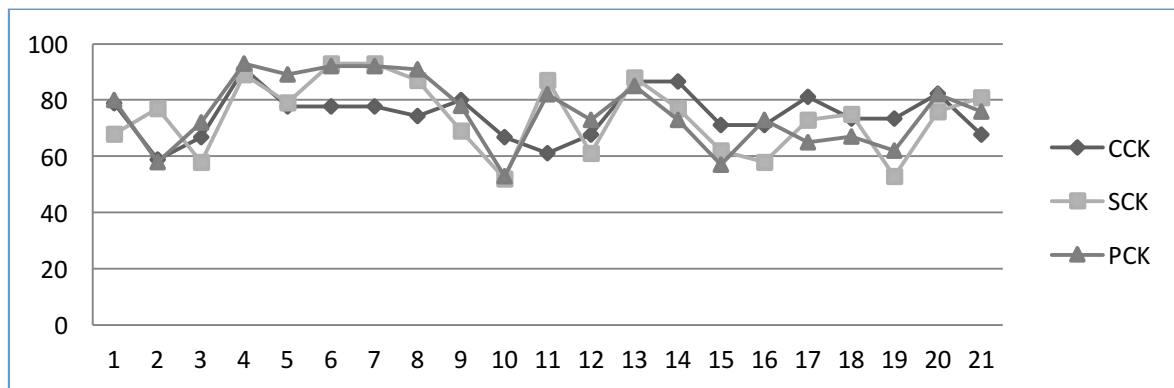
Figure 4. Graph of SCK Participants' Percentage Scores, N=21.

Examining the graph of the performance of each of the 21 participants for whom we had all three scores, suggests that the scores for SCK and PCK were more closely clustered for more of the participants than the CCK scores. (The lines joining the points are illustrative only; the actual dataset is comprised only of the three scores for each separate person (numbered 1 to 21)).

When examining the relationship for participants with and without SCK on the fall Methods course exam (PCK), Levene's Test for Equality of Variances showed there was no significant

variance between the subgroups. Therefore, an independent t-test was run on the data as well as 95% confidence intervals (CI) for the mean difference. It was determined there was a significant difference in results of PCK between those who had taken the Pre-professional year Mathematics for Teaching course (SCK) and those who had not ($t(138) = -2.472$, $p = .015$) with a difference of -8.196 (95% CI, -14.752 to -1.64). When subsequently examining the relationship for participants who had taken the new Mathematics for Teaching course (SCK) and those who had not, on the Mathematics Competency Exam (CCK), Levene's Test for Equality of Variances determined there was significant variance ($p = .010$) between the subgroups. Therefore, an independent t-test was run on the data and equal variances were not assumed, as well as 95% confidence intervals (CI) for the mean difference was also determined. It was determined there was a significant difference in results on the Competency Exam (CCK) between those who had taken the new Mathematics for Teaching course and those who had not ($t(55.006) = -2.581$, $p = .013$) with a difference of -2.813 (95% CI, -.629 to -4.997).

Table 2: Regression Model

Predictor of Methods course exam	Beta	Sig.
Competency Exam (CCK)	.265	.099
Mathematics for Teaching course (SCK)	.662	.000

R = .797, R squared = .634, N = 21

The regression model showed a highly statistically significant ($R^2 = 0.632$, $p = .000$) prediction of the fall Methods exam (PCK) when examining the participants who had taken the new Mathematics for Teaching course (SCK). It was determined that 63% of the variance on the fall exam (PCK) could be explained by the two factors. For this model, SCK ($\beta = 0.0662$, $t(18) = 4.333$, $p=.000$) significantly predicted scores on the fall Methods exam (PCK), while CCK ($\beta = 0.265$, $t(18) = 1.737$, $p = .099$) was not significant. Scores in the new Mathematics for Teaching course (SCK) had a greater impact on the fall Methods exam (PCK) scores (see Table 2).

Discussion

The data may shed some further light on the impact and importance of different aspects of mathematics knowledge on the development of pedagogical content knowledge. While one perception might be that learning about contexts, models and reasoning in elementary mathematics might best be done in methods courses (rather than 'mathematics' courses), we continue to argue for the crucial need for such learning during *mathematics* courses for teachers, in order to better support what can be subsequently achieved during methods courses. The current data may support such an assertion. Indeed, Mitchell, Charalambous, and Hill (2014) argue that an understanding of models and reasoning is an area particularly in need of support for many teachers.

The results suggest that the Competency Exam, used in the current analysis as a measure of CCK, had a relatively weak impact on performance at the Methods course level. We argue that "knowing the curriculum", particularly the more computationally-related aspects, is far from sufficient for teaching. Certainly, our previous qualitative observations of participants during the Methods course suggest that the more specialised the background of the participants, the greater the strides they may be able to make in terms of PCK development (Holm & Kajander, 2012). Without such conceptual understanding, learning about structuring a lesson to explore mathematical concepts is impossible; teachers are left with a rule-based treatment as their only lesson option. Indeed, we have found that prospective teachers without previously-developed knowledge of how to understand and represent mathematical content (SCK) are so significantly distracted by the mathematical challenges that they are often less involved in the pedagogical conversation taking place. Hence we find that while more knowledgeable Methods course participants are having discussions regarding

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lesson design, student responses, typical errors, and so on, the participants with weaker SCK are typically focused mainly on the mathematical concepts, missing out on the other ideas. This suggests that those with weaker SCK may well have reduced opportunities to develop PCK, and aligns with the results presented here. Hence the current results accurately reflect our anecdotal experiences while teaching the Methods course, regarding the importance of SCK in pedagogical development during elementary teacher education.

The study results suggest that the impact of a specialised course in mathematics for teachers, (SCK), does impact Methods course performance (arguably termed PCK here) in a highly significant way, and thus underscore the need for continued emphasis on *specialised* mathematical experiences for teachers. Generally, participants who did well in the Mathematics for Teaching course tended to do well in the Methods course, and additionally were also observed informally to demonstrate deeply pedagogical understandings during Methods course classes. In particular, several participants who had taken the Mathematics for Teaching course prior to beginning the Methods course and had done very well, both began and ended the Methods course in a very strong position. As course instructors, we observed generally that evolving confidence in the specialised mathematics seemed tightly aligned with pedagogical mastery. This data provides further support of the mounting evidence that it is the more conceptual and specialised mathematical knowledge that supports better teaching rather than only more general mathematics knowledge. The results also align with the Baumert et al. (2010) study, which found that the degree of teaching-specific mathematics knowledge of early secondary level classroom teachers made a measureable difference on student achievement, while general mathematics background of teachers did not.

At the institution where this study took place, there is a continued sense by non-mathematics education faculty and administration that specialised content knowledge, such as knowledge of models and reasoning, should be contained in Methods courses, rather than being thought of as 'mathematics'. Based on the MKT (Ball, Thames, & Phelps, 2008) model, and our own data, we continue to argue for the need for *mathematical* experiences specialised to teaching, in order to support richer pedagogical development. The data presented here contribute to such a stance.

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DIAGNOSING REASONING TO MEASURE GROWTH IN PRE-SERVICE MIDDLE-GRADES TEACHERS' FACILITY WITH FRACTION ARITHMETIC

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The present study extends recent advances coordinating research on cognition and psychometric modeling around fractions. Recent research has demonstrated that the Diagnosing Teachers' Multiplicative Reasoning Fractions survey provides information about distinct components necessary for reasoning in terms of quantities when solving fraction arithmetic problems. The present study (a) adds a new component of validity for the survey and (b) examines the utility of the survey as a measure of growth in pre-service middle-grades teacher's facility with fraction arithmetic as they completed a 1-semester content course. Results provide an existence proof that that the survey is sensitive to shifts towards more proficient reasoning.

Keywords: Rational Numbers, Research Methods, Teacher Education-Preservice, Teacher Knowledge

One central challenge for mathematics education is fostering teachers' and students' capacities to reason about multiplicative relationships in terms of quantities. Relevant topics include whole-number multiplication and division, arithmetic with fractions, proportional relationships, and linear functions. Reasoning with quantities is emphasized in recent curriculum standards documents (e.g., Common Core State Standards Initiative, 2010; National Council of Teachers of Mathematics, 2000) and recommendations for teacher education (e.g., American Mathematical Society, 2010; Sowder et al., 1998). These standards and recommendations place high value on developing conceptual understanding by solving and reflecting on solutions to problems couched in quantities.

A second central challenge is developing ways to coordinate research on cognition with the increasing variety of available psychometric models (e.g., Izsák, Remillard, & Templin, in press). As explained below, most recent applications of psychometric models to mathematics education research, especially to research on teachers, have relied on well-established item response theory (IRT) models to measure knowledge. These models report single scores that locate examinees on uni-dimensional continuous scales. A recently developed family of psychometric models, called diagnostic classification models (DCMs), trade continuous for categorical variables in exchange for measuring multiple dimensions within practical testing conditions. Instead of reporting single scores, DCMs report profiles of strengths and weaknesses on several dimensions simultaneously. Bradshaw, Izsák, Templin, and Jacobson (2014) reported on the Diagnosing Teachers' Multiplicative Reasoning (DTMR) Fractions survey that was developed for use with DCMs and measures teachers' capacities to reason about multiplication and division of fractions in terms of quantities. That study reported on content and item-level validity of the survey and results of analyzing a national sample of 990 in-service middle grades teachers.

The present study extends research on the DTMR Fractions survey by applying it directly to mathematics teacher education. In particular, we administered the survey at the beginning and end of a 1-semester content course focused on arithmetic with rational numbers and asked two questions:

1. To what extent were the DTMR Fractions profiles consistent with pre-service teachers' reasoning across multiple survey items and related tasks?
2. Did the distribution of DTMR Fractions profiles shift after the 1-semester course on number and operations and, if so, how?

The first question addresses additional aspects of validity not taken up by Bradshaw et al., and the second question asks what growth and change might be captured by the DTMR Fractions survey that would be obscured by measures designed for uni-dimensional IRT models.

Measuring Teachers' Mathematical Knowledge for Fraction Arithmetic

In the last 2 decades, researchers have made important advances conceptualizing teacher knowledge that supports student learning. Well-known examples include Shulman's (1986) knowledge categories, especially pedagogical content knowledge, and Ball and colleagues' (e.g., Ball, Thames, & Phelps, 2008) subsequent articulation of mathematical knowledge for teaching (MKT). New conceptualizations of teacher knowledge have spurred, in turn, new approaches to measuring teachers' mathematical knowledge (e.g., Baumert et al., 2010; Hill 2007; Kersting, Givven, Sotelo, & Stigler, 2010; Saderholm, Ronau, Brown, & Collins, 2010; Shechtman, Roschelle, Haertel, & Knudsen, 2010). Most of these measures have been developed for use with traditional, uni-dimensional IRT models.

In contrast to the teacher knowledge measures mentioned above, the DTMR Fractions survey captures information about four components for reasoning about fraction multiplication and division in terms of quantities. The solution outlined to the following problem illustrates what we mean by components of reasoning:

> A batch of brittle calls for ¼ of a cup of honey. Megan has 2/3 of a cup of honey. How many batches of brittle can Megan make?

The solution we present presumes a teacher providing opportunities for students to solve fraction division problems before learning a general numeric method, such as multiplying by the reciprocal of the divisor. Thus, the teacher and students have to reason with quantities directly, which is consistent with the curriculum standards and recommendations for teacher education mentioned above.

First, to see the opportunity for discussing division, a teacher would have to recognize that the Brittle problem asks the signature how-many-groups question for measurement division. To support students' reasoning with quantities in the problem, some drawn model would be useful. The rest of the solution we present makes use of double number lines, but our larger points are not dependent on this choice of drawn model. Figure 1a shows a double number line that uses lengths to depict cups of honey. One number line represents cups subdivided into thirds, and one represents cups divided into fourths. Juxtaposing the two number lines highlights the challenge that fourths and thirds do not subdivide one another evenly. Figure 1b illustrates how partitioning thirds into 4 parts and fourths into 3 parts creates a finer unit, twelfths, which subdivide both thirds and fourths. The final challenge is to interpret the mini-pieces in terms of the given situation. There are multiple candidates, including interpreting one mini-piece as a twelfth of 1 cup, as a fourth of 1/3 cup, and as a third of ¼ cup. Because the problem asks about the number of ¼ cups in 2/3 cups, ¼ cup is the appropriate referent unit: There are 8/3 ¼-cups in 2/3 cups.

This is not the only solution to the Brittle problem, but it does illustrate that constructing a solution requires multiple, constituent components of reasoning. The given solution highlights the ability to (a) recognize the appropriateness of an arithmetic operation for modeling a given problem situation, (b) use whole-number factor-product combinations as a resource for partitioning quantities, and (c) identify appropriate referent units for each number. A measure of knowledge for this domain designed for traditional, uni-dimensional IRT models would collapse information about these different components into a single score.
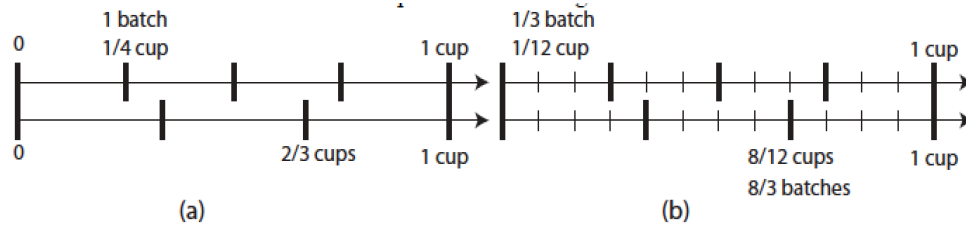
Figure 1. Reasoning with a double number line.

The DTMR Fractions survey consists of 27 items that measure these three components (termed appropriateness, partitioning and iterating, and referent unit, respectively) and one more, reversibility, which is important for solving partitive division problems. Some items measure just one of the four components, referred to as attributes. Other items measure more than one attribute at the same time. Nineteen of the items are multiple choice, and eight are constructed response. Figure 2 shows a released item that measures two attributes simultaneously: Selecting the correct answer choice, b, requires identifying the correct referent unit for 1/8 and partitioning intervals appropriately. The DTMR Fractions survey systematically elicited combinations of the four target attributes and was designed for use with DCMs, which report profiles of strengths and weaknesses on the multiple attributes instead of single scores on one overall dimension.

Ms. Roland gave her students the following problem to solve:

Candice has 4/5 of a meter of cloth. She uses 1/8 of a meter for a project. How much cloth does she have left after the project?

She had students use the number line so that they could draw the lengths. Which of the following diagrams shows the solution? Assume all intervals are subdivided equally.

Figure 2. DTMR Fractions item. From Izsák, Jacobson, de Araujo, and Orrill (2010). © 2010 by University of Chicago. All rights reserved.

Methods

Data for the present report come from a larger on-going study of pre-service teachers' reasoning about multiplication and division, fractions, and proportional relationships. We administered the DTMR Fractions survey to a cohort of 22 pre-service middle-grades mathematics teachers before and

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after a number and operations content course offered in Fall 2014. The course was part of a teacher education program in a large, public university in the Southeast United States and emphasized reasoning with quantities to develop conceptual understanding of multiplication and division with whole numbers and with fractions. A following algebra course, offered in Spring 2015, focused on proportional relationships and linear equations. Beckmann taught both courses. For the present report, we focus on data from the number and operations course.

The pre-service teachers completed the DTMR Fractions survey for the first time as the first homework assignment in the number and operations course (August 2014). They were told that the survey was a formative assessment and that they should simply do their best. Bradshaw analyzed the survey data (item responses) to estimate DTMR Fractions profiles for all 22 pre-service teachers. Given the size of the national sample ($n = 990$), the DCM analysis was limited to treating each attribute as a dichotomous variable. Bradshaw then selected six pre-service teachers with a variety of initial profiles to be focal participants for this study. She revealed the overall distribution of DTMR profiles to Izsák and Beckmann but not profiles for particular individuals. Thus, Izsák and Beckmann were blind to the profiles until the end of the study.

Izsák and Beckmann then conducted a series of six individual video-recorded interviews with each of the six focal participants. The interviews began during the number and operations course and continued into the algebra course. In addition to the interviews, we collected written artifacts (e.g., homeworks, quizzes, and tests) from all pre-service teachers in the course. The pre-service teachers completed the DTMR Fractions survey for the second time as the first homework assignment in the algebra course (January 2015). Bradshaw analyzed the item responses to estimate DTMR Fractions profiles again for all 22 pre-service teachers, but still reported only the overall distribution of profiles, not those for specific individuals.

Within 1 week after the pre-service teachers completed the DTMR Fractions survey for the second time, Izsák interviewed each of the six focal participants about their item responses. This was the third in the series of six interviews. Because the DTMR Fractions survey takes about 1.5 hours to complete, Izsák and Beckmann selected a subset of the items to address each of the four attributes, referent unit, partitioning and iterating, appropriateness, and reversibility. During the first half of the interview, Izsák went through the selected items and limited follow-up questions to ones like *How did you interpret the problem?*, *Why did you select the choice you did?*, and *For what reasons did you reject alternate choices?* He made clear that the pre-service teachers could change their answers if they wanted. Thus, the interview data were more reliable for inferring the focal participants' thinking at the time of the interview, not when they completed the survey for homework. (At the same time, the pre-service teachers did not experience instruction between completing the survey for homework and the interview that would have provided additional practice with the four attributes.) During the second half of the interview, Izsák took a second pass through the same items and asked more extensive follow-up questions. He also presented additional tasks not part of the DTMR Fractions survey to gather further information about the participants' use of the four attributes. Each interview lasted about 80 minutes.

Izsák and Beckmann then analyzed the interview data to diagnose profiles. For each participant, they analyzed the interview data item-by-item listing evidence for and against mastery of each attribute. (The term *mastery* comes from the psychometric literature, which we take as a synonym for *proficiency*.) They also included a category called "other comments." Izsák and Beckmann included evidence for and against mastery wherever they found it both in the first and the second halves of the interviews. Izsák and Beckmann first watched the interviews separately and then together to compare notes. Discrepant interpretations were discussed until resolved. Once Izsák and Beckmann had discussed the entire video, they looked at all evidence for and against mastery of each attribute. They used a 1 to indicate mastery, a 0 to indicate non-mastery, and X to indicate cases where there was too little evidence or there was a balance of contrary evidence for and against mastery of a particular

attribute. Thus, Izsák and Beckmann allowed themselves more flexibility in diagnosing profiles than was permitted in the DCM analysis. Izsák and Beckmann gave their expert diagnoses for all six focal participants to Bradshaw, who then revealed the profiles as determined by the DCM analysis.

Finally, we underscore that we take the diagnosis of attribute mastery (indicated by a 1) to mean that the person uses that particular attribute fairly consistently across situations. In contrast, we take a diagnosis of non-mastery (indicated by a 0) to simply mean that there was a lack of evidence for a particular attribute. This is not the same as saying a person does not "have" the attribute: A person might have but not demonstrate a particular piece of knowledge, even across multiple tasks, for a variety of reasons.

Results

Our first research question asked to what extent were the DTMR Fractions profiles consistent with pre-service teachers' reasoning across multiple survey items and related tasks? Answering this question extends prior research that focused on content and item-attribute validity of the DTMR Fractions survey. Content validity, that is confirming that the DTMR Fractions items did indeed address important aspects of reasoning about fraction arithmetic in terms of quantities, was established by six external reviewers (four mathematicians and two mathematics education researchers) who examined the items and a document explaining the four attributes. Item-attribute validity, that is confirming that each item addressed the intended attributes, was established through interviews with teachers as part of the item development process and confirmed by DCM analysis of item response data from the large national sample. Because teacher interviews focused only on a subset of items as part of the item development process, and before the national sample was collected, there was no way diagnose their profiles either through expert analysis of interview data or statistical analysis of item response data.

The interview data afforded comparison of profiles as diagnosed by experts (Izsák and Beckmann) reviewing performance across multiple tasks and as diagnosed by analyzing item responses using a DCM. When considering places where expert and DCM diagnoses did and did not align, it is important to remember that each was based on overlapping but not identical information. Expert diagnoses were based on a subset of item responses, item responses revised during the interview that might indicate false positives or negatives, and information from the second half of the interviews that contained additional tasks not on the DTMR Fractions survey. DCM diagnoses were based on responses to the complete set of survey items and statistical information about item characteristics.

Table 1 shows profiles for the six focal participants as diagnosed with the DCM analysis before and after the number and operations content course and as diagnosed by Izsák and Beckmann based on the interview described above. Each string indicates a diagnosis of mastery (1), non-mastery (0), or uncertain mastery status (X) for the four attributes in the following order: referent unit, partitioning and iterating, appropriateness, and reversibility. Given six participants and four attributes, Izsák and Beckmann considered 24 diagnoses. They assigned a 1 in 10 cases, a 0 in 7 cases, and an X in the remaining 7 cases. Of the 17 cases where they assigned a 1 or 0, their diagnoses and the profiles Bradshaw generated with the DCM analysis agreed in 13. The four discrepancies were concentrated in two pre-service teachers, Alex and Diana, and we explain them as follows. Based on explanations Alex provided during her interview, Izsák and Beckmann judged several of her incorrect responses to partitioning and iterating items and reversibility items to be false negatives. Thus, they attributed mastery of these attributes where the DCM analysis did not. For Diana, both the expert and the DCM diagnoses with respect to referent unit were ambivalent. The DCM diagnosis indicated a probability for mastery of about .6 (probabilities greater than .5 are assigned mastery in the DCM model) and tilted one direction; the expert diagnosis tilted the other direction. Finally, Izsák and Beckmann judged several of Diana's incorrect responses to partitioning

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and iterating items to be false negatives, leading to an expert diagnosis (mastery) that disagreed with the DCM diagnosis (non-mastery). Of the 7 cases where Izsák and Beckmann assigned an X, indicating uncertain mastery status, the DCM analysis diagnosed mastery in 5 cases and non-mastery in 2 cases. In part, these differences reflect the fact that Izsák and Beckmann were conservative in diagnosing mastery.

These results indicate significant consistency between the expert and DCM diagnoses, especially when, as stated above, we interpret non-mastery to mean lack of evidence for a particular attribute. The consistency gives us increased confidence that DTMR Fractions profiles do, in fact, reflect relative strengths and weaknesses in reasoning about fraction arithmetic in terms of quantities. Confidence in profiles was critical for our second research question.

Table 1: Focal Profiles Before and After the Number and Operations Course

	DCM Diagnosis Pre: August 2014	DCM Diagnosis Post: January 2015	Expert Diagnosis (Post)
Alice	[0000]	[0000]	[0101]
Jack	[0000]	[0000]	[000X]
Linda	[0111]	[1111]	[11X1]
Claire	[1111]	[1111]	[X111]
Diana	[0001]	[1001]	[01X1]
Kelly	[0000]	[1011]	[X0XX]

Our second research question asked whether the distribution of DTMR Fractions profiles shifted after the 1-semester course on number and operations and, if so, how? This question is important because a primary motivation for developing the DTMR Fractions survey was to create an instrument sensitive to diversity in reasoning and shifts in that reasoning. Figure 3 shows a significant shift in the distribution of profiles before (Fall 2014) and after (Spring 2015) the number and operations course. In Fall 2014, 9 of the 22 pre-service teachers were diagnosed as masters of none of the attributes and only three were diagnosed as masters of referent unit. In Spring 2015, while 6 of the pre-service teachers still did not demonstrate mastery of any one of the four attributes, 7 more demonstrated mastery of all four attributes. Furthermore, comparing profiles before and after for each of 22 pre-service teachers revealed that 13 moved to profiles with additional mastered attributes and that all pre-service teachers who demonstrated mastery of at least two attributes at pretest demonstrated mastery of all four attributes at posttest.

Furthermore, the acquisition of additional attributes reflected what was emphasized to greater and lesser extent in the number and operations content course. In particular, Beckmann placed less emphasis on using whole-number factor-product combinations as a resource for partitioning and iterating. Of the 16 pre-service teachers who had not mastered partitioning and iterating at the inception of the course, 5 (~31%) had mastered and 11 had not mastered this attribute at posttest. Meanwhile, overall gains on the remaining three attributes, which were given more attention in the course, were higher. Of the 19 pre-service teachers who had not mastered referent unit at the inception of the course, 10 (~53%) had mastered and 9 and not mastered this attribute at posttest. Of the 16 pre-service teachers who had not mastered appropriateness at the inception of the course, 7 (~43%) had mastered and 9 had not mastered this attribute at posttest. Of the 9 pre-service teachers who had not mastered reversibility at the inception of the course, 4 (~44%) had mastered and 5 had not mastered this attribute at posttest.

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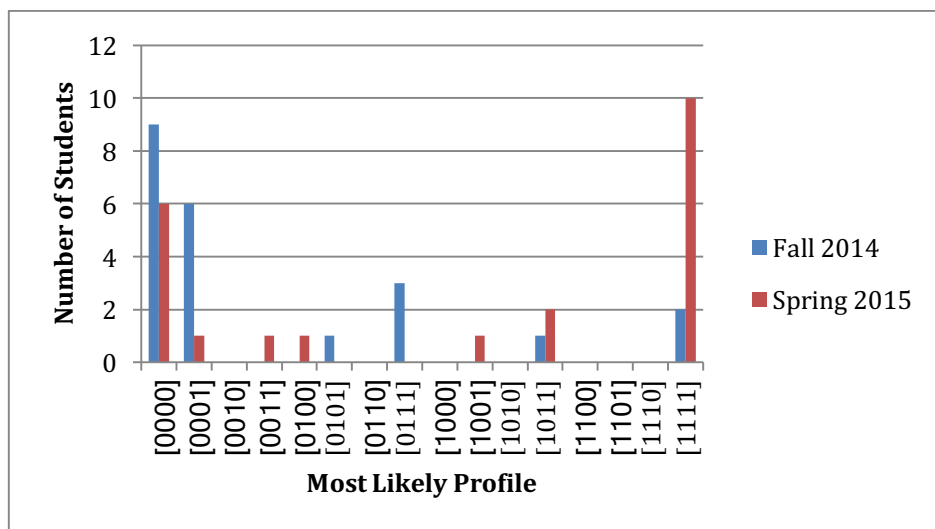
Figure 3. Distribution of DCM profiles before and after the Number and Operations content course.

We draw three main conclusions from these results. First, the shift in profiles from Fall 2014 to Spring 2015 behaved appropriately in the sense that in all but two cases when profiles shifted, pre-service teachers gained attributes. (In the two exceptions, pre-service teachers who demonstrated mastery only of reversibility on the pretest demonstrated mastery of no attributes on posttest). Second, the shift in profiles suggest that many pre-service teachers who entered the numbers and operations course demonstrating mastery of no attributes faced significant challenges developing all four components of reasoning with quantities. Teachers who had facility with at least some of these components tended to master the remaining ones during the course. Third, that shifts in profiles were consistent with what was given more and less emphasis in the course suggests that shifts in profiles reflected the pre-service teachers' opportunities to learn or develop different components of reasoning measured by the DTMR Fractions survey.

Discussion

The present study contributes to the development, validation, and application of measures that capture information about moment-to-moment mathematical reasoning, and does so in the critical mathematical domain of fraction arithmetic. We demonstrated (a) considerable agreement, or convergent evidence, between expert and DCM diagnoses of the four DTMR fractions attributes, and (b) that when pre-service teachers' profiles shifted, they added attributes in ways that reflected the mathematical emphases of their number and operations course. These results demonstrate the possibility of measurement that gets closer to moment-to-moment reasoning than can be captured by a single, summative score. In particular, we have used a survey consisting primarily of multiple-choice items to compare teachers in ways more nuanced than ordering them along a single dimension from less to more able or knowledgeable. Approaches like the one we report have the realistic potential of providing diagnostic information to teacher educators about those components of reasoning with which teachers are more and less facile. This can be useful formative and summative information. Future studies will be needed to see if our results generalize to other cohorts of pre-service teachers and other number and operations courses. Finally, although the present study is situated in teacher education, the potential benefits of multi-dimensional measurement extends to students as well.

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MULTICULTURAL MATHEMATICS DISPOSITIONS: A FRAMEWORK TO UNVEIL DISPOSITIONS TO TEACH CULTURALLY DIVERSE STUDENTS

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We used the Multicultural Mathematics Dispositions (MCMD) framework to analyze preservice teachers' (PSTs) dispositions to teach diverse student populations during a cultural awareness unit in a mathematics methods course. The framework is defined as a function of three constructs: openness, self-awareness/self-reflectiveness, and commitment to culturally relevant mathematics teaching. We selected two assignments from the unit as our data sources. Using the framework, we analyzed the ways in which PSTs expressed any of the MCMD constructs in each of the assignments. Our analysis found that self-awareness/self-reflectiveness was the most common dispositional construct expressed by the PSTs, which led to their openness. However, few PSTs committed to using culturally relevant pedagogy in their future classrooms. Additional coursework may be needed to prepare PSTs to teach culturally diverse student populations.

Keywords: Teacher Education-Preservice, Equity and Diversity

Introduction

As the number of Black and Hispanic students, English Learners (ELs), and students living in poverty enrolled in U.S. public schools continues to increase (Kena et al., 2014), preservice teachers (PSTs) need to be prepared to work effectively with all students. Several mathematics teacher educators (MTEs) agree that it is important for PSTs to learn how to recognize and build on students' cultural and mathematical backgrounds to meet the needs of an increasingly diverse student population (Foote et al., 2012; Gutiérrez, 2009; Kitchen, 2005; Leonard, 2008). PSTs need to be made aware of the relationship between culture and learning and the way mathematics classroom cultures act as a context that supports or constrains different forms of knowledge (Gutiérrez, 2013; Nasir & Cobb, 2007). Being aware of culture and how it interacts with learning is especially important in the mathematics classroom because mathematics is typically thought of as a subject that is culture free (Nasir, Hand, & Taylor, 2008). MTEs can help PSTs examine their beliefs about mathematics and mathematics pedagogy, and learn that mathematics classrooms are places of enculturation where certain social norms and practices are valued, while others are considered inadequate (Diversity in Mathematics Education Center for Learning and Teaching [DiME], 2007). MTEs can also help PSTs "take into account the diverse ways in which students understand and see mathematics rather than automatically discarding them as deficient or inappropriate simply because they are different from their ways of thinking" (White, DuCloux, Carreras-Jusino, Gonzalez, & Keels, 2016, p. 164). This requires PSTs to have a disposition that is conducive to teaching mathematics to diverse populations.

Conceptual Framework

Mathematics teachers' dispositions "are critically important because they underlie distinctions teachers are likely to make in moment-to-moment classroom activity" (Hand, 2012, p. 234). These dispositions play an important role in teaching practices and effectiveness. Thornton (2006) has asserted that it is imperative for teacher educators to find ways to develop PSTs' dispositions toward

multiculturalism. She defined dispositions as "habits of mind including both cognitive and affective attributes that filter one's knowledge, skills, and beliefs and impact the action one takes in classroom or professional setting" (p. 62). White, Murray, and Brunaud-Vega (2012) argued that "teacher disposition can provide a more comprehensive perspective towards the construction of a teacher's identity in the context of a multicultural classroom" (p. 33). Thus, PSTs need to develop culturally sensitive/critical dispositions in mathematics, which White et al. (2012) call *multicultural mathematics dispositions* (MCMD). The MCMD framework is defined as a function of three dispositional factors: openness, self-awareness/self-reflectiveness (SA/SR), and commitment to culturally relevant mathematics teaching. *Openness* is receptiveness to the role of culture in teaching and learning mathematics. It includes being open to (a) others' cultures in perceptions about teaching, learning, or doing mathematics; (b) the inclusion of culture in mathematics classrooms; and (c) the use of culturally relevant strategies to teach mathematics (Ladson-Billings, 1995). *Self-awareness/self-reflectiveness* is understood as (a) identification of one's own culture and perception of the differences between it and another culture, (b) awareness of one's own beliefs about the influence of culture on teaching and learning mathematics or mathematics classroom culture, and (c) the ability to think critically about those issues. *Commitment to culturally relevant mathematics teaching* is the explicit intention of teachers to use culturally relevant strategies in the classroom.

The constructs of MCMD enable researchers to characterize and analyze PSTs' dispositions to work with diverse student populations. Thus, we examined PSTs' MCMD during a cultural awareness mathematics unit we designed for mathematics methods courses (White et al., 2016). The research question guiding the study presented here is: How does the MCMD framework unveil PSTs' dispositions to teach culturally diverse students?

Methods

Participants

Three cohorts of PSTs who participated in the unit were enrolled in three different mathematics methods courses (*Elementary Mathematics Methods II*, *Middle Grades Mathematics Methods I*, and *Secondary Mathematics Pedagogy II*). Of the 60 PSTs, 48 were White females, 2 were Black females, 1 was a Black male, and 9 were White males. The distribution of the 60 PSTs enrolled in the three methods courses is shown in Table 1.

Table 1: Distribution of Preservice Teachers by Cohort, Race, and Gender

Cohort	White Males	White Females	African American Males	African American Females	Total
Elementary	1	26	0	1	28
Middle Grades	3	12	1	0	16
Secondary	5	10	0	1	16
Total		57		3	60

Cultural Awareness Unit

The goals of the cultural awareness unit were to help PSTs become aware of (a) the roles of teachers' cultures, students' cultures, and mathematics classroom cultures in students' mathematical learning; (b) stereotypes about who can do mathematics; and (c) strategies for infusing culture in mathematics classrooms. In particular, the cultural awareness unit encouraged PSTs to reflect on their own views about culture and to challenge borders that limit diverse students' access to mathematics in classrooms. We used Dumitrescu & Iacob's (2012) definition of *cultural awareness* as "recognizing that all people do not have the same cultural background. It signifies one's ability to stand back from oneself and become aware of both one's own culture and another's culture, i.e.

cultural values, beliefs, perceptions" (p. 122). The unit included an *Article Critique* (article search and reflection), *Class Discussions*, and a *Post Reflection* (post-discussion reflection). Table 2 details how each component is related to the MCMD constructs.

Table 2: Components of the unit and corresponding tasks

Component	Tasks	Development of MCMD
Article critique	▪ Search for an article and write a reflection about teaching or learning mathematics to students who are culturally different than themselves	▪ Self-awareness by identifying their own culture and comparing to others. ▪ Openness by learning culturally relevant strategies. ▪ Openness by learning how others do mathematics.
Class discussion	▪ Share cultures and strategies discussed in article. ▪ Define culture and create cultural tool list. ▪ Discuss how culture relates to mathematics classrooms norms. ▪ Discuss differences between stereotypes and generalization.	▪ Openness as defined above. ▪ Self-awareness/self-reflectiveness by reflecting on personal culture and experiences in the mathematics classrooms.
Post Reflection	▪ Write reflection on unit	▪ Openness and self-awareness as defined above. ▪ Commitment by encouraging them to adopt culturally relevant strategies.

Data Sources and Analysis

Data were collected from the PSTs' *Article Critique* and *Post Reflection* assignments. It should be noted that all the participants submitted the Article Critique, but only 56 of the 60 PSTs submitted the final assignment. A framework analysis (Ward, Furber, Tierney, & Swallow, 2013) method was used to analyze the data. Our analysis looked for patterns across the data sources to identify qualitative instances of PSTs' MCMD. As a group, we read through the papers several times and coded sections of text according to whether they demonstrated openness, SA/SR, and/or commitment. Next, we reread the coded passages to further characterize the PSTs' MCMD. More specifically, we wanted to know in what ways PSTs were open and SA/SR and what they were committed to doing in their future classrooms. Throughout the coding process, we discussed any discrepancies or questions until we reached consensus.

Results

We were able to identify the presence of MCMD constructs in contributions from most of the PSTs. As shown in Figure 1, our analysis of the *Article Critique* unveiled openness in 26 PSTs, SA/SR in 46 PSTs, and commitment in 6 PSTs. By the end of the unit, analysis of the *Post Reflection* showed evidence of openness in 42 PSTs, SA/SR in 16 PSTs, and commitment in 8 PSTs. We were unable to characterize any of the MCMD constructs in 12 PSTs and 7 PSTs for the *Article Critique* and *Post Reflection*, respectively. What follows are examples of the PSTs' MCMD for the two assignments.

| Distribution of PSTs by MCMD constructs in *Article Critique* assignment. | Distribution of PSTs by MCMD constructs in *Post-discussion Reflection* assignment. |
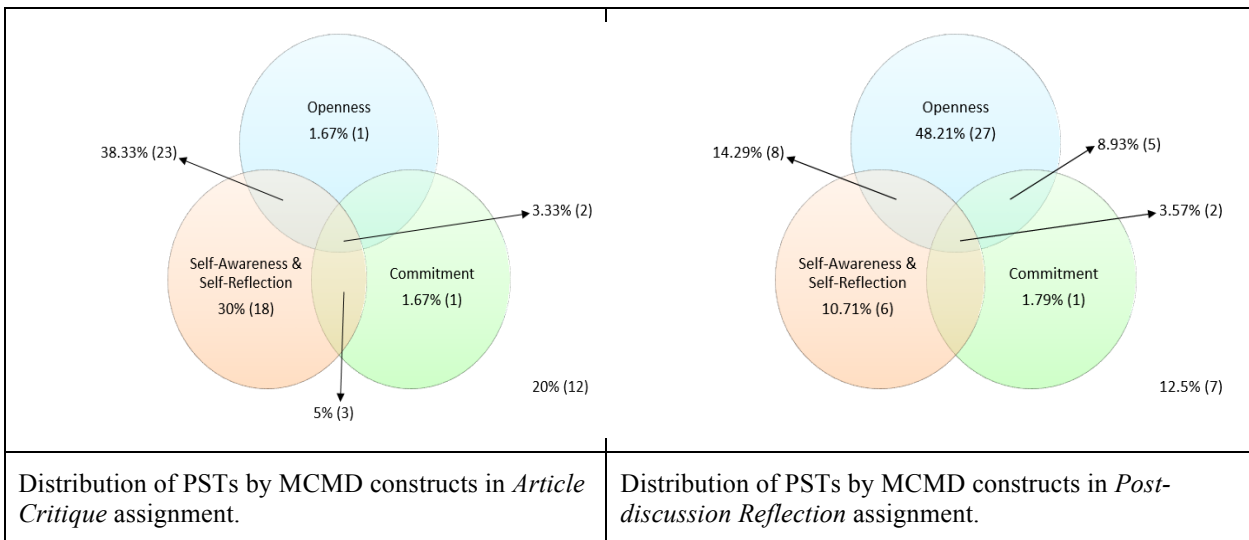
Figure 1. Distribution of PSTs by MCMD constructs.

Article Critique

The article critique was completed before the class discussion in order to limit our influence on PSTs' views on culture and mathematics. By asking PSTs to read about teaching and learning mathematics to culturally diverse students and to reflect on their own cultures, we were able to unveil aspects of PSTs' MCMD.

Openness. The majority of the PSTs were open to learning and including culturally relevant strategies to teach diverse students. A comment from Eva, an elementary PST, illustrates this form of openness:

> I need to learn ways in which I can reach students who are culturally different from me. In fact, I think this strategy could be used for children of all cultural types. Having students create their own math problems that relate to their experiences not only helps them remember how to solve the problems, but lets the teacher get to know the students and what they are interested in.

Several PSTs were open to the idea that culture has a role in the way people do and learn mathematics. The article critique assignment helped them realize that there is no one way to do mathematics and that different students may think about mathematics differently. For instance, Joyce, an elementary PST wrote,

> I was under the mistaken impression that numbers are just numbers, the same everywhere. The article pointed out, however, that symbols are often different in other languages, and words that we use to describe actions in math may be confusing, unfamiliar, or misleading ... Now that this has been brought to my attention, I will give it a lot more consideration as I am learning to be a teacher.

Self-awareness/self-reflectiveness. Most PSTs expressed awareness of aspects of their own culture and how these aspects differed from the cultures described in the article. Their comparisons focused on differences across race (White vs. Black and Hispanic), class (Upper middle class vs. Poor), and geography (Suburban vs. Urban or Rural). As Bella, a secondary PST wrote, "I attended a small private school with very little cultural diversity. As a result, I was not exposed to much diversity, especially linguistic diversity."

A few PSTs were self-aware and self-reflective of their own culture and their mathematical experiences. They reflected on their experiences in the mathematics classroom and compared them to those of students with different gender, language, and race. As Sophia, an elementary PST wrote,

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

I am a White female that was brought up in an upper middle class family. I always thought my mathematical learning experience was equal to my classmates until I read this article. The article mentioned that females do not receive the same equity as males. I had never thought about this before but it is true that in my high school mathematics classes the smartest people were always boys. It seemed like they always knew so much more than me which lowered my confidence in doing, mathematical problems. … Realizing how this made me feel, I cannot imagine what this does to people that are even more different than White, upper class males. For example, someone who is an African American, lower class female must be very intimidated during math classes.

Sophia's quote shows how she positioned her mathematics experience in the classroom at the intersection of her gender and race. Sophia came to realize how our experiences in the mathematics classroom are shaped by cultural factors and how different we are from the "White, upper class males" standard.

Commitment to culturally relevant mathematics teaching. Although most PSTs saw the value of using culture in the classroom, only a few expressed their commitment to use culturally relevant strategies to teach mathematics. PSTs who expressed a commitment to using culture in their future classrooms made comments similar to the following quote by Wade, a secondary PST:

One of the strategies from this article I would implement in my future math classroom is opening clear pathways for communication between students, parents and myself. Another strategy I would use is having material that applies to their culture and that students can relate to culturally. This strategy can be achieved by learning about your students and determining what the students are interested in and what they may deem culturally relevant.

Wade's quote suggests that the article critique assignment offered concrete examples that PSTs could access if they were considering using culturally relevant strategies in their own classrooms.

Post Reflection

The post reflection was the last component of the unit. PSTs reflected on their experiences writing the article critique and participating in the class discussions where they shared their ideas and listened to the ideas of their peers and us as instructors.

Openness. More than half of the PSTs showed openness to the idea that different cultures may think about and do mathematics differently and they have a role in supporting students' mathematical learning. A quote from Hannah, an elementary PST, is an example:

Looking at cultural diversity, especially through a mathematical lens, has changed the way I look at teaching. I thought that being culturally different really only affected a person in reading and maybe social studies, but not math. However, now I know that's not true. Different cultures view and do math in different ways. Some algorithms are completely different across the world and teachers need to be prepared to teach students with all these varieties of methods. Students who speak different languages cannot grasp a strong or a conceptual understanding of math because they cannot understand what is being taught. Overall, this focus has really opened my eyes and made me more aware of cultural diversity in the classroom. I know this will impact my methods and actions as a teacher.

As Hannah's quote suggests, the post reflection challenged her assumption that mathematics is a universal and culture-free subject. Hannah realized that culture has a prominent role in shaping the way people do and learn mathematics, and hence it is important to take cultural diversity into account when teaching mathematics.

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

Several PSTs also mentioned how listening to their peers helped them realize the role of culture in learning and teaching mathematics. These PSTs expressed that the unit opened their minds to the importance of cultural awareness for teachers. Hope, a secondary PST, described this effect:

> I think the biggest help for me was just listening to what other students in our class thought. … It is so great to be able to see so many different views. After this unit, I feel like I see a clearer need for cultural awareness in our classroom environments. Until it is pointed out, I think people have a hard time understanding how their culture may be so much different from someone else's. It is so important that we are aware of this as teachers. Learning about and being aware of our students' cultures will have us connect with them as well as help us to [not] unintentionally offend them.

Hope explained how listening to her peers helped her pay attention to different "key points" from the article critiques and others' points of view.

Self-awareness/self-reflectiveness. We found the most common characterization of PSTs' SA/SR included awareness of their role in creating an inclusive mathematics classroom culture, the impact of stereotypes on student learning, and the importance of teachers critically thinking about the intersection of various cultures -- teachers, students, classroom, and school -- in the classroom:

> This unit has helped me see that students and teachers bring their cultures into the classroom, and that these cultures affect how mathematics is learned. Teaching students with diverse backgrounds does not involve ignoring differences, but rather involves drawing on these differences. Students should not feel as though they must set aside their cultural identity when learning mathematics. [Justin, secondary PST]

> I feel that this exercise was one that every teacher should be exposed to because we see in our classrooms how teachers that have been around lack the ability to see outside their own experiences. Stereotypes are running rampant and are reinforced by accident everyday. We as future educators need to realize that impartiality is crucial to having a fair classroom and setting up a classroom culture that mirrors reality. [Derrick, secondary PST]

As Justin suggested, the unit helped PSTs become aware of the importance of incorporating cultural differences in the classroom rather than ignoring them so that all students feel included. Some PSTs were more reflective, as illustrated by Derrick's quote. These PSTs reflected on the negative impact of stereotypes on student learning and the importance of being able to "see outside of their own experiences" to avoid stereotyping students.

Commitment to culturally relevant mathematics teaching. Similar to what we found in the article critique, few PSTs showed some form of commitment. The PSTs who showed commitment in the post reflection were motivated to include diverse methods of doing mathematics, methods that come from different cultures. Paula, a middle grades PST, wrote

> I also learned how important mathematics is in different cultures. I will take this new knowledge and apply it to my classroom. I will be sure to research again the mathematics principles of cultures other than the American culture so that I can accommodate all of the students in my classroom. Even if the students in my class are from one culture, I will still include methods of instruction that other cultures are accustomed to. I believe this will make math more interesting for both me and my students. They will view math as a universal commodity that is helpful for real life applications.

Discussion

Establishing PSTs' dispositions toward culture in mathematics helps future teachers to understand "no culture is monolithic; every culture consists of multiple subcultures" (Leonard,

Brooks, Barnes-Johnson, & Berry, 2010, p. 267) and that culturally relevant teaching is useful. This study contributes to the growing literature on ways to help PSTs reflect on issues of equity in mathematics and to prepare them to effectively teach all students. We used the MCMD's three constructs – openness, SA/SR, and commitment— as a lens to unveil PSTs' dispositions to teach diverse students. Our analysis suggests that PSTs want to learn and use strategies for teaching mathematics and they benefit from discussions with peers to reflect on their ideas about culture. An interesting finding from this study was the inverse relationship between the openness and SA/SR constructs. Initially the percentage of PSTs expressing self-awareness was greater than the percentage with openness; however, this relationship had shifted by the post reflection at the end of the unit. This finding supports Mills and Ballantyne's (2010) claim that SA/SR evolves towards openness.

We believe the multiple opportunities for PSTs to express their MCMD during the unit supported PSTs' progression towards openness. In this study, PSTs' self-awareness of their own mathematics learning experiences, especially their reflections on traditional approaches they had experienced, seemed to support them to want to provide their future students with more opportunities to learn mathematics. Learning about their students' cultures and using them in the classrooms is a strategy PSTs became more aware of and open to learn. The class discussion was an opportunity for PSTs to learn about their peers' views about culture and their experiences learning mathematics, mathematics stereotypes, and the various cultural norms that exist in mathematics classrooms. The PSTs' post reflections unveiled that PSTs broadened their views on openness as they realized that they were different from their seemingly same peers. Their post reflections showed more openness, and their SA/SR involved a more critical analysis of their own experiences. We were not surprised that only a few PSTs were willing to commit to using culturally relevant strategies. PSTs do not have their own classrooms and have limited opportunities to try strategies in the practicum classrooms. Our analysis suggests that additional experiences, reinforcement and practice are needed to adequately prepare PSTs to teach for diversity, especially PSTs' commitment to using culturally relevant strategies.

We have used the MCMD framework to unveil PSTs' dispositions toward diversity and culture during a cultural awareness unit. This framework has also been used to examine the development of equity and social justice dispositions among PSTs (Chao & Murray, 2015). Further research is needed to examine MCMD in various contexts, how to support the development of MCMD, and the ways MCMD are enacted in classrooms with diverse student populations.

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TEACHLIVE™ REHEARSALS: ONE HBCU'S STUDY ON PROSPECTIVE TEACHERS' REFORMED INSTRUCTIONAL PRACTICES AND THEIR MATHEMATICAL AFFECT

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Scholars posit that descriptive education research that focuses on the instructional dynamic between teachers and students is perhaps one the most salient research topics that can improve learning and teaching. This case study seeks to describe prospective teachers' mathematical affect as they engage in "rehearse teaching" in TeachLivE™, a mixed-reality simulated classroom. Utilizing Goldin et al.'s (2011) engagement structures as evidence of mathematical affect, findings reveal that simulated rehearsals improve prospective teachers' reformed-based teaching and that this improvement may be related to their improved 'in-the-moment' affective states. This study potentially connects prospective teachers' beliefs and emotions as math learners with their behaviors and instructional praxes as novice math teachers.

Keywords: Affect, Emotion, Beliefs, and Attitudes, Teacher Education-Preservice, Equity and Diversity

Introduction

In the United States, policymakers, educational researchers and practitioners agree that one of the most important in-school predictors of mathematics student achievement is access to quality teachers (Ingersoll & Perda, 2010). It follows then that the precursor to teacher quality is quality pre-service teacher preparation (Ronfeldt, 2012). Quality pre-service teacher preparation includes rigorous foundational and methods of instruction coursework, opportunities to practice teaching, evaluation and feedback from expert mentor teachers and clinical faculty, revision of instruction and then more clinical practice to engender reflection, informed decision-making, and confidence in teaching performance (Chassels & Melville, 2009; Wilkins, 2002).

However, obtaining optimum opportunities to practice mathematics teaching in public schools remains challenging for teacher preparation programs, particularly in urban districts that serve underprivileged students (Ronfeldt, 2012). First, effective mathematics teachers who may serve as mentors must be recruited and developed; this can be difficult in urban districts where recruiting and retaining quality mathematics teachers is a challenge (Liu, Rosenstein, Swan, & Khalil, 2008; Khalil & Griffen, 2012). Second, the tenuous climate generated by the pressure of student performance on standardized testing narrows the window for innovative practice teaching (Beswick, 2006), particularly in urban districts that often face higher stakes with regards to student achievement on standardized tests (e.g. school closures and turn-around schools) (Sadovnik, O'Day, Bohrnstedt, & Borman, 2013). These conditions may then lead to a school climate that encourages more traditional ways of teaching and test preparation via scripted curriculum and less on teaching utilizing evidenced-based best practices often touted in teacher preparation programs (e.g. engaging students in reform-based teaching that is based on inquiry and conceptually challenging high cognitive demand tasks) (Ottmar, Rimm-Kaufman, Berry, & Larsen, 2013; Wilkins, 2002). Third, the advent of teacher evaluation systems has fostered concerns in potential mentor teachers who worry about the negative impact practice teaching may have on student performance on standardized tests, which in

turn may reflect poorly on their own teacher performance evaluations. Such poor evaluations may be more of a concern for elementary teachers who already struggle to teach mathematics (Ma, 1999) and whose negative affect has already been connected to their traditional instructional praxes (Wilkins, 2002).

Thus, for teacher preparation programs that seek to offer clinical experiences to prospective teachers, quality placements that provide the variety of resources and supports essential to pre-service teacher (PST) development of math teaching performance may be at a premium. Since research posits that clinical experiences are essential to teacher development, one solution to the shortage of quality placements may be simulated virtual classrooms. One key benefit of simulated clinical experiences is the feedback PSTs receive that encourages reflection and critical analysis of their teaching performance. The importance of this facet of practice teaching has been repeatedly emphasized in the literature on feedback and improved teaching performance among student teachers (e.g. see Voerman, Mejier, Korthagen, & Simons, 2012). Furthermore, Akkuzu's (2014) study illustrates that quality feedback not only improves teaching performance but that it also enhances preservice teachers' affect, specifically their self-efficacy beliefs about teaching. This research is noteworthy as other studies highlight the influence negative affect can have (e.g. low self-efficacy and anxiety) on novice elementary teachers' experience while teaching mathematics.

This paper seeks to suggest that simulated teaching experiences such *TeachLivE™* may serve as a worthy substitute for live classroom practicum teaching as a means of developing PST's instructional practices. Furthermore, simulated classrooms may ultimately be more conducive to improving overall performance of pre-service teachers prior to the student teaching experience (Dieker, Rodriguez, Lignugaris, Hynes, & Hughes, 2014). To examine the potential impact of simulated field experiences on teacher practice, the authors explored how *TeachLivE™* offers an opportunity to observe and respond to PSTs' "in-the-moment" affective behaviors patterns while the PST engaged in rehearse teaching of a math lesson. Additionally, the authors seek to further understand *how* improved teaching knowledge and practice may be related to affect among PSTs (Wilkins, 2002).

Perspectives

This paper is grounded in two bodies of theory and research: the work on prospective teachers' rehearsals as clinical practice (Lampert, Franke, Kazemi, & Crowe, 2013) and on powerful mathematical affect (Goldin, 2014; McLeod, 1992). First, this study analyzes PST reformed-based teaching while they "rehearse" teaching a math lesson in a simulated clinical environment (Dieker et. al, 2014; Lampert et al., 2013; Ronfeldt, 2012). Lampert and colleagues describe rehearsals as "a social setting for building novice' commitment to teach ambitiously" where the motivation to do so "depends on the social circumstances in which one learns and develops an identity" (p.227). This is consistent with Ronfeldt's (2012) assertion that optimal conditions for clinical work are essential as they are positively linked to improved teacher retention and student achievement. Furthermore, opportunities that permit PSTs to re-teach a lesson after receiving feedback and making revisions was found to benefit PSTs praxes in becoming more student-centered and reform-based (Chassels & Melville, 2009; Ganesh & Matteson, 2010). While, rehearsals of teaching is optimally conducted in an actual classroom where PSTs' can shape teaching identities within conditions that provide resources and support, such conditions are challenging to identify within inner cities (Ronfeldt, 2012). Therefore, this research explores the viability of a simulated classroom for rehearse teaching in an 'optimal' setting as a mean of examining simulated rehearsals' potential to improve PSTs' teaching performance.

Second, this paper utilizes Goldin, Epstein, Schorr, and Warner's (2011) "engagement structures" as evidence of PSTs' "in-the-moment" affective states experienced when interacting with mathematics. This paper posits that while "engagement structures" (ES) were first theorized to

characterize inner-city's "behavioral/affective/social constellation," this theory can *also* describe pre- and in-service "in-the-moment" teacher affect (Khalil & Johnson, 2016). Evidence of nine of Goldin and colleague's engagement structures were noted (e.g. *get the job done, look how smart I am, check this out, I'm really into this, don't disrespect me, stay out of trouble, it's not fair,* and *let me teach you*), where evidence of each structure's "in-the-moment" transaction was unpacked via seven strands (e.g. *goal or motivating desire, patterns of behavior, affective pathways, expression of affect, meanings encoded by emotions, meta-affect, self-talk or inner-speech*) (Goldin et al., 2011, p. 549). Exploring more about PSTs' "in-the-moment" affective states while teaching may lead to: a) further understanding of PSTs' affective traits, which are often negative (e.g. math anxiety and low self-efficacy) among elementary teachers teaching math, and b) improving PSTs' affective traits in an effort to encourage further reformed-based innovative teaching practices linked with positive affective traits (Wilkins, 2002).

Data and Methods

Study Design

This study draws on a larger mixed-method study which sought to explore the effect of pre-service teachers engaging in a lesson study project in which they planned and rehearsed a math lesson, received feedback that informed revisions of their lesson plan, and then taught the revised lesson (Hiebert, Morris, Berk, & Jansen, 2007). The PSTs worked through multiple cycles of lesson planning prior to their rehearsals. Data collected in the larger study included pre and post measures of PSTs' *Mathematics Teaching Self-Efficacy Survey* (MTEBI scores; Enochs, Smith, & Huinker, 2000), qualitative and quantitative video analysis, PSTs' reflections, as well as other course artifacts.

The lesson study took place during the fall of 2014 in an elementary mathematics methods and practicum course at Howard University, a Historically Black University with a mission to educate underserved populations of color. The first author served as the mathematics teacher educator for the four-credit mathematics methods class. Prospective teachers were required to spend 32 hours in the classroom and 32 hours in clinical settings that offered extensive opportunity for feedback (e.g. university supervisor, cooperating teacher, teacher-educators). The teacher educator asked PSTs to design a 90-minute lesson based on one CCSSM standard, as research shows that PSTs' focus on how to teach as opposed to what to teach when a topic is chosen for them (Deiker et. al, 2014). Fractions (Lee & Boyadzhiev, 2013) were chosen as a conceptually challenging topic and PSTs were instructed to design their lesson plan using the 5-E Learning Cycle (Bybee et al., 2006). PSTs received feedback on lesson plans multiple times, as teaching fractions in a conceptually engaging way is challenging.

For this paper's focus on *TeachLivE™* rehearsals, 11 PSTs (10 females; 1 male; all African-American) participated in the portion of the study reported here. PSTs took the course twice a week for two hours. All participants were 3rd year (Junior Level) students who were completing the requirements of a Bachelor's Degree for an Elementary Education Major. PSTs were asked to choose 15 min from their lesson plan to rehearse in *TeachLivE™*, as it has been shown that 7 minutes practicing specific teaching strategies with feedback can improve teaching performance. PSTs videotaped rehearsing the 15-minute segment of instruction in TeachLivE™ then received feedback from two teacher educators. PSTs reflected upon their experience of lesson planning and teaching, revised a portion of their lesson plan to reteach in TeachLivE™, and then rehearsed the same 15-minute segment. This amounted to 30 minutes of rehearsal teaching and two rounds of feedback from two instructors).

Research Question and Data

The primary question driving this study is, "in what ways can rehearsals in a simulated clinical

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environment like *TeachLivE™* be used as a tool to develop prospective teacher practice and powerful mathematical affect?" Qualitative data was collected and analyzed to explore the patterns and themes of PSTs' instructional practices and affective states while teaching in TeachLivE ™. Data analysis involved both a) contextual analysis of each PST's reflections, videos, and video transcripts, and b) cross-teacher analysis to compare evidence of each data point among PSTs. To establish inter-rater reliability, all coding was completed by two research assistants (Creswell, 2009). The researchers first coded videos, video transcripts, and reflections deductively by searching for instances of the 9 engagement structures, then re-coded to find evidence of each structure's "in-the-moment" transaction by searching for the aforementioned 7 strands that comprise an engagement structure (Goldin et al., 2011).

Additionally, quantitative data analysis was used to measure PSTs' reformed instructional practices. To establish inter-rater reliability, two research assistants used the Reformed Teacher Observation Protocol (RTOP; Piburn et al., 2000), observed videos of the rehearsals twice to assign scores that represent PSTs' instructional practices. RTOP provides a standardized means for detecting the degree to which classroom instruction/teaching praxes is learner-centered or engaged versus teacher-centered. RTOP includes five subscales. First, lesson design and implementation (LDI) seeks to measure what the PST intends to do. Items examine how the PST organizes the lesson to honor students' preconceptions constructed from every day experiences or previous instruction and examines how the PST creates opportunities to explore aspects of the topic prior to formal instruction. Second, the propositional pedagogic knowledge (PPK) seeks to measure what the PST knows, and how well s/he is able to organize and present material in a learner-oriented setting. Third, the procedural pedagogic knowledge (PK) seeks to measure what the student avatars did and how engaged they were in critical thinking skills advocated in the CCSSM standards of practice. Fourth, the student-student interaction (SSI) subscale measures the type of interactions among students and how the PST facilitates such interactions. Finally, the fifth subscale measures student-instructor interaction (SII), and how a PST creates learning environments where students are able to take risks asking questions. Questioning provides students the opportunity to exercise executive control over their learning process, empowers their learning, and increases their overall learning gains (Boykin & Noguera, 2011).

RTOP is one of the few validated observation tools that measure reformed-based teaching (also referred to as standards-based teaching; see Ottmar, Rimm-Kaufman, Berry, & Larsen, 2013 for further details). Reform-based teaching advocates that classes be "*taught via the kinds of constructivist, inquiry-based methods advocated by professional organizations and researchers*" (Piburn et al., 2000). Reform-based teaching is a paradigm adopted by the standards movement due to its goals of shifting traditional teacher-centered lecture-driven instruction to student-centered, activity-based learning which encourages collaboration among students (Ottmar, Rimm-Kaufman, Berry, & Larsen, 2013). The original RTOP protocol was modified to adjust for the limitations of the simulated rehearsals. Figure 1 below explains the modified interpretation of RTOP Scores (adapted for the simulated rehearsal experience).

Understanding TeachLivE RTOP Scores (Adapted for the Simulated Teaching Experience)		
Traditional Lecture	Active lecture	Active Learning
RTOP Scores 0-22	RTOP Score: 23—38	RTOP Score: 38+ (out of 77)

Figure 1. RTOP Score Meanings.

Results and Discussion

A descriptive analysis revealed that preservice teachers had an average RTOP score of 41.5 in their first teaching rehearsal in TeachLivE. This improved to an average score of 47.56 during the

second rehearsal in TeachLivE. According to figure 1, PSTs are implementing inquiry-based methods that promote active student-centered learning on average. Table 1below provides the descriptive statistics for RTOP and the respective RTOP subscales at Time 1 and Time 2.

Table 1: Descriptive Statistics

Subscale	N	Minimum	Maximum	Mean	Std. Deviation
LDI1	10	5	14	7.50	2.64
LDI2	9	4	14	8.22	3.19
PPK1	10	6	13	9.60	2.32
PPK2	9	5	14	11.11	3.14
PCK1	10	2	la11	7.90	2.81
PCK2	9	3	11	9.00	2.50
SSI1	10	5	14	7.50	2.72
SSI2	9	5	16	8.67	3.54
SII1	10	5	15	9.00	3.16
SII2	9	5	15	10.56	3.17
RTOP Time 1	10	23	67	41.50	12.47
RTOP Time 2	9	22	68	47.56	13.95

Improved Reformed Teaching: Evidence from Rehearsals

A paired-samples t-test was conducted to compare each subscale score of the RTOP at Time 1 and Time 2. There was a significant difference in Propositional Knowledge between Time 1 (M=9.60, SD=2.32) and Time 2 (M=11.11, SD=3.14); $t(8)$ = -2.600, p = .032. While scores did significantly improve, PSTs' propositional knowledge is one of the greatest challenges observed during the lesson study and required a large part of the class time. This is not unusual, as elementary PSTs need support for constructing their understanding of teaching mathematics from a procedural approach to a conceptual one. There was a significant difference in Student-Student Interaction between Time 1 (M=7.50, SD=2.72) and Time 2 (M=8.67, SD=3.54); $t(8)$ = -2.443, p = .040. PSTs were challenged during Student-Student Interaction due to the "behavioral" level of discipline we requested for the PSTs in the simulation (a level 3 typical of "urban" classrooms). Most of the PSTs did well despite this challenge. There was a significant difference in Student to Instructor Interaction between Time 1 (M=9.00, SD=3.16) and Time 2 (M=10.56, SD=3.17); $t(8)$ = -2.490, p = .038. Specifically, the results suggest that PSTs' Student-to-Instructor Interaction was a key strength of candidates. This was largely due to PSTs' affect. The affective domain includes a host of constructs, such as attitudes, values, beliefs, opinions, interests, and motivation. There was not a significant difference in LDI and PCK between Time 1 and Time 2. Table 2 presents the significant findings from the paired-samples t-test.

A paired-samples t-test was conducted to compare overall RTOP scores at Time 1 and Time 2. There was a significant difference in the RTOP scores between Time 1 (M=41.78, SD=13.19) and Time 2 (M=47.56, SD=13.95); $t(5)$=-2.550, p = .034. These results suggest that TeachLivE rehearsals, as part of an overall lesson study, may offer a venue for improving PSTs' teaching performance. Table 2 presents the results from the paired-samples t-test.

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Table 2: Paired-Samples *t*-test Results

		Mean Difference	T	df	Siq. (2-tailed)
Pair2	PPK1 - PPK2	-1.44	-2.600	8	.032
Pair4	SSI1 - SSI2	-1.11	-2.443	8	.040
Pair5	SII1 - SII2	-1.44	-2.490	8	.038
Pair6	RTOP1- RTOP2	-5.76	-2.550	8	.034

Powerful Mathematical Affect: Evidence from Rehearsals

Rehearse teaching proved to be a turning point for many prospective teachers. Data collected from PSTs' journal reflections indicate that the positive affective experience of their first *TeachLivE™* rehearsal boosted their belief with regards to their ability to learn lesson planning and teaching mathematics. Some of the ES were characterized by belief systems, such as self-efficacy and self-identity while others were characterized by behaviors oriented toward fulfilling emotions. PSTs' meta-affect and affective pathways were also "strands" that helped unpack the emotions and behaviors in each ES, particularly with regards to understanding "the sequence of emotional states interact[ing] with heuristics during [lesson planning and rehearsing]" (Goldin, 2000). These emotions served as "AHA" moments signaling to the PSTs' belief that they can in fact teach a lesson on fractions (Liljedahl, 2005).

One of the most prominent engagement structures observed in PSTs' *TeachLivE™* rehearsal was *Get the Job Done*. This structure was exhibited in their focus to persevere through an explanation and in their efforts to be comprehensive in their planning. This persistence may be in deference (a meta-affect) to instructions that required a 15-minute rehearsal; instructions they felt obligated to comply. As one PST reflects, "*The only thing I got out of teach live was the fact that you can never have too much. You should plan for a lot and try your best to get through it all. Have a multitude of questions prepared for all kids who may or may not be engaged.*" Similarly, another PST reflects, "*If I over complicated them (fractions) I might not be able to accomplish them,*" while another contends she needs to "*write out some of [the] key questions to make sure that [she] asks students everything [she] want them to think about.*" Indeed, the "motivating desire for task completion [was to] evokes more procedural, time-efficient strategies" (Goldin et al, 2011; p. 552).

Unsurprising, PSTs also exhibited the ES *Let me teach you*, where they verbalized the lesson objectives by using "I" statements to demonstrate to the avatar students and their instructors their ability to teach math. For example, through a series of scaffolded questions, several PSTs exhibited their *motivating desire* (mastery-approach goal) was to help a confused student understand how to transform an improper fraction into a mixed fraction. In a similar way, PSTs exhibited the ES *Check this out,* which may be linked to PSTs' intrinsic motivation of meeting the challenge to provide clarity to difficult math problems. Thus, PSTs' *motivating desire* to obtain an *intrinsic reward* was observed when PSTs assessed CJ, an avatar who struggled the most during the lesson, or Maria, another avatar who appeared disengaged most of the time. Several PSTs demonstrated *external emotions* such as excitement and enthusiasm, which was evident in their tone of voice.

One of the most frequently observed ES in the data was *Stay out of Trouble*, where PSTs avoided interactions that may lead to distress. For example, one PST avoided feeling vulnerable in her inability to answer question (performance-avoidance goal); perhaps this was due to her initial low self-concept. A second frequently observed ES was *Don't disrespect me*, which was illustrated in the way PSTs commanded respect in a no-nonsense tone and nonverbal cues that indicated their intolerance towards distractions that detracted from meeting their lesson objectives. For example, several PSTs began calling on student avatars randomly, thereby requiring the avatars to remain focused throughout the lesson. This ES was markedly evident even within rehearsals of short

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duration and was characterized by PSTs displaying respect for everyone's dignity and by providing a "safe" space for students to ask mathematics questions without fear about belittlement. One PST reflected in a "self-talk" tone, "*Although I was able to shut all these [student distractions] down within the first few minutes, I heard that they were worse with some of my classmates. I see how important it is to incorporate tasks that really engage the students so you can have their undivided attention for the time that you have.*" It is noteworthy that PSTs with lower self-efficacy (as it pertains to the context of modeling operations with fractions) may be linked with two additional engagement structures including *Its Not Fair* and *Pseudo Engagement.* Regarding *Its Not Fair*, PSTs expressed disappointment in their journals over rehearsing with avatars as opposed to K-12 students in the field and described their frustration with *TeachLivE's™*, which limited their proximity to students. *Pseudo Engagement* was observed among PSTs who exhibited apathetic behavior during teaching and relief when teaching sequences ended. Unsurprising, the findings revealed that PSTs often exhibited the *Let me teach you* engagement structure along with the *Look how smart I am* and *Check this out* engagement structures. These ES tended to link to PSTs' higher self-efficacy, as evidenced by language in PSTs' journal reflections and higher MTEBI scores.

Implication

Rehearse teaching in *TeachLivE™* proved to be a turning point for many prospective teachers as it allowed them an opportunity to practice teaching, which in turn improved their affect and confidence in teaching, in general, and in teaching mathematics, in particular. Further, their overall improved reformed-teaching mirrored the improved affective states as evidenced by the "in-the-moment" affective constellation in the engagement structures. Additionally, despite instances of less positive affective states, PSTs' RTOP scores did demonstrate a statistically significant increase in scores that nudged PSTs from active teaching to active learning between their two rehearsals. This paradox of instances of low affect but high teaching performance is intriguing and suggests that low affect does not always translate into poor teaching ability, just as less positive engagement structures for students does not necessarily translate into poor performance in mathematics (Goldin et al., 2011). This outcome is worthy of further research.

Rehearsals with feedback in a simulated learning environment also enabled prospective teachers to enact instruction in controlled settings and afforded university faculty opportunities to provide immediate feedback after rehearsals. These experiences helped prospective teachers transform learning into practice, which is especially important for prospective elementary teachers who experience low confidence in mathematics settings. It also provided a venue for university faculty to closely mentor students thereby overcoming some of the constraints imposed by live classroom rehearsals. This study's results suggest that simulations may prove to be viable alternatives for rehearse teaching in clinical settings where optimal conditions cannot be secured. Simulations may also provide candidates with early mentoring opportunities that build self-confidence while also reducing the burden of placements on already taxed schools. With optimal clinical settings at a premium in rural, urban and suburban school districts, alternative modes for rehearse teaching need further investigation.

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MEASURING SHIFTS IN REASONING ABOUT FRACTION ARITHMETIC IN A MIDDLE GRADES NUMBER AND OPERATIONS CONTENT COURSE

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The present study extends recent advances developing and applying measures of mathematical content knowledge for teaching. Recent research has demonstrated that the Diagnosing Teachers' Multiplicative Reasoning Fractions survey provides information about distinct but related components necessary for reasoning in terms of quantities when solving fraction arithmetic problems. The present study adds a new component of validity for the survey by examining the extent to which one pre-service teacher's growth in reasoning about fraction arithmetic, as indicated by assessments she completed for a middle grades numbers and operations course, was reflected in her performance on the survey. Results provide an existence proof that the survey is sensitive to shifts towards more proficient reasoning.

Keywords: Mathematical Knowledge for Teaching, Measurement, Teacher Education-Preservice

Recent curriculum standards documents (e.g., National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010; National Council of Teachers of Mathematics, 2000) and recommendations for teacher education (e.g., American Mathematical Society, 2010; Sowder et al., 1998) have placed high value on developing conceptual understanding by solving and reflecting on solutions to problems embedded in situations. These standards and recommendations have led to two critical challenges for mathematics teacher education addressed by the present study. First, a body of past research (e.g., Ball, 1990; Izsák, 2008; Ma, 1999) has reported persistent difficulties preservice and inservice teachers have had explaining fraction arithmetic in terms of quantities and has demonstrated a need for teacher education to attend to this critical content. Second, the field has been striving to develop measures that target mathematical knowledge needed for teaching practice (e.g., Bradshaw, Izsák, Templin, & Jacobson, 2014; Hill, 2007; Izsák, Jacobson, de Araujo, & Orrill, 2012; Kersting, Givvin, Sotelo, & Stigler, 2010). For us, addressing both challenges simultaneously means developing and utilizing measures that capture information about teachers' capacities to construct chains of reasoning for explaining fraction arithmetic in terms of quantities.

The present study addresses both challenges identified above by examining the first application of the Diagnosing Teachers' Multiplicative Reasoning (DTMR) Fractions survey to study growth and change in a cohort of preservice teachers. The important feature distinguishing the DTMR Fractions survey from the Learning Mathematics for Teaching (LMT) measures of mathematical knowledge for teaching (e.g., Hill, 2007) is that it is multi-dimensional. The LMT measures are used to estimate a single score on a continuous scale that can be interpreted as an overall measure of ability with respect to the targeted mathematical content. In contrast, the DTMR survey provides information about four components of reasoning (illustrated with an example below) that are well established in the theoretical and empirical research literature on children's and teachers' reasoning about fraction arithmetic in terms of quantities. The trade-off for multidimensionality is that the DTMR components are dichotomous variables. By measuring components of reasoning, the DTMR survey has potential to capture nuanced information about teachers' reasoning and ways their reasoning can shift during content courses.

In prior work, Bradshaw et al. (2014) reported on the development of the DTMR Fractions survey and analyzed data from a national sample of 990 in-service middle grades teachers. The

results established the content validity and psychometric properties of the survey. In the present study, we administered the DTMR Fractions survey before and after a semester-long course on number and operations that gave significant attention to reasoning about fraction arithmetic in terms of quantities. The course was offered to 22 preservice mathematics teachers in a middle grades program. The goal of the study was not to evaluate the effectiveness of the course but rather to examine a form of convergent validity around growth and change in the preservice teachers' reasoning. In particular, our research question asked whether or not the reasoning that preservice teachers demonstrated on course assessments was consistent with shifts in their reasoning as indicated by their performance on the DTMR Fractions survey. Thus, this study was a test case examining the extent to which it is possible to measure shifts in multiple components of teachers' reasoning using psychometric methods.

Measuring Teachers' Reasoning About Fraction Arithmetic in Terms of Quantities

As indicated above, solving fraction arithmetic problems in terms of quantities involves multiple components of reasoning. The solution outlined to the following problem illustrates three of the four components measured by the DTMR Fractions survey.

> A batch of brittle calls for $^1/_4$ of a cup of honey. Megan has $^2/_3$ of a cup of honey. How many batches of brittle can Megan make?

The solution presented below presumes that a teacher will not employ a general numeric method, such as multiplying by the reciprocal of the divisor, but instead will reason directly with the quantities of cups and batches to solve the problem.

First, to see the opportunity for discussing division, a teacher would have to recognize that the brittle problem asks a how-many-groups question, the signature for measurement division. We use the term *appropriateness* to refer to selecting an arithmetic operation that can model a given problem situation. Next, the teacher might produce a drawn model. The double number line shown in Figure 1a, where lengths depict cups of honey, is one possibility. Our next point about partitioning is not dependent on this choice of drawn model. Juxtaposing the two number lines highlights the challenge that fourths and thirds do not subdivide one another evenly. Whole-number factor-product relationships are useful for overcoming this challenge. In this case, 12 is a common multiple of 3 and 4. Figure 1b illustrates how twelfths simultaneously subdivide thirds and fourths of cups and thus provide a finer unit with which to compare the two. Thus, teachers must be skillful at *partitioning* quantities, often using factor-product combinations as a tool. A related aspect involves *iterating* the resulting mini-piece (8 times in this example).
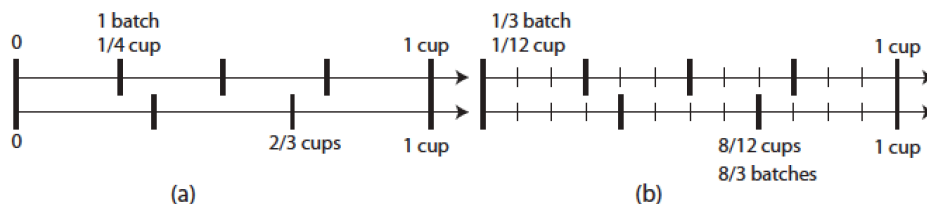
Figure 1. Reasoning with a double number line.

Finally, the teacher must interpret the mini-pieces in terms of the given situation. There are multiple candidates, including interpreting one mini-piece as a twelfth of 1 cup, as a fourth of $^1/_3$ cup, and as a third of $^1/_4$ cup. Because the problem asks about the number of $^1/_4$ cups in $^2/_3$ cups, $^1/_4$ cup is the appropriate referent unit: There are $^8/_3$ $^1/_4$-cups in $^2/_3$ cups. Thus, the teacher must be clear about the *referent units* for all numbers used in the solution. The DTMR Fractions survey provides information about teachers' facility with appropriateness, partitioning and iterating, referent units,

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and a fourth component, reversibility, that is important for solving partitive division problems. Reversibility has to do with constructing the one whole given a fractional amount of the whole (e.g., $^3/_5$ or $^7/_5$).

The DTMR Fractions survey consists of 27 items that measure the four components of reasoning just discussed. We used a psychometric model called the log-linear cognitive diagnosis model (LCDM) to analyze the item responses and estimate profiles that indicate "mastery" or "non-mastery" of appropriateness, partitioning and iterating, referent units, and reversibility. (The term *mastery* comes from the psychometric literature, which we take as a synonym for *proficiency*.) The LCDM is one member of a recently developed family of psychometric models referred to as diagnostic classification models. The DTMR Fractions survey is one of the first practical applications of these new models.

Methods

Data for this report comes from an on-going study of preservice teachers' reasoning about multiplication and division, fractions, and proportional relationships. As part of the broader study, the project team administered the DTMR Fractions survey to a cohort of 22 preservice middle-grades mathematics teachers before and after a number and operations content course offered in Fall 2014. The course was offered as part of a teacher education program at a large, public university in the Southeast United States. The course emphasized reasoning with quantities to develop conceptual understanding of multiplication and division with whole numbers and with fractions. A project-team member taught the course. We administered the DTMR Fractions survey the first week of the course (August, 2014) and again at the beginning of the following semester (January, 2015). In addition to DTMR survey data, we collected all course assessments (quizzes and tests) from all preservice teachers in the course. The preservice teachers completed these assessments between mid-September and mid-December.

We selected six preservice teachers for more detailed study based on their initial fractions profiles as determined by the LCDM analysis. Of the six, the LCDM analysis indicated that Kelly's (a pseudonym) profile shifted along the most (3) dimensions. Thus, we selected her as an initial case to examine the extent to which her growth in reasoning about fraction arithmetic, as indicated by her assessments completed for the number and operations course, was consistent with her shift in profile. The first author first compiled a list of all assessment problems that provided opportunities for Kelly to use at least one of the four components of reasoning—appropriateness, partitioning and iterating, referent units, and reversibility. The compiled problems came from two quizzes, three tests, and the final exam. The first and third authors then individually analyzed Kelly's work on the selected problems, listing evidence for or against facility with each component. The first and third authors then looked at the problems together and compared their lists of evidence for or against facility with each component. Discrepant interpretations were discussed until resolved. The second author then confirmed the first and third authors' analysis, discussing discrepant interpretations with the first and third authors. We then compared Kelly's reasoning as evidenced in her course assessments with the shifts indicated by her performance the DTMR Fractions survey before and after the course.

Results

According to the LCDM analysis, Kelly was not a master of any of the four DTMR components at pretest. Her probabilities of mastery for each component at the beginning of the number and operations course were appropriateness (.01), partitioning and iterating (.01), referent units (.00), and reversibility (.15). At posttest, each of these probabilities increased: appropriateness (.98), partitioning and iterating (.30), referent units (.52), and reversibility (.86). This suggests that by the end of the course Kelly had made significant gains on appropriateness and reversibility, that there was conflicting information about her facility with referent units, and that she still struggled with

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partitioning and iterating. Our analysis indicates that shifts in Kelly's performance from DTMR pretest to DTMR posttest was largely consistent with her performance on the assessments she completed for the numbers and operations course, and that places where her performance diverged corresponded to places where the DTMR Fractions survey was less well aligned with the course assessments.

Shifts in Kelly's Reasoning about Appropriateness

Kelly's performance on assessments for the numbers and operations course indicated that she used tools developed in the course to make appropriate determinations about when and where multiplication and division could be used to model problem situations. These tools were an explicit meaning for multiplication and a form of ratio table. Oftentimes Kelly was explicit about using these two tools. Figure 2 shows one example of her work on a test given in December, near the end of the course. The question presented four word problems and asked whether each one could be modeled by $^2/_3 \div ^3/_4$, $^3/_4 \div ^2/_3$, or neither. Kelly identified the appropriate operation in all but the second example.
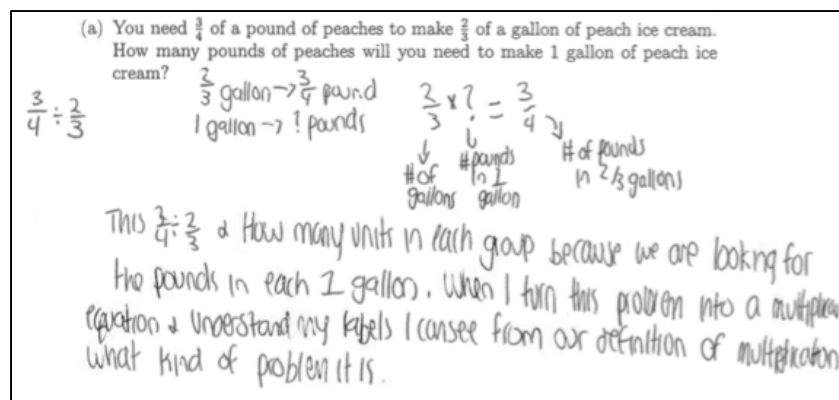
Figure 2. Kelly uses tools from class to identify partitive division (December).

Figure 2 shows Kelly's work on the first word problem. The ratio table in the top left of Kelly's work and the multiplication equation in the top right illustrate the tools from class. In particular, the course instruction used multiplication with an unknown factor, represented in Kelly's work with a question mark, to represent division situations. The units Kelly attached to the $^2/_3$, the question mark, and the $^3/_4$ also followed course instruction closely. For reasons that are not clear, Kelly answered part (b) of the same question incorrectly. The only difference was the question, which asked, "How many gallons of peach ice cream can you make from 1 pound of peaches?" Although this is also partitive division, Kelly set it up as a measurement division problem and got confused. The final exam presented six word problems and asked which could be modeled by multiplication, division, or subtraction, again with the fractions $^2/_3$ and $^3/_4$. Kelly again used ratio tables and multiplication equations like those shown in Figure 2 to identify appropriately all instances of division (partitive and measurement). She also identified appropriately instances of multiplication but then changed her answers. Thus, by the end of the course, her recognition of appropriate operations was largely, but not completely, accurate.

The DTMR Fractions items that measured appropriateness asked whether situations described in word problems could be modeled by multiplication or division. (The test is secure, so we cannot provide specific items.) Kelly did not explain how she selected her choices to the DTMR items, so we cannot be sure if she used ratio tables or the meaning of multiplication illustrated in Figure 2. Nevertheless, her use of the explicit meaning for multiplication and ratio tables on course assessments as instructed in class suggested the number and operations course played a key role in her improved performance; and the contrast between her performance on the DTMR pretest and

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posttest reflected genuine improved facility identifying which arithmetic operations can be used to model situations described in word problems.

Shifts in Kelly's Reasoning about Partitioning and Iterating

Kelly's performance on assessments for the numbers and operations course indicated frequent use of common denominators when partitioning quantities. Figure 3 shows one example of her work from a test given in September. This example occurred 2 weeks after partitioning had been introduced for the purpose of generating equivalent fractions. The question presented a number line showing the locations of 0 and $^2/_7$ and asked for the locations of $^1/_7$ and $^1/_5$. Kelly began her explanation by stating: "In order to have equally spaced tick marks we must first find a common denominator with $^1/_7$ and $^1/_5$." We did not assign much significance to the uneven spacing of her tick marks.
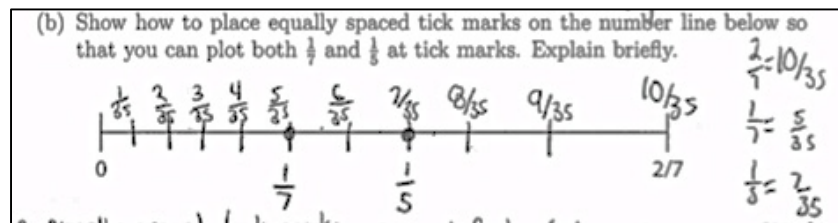
Figure 3. Kelly uses common denominators to partition (September).

Kelly's consistent attention to common denominators served her well in some, but not all, situations: She continued to focus on common denominators as a guide for partitioning when problems called instead for partitioning using common numerators. Partitioning by common numerators is useful for some methods for solving partitive division problems with drawings. This aspect of partitioning was not emphasized in the course. Figure 4 shows Kelly's work on a test given in December, near the end of the course. She generated an appropriate partitive division word problem as asked, again using the meaning for multiplication and the ratio table discussed above. As part of her explanation, she stated "I must find a common denominator of 4" when, in fact, partitioning $^1/_2$ into two equal parts comes from the least common multiple of the numerators for $^1/_2$ and $^2/_3$.

The DTMR Fractions items that measured partitioning and iterating presented a mix of situations calling for partitioning by common denominators and by common numerators. On the DTMR posttest, Kelly demonstrated continued use of partitioning by common denominators but missed nearly all items for which attention to common numerators would be useful. Thus, Kelly's modest increase in performance from the DTMR pretest to posttest reflected her restricted attention to common denominators when partitioning.

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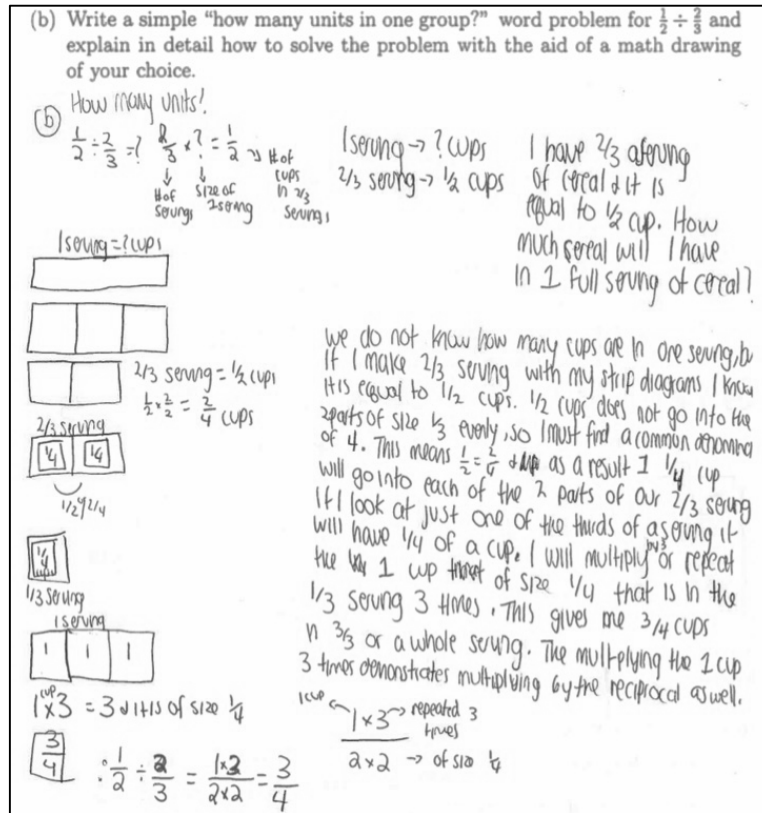
Figure 4. Kelly's reasoning about partitioning and iterating and reversibility (December).

Shifts in Kelly's Reasoning about Referent Unit

Kelly's performance on assessments for the numbers and operations course indicated inconsistent reasoning about referent units. At times she gave clear, correct explanations for the referent units in multiplication and division problems. Figure 5 shows one example where Kelly demonstrated taking part of a part correctly when explaining the meaning of $^2/_3 \times {}^4/_5$. From her written work we infer that she drew $^4/_5$ of one whole first and then took $^2/_3$ of the $^4/_5$. There were also examples where she assigned appropriate referent units to the quotient in both partitive and measurement division problems. On a test in December, Kelly wrote the following when asked for a measurement division problem that illustrated $^1/_2 \div {}^2/_3$: "One serving of cereal is $^2/_3$ cups. How many servings of cereal will I have in $^1/_2$ cups?" When solving her problem, Kelly was explicit about converting the given fractions into $^3/_6$ and $^4/_6$ and using $^4/_6$ as a new unit.
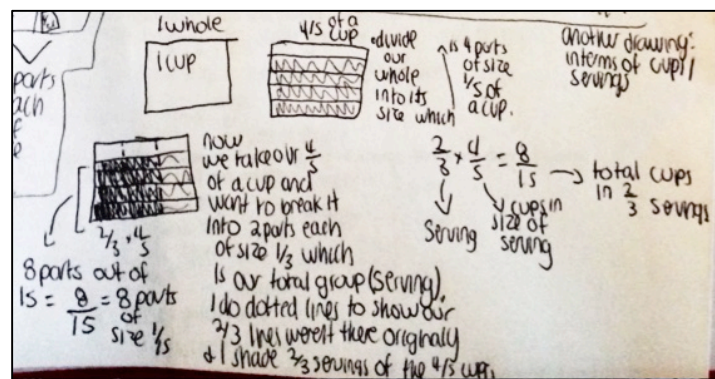
Figure 5. Kelly's reasoning about referent units for fraction multiplication (October).

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At the same time, Kelly's incorrect reasoning about units was evident in several places, including some of her work on the final exam. One problem presented a picture of a 5-part strip with three parts shaded. The problem stated that the entire 5-part strip represented $^7/_2$ acres of land and asked whether, according to the meaning for multiplication developed in the course, the 3 shaded parts represented $^7/_2 * ^3/_5$ (incorrect) or $^3/_5 * ^7/_2$ (correct). Ostensibly, this problem was about appropriateness, but a central feature of Kelly's work was incorrect referent units. In particular, she divided each of the 5 parts in half, creating 10 parts, and argued that the shaded region showed $^6/_{10}$, not the $^{21}/_{10}$, which she knew was the product. Thus she conflated $^1/_{10}$ of $^7/_2$ with $^1/_{10}$ of the whole.

The DTMR Fractions items that measured referent units for multiplication and division situations often presented a complete number sentence and several drawings indicating different choices for the referent unit for the product or the quotient. Kelly got some items correct while missing others. Thus her reasoning appeared inconsistent both on course assignments and on the DTMR survey.

Shifts in Kelly's Reasoning about Reversibility

Kelly's performance on written assignments for the numbers and operations course indicated her ability to start with a proper or improper fraction and construct the relevant whole, which is the sense of reversibility used in the DTMR Fractions survey. The examples in Figures 4 and 6 illustrate Kelly's reasoning with reversibility after the Common Core State Standards definition for fraction, which is based on iterating a unit fraction, was introduced in late August. Figure 6 shows Kelly's work on a quiz in September. Given an array of X's that was $^4/_3$ of another array, Kelly partitioned by 4 to find $^1/_3$ of the given array. She explained that "The unit whole of a fraction of $^1/_3$ would be 3 equal parts of each of size $^1/_3$." In the rest or her work (not shown) she then redrew the first three columns of circled X's shown in the figure. In Figure 4 she correctly reasoned that if $^1/_2$ cup makes $^2/_3$ of a serving, then $^1/_4$ cup makes $^1/_3$ of serving. She completed her solution by explaining that since 3 one-thirds make a serving, 3 one-fourth cups must be needed for one serving. On the final exam, she wrote a partitive division word problem for $^1/_3 \div ^4/_5$ and solved using the same method as the one used for the problem shown in Figure 4. We conjecture that the introduction of the Common Core State Standards early in the course provided Kelly a new perspective on fractions that allowed her to solve reversibility problems quickly.
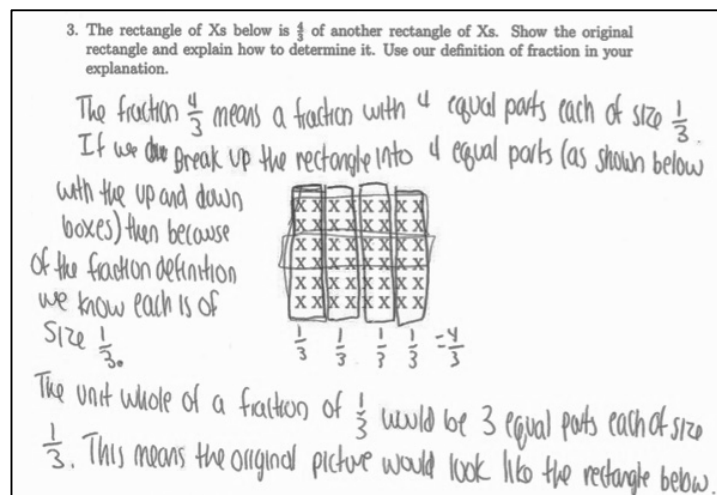
Figure 6. Kelly's reasoning about reversibility (September).

Discussion

The results we present add to an accumulating body of evidence that the DTMR Fractions survey is a valid measure of teachers' reasoning about fraction arithmetic in terms of quantities. Past results

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established the content validity and psychometric properties of the survey. The present study examines the sensitivity of the survey to growth and change in reasoning during a one-semester content course on number and operation. In particular, our results suggest that changes in Kelly's reasoning about fraction arithmetic, as evidenced by her use of tools and definition for fractions introduced during the course, were reasonably consistent with the shift in her performance on the DTMR survey before and after the course, at least in those areas where the content of the survey and course were closely aligned. Further analysis of the remaining dataset is ongoing, and we will present results of that analysis in the near future. The results we present here are a critical next step in developing measures that capture information about moment-to-moment reasoning, narrowing the gap between information about teachers that can be ascertained through large-scale surveys and detailed case studies of problem-solving performance. Narrowing this gap is critical for measuring growth in reasoning necessary for enacting current curriculum standards (e.g., National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010; National Council of Teachers of Mathematics, 2000). Since the approach to measurement we used, based on diagnostic classification models, can be applied to other content areas and to students as well as teachers, the implications for psychometric models as tools for research and practical applications are broad.

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EXPLORING PROSPECTIVE TEACHERS' WRITTEN FEEDBACK ON MATHEMATICS TASKS

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This cross-case analysis of three distinct letter writing activities between prospective teachers (PTs) and K-12 learners provides insights into PTs' levels of feedback according to Hattie and Timperley's (2007) framework. PTs' written feedback from letter writing activities conducted in teacher preparation programs were analyzed using this framework in order to describe feedback provided to K-12 learners by PTs. Crossing the borders of programs at three different U. S. universities, elementary and secondary teacher preparation, and elementary, middle, and high school learners, evidence indicates that PTs provided feedback focused on self, task, and process but rarely on self-regulation. Further, they did not adequately address incorrect answers and redirected student thinking as often as building on student thinking. Recommendations for teacher preparation activities that address the development of feedback practices are provided.

Keywords: Teacher Education-Preservice, Instructional Activities and Practices

Purpose for the Study

Providing feedback positively impacts student learning (Hattie & Timperley, 2007; Shute, 2008; Black & Wiliam, 1998) and mathematics performance (Fyfe, DeCaro, & Rittle-Johnson, 2015). Attention to the development of prospective teachers' (PTs') feedback practices can therefore impact student learning once PTs enter the classroom. Hattie and Timperley's (2007) review of feedback research offers a framework for describing feedback which asserts consideration of the goals, progress, and what is needed to reach the goals are required for feedback to be effective. This framework categorizes feedback as information on the self, process, task, and self-regulation and describes the effectiveness of each type. Considering evidence of the potential impact on student learning when effective feedback is incorporated, we explored the ways in which PTs give feedback. Specifically, what levels of feedback (Hattie & Timperley, 2007) do PTs provide in written comments to students about their performance or mathematics learning? To answer this question we analyzed PTs' written responses to mathematics students about their solutions to mathematics tasks. Findings from the analysis provide insight into levels of feedback given by PTs and suggest activities which develop PTs' handling of incorrect responses, attention to self-regulation, and skills in building on students' mathematical thinking are needed.

Perspective

Black and Wiliam (1998) described written feedback as a formative assessment practice used to move students toward instructional goals and significant in supporting student learning. Many factors (Hattie & Timperley, 2007; Shute, 2008; Wiliam, 2011), including the use of praise (Brophy, 1981), correlate to student learning. The use of feedback to learn is a function of how students attend to and interpret the feedback (Wiliam, 2011). Students' reflections on teachers' feedback illustrate that emotional responses to feedback mediate how the information impacts learning (e.g. Hargreaves, 2013; Havnes, Smith, Dysthe, & Ludvigsen, 2012). These factors illustrate that feedback practices are relational (Fletcher, 1998) and highlight the need to understand PTs' approaches to feedback. We view written feedback as a teaching practice that supports relational interactions between mathematics teachers and students and creates opportunities for student learning in mathematics.

While no studies have examined feedback as a relational practice, meta-analyses have highlighted the effects of feedback (Wiliam, 2011) and defined feedback in various ways: as

"information communicated to the student that is intended to modify his or her thinking or behavior for the purpose of improving learning" (Shute, 2008, p. 154); or as "information generated within a particular system, for a particular purpose" (Wiliam, 2011, p. 4). Wiliam's (2007) discussion of providing feedback in mathematics identified *comments only* feedback as having the potential to positively impact students' attitudes toward mathematics. Student characteristics were shown to mediate these effects, thereby highlighting the role of the teacher's relationship with the student in the practice of giving feedback. We draw on these definitions to create a relational view of feedback as information generated within an instructional system and communicated to the student with the intention of improving learning or performance.

A logical extension is that the feedback must be taken up by the learner in order to be effective. For this to occur the feedback must be seen as supportive of the student or the feedback likely remains a suggestion rather than a call to action. We find that PTs act relationally in letter writing, working not only to build mathematical performance, but also to show care by drawing on interests of students and their thinking to create feedback. Although efforts may begin haphazardly, Crespo's (2002) work illustrates that these efforts can be supported to encompass attention to the self as well as mathematics involved in the task.

Effective feedback answers these questions:	
Where am I going? (the goals)	Feed Up
How am I going?	
Feed Back	

↓

Each feedback question works at four levels:

↓

Task Level	**Process Level**	**Self-Regulation Level**	**Self Level**
How well tasks are understood/ performed	The main process needed to understand / perform tasks	Self-Monitoring, directing, and regulating of actions	Personal evaluations and affect (usually positive) about the learner

Figure 2. Feedback Model (Hattie & Timperley, 2007, p.87).

Hattie and Timperley's (2007) "model of feedback to support learning" (Figure 1) informs discussions about why "particular kinds of feedback promote learning" while others do not (p. 86). Focusing on PTs' feedback to students about their mathematics task work, we address the question "How am I going?" (p. 89) and use this framework to describe PT's feedback at four levels: task level (FT), process level (FP), self-regulation level (FR), and the self level (FS).

We use these levels to develop a profile of PTs' feedback approaches to answer the question: What levels of feedback do PTs provide in written comments to students about their performance or mathematics learning? Each of the authors independently designed opportunities for PTs to gain insights into students' thinking through letter writing, which we considered an approximation of practice (Grossman et al., 2009). PTs provided written feedback to students about their responses to mathematics tasks in the form of letters. Existing research on letter-writing (e.g. Crespo, 2002) has

focused on the types of tasks PTs select or create. We instead describe levels of feedback PTs provide in their responses to students' mathematics.

Modes of Inquiry

We analyzed PTs' letters from three university teacher education programs in the US. All three letter writing activities involved PTs providing feedback to students on work from mathematics tasks to support mathematics learning. The contexts are described in Table 1.

Table 1: Descriptors of the Three Letter Writing Contexts

	Context 1	Context 2	Context 3
Setting	• Rural Midwest • Elementary Education • Mathematics methods	• Suburban Southeast • Secondary Education • College Geometry	• Small Midwest City • Elementary Education • Mathematics methods
Participants	• 19 PTs • 19 Fourth grade students	• 12 PTs • 12 Ninth grade students	• 100 PTs • 76 Sixth grade students
Structure of Task	• 5 letters exchanged between PTs/students • One PT paired with one student • Different mathematics tasks were addressed	• 5 letters written by PTs with 4 student responses • One PT paired with one student • One problem situation addressed through all correspondence	• 1 letter written by PTs (draft and final letter) • 1-3 PTs paired with one student • One mathematics task was addressed
	• PTs wrote a self-report focused on difficulties and insights in corresponding with children	• PTs wrote a self-report focused on PTs abilities in eliciting student thinking and their reflections on learning about feedback	• PTs revised letters after having received written feedback provided by the instructor
Data	• Exchanged letters • PT self-reports	• Exchanged letters • PT self-reports	• PTs' letters • Students' work

Similarities cut across the borders that described the three contexts related to settings, participants, structure of the task, and data, and yet each context maintained unique features. This combination of common and unique characteristics across the contexts provided for data analysis and interpretation that considered how these characteristics might allow for similarities and differences in results and, in turn, better inform the development of PTs' feedback practices.

Hattie and Timperley's (2007) four levels of feedback (FT, FP, FR, and FS) were used to code PTs' feedback. Several coding iterations were conducted while the researchers clarified meanings of the levels of feedback in each context and appropriate uses of codes with respective data. In each context, student responses to tasks were used as a data source to clarify ambiguous

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statements in PTs' feedback. For example, a PT reflected, "I praised her on her unique thinking strategies that helped her arrive at the correct answers." Unsure of the PTs' meanings for "praise" and a "unique thinking strategy," we searched the PT's letters for evidence of praise for a thinking strategy. The "unique thinking strategy" referred to specific approaches taken by the student such as the following: "I noticed that you knew that you could the (sic) use multiple units to measure the different items." Referring to other data sources clarified the coding of the use of praise as FS and the "unique thinking strategy" as FT. Data chunks that contained more than one level of feedback were labeled with more than one code. Categories within each level (i.e., non-specific and specific praise within FS in Context 1) were developed for each context. We then compared findings from the three groups of PTs to identify commonalities and differences in levels of feedback given, which will be presented in the cross-case discussion.

Results

Findings from Context 1

A majority of PTs in this context provided feedback in the form of FS or FT, with a few providing FP and none acknowledging FR. Twelve of 19 PTs identified praise or FS as a component of their feedback. One PT suggested she used general praise such as "good job." Two other categories of praise linked FS to FT. Five of the PTs described *non-specific praise* of the student's responses (e.g., "I liked how you answered that question"). PTs who used non-specific praise described preserving self-esteem or motivating the student as a rationale for this approach. The six remaining PTs described using *specific praise* that identified what the child had done well. One PT described this approach "My feedback to my Pen Pal was always very encouraging and included things like, 'I like that you did . . .'." These PTs valued providing task specific feedback with praise and were aware they were being specific about what was valued. For example, "I would make sure to write compliments about what they saw in the graph and then explained why that was good that they saw that." With the exception of the PT providing praise alone, the remaining PTs coupled FS (praise) with FT, such as whether the answer was correct, or FP, focusing on processes involved in mathematics. One PT wrote,

> I even broke down how I thought he was thinking about something, so that it was obvious that we were on the same page, that I liked seeing how he did it, and also to give him an example of how to be descriptive as he thinks and writes about math.

Ten PTs were aware that they were providing FT and explained their approaches. Three PTs identified providing feedback whether answers were correct or incorrect. Remaining statements about FT highlighted the PTs' need to focus on what was correct or the strengths of the student's work such as "I told him what he did well on and if he didn't get something right I would explain it so hopefully he would understand." These PTs described a desire to explain how a student might be able to think about the problem, as in the response above, or to extend the student's thinking by suggesting particular strategies. For example, when a student incorrectly interpreted a graph, the PT "recognized the method he used and then elaborated on his thinking," suggesting that the child "look at what the X and Y sides on the graph represent." Overall, the PTs were conscious of not explicitly stating students' answers were wrong, what some called "negative feedback." Instead, they provided "corrective feedback" (Hattie & Timperley, 2007, p. 91) such as solutions or suggestions, as in the graphing example.

Six PTs provided FP. One PT reflected that he always commented on the child's strategies to "help her become more aware of the different thinking methods that you can use to arrive at an answer." His feedback focused on describing the student's work as general strategies. For example, when the student gave an estimate of a measurement and a result of her measurement, the PT

described this as estimating before measuring or computing. Remaining PTs focused on "motivating" students to write more about their thinking or the numbers they choose: "I prompted her to be more specific and write about the specific numbers that she sees." PTs' focus on eliciting detail was motivated by a need to understand the child's mathematics. They shared: "I asked how he was seeing the math. The hope was that it was clear to the student that I wanted him to be detailed and descriptive so that I understood how he was seeing it." PTs' reflections suggest they provided FP to gain evidence of student thinking rather than to support the child's thinking. The PTs' need for more evidence resulted in feedback on students' processes.

Findings from Context 2

Data in the second context revealed that PTs provided feedback in the areas of FS, FT, and FP but that they did not provide FR. Thirty-seven instances of praise (FS), often addressing student's effort, were identified in the letters. Eighteen instances were non-specific statements about self (e.g., "I appreciate your hard work so far. You are making great progress."). And 19 instances included specific statements about the process (e.g., "I really like the way you used your knowledge about the angle measures of a triangle to find all three angle measures.").

PTs' FT (55 instances) attended to two particular areas: helping students draw a picture of the problem situation, and addressing a particular concept need to solve the problem. This letter writing exchange required students to draw a model to construct a solution. PTs attended to the difficulties students had drawing the model. For example, a PT wrote:

> The pirates are facing each other in the locations the problem gives. One last thing, be sure to pay attention to any geographical locations that are given. You can use the compass at the bottom of the map to help. I have highlighted these things for you!

After addressing the model, PTs made efforts to move students toward ideas that focused on concepts that would aid in solving the problem. For example, another PT wrote:

> You are correct that angles do not have the same measure. You are also correct that you know the two angle measures, but let's think about AA for a minute. You said that it was a congruence postulate, but if I had two triangles as shown below with the given angles they do not appear to be congruent triangles. Why is that? AA is a postulate, but it is not a congruence postulate. Keep working you are almost there.

In the area of FP (47 instances), PTs tended to give feedback that either used student statements as the basis for further questions (*building on*) or acknowledged the student statements and then asked a question that was not directly related (*redirecting*). For example, one PT suggested a direction the student should pursue, building on from the student's statements: "In your last letter, it looks like you created a triangle in your map. I like where you were headed with that. Perhaps that is what you should be looking for moving forward." In contrast, another PT redirected the student:

> What you did was draw two parallel lines, but why do you need them? Are we able to get rid of them and still work the problem? With getting rid of the parallel lines . . . we can now connect point A and point B, since that is where the two pirates are.

Findings from Context 3

Feedback written by PTs in Context 3 addressed FS (93 instances), FT (140 instances), FP (90 instances), and FR (7 instances). Much of the PTs' feedback included some form of praise (FS) and was more often than not connected to FT or FP. Praise related to the task addressed correct answers, such as "4/6 is a correct answer! Very good." Praise related to the process addressed strategies used by students: "Your strategy … is a smart way to approach this problem." PTs also acknowledged students' responses without specific use of praise, such as "I can see that you understood what each

fraction is." Two feedback responses included praise focused on FR such as "It was good to see that you tried a different approach at first, and when it did not work you were able to adapt your approach."

PTs provided feedback on the task (FT) that identified correct or incorrect answers (e.g., "Your answer is correct!" and "You are thinking of numbers between the 2 fractions, but 4/5 is actually bigger than 5/8 & 3/4"), and also identified when students did not provide an answer (e.g., "Unfortunately, I am unable to find your answer. What fraction falls between 10/16 and 12/16?"). Additionally, PTs encouraged students to consider more than one answer (e.g., "… could you find another answer that would fit? Or, is 11/16 the only possible answer?").

Feedback on process (FP) requested more information from students, or made suggestions to provide direction for students. A PT wrote, "… could you explain this pattern to me?" Other PTs suggested students visualize relationships among fractions by drawing shaded regions, or number lines. One student had a list of 5/8, 6/8, 7/8, 8/8, 3/4, 4/4 with no explanation, and the PT wrote:

> … start by showing the two fractions as pies so that you can see how they compare. Split the pies into parts to show 5 parts out of 8 and 3 parts out of four. To find one fraction how could you make two pies of the same size using fractions that are still equal to 5/8 and 3/4?

PTs provided FP that clarified faulty procedures in students' attempts to solve the task (e.g., "However, not all numbers between the numerators and the denominators will be between 5/8 and 3/4. For example, 7 is between 4 and 8 for the denominator, but 4/7 is not between 5/8 and 3/4."). PTs also provided suggestions to build on students' strategies. For example:

> Your number line shows us the fourths and eighths for a denominator. However, you can use this number line to show even more possible answers by including sixteenths (1/16, 2/16, 3/16 and so on) and twenty-fourths (1/24, 2/24, 3/24, and so on).

Sometimes (33 instances) PTs' feedback on process (FP) redirected students to strategies not related to the students' work:

> Although there is a pattern of numbers between numerators and denominators, . . . I would suggest to you to find what the decimals are of both fractions, and then try to make up different fractions that would give you a decimal in between both of those two numbers.

PTs were prompted to provide specific feedback to students. The findings illustrate that PTs attended to instructions with providing specific FT and FP. While FT attended to correctness and progress on completing the task, the FP requested more information from students to explain their thinking, suggested a direction that built on students' solutions, or redirected students to a different approach to the task.

Cross-Case Findings

Looking across the contexts we can answer the question: What levels of feedback do PTs provide in written comments to students about their performance or mathematics learning? Across the contexts PTs provided FS, FT, and FP, with a few instances of FR in Context 3.

Praise (FS) was present in the data from all the contexts with differences in the way it was used. The majority of the PTs in Contexts 1 and 3 provided praise either in relation to the task or to the process. In Context 1, PTs were aware of their use of praise, highlighting it as part of the design of feedback. In Context 2, praise of effort was common. We hypothesize that these differences in the use of praise are the result of the activity contexts. In Context 2, PTs were writing to high school students whose answers were generally incorrect. For those PTs, effort was the only element of the response warranting praise. Secondary PTs' attention to effort in feedback is consistent with the findings of Norton and Kastberg (2012). We hypothesize that secondary PTs might attend to effort in

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constructing responses to problem solving situations because of their awareness of the role of effort in successful problem solving. In contrast, PTs in Contexts 1 and 3 may not have viewed problem situations as requiring sustained effort.

PTs in elementary programs may initially share general praise as shown in Crespo's (2002) findings, but for PTs in Contexts 1 and 3, praise tended to be associated with either FT or FP. Some PTs in Context 1 praised correct answers in non-specific ways. Other PTs, including those in Context 3, tended to combine praise with specific FT or FP on student responses.

While studies of teacher praise suggest that it is not useful to students, PTs may be using praise in letter writing to communicate care. Unlike oral praise that is meant to reinforce behavioral aims (Brophy, 1981), PTs identify praise as a way to demonstrate care and support effort. Particularly where PTs are unfamiliar with the students or the academic goals of the students' teachers, their feedback seeks to reach out to the students to make a connection and then to attend to feedback on the task and, in some cases, processes involved in completing the task. This finding is consistent with emerging use of reform practices across letters Crespo (2002) identified in PTs' teacherly talk.

PTs in all three contexts provided FT. Attention was paid to the student's correctness, and as in Crespo (2002), PTs were explicit when students provided correct answers, but not when answers were incorrect. Instead, PTs focused on task specific approaches or concepts that could lead to correct answers. In Study 2, PTs suggested ways in which students could create a model to support their reasoning. In all three contexts PTs provided specific examples or suggestions the students might use to solve the problem or move toward a correct answer.

PTs in all three contexts attended to FP; however, significant differences were identified. Some PTs in Contexts 1 and 3 requested further evidence from students with those in Context 1 suggesting that further evidence was needed to gain insight into student reasoning. PTs in Contexts 2 and 3 providing FP suggested processes that drew from PTs' inferences about processes students were using. This feedback was described as *building on*. Other PTs suggested processes that were not aligned with student work. This FP *redirected* students to use a new process, perhaps one preferred by the PT.

Discussion

Research shows that practicing teachers tend to give feedback that is limited to correct or incorrect responses, limiting the educative impact that productive feedback practices can afford (Crooks, 1988). Preparation of PTs in the area of effective feedback practices, based upon their current understandings, can positively impact their ability to make use of this instructional tool. Across the three contexts we see FS, FT, and FP. Some of these uses of feedback are identified as effective (Hattie & Timperley, 2007; Wiliam, 2007) and others are not. To build experience with and insight into the use of written feedback, PTs need opportunities to craft such feedback. Our analysis illuminates the PTs' attention to giving feedback as a relational practice. Attention to students as people whom the teachers did not know may have motivated them to praise the children to build relationships. PTs in Context 1 were aware they were using praise but identified it as motivating persistence and emphasizing the child as a unique person. FT focused on explicit attention to correct responses without the same explicit discussion of incorrect responses. Support for extending or repairing solutions in the form of suggestions or explanations of what to do in the context of the task were provided in FT, yet without explicit attention to incorrect responses students may be confused about the feedback and disregard it (Hattie & Timperley, 2007). FP tended to acknowledge the process used by the student, but was as likely as not to redirect the student to follow a process not related to his or her work.

With these findings in mind, mathematics teacher educators designing instructional activities for the development of feedback practices should be focused on discussions of the role of praise in the instructional system, handling of incorrect and incomplete responses, and building on to student

processes. PTs have ideas about how to give feedback on the self, task, and processes that can launch discussions of the benefits and drawbacks of various levels of feedback. After initial feedback to student work is crafted by PTs, mathematics teacher educators can discuss the affordances and limitations of various feedback responses in light of professional guidelines (e.g. Wiliam, 2007). In addition, inattention to FR in all but a few responses across the contexts suggests that instructional activities should be developed that attend to self-regulation as a factor in student performance and how to provide feedback on self-regulation. Crossing the borders of our own programs and contexts allowed us to think more deeply about the development of feedback practices of prospective teachers.

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CRITICAL MATHEMATICS EDUCATION: EXTENDING THE BORDERS OF MATHEMATICS TEACHER EDUCATION

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This study describes efforts at two institutions to integrate critical pedagogy within the context of two mathematics content and pedagogy courses for K-8 pre-service teachers (PSTs). The purpose of the curriculum within these courses was to focus PSTs' attention on how issues pertaining to social justice may be taught within mathematics contexts. The desired goals were for PSTs to: 1) come to appreciate individuals within their own communities as valid practitioners of mathematics; and 2) come to understand the responsibility that they, their school districts, curriculum developers, and others bear to ensure that all students have equitable opportunities to learn mathematics.

Critical Mathematics Education: Transforming Ideology

Mathematics is often taught, and learned, within the confines of a classroom where, in the classical model of mathematics education, the teacher holds the power to teach, and the student receives the information being taught. Mathematics is mostly perceived and presented as an elite body of knowledge far removed from the "lives and ways of living of the social majorities in the world" (Fasheh, 2000, p. 5). Yet, mathematics is woven within all aspects of our world and should be taught, and engaged with, through the lens of a more global perspective. There is a need to institute change in teacher education to provide a counter-narrative to traditionally-held perceptions of mathematics and its pedagogy. Mathematics education instruction needs to be constructed so that individuals may develop both cognitive and social understandings of their world (Wood, 2001). According to Bigelow and Peterson (2002):

> [we need to hold in] our minds and in our classrooms the big global picture. The world is a web of relationships. To be truly effective, every effort to make a difference needs to be grounded in that broader analysis. (p. 7)

Skovsmose (1985) states that when implementing critical education pedagogy, two criteria-subjective and objective, need to be used when selecting problems for the classroom. The subjective requires that the problem appears relevant to the students and within their conceptual understanding. The objective requires using data and detail to view an existing social issue unbiasedly to build deeper understanding. The integration of mathematics and social justice provides the potential to have meaningful conversations about issues impacting local communities, as well as the world, and prepares individuals for citizenship.

This research study inquires how we can teach mathematics that helps learners attain the content proficiencies, but beyond that, equips them with the empathy, consideration, and skills necessary to appreciate all individuals that comprise their community and to see these people as capable of learning and doing mathematics. We seek to merge the research domains of mathematics and critical mathematics education to design courses that question and extend the borders of academic mathematics to include mathematics for social justice. In this paper, we describe curricular efforts specific to two teacher education courses that address goals for PSTs to: 1) come to appreciate individuals within their own communities as valid practitioners of mathematics; and 2) come to understand the responsibility that they, their school districts, and others bear to ensure that all students have equitable opportunities to learn mathematics.

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Methodology

Setting and Participants

Two mathematics teacher education courses were taught by the authors at two distinct four- year institutions, one in the midwest and the one in the southeast U.S. PSTs pursuing licensure to teach K-8 enrolled in these courses. The first class is an elective class for elementary PSTs that focuses on issues of social justice in a mathematics education context. This class focused on assignments that required PSTs to explore social justice problems involving mathematics in a global context. The second class was a required mathematics content class for elementary PSTs. The second class included components focused on projects aimed at developing PSTs' awareness of individuals within their communities doing mathematics in local contexts. The key course goals, course activities, and outlines of the course structures were designed collaboratively.

Research Methods

Quantitative. There were two aspects to the methodology of the study. First, to gain insight into PSTs' views concerning multiculturalism and diversity, we administered the scale, *Teachers Sense of Responsibility for Multiculturalism and Diversity (TSR-MD)* (Silverman, 2009) to both classes as a pre-post test to determine if the different approaches to class activities led to changes in the PSTs' scores on this instrument. The instrument requires students to respond to a set of Likert scale questions. The Wilcoxon signed-rank test was conducted on each of the questions in the scale. The significant results with their effect sizes are discussed below.

Course 1 ($n = 23$, (Spring and Fall 2015) was the elective course aimed at social justice in the context of mathematics education. The possible responses for Questions 1 – 74 were: 1 = Strongly disagree, 2 = Disagree, 3 = Moderately disagree, 4 = Moderately agree, 5 = Agree, 6 = Strong agree. Initially, some students seemed to interpret Question 1, *In general, race is unimportant to me*, as "it would be best not to see race in teaching". On the post-test, students interpreted this question differently, beginning to develop a novice understanding of why a teacher should see and accommodate racial diversity. A Wilcoxon signed-rank test determined that there was a statistically significant decline in the number of PSTs who responded positively to this statement on the post-test ($Mdn = 3.0$) compared to the pre-test ($Mdn = 5.0$), $z = -2.06$, $p = .039$, $r = .43$. For Question 8, *In general, schools are responsible for addressing the differences among races*, there was a significant increase in the number of students on the post-test ($Mdn = 4.0$, $s = 1.3$) who saw schools as being responsible for addressing difference between the races, as compared to the pre-test ($Mdn = 4.0$, $s = .96$), $z = 2.03$, $p = .042$, $r = .42$. Regarding Question 10, *My students' economic class plays a role in my teaching*, there was a significant change in how PSTs perceived how students' economic class should affect their teaching as evidence in the post-test ($Mdn = 2.0$) compared to pre-test ($Mdn = 4.0$). There appeared to be a shift toward thinking that economic class should not impact teaching and learning, which seemed somewhat paradoxical given the class focus. Test results were significant, $z = -2.09$, $p = .037$, $r = .43$. On Question 11, *It is my responsibility to ensure various economic classes are represented in my teaching*, the post-test ($Mdn = 5.0$, $s = .1$) compared to the pre-test ($Mdn = 5.0$, $s = .89$) showed more positive than negative differences indicating that PSTs were assuming more personal responsibility for ensuring that various economic classes were represented in their teaching, $z = 2.07$, $p = .039$, $r = .43$. Regarding Question 20, *Gender is relevant to learning*, post-test results ($Mdn = 5.0$, $s = 1.3$) compared to pre-test results ($Mdn = 5.0$, $s = 1.5$), yielded significance, $z = 2.41$, $p = .016$, $r = .5$, pointing to an increase in PSTs viewing gender as being important for teaching and learning. For Question 21, *Curriculum developers are responsible for ensuring various genders are represented in the content curricula*, the post-test results ($Mdn = 5.5$, $s = .87$) in relation to the pre-test ($Mdn = 5.0$, $s = 1.11$) showed significant changes in scores demonstrating that PSTs placed more responsibility on curriculum developers to represent genders in the curriculum ($z = 2.14$, $p = .032$, r

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= .45). Question 27 asked PSTs to respond to: *It is my responsibility to ensure various faiths are represented in my teaching*. The Wilcoxon signed-rank test indicated a significant increase in PSTs assuming responsibility for representing different faiths in their teaching (post-test, *Mdn* = 6.0, *s* = 1.07; pre-test, *Mdn* = 5.0, *s* = 1.3) with *z* = 2.40, *p* = .017, *r* = .5. For Question 29, *Curriculum developer are responsible for ensuring various faiths are represented in the content curricula*, post-test (*Mdn* = 5.0, *s* = 1.3) compared to pre-test (*Mdn* = 4.0, *s* = 1.4) indicated that PSTs came to place more responsibility on curriculum developers to address diversity of faiths in the curriculum (*z* = 1.99, *p* = .047, *r* = .42).

In the second course, a required mathematics content class for elementary PSTs (n = 14, Spring 2015), we also observed some significant results on the TSR-MD (Silverman, 2009). However, all of these results should be approached with great caution, given the small sample size. For Question 7, *Various races need to be represented in teaching only if students from those races are present in the classroom*, responses on the post-test (*Mdn* = 4.0, *s* = 1.7) compared to pre-test (*Mdn* = 2.0, *s* =.1) were interesting as all of the changes (*n* = 8) were positive, with no negative changes and 6 ties in rank. PSTs seemed to register a belief that different races should be portrayed on the basis of who was in the class, which was counter to the goals of the class. Test results were significant, *z* = 2.54, *p* = .011, *r* = .68. Regarding Question 49, *In general, culture is unimportant to me*, test results were significant with *z* = 2.5, *p* = .01, *r* = .67. The post-test results (*Mdn* =2.0, *s* = 2.0) compared to the ranks of the pre-test (*Mdn* = 1.0, *s* =.61) indicated a change in how PSTs regarded culture in teaching. For Question 50, *My students' culture plays a role in my teaching*, post-test ranks (*Mdn* = 5.0, *s* =.96) compared to pre-test ranks (*Mdn* = 6.0, *s* = .5) showed a slight increase in the PSTs' understanding of their responsibility in including culture in their teaching (0 positive differences, 7 negative differences, and 6 ties) with a significant test result (*z* = -2.5, *p* = 0.14, *r* = -.67). For Question 53, *Curriculum developers are responsible for ensuring that various cultures are represented in the content curricula*, the post-test ranks (*Mdn* = 5.0, *s* = .83) compared to the pre-test (*Mdn* = 5.5, *s* = .65) indicated a slight increase in PSTs' beliefs that curriculum developers are responsible for including various cultures in the curriculum (*z* = -2.33, *p* = .02, *r* = -.63). (For Questions 75-89, the Likert responses were: 1 = Nothing, 2 = Very little, 3 = Some, 4 = Quite a bit, 5 = A great deal). On Question 75, *How much can you do to ensure diverse students learn about their own cultural heritage in your class?* the post-test ranks (*Mdn* = 4.0, *s* = .61) versus the pre-tests (*Mdn* = 4.0, *s* = .62) showed five positive differences, zero negative differences, and nine ties. The significance test was significant with *z* = 2.23, *p* = .03, and *r* = .60. Finally, for Question 76, *How much can you do to teach your students about cultural conflic*t? The post-test results (*Mdn* = 5.0, *s* = .76) compared to the pre-test results (*Mdn* = 4.0, *s* = .83) showed six positive differences, zero negative differences and eight ties, indicating a modest increase in PSTs assumption of responsibility for teaching students about cultural conflict. The test was significant, *z* = 2.33, *p* = .02, and *r* = .63.

The differences in the number and type of significant results for the two courses can be accounted for, in part, by the differences in the foci of the two courses and the emphases of the assignments within the courses, as explained below. However, we also have concerns that the wording of the TSR-MD may have sometimes confused the PSTs. In order to explore some of the results that are confounding given the goals of the courses, follow-up interviews to clarify PSTs answers would be helpful.

Qualitative. We used narrative inquiry as a research method (Creswell, 2008) to organize, present, and analyze the qualitative data, which involved a detailed examination of the field texts (e.g., course activities, projects) produced by the PSTs. In choosing this method, we "adopt a particular view of experience as phenomenon under study" (Connelly & Clandinin, 2006, p. 375). Like other qualitative research methods, well defined boundaries cease to exist in narrative inquiry. In our narratives, we consciously choose and present data that will best help us address the central goal of this paper. The two teacher education courses shape the contextual setting and the students

(PSTs) enrolled in these courses are an integral part of the study. We inquire into our practical and professional experiences grounded in this context, enact dual roles as teacher educators and researchers, and retell a story that unravels connections between theoretical ideologies and practical enactments. The story is structured around a chronology of events that include: a) course design and enactment; b) documentation of field texts; and c) our analyses and interpretation of field texts in relation to the chosen theoretical domain.

The course designs aimed to establish an equitable community of mathematics learners as they explored ways to better understand mathematics and the potential it holds to explore social justice issues. The key course goals were to help PSTs: a) begin to build a critical understanding of the world; b) enlist mathematical tools that support and promote critical understanding of social issues; and c) understand that a global perspective helps us to know ourselves, our community, and our fellow human beings. As members of a community of learners, PSTs were required to engage in class activities, listen to and learn from their colleagues, and respect the opinions of other participants. To establish specific connections to critical mathematics education, participants were asked to think more deeply about the types of tasks that are usually presented in mathematics curriculum; to attend to and explore the connections between a given mathematical task and the local and global contexts. Course content was delivered through practical and research components. As part of the practical component, each week, PSTs participated in at least one content exploration activity set in a social, cultural, and political context. As part of the research component, PSTs were required to identify and investigate a social practice and establish connections to school mathematical topics.

We highlight two course activities (field texts) collected from PSTs work from the two courses. The first activity, required participants to use the Gapminder software tool (www.gapminder.org) and thus help them probe their beliefs and perceptions of the developed and the developing nations of the world. This activity enabled participants to uncover, understand, and question their perceptions about various nations. In particular, they were asked to complete the following tasks:

1. Identify a research question.
2. Explore: a) Describe how the data was analyzed using the Gapminder software.
   b) Interpret the graphs and tell the story. c) Discuss plausible reasons that will explain the trends that were noted in the graph.
3. Reflect: In light of the presented data and its analysis, reflect on the research question.
   a) Suggest possible actions that could be initiated by various agencies to address the global issue. b) As a citizen of the world, how do the findings this concern you? What can you do about it? c) Discuss ways in which prospective teachers can engage students in a deeper analysis of global or local issues using the lens of statistics and through the use of the Gapminder software.

We provide one PST's response here.

PST Research Question: What is it that causes some countries to have such a high child mortality rate, when others don't even have to think about the possibility of their children not making it? I decided to analyze three countries from different locations all over the world: the United States of America, Colombia, and Botswana; all three are on different continents with different ways of life, and overall very diverse economic levels. I was curious to find why with the modern medicine we have is there still a reason to have any child mortality at all.

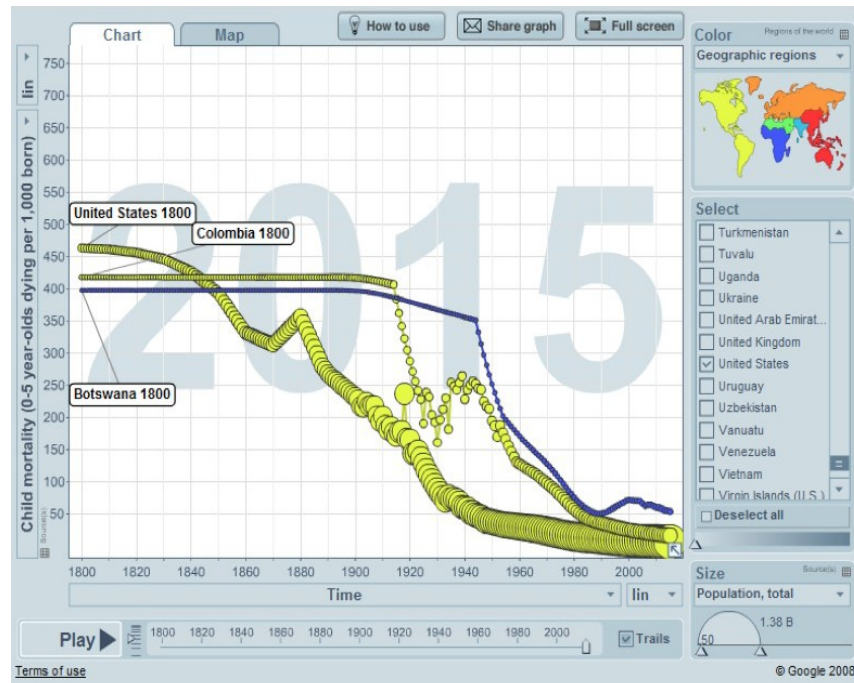Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

Figure 1. Gapminder Graph: Children Mortality -USA, Colombia and Botswana.

Interpretation: After looking at all three graphs and countries I've learned a lot about what child mortality means in relation to the world around it. It's so interconnected with the rest of the countries issues, no matter if it's what society thinks of those around them or major issues such as the government changing power or parties, this is always being affected, like a ripple effect. It was horrifying to see the starting numbers and think about what those parents had to go through if their child played a part in that statistic, but I did find it comforting to see that even the countries I thought would still be very bad have improved so very drastically. I did catch myself losing touch with what the statistic meant though, sometimes when I would write out the numbers it was like I was almost desensitized with what that number was saying. The gravity of children dying is intense and I think sometimes when we're shown so many numbers like that it's hard to remember what they are truly portraying to us.

Reflection: I think that national agencies could initiate programs educating not only the adults of these countries but the children on health and medicine and prevention. This would help ensure that when these children grow up they know even more about how to help their own children, because if no one learns how to prevent it, these health care issues or diseases will evolve till they can't be controlled at all. I also believe that any action we do or someone does to help these people, needs to be done with these people, they aren't less than us or weaker, they are equals to those helping them.

As a citizen of the world these findings concern me that not all countries have access to the medicine, healthcare, or even knowledge about these issues, and for some to have had them for so long and not reached out to help makes my stomach hurt. ... I understand that the whole situation is more complicated than that involving the governments but I think that anyone can relate to seeing children and the thought of them not making it till they are 5, not even starting kindergarten, is horrifying.

I believe that teachers could integrate things like Gapminder into the classroom in many ways. It's a great way for children and students to start seeing what these numbers actually are and how

they compare to what we have as Americans. It could definitely show them what they take for granted everyday but I also believe in order for them to truly dive deeper into what this mean and talk about the statistics in a local, national, and global level, they need to fully understand what each means and that it isn't just a number, we can't let them desensitize themselves to what they are seeing or they will never understand the weight that these numbers hold.

Through the Gapminder exploration, Skovsmose' criteria of subjective and objective problem posing related to critical mathematics pedagogy are engaged. Students were able to define a problem, that they deemed important to society, and then utilize unbiased information provided through Gapminder to identify issues that influence the overall trends depicted in data.

An activity from the second course required participants to select a topic and complete a project emphasizing that mathematics is a human activity. PSTs identified a personally meaningful everyday activity, collaborated with individuals who were insiders to this activity, and reflected on questions such as: What counts as knowledge in school mathematics? Who are viewed as experts of knowledge? Whose mathematical expertise is valued? Project examples of some projects include firefighter's mathematics, lunchroom mathematics, mathematics of curve stitching, mathematics of a seamstress, and mathematics of a construction worker.

We provide a brief overview of one PST's research project titled *My mother, a lunch lady and our mathematics*. Sam, a prospective middle school teacher, chose her mother, Pam, as her inspiration and resource. Pam has worked in the school lunch industry for seven years and has progressed from an assistant cook, to head cook at a private school, and currently is the assistant cook at a public elementary school. Her role is to help develop lunches on a monthly basis, help prepare the daily lunches, and help serve the food as the students arrive. In collaboration with her mother, Sam developed a mathematical modeling task situated in the school lunchroom context.

Ms. Pam is a head cook in a school cafeteria. She must follow a set of guidelines to ensure that students receive the right amount of each food group per day. There is also a regulation that lunches must be within a certain calorie range.

4. Create a weekly lunch plan that Ms. Pam would approve.
5. On a given day, how much of each food type would Ms. Pam have to prepare?

Sam enacted this activity with a group of 5th graders. We present 5th graders' work specific to question (1) and Sam's commentary. After beginning work on the assigned task, students realized that they needed additional information regarding nutritional and serving size regulations, which Sam provided. Figure 2 captures students' initial responses to the two questions (first model).

Figure 2. Lunch Menu - Trial One.

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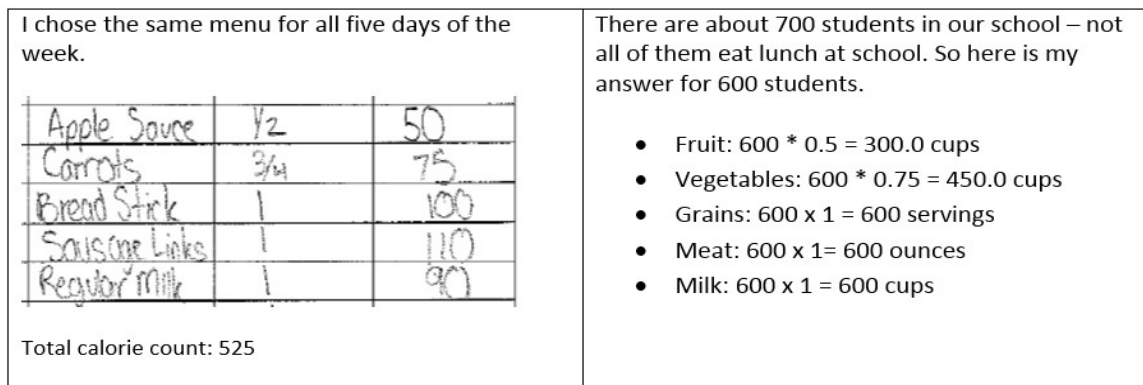
Upon a review of this work, Pam noted that this simplistic plan neither fulfilled the nutritional requirements nor the recommended serving size regulations. She challenged Sam and the students to experiment with different menu options, interpret their solutions in the context of the problem, and to be explicit in communicating their thinking. Students created another model, this time using five different menu options, one for each day of the week (see Fig. 3).

Food type	Fruit			Veg			Grains			Meat			Fluid			Total Calorie count
	Name	SS	CC	Name	SS	CC	Name	SS	CC	Name	SS	CC	Name	SS	CC	
Mon	Apple sauce	1/2	50	Carrot	3/4	75	Bread	1	100	Sausage	1	110	Reg. Milk	1	90	425
Tue	Banana	½	75	Peas	3/4	60	Roll	1	65	Ham burger	1	185	Choc. Milk	1	90	570
Wed	Grapes	½	60	Broccoli	3/4	80	Bun	1	150	Grilled chicken	1	120	Str. milk	1	90	500
Thu	Orange	½	45	Green beans	3/4	65	Hotdo g bun	1	125	Breaded chicken	1	160	Reg. milk	1	90	485
Fri	Apple juice	½	45	Green beans	3/4	65	Bread	1	70	Ham burger	1	185	Choc. milk	1	90	455
Weekly serving	2 1/2		Weekly serving	3 3/4		Weekly serving	5		Weekly serving	5		Weekly serving	5			

Figure 3. Lunch Menu - Trial Two.

However, as they revisited the task guidelines, students realized that Ms. Pam would not approve their lunch menu. They realized that they still had not attended to the nutritional requirements and the serving size regulations. In the students' next iteration, they made three changes: a) Make changes to the grain / meat item so as to bring the serving size up; b) Make no changes to the milk/ veg. items since they conform to the serving size regulation (if possible, add fruit); and c) All changes must increase the calorie count but not exceed 700. Students realized that a revised model with these changes would certainly get Ms. Pam's approval. At this point in her project work, Sam began to notice how her mother emerged as a prominent figure in this discussion. Sam began questioning her initial beliefs about who can be considered as sources of mathematical knowledge. She began to understand the meaning of learning as, "a truthful collaboration in which all parties come both as learners and as resource" (2, 1998). Sam noted, "Through my interactions with my mother, I learned the lunchroom is full of hidden mathematical concepts. She helped me realize the potential for using them to design math activities." Sam also began to realize that while her mother did do meaningful mathematics, and appeared to be very confident about her knowledge of her practice, "she seemed to dismiss this knowledge as if there was nothing to it." Sam began to see that this dismissal was not warranted. Sam could instead see that her mother, a working class lady, was able to build on her lived experiences, to create a mathematical task that challenged the students. Sam explained, "Upon reflection, I see that I have been oblivious to my own mother's expertise because I truly did not believe that she is a source of knowledge. I have grown confident in teaching modeling to children as I no longer believe that these are abstract concepts for only a select group of students."

Implications for Mathematics Teacher Educators

Both the quantitative and qualitative analyses indicate preliminary results that suggest that through a deliberate scaffolding of course activities and projects, mathematics educators can help PSTs learn to appreciate individuals within their own local communities as practitioners of real mathematics and can leverage the expertise of these individuals in their own classrooms. This may

build confidence within diverse groups of students that they, too, can engage in meaningful mathematics and can build self-confidence within the community about what they have to offer students, mathematically speaking. The results also speak to the power of using sources of real-world data, such as Gapminder, to help PSTs learn to use mathematical tools to explore problems of a global nature and make data-driven recommendations to help solve them. The results suggest that there are concrete ways that we can help PSTs learn to use mathematics to act locally and globally and become willing to assume responsibility for doing so.

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GREATER NUMBER OF LARGER PIECES: A STRATEGY TO PROMOTE PROSPECTIVE TEACHERS' FRACTION NUMBER SENSE DEVELOPMENT

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Prospective teachers (PTs) need opportunities to develop fraction number sense, yet little research has explicated how this development occurs. Our research team collaboratively designed a task targeted at helping PTs develop fraction number sense through an exploration of fraction comparison strategies. This paper focuses on developing one particular strategy, which we call Greater Number of Larger Pieces (GLP). We argue that understanding this strategy has the potential to support PTs' number sense, particularly in regards to the measure interpretation of fractions. Analysis of data from two iterations of this task (implemented by five mathematics teacher educators at five US institutions with 124 PTs) showed an improvement in the task's ability to naturally elicit the GLP strategy from PTs. We share our task, results from each iteration, and discuss modifications that we believe led to increased usage of the GLP strategy.

Keywords: Teacher Education-Preservice, Mathematical Knowledge for Teaching, Rational Numbers, Teacher Knowledge

Introduction and Background Information

Researchers have argued that number sense is an important part of the mathematical knowledge needed for teaching (Ball, Thames, & Phelps, 2008; Tsao, 2005), and thus, its development should be an integral component of the mathematical preparation of prospective teachers (PTs). Given the prevalence of fraction topics across the K-8 mathematics curriculum, we focus our work on the development of fraction number sense, which Lamon (2012) defines as "an intuition that helps [students] make appropriate connections, determine size, order, and equivalence, and judge whether answers are or are not reasonable" (p. 136). Lamon argues that this intuition is especially important for teachers to develop, as they will need it to evaluate the appropriateness of student reasoning. Yet research shows that PTs often exhibit particularly weak and procedurally-oriented thinking in terms of fractions (Tobias et al., 2014; Yang, Reys, & Reys, 2009).

In order to work proficiently with fractions, students and teachers should be familiar with a variety of fraction interpretations, rather than focusing solely on the traditionally-taught *part-whole* interpretation (Kieren, 1976; Lamon, 2012; Thompson & Saldanha, 2003). Watanabe (2007) states that "when students' understanding of fractions is limited to the part-whole meaning, it is doubtful that they understand fractions as numbers" (p. 57). Busi and colleagues (2015) note, for example, that considering 3/5 as three parts out of five can lead to a non-sensical interpretation of improper fractions, such as 7/5, as seven parts out of five.

Recommendations in the *Common Core State Standards for Mathematics* (CCSSM) in the United States (National Governors Association [NGA] & Council of Chief State School Officers [CCSSO], 2010), and throughout international curricula (Son, Lo, & Watanabe, 2015; Watanabe, 2006; 2007), suggest that viewing a fraction as a measure can help overcome some of the limitations of the *part-whole* interpretation. This view is exemplified by the following third-grade CCSSM content standard (3.NF.A.1): *Understand a fraction 1/*b *as the quantity formed by 1 part when a*

whole is partitioned into b *equal parts; understand a fraction* a/b *as the quantity formed by* a *parts of size 1/*b. The *fraction-as-a-measure* interpretation can support children's development of fraction addition and subtraction knowledge (Son, Lo, Watanabe, 2015) by helping students recognize how adding fractions with like denominators is adding additional iterations of the unit fractions that comprise the original fractions. For example, if one sees 3/7 + 2/7 as 3 pieces of 1/7 and 2 pieces of 1/7, then the combined result of 5 pieces of 1/7, or 5/7, makes sense. Such understanding could help students avoid a common error of adding across the numerators and denominators, i.e., 3/7 + 2/7 = 5/14 (Mack, 1995). These ideas can then be extended to the addition and subtraction of fractions with unlike denominators (McNamara, 2015).

A Task to Develop Fraction Number Sense

This paper reports on our efforts to help PTs develop fraction number sense, including the ability to interpret fractions as measures, through the study of comparing and ordering fractions. A group of six mathematics teacher educators developed and enacted a fraction comparison task, based on a task designed for fifth-graders, in their mathematics content courses for PTs (Tobias et al., 2014). Our goal was to help PTs shift their perspectives on fractions from a *part-whole* to a *measure* interpretation, and in doing so, begin to see fractions as numbers. Research and policy recommendations highlight the importance of providing learners with repeated opportunities to grapple with problems and generate their own solution strategies instead of applying a strategy made explicit by an instructor (Conference Board of Mathematical Sciences [CBMS], 2012; Hiebert & Wearne, 1993; Stein, Grover, & Henningsen, 1996). To this end, we created a task consisting of ten fraction comparison problems designed to help PTs develop understandings of several fraction comparison strategies beyond *common denominators*, the strategy with which they are traditionally most familiar (Olanoff, Lo, & Tobias, 2014).

Although PTs were given the freedom to construct their own fraction comparison strategies, the task was designed to support the development of several predetermined fraction comparison strategies identified in the literature, specifically: *Same Size Pieces* (SSP; also known as common denominators), *Same Number of Pieces* (SNP; also known as common numerators), and *Comparing to a Benchmark Value* (BV) (Lamon, 2012; NGA & CCSSO, 2010). Given our goal of helping teachers develop understandings of fractions as measures, we sought to introduce an additional strategy that we refer to as *Greater Number of Larger Pieces* (GLP). In this strategy, one uses the measure interpretation of fractions to consider each fraction as a certain number of equal-sized pieces. For example, when comparing 18/25 to 16/27, one can interpret 18/25 as eighteen fractional pieces each of size 1/25 and 16/27 as sixteen pieces each of size 1/27. Since pieces of size 1/25 are larger than those of size 1/27 and there is a greater number of them (18 > 16), one can conclude that 18/25 > 16/27. The use of the GLP strategy requires the simultaneous coordination of two quantities - one referring to the *number* of fractional pieces and one referring to the *size* of those pieces.

Below, we describe the implementation of two iterations of our fraction comparison task in mathematics content courses for PTs. We present data analysis focused on PTs' abilities to successfully implement the GLP strategy to solve individual fraction comparison problems and the task's ability to elicit the strategy from PTs. We will discuss GLP-related modifications that were made to the first version of the task based on our data analysis and the effects of those changes. For a more detailed explanation of the task design process, data analysis, and findings pertaining to the task as a whole, see Thanheiser et al. (2016).

Task Version 1

The first version of our task included ten fraction comparison problems designed to elicit our four targeted fraction comparison strategies noted above: SSP, SNP, BV, and GLP. Some problems also required finding equivalent fractions (EF) in conjunction with one of the four strategies. PTs

were instructed to circle the larger fraction and to provide a sense-making justification for their choice. Table 1 presents each fraction comparison problem, as well as the strategy we intended for each problem to elicit.

Table 1. Task Version 1: Ten fraction comparison problems with intended strategies (bracketed numerals following BV indicate intended benchmark values). Underlining indicates the greater fraction.

Problem	Fractions to compare	Intended strategy		Problem	Fractions to compare	Intended strategy
#1	1/2 vs. 17/31	BV [½], EF-SSP, or EF-SNP		**#6**	13/15 vs. 17/19	BV [1]
#2	2/17 vs. 2/19	SNP		**#7**	5/6 vs. 6/5	BV [1]
#3	4/7 vs. 9/14	EF-SSP		**#8**	7/10 vs. 8/9	GLP
#4	3/7 vs. 6/11	BV [½] or EF-SNP		**#9**	1/4 vs. 25/99	BV [¼] or EF-SNP
#5	8/9 vs. 12/13	BV [1]		**#10**	24/7 vs. 34/15	BV [3]

The data analysis included in this paper will focus on problem #8 (7/10 vs. 8/9) since this was the only problem intended to elicit the GLP strategy.

Task Implementation and Data Collection

We launched the task above by asking PTs to list everything they knew about 7/8. We also gave them the following two prompts to work on in small groups:

1. Keeping the denominator the same, find 3 fractions that are greater than 7/8, and 3 fractions that are less than 7/8.
2. Keeping the numerator the same, find 3 fractions that are greater than 7/8, and 3 fractions that are less than 7/8.

Following small group work, each instructor facilitated a brief whole-class discussion in which PTs articulated and justified their thinking.

One of the primary goals of the launch activity was to help PTs begin the transition towards interpreting fractions as measures. For example, many PTs' responses focused on the idea of 7/8 meaning 7 out of 8 parts, which gave instructors the opportunity to highlight additional interpretations, such as 7 pieces of size ⅛. Furthermore, some instructors asked PTs to justify their answers to the prompts above to solidify the relationship between fractions that have the same denominators and those with same numerators.

Following the launch, PTs were given handouts containing the ten comparison problems and instructed to work on the problems either individually or in small groups, without the use of calculators. PTs were encouraged to think beyond using the SSP strategy and apply their understanding of fractions as numbers to find alternative ways of determining the larger fraction in each pair. Instructors allowed the PTs to work on the task for 30-60 minutes before bringing them all together for a class discussion on their solutions and strategies. PTs' written work on the task was

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collected and copied before the whole group discussion where individual strategies were explicated. This student work on the task prior to the whole class discussion comprises the data for this paper.

Data Analysis and Results - Round 1

Version 1 of our task was implemented in the spring of 2013 by three authors with 61 PTs in four mathematics content courses for elementary PTs across three universities. PTs' written work was analyzed for correctness of solutions, strategies applied, and quality of explanations. While this task proved successful at helping PTs elicit many of the intended fraction comparison strategies, it was not particularly successful at eliciting GLP. In fact, while 51 of the 52 (98%) PTs who answered problem #8 (7/10 vs. 8/9) were able to correctly identify the greater fraction, only three responses (6%) used the intended (GLP) strategy. An additional 10 responses (19%) included the use of valid strategies, such as finding common denominators or converting the fractions to decimals or percents. However, we found that the remaining 75% of responses offered incorrect or incomplete reasoning, e.g., claiming that 8/9 > 7/10 because 8/9 is "close to 1," while 7/10 is "3 pieces away." While this line of thinking does leverage some intuition about a fraction's magnitude, it does not provide a reasoned explanation for how one knows that 8/9 is closer to 1 than 7/10. Moreover, it does not attend to the *fraction-as-a-measure* interpretation, which is a necessary component of the GLP strategy. Additionally, data revealed that the three PTs who developed GLP were in only two of the four classes; thus the eventual presentation of the strategy in the other two classes had to come from the instructors, as opposed to the knowledge being constructed and shared by the learners.

These results led to us hypothesizing two reasons for problem #8's inability to aid in the natural emergence of the GLP strategy. First, given the rate at which algorithmic procedures were successfully applied, we believe our choice of fractions (7/10 and 8/9) did not compel PTs to reason about the number and size of pieces. It appears that PTs were familiar enough with the chosen fractions to use their intuition about their magnitude to determine the larger fraction. Many PTs applied their understanding of 7/10 = 0.7 or 70% and 8/9's "closeness to 1" to justify that 8/9 > 7/10. It seemed that the PTs did not see a need for additional reasoning or a more robust justification. Second, we believe that the GLP strategy may be more difficult to apply than other fraction comparison strategies (such as SNP) because it requires the simultaneous interpretation and coordination of both the numerators and denominators.

Task Modifications

Since the first version of our task was not as successful in eliciting the GLP strategy as we had hoped, we added three new comparison problems to the task with the goal of providing more opportunities for PTs to think about and develop GLP on their own. First, we added 2/9 vs ⅜ (problem #14), which can be solved using a variety of strategies including the GLP strategy. Eliciting multiple solution strategies is known to be an important characteristic of effective tasks (Stein, Grover, & Henningsen, 1996). Second, we added 2/7 vs ⅜ (problem #11), which is purposefully similar to 2/9 vs ⅜, but cannot be solved using GLP. We added this problem to provide an opportunity for PTs to determine when GLP is and is not applicable, as knowing when a particular strategy is and is not appropriate to use can further deepen PTs' understanding of the strategy (Borich, 2011). Third, we added 18/25 vs 16/27 (problem #15), specifically choosing fractions with larger numerators and larger, relatively prime, denominators to deter PTs from using computation-heavy strategies (e.g., common denominators or common numerators), and instead look for more efficient strategies. A total of five new items were added to the task, resulting in a second iteration containing 15 problems. Problem #8, the original GLP problem (7/10 vs. 8/9), was left unmodified.

We also revised our implementation of the launch activity to better elicit the *fraction-as-a-measure* interpretation. Instructors spent more time discussing PTs' explanations of 7/8 and pressing PTs for complete "sense-making" explanations for why certain fractions were greater than or less

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than 7/8. As one instructor noted in her instructor memo, "I really took my time on this and pressed and probed a lot more than I did last spring" (Hillen, instructor memo). For example, explanations such as "7/9 is farther away from 1 than 7/8" were no longer deemed satisfactory; instead, instructors pushed PTs to explain how they knew this was true. When pressed to explain why 7/8 > 7/9, PTs began to recognize that both fractions have the same number of fractional pieces (7), but since eighths are larger than ninths, 7/8 must be greater than 7/9. Similarly, when explaining why 9/8 > 7/8, PTs recognized that 9/8 has a greater number of eighths. In this way, the launch provided opportunities for PTs to begin thinking about fractions as measures.

Data Analysis and Results - Round 2

The second iteration of the task was implemented during the fall of 2013 by four authors (two of whom were involved in the first implementation) in classrooms at four US institutions with a total of 63 PTs. Compared to the 3 GLP-related responses we received on the first version of the task (6% of PTs who answered #8), nearly 15% of the 61 PTs who answered problem #8 on the second version correctly applied the GLP strategy. PTs' success rates were even greater for the two newly-added GLP comparison problems #14 (2/9 vs ⅜) and #15 (18/25 vs 16/27). Of the 52 PTs who answered #14, over 21% used GLP reasoning; and, while only 46 PTs answered #15, the last item on the task, over 41% of them applied GLP.

Table 2. A comparison of success rates for PTs' application of the GLP strategy on the first and second iterations of the task. (N=61 for Iteration 1; N=63 for Iteration 2)

Iteration	Problem #	GLP Problem	# of PTs who answered the question	% of responses received with correct answers	% of responses received using GLP
1	#8	7/10 vs 8/9	55	98.0%	6.0%
2	#8	7/10 vs 8/9	61	96.7%	14.8%
2	#14	2/9 vs 3/8	52	96.2%	21.2%
2	#15	18/25 vs 16/27	46	95.7%	41.3%

It should be noted that Problems #14 and #15 were the last two comparison problems on the 15-item revised task, so low completion rates may be related to PTs not having enough time to attempt them. However, it might also be that PTs intentionally left these items blank because they were unsure about how to approach them. Regardless, we believe that the major increase in the application of GLP on #15 was at least partly due to the problem being specifically designed to discourage other comparison strategies. With the comparatively large numerator and denominator values, and the selection of relatively prime denominators, the use of the common denominator and common numerator strategies becomes quite tedious, especially without the use of the calculator. As such, it appears that PTs were more compelled to seek alternative strategies for solving #15 than #8 or #14, and ended up applying the *fraction-as-a-measure* interpretation by considering each fraction as a number of equal-sized pieces.

Discussion and Conclusions

The data provide evidence that the second iteration of the task was more successful in eliciting the GLP strategy. Out of the 63 PTs who worked on the second iteration of our task, 21 of them (33%) used GLP on at least one problem, 19% used it on two, and 8% used it on all three of the

applicable problems. These findings are in contrast to the first task iteration, during which only 5% of PTs who worked on the task attempted to use GLP. Additionally, at least two PTs in each of the four classes discovered the GLP strategy on their own, which provided an opportunity for PTs to lead discussions about the strategy with their classmates.

One task modification strategy that has potential for supporting PT learning is creating problems that lend themselves to particular solution strategies while discouraging the use of alternate strategies. Problem #15 (18/25 vs 16/27) is an example of such a problem; PTs were most successful in correctly using the GLP strategy here. Although the benefits of constructing mathematical tasks that elicit multiple strategies are well known (Stein, Grover, & Henningsen, 1996), our work suggests a nuanced approach to task design. In order to elicit particular ideas or strategies, there may be a benefit to constructing some problems that narrow the field of possible solution strategies to those under investigation. In this way, PTs are forced to abandon certain strategies that may not support reasoning and sense making.

Nevertheless, even for problem #15 the majority (58.7%) of the 46 PTs who answered the question opted to use alternative fraction comparison strategies including applying common denominators or common numerators, and comparing the distance of the fractions from ½ or 1. Five PTs (11%) correctly identified the larger fraction, but gave no explanation for their answers. It is interesting to note that seven PTs (15%) showed evidence of some GLP-related thinking, but either their thinking was incorrect or their responses were too incomplete to conclude that the GLP strategy had been successfully applied. For example, some PTs recognized that pieces of size 1/25 are larger than pieces of size 1/27, but they did not attend to the number of fractional pieces in either fraction. Others recognized that 18/25 has a greater number of fractional pieces than 16/27, but did not address the size of those pieces. This result supports the contention that the GLP strategy is challenging for PTs because it requires simultaneous coordination of both the number (numerator) and size (denominator) of the fractional pieces in each fraction being compared.

On a positive note, follow-up analysis suggests that this task can serve as a useful introduction to the GLP strategy. Data from final exams in three of the four classes showed that 78.6% (44 out of 56) of the PTs were able to correctly use a measure interpretation of fractions to justify why the GLP strategy cannot be used to compare fractions such as 27/29 and 31/33.

The evolution of PTs' fraction number sense is critical to their development of mathematical knowledge needed for teaching (Lamon, 2012). While this idea is clearly detailed in the literature, there is little research pertaining to *how* teacher educators can help PTs develop this knowledge. The results of our study provide support for the use of the GLP strategy as one way in which teacher educators can begin to facilitate PTs' fraction number sense. We also provide task design suggestions for eliciting the GLP strategy. Though we found more PTs successful on the second iteration of the task, we recognize that the GLP strategy is complex and may not be learned easily. Thus, future research will be needed to determine additional ways to increase the number of PTs that develop the GLP strategy as well as designing, implementing, and analyzing tasks that will encourage PTs to apply their fraction number sense to develop conceptual and sense-making methods for operating with fractions.

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PERSPECTIVES ON TEACHING MATHEMATICS AND SCIENCE IN HISTORICAL AND CULTURAL CONTEXTS

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This qualitative and descriptive study examines the evolution of secondary preservice teachers' views on teaching and learning mathematics and science in historical and cultural contexts. Data were collected throughout participants' enrollment in a semester-long course entitled, Perspectives on Science and Mathematics, which is taken in conjunction with student teaching. Data sources included university classroom observations and field notes as well as preservice teachers' verbal and written responses to class discussions, reading assignments, and course activities. Common themes and categories of response were derived from the triangulation of data to include prospective teachers' critical reflections on teaching and learning. The paper ends with a discussion of findings and concluding remarks.

Keywords: Teacher Education-Preservice

Introduction

Mathematics and science are dynamic studies of patterns and relationships that evolve as people participate in and contribute to cultural and historical activities (Wheatley & Reynolds, 1999). "It is estimated that 95% of mathematics known today has been produced since 1900" (Berlinghoff & Gouvea, 2004, p. 53). Since mathematics and science are connected to people's culture and history, many educators are convinced that the teaching and learning of these disciplines can be made more relevant and meaningful to learners if embedded in context. As a consequence, increased attention has lately been given the study of ethnomathematics and its role in helping to clarify the nature and development of mathematical knowledge (Ascher, 1994; Bishop, 1991; D'ambrosio, 2001; Frankenstein, 1995; Nunez, 1992; Orey & Rosa, 2001).

Broadly defined, ethnomathematics involves the integrated study of relationships among mathematics, culture, and history. Its classroom application is predicated on the belief that students' understanding and appreciation of mathematics and science will be enhanced when content is presented in relevant contexts. Support for this idea stems from research which indicates that brain evolution, cultural interaction, and communication all play a major role in the development of mathematical and scientific knowing and understanding (Lakoff & Nunez, 2000; Lave & Wenger,1991; Lerman, 2000).

Presumably, teachers' understanding of what comprises mathematics and science may exert a strong influence on the ways in which they think, reflect, plan, teach, and provide opportunities for their students to learn these subjects. The National Council of Teachers of Mathematics (NCTM, 2000, 1991, 1989) holds that mathematics and science are something people do. They have broad contents encompassing many fields that need to be at the command of all students in a technological society. "Mathematics [and science] are the greatest cultural and intellectual achievements of human kind, and citizens should develop an appreciation and understanding of that achievement" (NCTM, 2000, p. 4).

Methodology

This qualitative and descriptive study is guided by constructivist inquiry (Guba & Lincoln, 1989, 1994; Lincoln & Guba, 1985; McCracken, 1988). In this sense, the study is context specific (i.e., preservice secondary mathematics and science teachers' views on teaching and learning mathematic and scientific content as embedded in the historical and cultural context of a state-supported, urban,

university located in Midwestern America). Data sources include university classroom observations and field notes as well as preservice teachers' verbal and written responses to class discussions, reading assignments, and course activities. Data collection and data analysis occurred simultaneously throughout a semester-long course that aims to provide an overview of the history of science and mathematics and to enable future teachers to enact these historical perspectives and contexts throughout their pedagogy. Based on emergent patterns and themes, several factors were identified as being important considerations in promoting the integration of historical and cultural contexts when planning for and delivering mathematics and science instruction.

All participants in this study were enrolled in a course entitled *Perspectives on Science and Mathematics.* Taken concurrently with student teaching, the design and content of this course has been influenced by work of two curriculum theorists—Doll's (1993) *A Post-Modern Perspective on Curriculum* and Grundy's (1989) *Curriculum: Product or Praxis*. Briefly, Doll (1993) asserts that "curriculum is a process, not of transmitting what is absolutely known, but of exploring what is unknown; and through exploration students and teacher "clear the land" together, thereby transforming both the land and themselves" (p.155). Further, he continues by saying, "a constructive curriculum is one that emerges through the action and interaction of the participants; it is not one set in advance, except in broad and general terms (p.262). Similarly, Grundy (1989) maintains that

> Curriculum is a cultural construction. It is not an abstract concept which has some existence outside and prior to human experience. Rather, it is a way of organizing a set of human educational practices (p. 5).

From these two compatible perspectives, course assignments and activities were developed with the qualified hope that preservice teachers would make their own roads by walking them (Horton & Freire, 1990)—autonomously finding creative and meaningful ways to design, develop, and implement lesson plans based on the pedagogical premises of ethnomathematics.

Importance of Critical Reflections for Teaching and Learning

Reflection is an important tool in the repertoire of any good teacher. The power and influence of critically reflecting on reading assignments, classroom discussions, and activities cannot be underestimated. As the preservice teachers who participated in this study delved into critical reflections and expressed their thoughts and views openly and freely, they became more observant of their own practice. They made what were previously obscure connections more visible and tangible. In addition, by emancipating themselves from being evaluated by the instructor or by their peers, they were able to deconstruct their previous views and reconstruct their new ideas. For example, in a final written reflection, one preservice teacher noted,

> Without the critical reflections, I may have never noticed the issues of my teaching approach or found solutions to problems that I will most likely face in the future. In other words, these reflections have helped me plan ahead for my career and have been invaluable.

Thinking critically, not only on readings and class discussions, but also on lesson planning and presentations, played a vital role in enhancing teaching performance.

All participating preservice teachers agreed that if implemented carefully, thoughtfully, and purposefully, high school students would benefit from learning mathematics and science content from historical and cultural perspectives. One such benefit mentioned by the preservice teachers was student motivation to learn. Another benefit mentioned by interns was the opportunity such study provides for students to celebrate their respective cultures' contributions to science and mathematics. Take for instance this preservice teacher's comment:

In my Algebra I class, my kids asked "who came up with this, anyways?" and I had the chance to tell them a brief history of Algebra, and how it was an Arabic invention. One of my students who happened to be Arabic proudly claimed "no wonder I'm so good at it.

Another preservice teacher had this to say about how a mini-lesson she presented on history and culture to students who were struggling with mathematical notations and expressions increased their interest and persistence in problem solving: "By having this kind of information readily available when students show curiosity, I was able to get them engaged fairly quickly, and they surprised themselves by doing math that they thought was way beyond them." In short, preservice teachers were convinced that teaching mathematics and science in cultural and/or historic context would engage and deepen their students' understanding as they begin to perceive mathematics and science as a living, changing body of contributions and ideas. Preservice teachers felt that teaching content in context also led to an enhanced sense of self-efficacy as their students began to believe that they could overcome obstacles in solving problems just as mathematicians and scientists have done in the past.

Teachers' Realizations

There were three emerging realizations expressed by preservice teachers throughout the semester, particularly during classroom discussions and dialogues. These three realizations were 1) the marginalization of the mathematic and scientific contributions of non-western, non-white, and non-male individuals, 2) the importance of acquiring the knowledge and background needed to teach mathematics and science from cultural and historical perspectives, and 3) the need for an epistemological change in the teaching and learning science and mathematics.

The notion of how marginalized non-western, non-white, and non-male contributions to mathematics and science are in western society emerged several times during class discussions and dialogues. As one preservice teacher commented during a class discussion:

One realization I made is that young women do need particular attention when teaching math and science. I found that it makes a difference to them if they feel included in the subjects. For example, scientists such as Marie Curie and Rosalind Franklin contributed so much to our current understanding of science, and it is a disservice to young women to ignore contributions of women in a high school science classroom.

In support of this perspective, still another preservice teacher stated: "The most prominent realization is that mathematics and science did not start from Greece. If we look around carefully for evidence, we find contributions of many people from many cultures." In short, as the course progressed, the preservice teachers began to appreciate how much a historical and a cultural context can play in understanding and appreciating scientific advances.

The second realization mentioned by preservice teachers during class discussions was that they personally had no recollection of ever studying mathematics or science from a historical or cultural perspective; albeit, everyone agreed that it would have been useful to do so. Generally speaking, teachers teach the way they were taught. Hence, for lack of exposure and insufficient background knowledge, most shy away from including history and culture in their teaching of mathematics and science. This lack of knowledge limits learning opportunities for all students. Even when teachers do attempt to provide a historical context in their lesson planning and implementation, it is often pseudo- or quasi-history (Matthews, 2014). Unfortunately, this is the extent to which most teachers go when incorporating history and culture in their teaching. Hence, teachers' lack of existing knowledge was repeatedly identified as a major impediment to using the history and culture of as a means of engaging students in mathematics and science. On the other hand, as one preservice teacher

observed, "Anybody can put together a lesson plan, but it's the teachers who take the time to understand the background behind their lessons that are the most passionate about what they teach."

The third realization mentioned by several preservice teachers was a noted change in their own epistemological beliefs to include constructivism as a way of knowing and understanding mathematics and science. As one preservice teacher noted, "I believe strongly in a constructivist, problem-based, inquiry approach to teaching science. I find that adding historical and cultural components to my lessons will only serve to strengthen student comprehension and growth, as it brings command, application, and self-efficacy of student-centered learning to new heights." Constructivism asserts that understanding is an activity of the learner. It places the learner at the center of the activity and the teacher as a facilitator of learning. Thus, understanding is built by the learner from his/her experiences as they participate in and contribute to classroom activities in mathematics or science.

> When I first had the idea of teaching mathematics, I was set on teaching the basics, formulas, and everything else that math has to offer. I was very narrow-minded in that regard because that is how I was taught throughout my academic career. But after taking many educational classes, specifically this one, my mind has broadened. (Preservice Teachers Classroom Presentation)

From a constructivist perspective, many preservice teachers mentioned obstacles they faced as they tried to integrate history and culture into their lessons. Overcoming these obstacles certainly requires creativity and a change in epistemology. All preservice teachers mentioned that they needed to gain additional knowledge and experience in order to successfully integrate history and culture into their teaching of mathematics and science. They further recognized this as a virtuous goal that they have not fully achieved.

Challenges of Teaching from Cultural and Historical Perspectives

The preservice teachers who participated in this study expressed a variety of concerns, struggles, obstacles, and questions relative to teaching mathematics and science from historical and cultural perspectives. One such issue was the time it takes to prepare and implement lessons from these perspectives. As summarized by one student during a class discussion:

> I am still struggling with how to incorporate historical and cultural perspectives in a lesson plan, when I do not even have time to present the bare minimum in terms of mandated standards set before us. I feel as if I am in a constant state of 'catching up' when it comes to teaching a particular lesson.

Still another time-related matter mentioned by preservice teachers throughout the semester was the more formidable required adherence to state-mandated standards and standardized tests: "When it comes to Common Core Standards, the history of mathematics or science is not a part of those standards and can't take up too much of class time trying to explain" (Classroom Discussions). Similarly, as expressed by another preservice teacher, "As a teacher, my hands are tied by the standards that are mandated by the state, and it is frustrating that the people calling the shots typically have little to no experience in the classroom" (Classroom Discussions).

Teachers in many schools today are faced with the daily dilemma in deciding how much attention to devote to important mathematics and science content that students need to know and knowledge and skills required to pass state-mandated tests. Teachers and schools are judged on their students' performance on those tests. As a result, test content has a great influence on teachers' instructional time. Most teachers are forced to teach to the tests. However, as evident in the following comment, some of the preservice teachers were convinced that with all these limitations they can and will transform their curriculum planning and instruction by incorporating history and culture into classroom activities.

I believe if something is important to me as a teacher, then I will make time to do it. If incorporating history into my lessons and activities is truly important to me, I will find a way to integrate it in a way that supports the standards and does not cut into my limited time. I am 100% committed to making significant changes to my math curriculum in order to incorporate historical and cultural perspectives. (Classroom Discussions)

Access and equity in mathematics and science classrooms with respect to the availability of technology such as computers and smartboards for teaching and learning was another dilemma and concern expressed by several preservice teachers. This constraint limits meaningful learning opportunities for all students in this technological society. This limitation has a social justice implication. "Technology has changed the way we think and learn, so as educators we must adjust to accommodate for the young minds we are educating. Unfortunately, inner-city schools don't have adequate resources to educate youth" (Preservice Teacher's Final Written Reflection). Yet another expressed concern was finding ways to engage urban youth. As expressed by one participant, "My biggest concern and hurdle is how to take a historical or cultural issue and making it relatable or interesting to a group of students living in poor communities."

A particular challenge mentioned by preservice biology teachers was the conflict between science teaching and religious beliefs, particularly on the topic of evolution. Dialogues emerged several times in the classroom relative to using strategies for solving this type of problem. Some participants believed that they should take a non-confrontational approach to this sensitive issue:

I think at this point I will just teach what employers want to be taught at their schools. I realize that certain topics such as evolution, science and religion are still considered 'Taboo.' Therefore, they should be carefully navigated and if possible should be avoided. (Classroom Dialogues)

Another participate agreed with this non-confrontational approach by saying:

In schools, there's a separation of church and state, so do I continue to elaborate when students have questions or do I discard the questions and say that 'this is not a part of your standards, therefore, we must not discuss it?' Till this day, I have not addressed this concern. I avoid touchy subjects because I do not want my students to be uncomfortable or create a cognitive conflict between their own views and what's scientifically proven. (Classroom Dialogues)

Some participating teachers challenged this non-confrontational approach by suggesting that instead of avoiding and/or ignoring this situation, they must confront the issue and reconstruct it with their students:

I have one student in particular who despite understanding the evidence and recognizing its validity, still refuses to accept evolution as a reality. Her rationale is entirely theological. I am uncomfortable with allowing a student to miss out on such an important theory. As such, I was able to differentiate instruction to spend some time working with her so that I could learn her perspective. It is still work in progress. (Classroom Dialogues)

Another preservice teacher agreed with this approach. He mentioned that when students raise sensitive questions, teachers need to be prepared to respond thoughtfully. He said, "As a teacher I owe it to my students" (Classroom Dialogues).

Discussion

Professional transformation was one of the important realizations achieved by the preservice teachers who participated in this study. Other realizations included 1) the marginalization of non-western, non-white, and non-male contributions to science and mathematics; 2) teachers' lack of necessary knowledge and background experience for teaching mathematics and science from cultural and historical perspectives, and 3) a needed change in epistemology. Summarizing the impact of the

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course on professional development, one preservice teacher wrote, "I became aware of the fact that the idea of an effective classroom is not for all students to have the same outcomes, but more so to create an environment that is friendly, welcoming, and full of historical and cultural aspects that celebrate students' diverse backgrounds, and ultimately creates an equitable learning community." As the preservice teachers engaged in reflective processes, they became better communicators and demonstrated their growth in critical thinking. In addition, as they prepared lessons and presented them to their peers in the university classroom and shared some of their high school teaching experiences, they showed a great deal of professionalism relative to teaching and learning mathematic and scientific content embedded in historical and cultural contexts.

Primarily, the biggest issue teachers have with ethnomathematic and ethnoscientific teaching is the strict standards that are often set by school districts, states, and nationwide education agendas. As a continuing trend in education, schools continue to urge the importance of high marks on standardized tests, and with this as the primary focus, schools set their standards and develop curriculum to maximize test score results. Clearly, teaching the cultural and historical aspects of mathematics and science along with curriculum that has already been set by schools might be deemed unacceptable or a waste of time by some. Given an increasingly wide breadth of material to "cover," administrators may have issues with this approach and, in fact, find it to be too time consuming. Similarly, incorporating history and culture into a mathematics or science lesson might prove a daunting task for teachers in both the time needed for researching and planning such lessons as well as the extra time needed for instructional delivery. Despite these obstacles, participants felt the added effort was worthwhile as evidenced in this preservice teacher's final written reflection:

> With all of these issues such as high stakes assessments, pressure from administration, lack of time and resources, and varying student achievement, I have come to the conclusion, with much ease, that teaching with history and culture in mind to create a strong, student-centered environment is well worth these troubles.

A growing concern in the educational community is that students are losing the ability to create, develop, test, and ultimately think abstractly about mathematics and science content. By incorporating the incredible stories of mathematicians and scientists alike, teachers will elicit an intrinsic curiosity among students and a newfound drive to think critically. Furthermore, teaching with history and culture incorporated into the curriculum will challenge students to confront their own internal struggles with individuality and diversity. As Kragh (1992) put it:

> In an educational context, history will necessarily have to be incorporated in a pragmatic, more or less edited way. There is nothing illegitimate in a pragmatic, more or less edited way. There is nothing illegitimate in such pragmatic use of historical data so long as it does not serve ideological purposes or violates knowledge of what actually happened. (p. 360)

In using the robust history and culture of mathematics and science, teachers can create a learning environment that celebrates individual differences and ultimately promotes positive personal growth and development for all students. "Historical investigations not only promote the understanding of that which is now, but also bring new possibilities before us" (Mach, 1996, p. 316). In addition, teaching from this perspective may help establish a stronger bond between teachers and students. This caring relationship is crucial to the learning process.

Closing on a more personal note, I would simply note that while teaching *Perspectives on Science and Mathematics* for the past six years, it has been a transformative experience to observe the growth of my preservice teachers in terms of their attitudes and professed commitment for teaching and learning mathematics and science in historical and cultural contexts. Their professional transformation has, I believe, helped the interns to begin making their own roads by walking.

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WHAT KNOWLEDGE AND SKILL DO MATHEMATICS TEACHER EDUCATORS NEED AND (HOW) CAN WE SUPPORT ITS DEVELOPMENT?

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Refocusing teacher preparation on the practice of teaching calls for new and different knowledge and skill. We describe the types of knowledge and skill that teacher educators need in order to teach a practice-focused mathematics methods course and report on one program's exploration of possible supports to develop this knowledge in teacher educators.

Keywords: Instructional Activities and Practices, Mathematical Knowledge for Teaching, Teacher Education-Preservice

In recent years, scholars and teacher educators have called for teacher preparation to focus more directly on the practice of teaching (Ball & Forzani, 2009; Grossman, Compton, Igra, Ronfeldt, Shahan & Williamson, 2009; Lampert & Graziani, 2009). They emphasize that preservice teachers (PSTs) need to learn to do key aspects of the work of teaching, rather than just talking about teaching or analyzing someone else's teaching. This call has spurred important changes in both the design and implementation of teacher education. For example, in some teacher education programs, course content has shifted to the teaching of specific "high-leverage practices" (Ball, Sleep, Boerst & Bass, 2009; Davis & Boerst, 2014; McDonald, Kazemi, & Kavanagh, 2013). Teacher education pedagogy is also shifting, in particular, to incorporate "pedagogies of enactment" (Grossman et al., 2009), which include "approximations of practice" such as coached rehearsals (Kazemi, Ghousseini, Cunard, & Turrou, 2015).

This shift toward practice-based teacher education occurring in many programs, combined with influxes of new mathematics teacher educators (MTEs), creates a need to focus on the preparation and professional learning of MTEs. The work of teacher education is complex. MTEs must support PSTs in developing deep and usable knowledge and skill. They need to understand the content themselves and in more specialized ways to support teacher learning (Superfine & Li, 2014), be able to manage the teaching of integrated content, and be able to provide specific and detailed feedback aimed at improving teachers' practice (Van de Ridder, Stokking, McGaghie, & Cate, 2008). Our teacher education program, like many, must prepare novice MTEs to *do* the work of teacher education while simultaneously supporting the learning of PSTs. This work is different in important ways from K-12 classroom teaching. Not only must MTEs be able to enact teaching practices such as eliciting student thinking or leading mathematics discussions, in practice-based teacher education MTEs must be able to decompose teaching practice (Grossman et al., 2009) and support PSTs' learning of these practices. The work of teaching PSTs to successfully engage in these practices entails being able to identify and talk about specific skills and techniques for carrying out the practice as well as appraising PSTs' enactment of the practice. This is different than successfully engaging in the teaching practice.

Less attention has been paid to the knowledge and skills that MTEs themselves need to teach teaching practice; however, the findings we do have suggest that MTEs may be underprepared. For example, in a study of 293 practicing university-based teacher educators, Goodwin and colleagues

(2014) found that most respondents felt underprepared for the work, reporting "happenstance in becoming engaged in teacher education" and a "lack of explicit development of teaching skills or pedagogies related to teacher educating" (p. 291). It is crucial to focus on the professional learning of MTEs for a number of reasons. Practice-based teacher education places increasing demands on MTEs, both in their instruction and their appraisal of PSTs' skills. Second, in our context, our doctoral students will be among the next generation of MTEs, and we want graduates of our doctoral program to be prepared to support the learning of PSTs. Third, within our specific teacher education program, we seek ways to prepare and support MTEs to enact teacher education in ways that are aligned with the goals of the program. Across the board, MTEs' skills matter for the outcomes of teacher education and such attention is crucial to achieving the desired outcomes.

This paper is focused around two fundamental questions in the preparation of MTEs. First, what knowledge and skill is demanded of MTEs to engage in practice-based teacher education? Second, what structures are useful for supporting novice MTEs' learning of such knowledge and skill? We use the case of an elementary mathematics methods course to consider these questions.
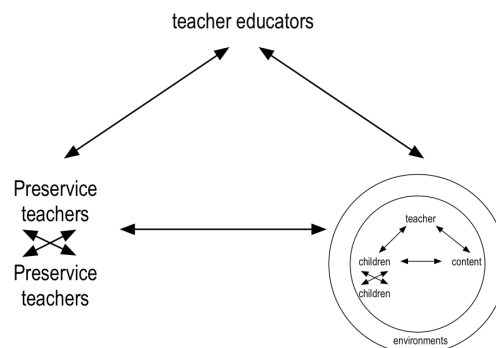

Figure 1. Instructional triangle for teacher education (D. L. Ball, personal communication).

Theoretical Framework

Teacher education is something that people *do;* it is not merely something to *know.* MTEs must use knowledge flexibly and fluently as they interact with PSTs, with the aim of helping PSTs become proficient with teaching. This interactive view of instruction can be portrayed using the "instructional triangle" (Cohen, Raudenbush, & Ball, 2003). Conceptualizing the work of teaching PSTs as interactions among teachers (teacher educators), students (preservice teachers), content (K-12 teaching practice), and environments (see Figure 1) has important implications for the identification of skills and knowledge that MTEs need to enact practice-based teacher education. For example, MTEs' knowledge of content, particularly teaching practice, must go beyond being able to enact teaching practices to include being able to identify and decompose such practices, talk about the practices and ways of enacting them, and see different ways of enacting the practice, all of which must adhere to the articulation of the practice. Further, MTEs may need specialized knowledge of mathematics content for teaching teachers (Superfine & Li, 2014). They also need knowledge of their students including the skills that they bring to teacher education, the ways in which they are likely to interpret particular practices, and progressions of development with such practices. Further, because the environment is crucial to the work of teaching, MTEs must consider environments beyond their own classrooms such as field placement classes in which there are mentor teachers with particular orientations to teaching and supporting PSTs' learning. Thus, teaching practice-based courses requires the integrated use of knowledge and skills in particular contexts of instruction.

Context

Over the last decade our teacher education program has engaged in a collective redesign of its undergraduate elementary teacher education program, centered on an effort to focus more directly on practices of teaching (see Davis & Boerst, 2014). The mathematics methods course has been collectively developed by a rotating group of members (see Ball et al., 2009 for a description of the collective work of the planning group) to focus on preparing PSTs to teach K-6 mathematics skillfully to their students. The course works on four teaching practices with a mathematics focus: (1) explaining core content; (2) leading discussions; (3) assessing students' knowledge and skills; and (4) planning instruction. The course develops PSTs' skill with these practices while simultaneously developing mathematical knowledge for teaching. In this paper, we examine the practice of explaining core content with a particular focus on representing and connecting mathematical ideas.

Explaining core content focuses on the work a teacher does to provide all students with access to fundamental ideas and disciplinary practices. There is much for PSTs to learn about this work, including: "strategically choosing and using representations and examples to build understanding and remediate misconceptions, using language carefully, highlighting core ideas while sidelining potentially distracting ones, and making one's own thinking visible while modeling and demonstrating" (TeachingWorks, 2015). In teaching, explanations are often co-constructed with students; however, being able to co-construct an explanation requires knowing what is involved in building a mathematical explanation. Learning to do that oneself is an important part of understanding the practice of explanation well enough to help students learn to explain. We work on the practice of explaining core content across the course with a focus on explaining computational algorithms and their connections to representations in order to build PSTs' mathematical knowledge for teaching.

Mathematics Teacher Educator Knowledge and Skills for a Practice-Based Approach

The instructional triangle illuminates the complexity of the work of teaching PSTs. In this paper, we consider a subset of the knowledge and skill called for by these complex interactions – specialized knowledge of teaching, with a focus on the instructional practice of explaining core content with representations. We then turn to identifying teacher education practices and pedagogies that can be used to support PSTs' learning to teach.

Specialized Knowledge for Teaching Teachers: The Case of Explaining Core Content

To enact the type of practice-based approach to teaching "explaining core content" described earlier, MTEs need to draw on different types of specialized knowledge that could be viewed as parallel with types of mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008), that is – –common content knowledge, knowledge of content and students, and knowledge of content and teaching. However, the "content" in this case is the work of mathematics instruction, and specifically the work of explaining core content.

For MTEs, *common content knowledge* (CCK) can be viewed as the knowledge that mathematics teachers themselves hold. Consider the case of explaining subtraction with multi-digit numbers. The common content knowledge for teaching PSTs to explain multi-digit subtraction includes the mathematical knowledge for teaching (MKT) that would allow PSTs to engage in this instructional practice. This includes knowing which key mathematical ideas to highlight (e.g., importance of making equivalent trades), common student errors in a particular domain (e.g., challenges with regrouping across zero), and how different representations can be used to illustrate particular ideas (e.g., how bundling sticks can be used to show regrouping). Common content knowledge for MTEs might also include knowledge of the instructional practice itself, such as knowing characteristics of a good explanation (Leinhardt & Steele, 2005). In other words, MTEs need the knowledge that they are helping PSTs learn; however, just as teachers need more than common content knowledge of

mathematics (Ball et al., 2008), MTEs also need more specialized knowledge for teaching mathematics instruction.

MTEs would also need *knowledge of content and students* (KCS), which is characterized by knowledge of interactions between their students (in this case PSTs) and content, which in this example would be the instructional work of explaining core content. Knowledge of content and students includes knowing common errors that PSTs tend to make when engaging in the instructional practice. For example, when PSTs explain subtraction with regrouping using base 10 blocks, we have found that they often use language imprecisely when talking about "taking away" and "trading." Additionally, PSTs often attempt to explain using base 10 blocks by first solving the problem with the materials and then showing the algorithm instead of conducting them simultaneously to make clear the meaning of the steps of the process. Knowledge of content and students also includes knowing what parts of an instructional practice tend to be most difficult for PSTs such as explaining the equivalency of trades when regrouping.

Another type of specialized MTE knowledge can be described as *knowledge of content and teaching* (KCT). This includes knowing the types of tasks and representations that are useful in helping PSTs learn a particular part of mathematics instruction. In the case of explaining core content, KCT includes knowing which mathematical content domains might be productive choices for PSTs, who are just beginning to learn this practice (e.g., explaining subtraction with regrouping is more accessible than explaining long division). KCT also encompasses knowing the characteristics of video examples that might be useful in illustrating aspects of a practice.

These illustrative examples highlight that, while MTEs need the same knowledge that skilled teachers need to engage in mathematics instruction, they also need an additional layer of knowledge that can be viewed through interactions inside an instructional triangle in which PSTs are the students and the content is mathematics instruction (See Figure 1).

Teaching the Instructional Practice of Explaining Core Content

In addition to specialized knowledge, a practice-based approach also demands new skills on the part of the MTE. To illustrate some of these new pedagogical skills, we consider two different pedagogies of practice that are used in our teacher education program to support PSTs in learning to explain core content as well as a practice of teacher education which has new demands inside of practice-based teacher education[1]. These are not meant to be a comprehensive list of ways to support PSTs; rather, these serve as examples of the work that a practice-based approach demands on the part of the teacher educator and draws on the types of specialized knowledge illustrated in the previous section.

Modeling. In a practice-based approach to teacher education, "modeling by the MTE" is a key pedagogy (McDonald et al., 2013). This involves the MTE engaging in the instructional practice in front of PSTs to both demonstrate the practice and provide meta-commentary throughout to narrate and make visible the instructional work and decision-making. For example, to model the practice of explaining core content, a MTE could explain subtraction with regrouping using base 10 blocks and simultaneously comment on the instructional work (e.g., making a meta-comment about the decision of how to represent the numbers and base 10 blocks in parallel to highlight connections). This teacher educator pedagogy requires that the MTE be able engage productively in the practice of explaining core content, drawing on the common content knowledge of teachers described above; however, this pedagogy of practice also requires new and different skills not required by teachers themselves. In this case the skill of simultaneously engaging in a teaching practice while narrating and commenting on the instructional work involved. This additional layer involves skills such as deciding what to highlight (or not) about the instruction and how to describe the work in meaningful ways. These MTE skills draw on the specialized knowledge described above; for example, when modeling the

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practice of explaining core content, a MTE might decide what to highlight based on their knowledge of what common errors PSTs tend to make when explaining that core content.

Rehearsal. Another pedagogy of practice that demands new work on the part of the MTE is facilitating coached rehearsals with PSTs (Kazemi et al., 2015). Consider the case of facilitating rehearsals of explaining core content. First, this requires the MTE to establish the norms and culture for working together on practice, as PSTs actually engage in explaining core content in front of the class. This likely builds on other relational work but adds new demands as the MTE asks PSTs to share their teaching publically. Another set of skills is required for facilitating the rehearsals themselves. For example, the MTE must decide when to interrupt a rehearsing PST, what to comment on in the performance, and how to coach (e.g., asking a question, making a suggestion, commenting on what was productive). These skills of providing in the moment feedback draw on a MTEs' specialized knowledge as they simultaneously analyze instruction in the moment and determine what might be most productive to coach on.

Providing feedback on pre-service teacher enactments. A teacher education practice used in our program to support PSTs to explain core content is the work of providing formative feedback on PST enactments). Similar to the work involved in facilitating rehearsals, this practice involves determining which parts of a performance to provide feedback on and how. This rests on the MTE being able to recognize productive, problematic, or incomplete aspects of an enactment and draws significantly on the types of specialized knowledge described earlier. Additionally, MTEs need to determine how to ground their feedback in whatever rubrics or decompositions of the instructional practice are used in the course. This work on the part of the MTE is different from giving feedback on PSTs' reflections about their practice (which is also important), as it demands that the MTE interact with and directly give feedback on enactments of practice, rather than on a PST's skill with analyzing and reflecting on his or her own practice.

Given the demands of the coordination of the specialized content knowledge and pedagogies of practice required to teach a practice-based mathematics methods course, we sought to design and study supports that would enable novice MTEs to build their knowledge and skills while simultaneously supporting their PSTs in learning to teach mathematics.

Structures for Supporting MTE Knowledge and Skills for a Practice-Based Approach

To explore the ways in which support could be provided to novice MTEs, our program uses a set of structures to build the specialized knowledge and pedagogical skills demanded in the teaching of a practice-based methods course. We first describe the overarching organization of the work and then closely examine three of the structures that supported our new MTEs.

Planning group participation

To teach the mathematics methods course, novice MTEs participate in a planning group. The group consists of experienced MTEs, many of whom are not currently teaching the course but are invested in either course development or MTE development, and novice MTEs. The group meets several times prior to the start of the course and then once per week throughout the duration of the course. The group is facilitated by a lead MTE who is an experienced MTE and is also responsible for teaching the lead section of the course. All members of the group observe the lead section with an eye to identified areas of focus. Following the observation of the lead section, members debrief the class with regard to the observation focus areas.

The goal of the group is three-fold. First and foremost, we seek to ensure that the course is designed and taught to consistently provide PSTs with learning opportunities that support their development as elementary mathematics teachers. Second, we provide opportunities for new MTEs to build specialized content knowledge and pedagogical knowledge to ensure that all PSTs are receiving instruction that will allow them to engage in the mathematics teaching practices. Third, we

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use the teaching of the lead section to adjust the plan or the materials in response to what we learn from the first teaching of the shared plan.

Detailed lesson plans. Prior to each meeting, planning group members review detailed lesson plans and decompositions of practices involved in the upcoming class. The lesson plans include scaffolds to support MTEs in the areas of specialized knowledge, pedagogies of practice, and teacher education practices. An excerpt from a lesson plan is shown in Figure 2 to illustrate the level of detail provided. This plan supports MTEs' knowledge of content and students by noting the areas in which PSTs may need support and explicitly naming what is to be worked on in the modeling. CCK is also supported through the description of the modeling and in the notes.

Activity/task	Detail	Notes
Modeling the standard algorithm for subtraction with bean sticks: Comparison • Distinguish between the take-away and comparison meanings of subtraction • Understand the meaning of the standard algorithm for subtraction and be able to explain why it works: • Begin to problematize the language that is traditionally used with the standard algorithm (e.g., borrowing, you can't take a bigger number from a smaller number) • Continue to develop skills with, and explicit knowledge of, what is involved in modeling computational procedures: e.g., physically placing materials; attending to language and using the language of the materials; coordinating use of the materials with talk and recording	Comparison interpretation: (15 minutes) Explain that we are now going to shift our focus to the comparison interpretation of subtraction. Model the standard algorithm again, with the same numbers (52 − 13), but this time using a **comparison** interpretation. Do NOT use the modeling checklist for this first performance. Elicit a few comments from the group. Ask interns to model the standard algorithm, with the same numbers (52 − 13), using a **comparison** interpretation. Suggest they start by laying out both 52 and 13. Encourage interns to use the modeling checklist. (If interns finish early, try another problem like: 60 − 27; 48 − 25 etc.) If time permits, invite an intern to come to and model for the group. Alternative, discuss questions/challenges which arose when working with a partner.	*Modeling the comparison tends to be more complicated and it is important for interns to see a complete performance.* *The comparison can be modeled by matching, either by drawing lines or by removing the matching pairs. Both have issues to consider: When doing the matching by drawing lines, the answer may not be easy to find — the answer is the amount that remains unmatched. And with taking matches off the board you lose the original numbers in the problem.* *You might comment that comparison is typically more difficult for children.*

Figure 2. Lesson plan for modeling subtraction using the standard algorithm.

Planning group structure. Each meeting is structured to first provide MTEs an opportunity to debrief both the observation of the lead section and the teaching of their own section. These debriefs are focused on questions designed by the lead to highlight the key work of MTEs in this context. For example, the lead MTE might ask group members to observe a class session in which multiple pedagogies of practice are being used to identify the pedagogies and consider the features of each and how they are supporting PSTs' learning. This type of question brings to the fore the questions that MTEs must consider when determining how to teach particular practices to PSTs as well as the key features of the pedagogies used.

Following the debriefing, MTEs engage in work to prepare for the next class. The structure of this work varies depending upon the mathematical content and required pedagogies and teacher education practices for the class. Structures include framing and walking through particular activities (e.g. setting up the goals and purpose of modeling the explanation of subtraction using a particular representation and talking about what should be highlighted for PSTs), providing space for discussion of questions that MTEs may have about the lessons, rehearsing sections of the lesson, and discussing annotated videos. Each of these structures was designed to support novice MTEs' content knowledge and development of skill with teacher education practices and pedagogies of practice. The primary (noted by P) and secondary (noted by S) uses of each of the structures in shown in the table below. Descriptions of two key structures that are specifically designed to support new MTEs' understanding of and ability to engage in pedagogies of practice and teacher education practices follow.

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Table 1: Support structures and purposes for MTE learning (primary, P; secondary, S)

	Modeling	Rehearsal	Feedback	Specialized knowledge
Decompositions of practices	S		P	P
Lesson plans	S	S		P
Walk-throughs	P	P		P
Question sessions	S	S	S	P
MTE rehearsal	P	P		P
Annotated videos	S	S	P	P
Observation	P	P		P
Debrief	S	S		P

MTE rehearsals. MTE rehearsals are used in two different ways. To build skill in teacher education pedagogy of modeling, we have novice MTE rehearse the modeling that will take place in front of the PSTs. One experienced MTE runs the rehearsal while other MTEs participate in the role of PSTs. The experienced MTE pauses the rehearsal at strategic moments to support the novice MTE and the group as a whole in engaging in the metacognitive work of MTEs such as considering what should be highlighted (and how) for PSTs, at which moments meta-comments should be made for PSTs, and how best to represent the connection between the mathematical notation and the representations. This collective work provides support for MTEs in understanding the pedagogy as well as the pedagogical content knowledge and knowledge of content and students required to engage in the pedagogy with particular content.

We also rehearse running a rehearsal with PSTs. It is structured slightly differently, but provides many of the same supports. In rehearsal of rehearsal, an experienced MTE serves as the PST who is engaging in the practice of explaining core content, the novice MTE serves in the role of the MTE who is running the rehearsal with the PST, and another experienced MTE runs the rehearsal, giving feedback to the new MTE on their choices for pausing the rehearsal, the feedback they give to the PST, etc. In this case, the MTE serving in the role of the PST designs their performance to highlight common ways that PSTs approach explaining the content, including challenges with coordinating between representations and common language issues. This design provides opportunities for MTEs to develop their own KCS at the same time as developing their skill with the pedagogy of rehearsal.

Annotated videos. The work of providing feedback to PSTs involves not the ability to identify key parts of the practice, but an ability to align feedback with both the decomposition of the practice that is being used with the PSTs and with the PSTs' progression of development expected at the time of the feedback. One way to support novice MTEs with this work is through the use of annotated video. Videos are annotated by experienced MTEs to provide feedback to PSTs. Initially, novice MTEs watch the videos with annotations in an attempt to notice and justify on what and how the MTE provided feedback to the PST. This initial experience supports new MTEs in developing a sense of how to align feedback with the decomposition and how to choose what to give feedback on. Later MTEs watch videos without annotation then annotate the videos themselves as if they were giving feedback to the PSTs. Their annotation is then compared with the experienced MTE's annotation. Novice MTEs are provided with opportunities to discuss decisions that experienced MTEs made when providing feedback.

Discussion

The knowledge and skills of teacher educators are crucial for the success of teacher education. Our paper focuses on one particular element of content taught in an elementary mathematics methods course and unpacks the knowledge and skill that MTE need to support the learning of PSTs. We

describe several structures that we have found useful when supporting novice MTEs' skills and knowledge development. Importantly, our analysis shows that being an experienced K-12 classroom teacher is by itself insufficient for teaching practice-based teacher education courses. As a field, we must plan deliberately for the development of novice MTEs to realize the goals of practice-based teacher education. Further, although we focus on novice MTEs in this paper, because of the context of our work, we believe that experienced MTEs who have not taught practice-focused courses need to develop additional skills to enact practice-based teacher education. As a field, we must put increased focused on the preparation of teacher educators and this paper offers an analysis of one teacher education program's attempt to do so.

Endnotes

[1]In our view, there is an important distinction between teacher education practices (such as eliciting PST thinking, assessing PSTs' work, providing feedback) and pedagogies of practice (such as rehearsal) that are activity structures in the context of teacher education. We take up this distinction in other work.

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"YOU CAN'T GO ON THE OTHER SIDE OF THE FENCE": PRESERVICE TEACHERS AND REAL-WORLD PROBLEMS

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Our study investigates preservice teachers' perceptions of real-world problems; their beliefs about teaching real-world contexts, especially ones sociopolitical in nature; and their ability to pose meaningful real-world problems. In this paper we present cases of three preservice teachers who participated in interviews that probed their thinking about real-world problems, and asked them to create problems they would consider using in their future classrooms. We use the three cases to propose a potential trajectory for preservice teachers from ambivalence to certainty about teaching mathematics through real-world and controversial contexts.

Keywords: Teacher Education-Preservice, Equity and Diversity, Problem Solving

Introduction and Review of the Literature

In this paper we present work that is part of a larger study investigating different aspects of preservice teachers' (PSTs) relationship to real-world problem solving and problem posing. Here we place an emphasis on PSTs' problem posing and draw on their thinking about contexts they would (and would not) use in their teaching to inform the steps that we, as mathematics teacher educators (MTEs), need to take in order to prepare teachers to teach through real-world and controversial issues. The border-crossing metaphor applies to our work in at least two ways: we want PSTs to cross from posing superficial real-world problems to posing authentic ones; and we want them to cross from "safe" topics that do not challenge unjust systems to those that draw from sociopolitical and potentially controversial contexts.

Real-world problems have recently come into the spotlight in mathematics education due to their treatment in the Common Core State Standards for Mathematics (CCSSM) (CCSSI, 2010). The CCSSM introduced modeling not only as a content strand in high school mathematics, but also as one of the eight essential mathematical practices to be developed in all mathematics classrooms at all grade levels. According to the document, "mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace" (CCSSI, 2010, p. 7). However, the types of problems recommended by the CCSSM are not the norm in school mathematics, where real-world problems are frequently understood to mean textbook story problems, with contrived contexts and neat solutions that combine the numbers given in the problem in an algorithmic manner. Verschaffel, Greer, and De Corte (2000) describe textbook problems as "routine applications without judgment or any higher level thinking skills" (p. xiii) and "artificial, puzzle-like tasks that are unrelated to the real world" (p. xv). As an alternative to scripted textbook problems, researchers suggest realistic real-world problems, as they can enhance student performance (Boaler, 1993; Verschaffel, Greer, Van Dooren, & Mukhopadhyay, 2009) and contribute to a bigger picture of mathematics (Blum, 2011).

These real-world contexts recommended by research (and the CCSSM) are rich and varied, and include examples from science, engineering, and business among others (Common Core Writing Team, 2011), but do not explicitly include applications of mathematics that emphasize (a) connecting mathematics to students' lives and backgrounds in meaningful ways and (b) using mathematics to critically analyze our world and challenge injustice. Real-world applications that exclude the aforementioned (a) and (b) can certainly improve student access to rich mathematics and their academic achievement, but do not help develop their identity or empower them as citizens

(Gutiérrez, 2007). Frankenstein (2009) argues instead for *real* real-world problems, which "use mathematics ideas in struggles to make the world better" and, along with Gutstein (2006), promotes mathematics as a tool for reading the world, explicitly focusing on issues of social justice, including, but not limited to, income inequality, education funding, homelessness, institutional racism. The use of critically oriented real-world problems is essential in mathematics teacher education, as it helps PSTs see mathematics as a window into the worlds of their future students and a mirror into their own (Gutiérrez, 2007). Through engaging with *real* real-world problems, PSTs gain an understanding of the circumstances students live in, the relevance of mathematics to their own lives, and the power of mathematics in reading the world and developing agency.

In addition to solving real-world problems, PSTs need to be able to write them as well (Gonzales, 1994). Because problem posing is typically not emphasized in mathematics education, PSTs encounter difficulties when posing real-world problems for the first time. They typically create problems that can be solved in only one way and in a single step (Crespo, 2003); limit contexts to time, food, and money (Gainsburg, 2008; Lee, 2012); and are unrealistic and can even include objects like unicorns and aliens (Lee, 2012).

Our work investigates real-world problem solving and posing in the context of PSTs' beliefs and knowledge about connecting mathematics to students' lives and about using mathematics to critically analyze the world. We have developed a survey that measures preservice teachers' beliefs about engaging in mathematics teaching based in real-world contexts (Simic-Muller, Fernandes, & Felton-Koestler, 2015); and have conducted interviews in the first author's mathematics content courses for K-8 PSTs, investigating their thinking about different types of real-world contexts. In this paper we focus on examples of real-world and controversial issues that PSTs were asked to create in the interviews. In particular, we present cases of three PSTs and propose a trajectory, based on their interview responses, towards competency in teaching mathematics through authentic problems about complex issues.

Methods and General Observations

Our 35-item survey investigates PSTs' beliefs about teaching through real-world problems (6 items) as a whole, as well as three other subscales: controversial issues (8 items), injustices (11 items), and family backgrounds and community practices (10 items). We used a 1 (Strongly Disagree) to 5 (Strongly Agree) Likert scale for each item. These categories are present in a survey we have created and conducted with 127 PSTs thus far. The survey design was informed by literature and our experience working with preservice teachers (see, in particular, Felton-Koestler, 2015). We also sought comments from seven experts in the area to ensure content validity. We highlight a few items in Table 1. The questions corresponding to the other three sub-categories closely match the ones in Table 1.

Table 1: Sample Survey Questions

1.	When I teach mathematics, I will make connections to REAL-WORLD SITUATIONS.
2.	I am interested in learning how to make connections to REAL-WORLD SITUATIONS.
3.	When I teach mathematics, I will focus on mathematical concepts (e.g., addition and subtraction, geometric shapes, etc.) and not worry about using REAL-WORLD SITUATIONS.
4.	When teaching mathematics, REAL-WORLD SITUATIONS can distract students from learning the important mathematical concepts.
5.	An advantage to teaching mathematic with REAL-WORLD SITUATIONS is that they help students learn about the world around them.
6.	Teaching mathematics with REAL-WORLD SITUATIONS helps students learn the mathematical concepts better.

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In the survey and interviews, we explained controversial issues as "topics that will likely be viewed as contentious or debatable. Not everyone agrees on what topics are controversial, but some examples might include the costs of the war on drugs, government spending, funding for schools, or climate change."

We have been recruiting preservice teachers nationwide for the survey, but have currently mostly collected data from the first author's two-part mathematics content course for K-8 PSTs. In the course, PSTs see some examples of contexts investigated in the survey prior to responding to it (e.g. they might have investigated the affordability of housing by minimum wage-earners, or the gender wage gap), and after the survey is administered (e.g. the racial make-up of the Congress or the interest rates when renting to own); but the primary focus of the course is on developing mathematical content knowledge.

Based on the 127 survey responses, 96% of the respondents agree or strongly agree with making connections to real-world contexts in their teaching. Unsurprisingly, it is much more difficult for PSTs to agree with using controversial topics, with only 24% responding with "agree" or "strongly agree." However, both the survey and interviews indicate that PSTs are generally curious about controversial issues: in the survey 64% (agree or strongly agree) said they were interested in learning how to make connections to controversial issues.

In order to better understand survey responses, the first author also interviewed nine students who were, with one exception, taking a class with her for the first time, and had had no known prior exposure to issues of social justice in the context of mathematics. The interviews took place after the end of the semester. PSTs volunteered for the interviews, and represented a wide range of interests and openness to mathematics teaching and social justice. The interviews varied in length, and followed a semi-structured format. The interview questions investigate survey topics at more length, for example:

> Some people think it is important to teach mathematics by making connections to **real-world situations**. What does that mean to you? [Can you give me some specific examples? Can you give me more specifics? How would you teach it? Which mathematics would you use? What issues are relevant to you?]

The interview protocol contains similar questions about controversial issues, injustices, and family backgrounds or community practices. For the purposes of this paper, we will consider questions related to real-world situations and controversial issues, as defined above. The first two authors coded the interviews for common themes.

While all interviews provide valuable insights, we decided to focus our attention to three PSTs: Mirinda, Briana, and Laura (pseudonyms). We began the data analysis for this paper by looking at the codes related to mathematics content present in the interviews, and extracting all relevant quotes from all nine. While rereading and organizing the quotes, we noticed that Briana's statements often emerged as significant; she was therefore the first PST chosen for a case study. We next included Laura, who we identified in prior research (Simic-Muller et al., 2015) as more advanced in her thinking about real-world and controversial issues than her peers. We realized, when comparing their responses, that there was a progression from Briana's to Laura's responses, and finally decided to include Mirinda's case to complete the progression. We identified four common themes that emerged in all three interviews: the ability to imagine relevant contexts for real-world problems, perceptions of children's ability to engage with complex real-world contexts, beliefs about the role of controversial issues in the classroom, and interest in teaching through real-world and controversial issues. We describe the range of views among the three PSTs in each of the four themes through quotes from their interviews and our interpretation of their quotes, and hypothesize that these may be a potential trajectory for PSTs engaged in course work emphasizing real-world and sociopolitical connections in mathematics.

Three Cases

Mirinda: Beginning the Journey

Mirinda is a quiet student who wants to teach third grade, and who performed well in the course, but seldom made contributions to class discussions or approached assignments in original ways. Her interview is not a departure from her typical classroom demeanor. Mirinda's interview displays characteristics we think of as emergent in the development of a positive disposition toward teaching through real-world and controversial issues. The four themes in her interview are discussed below.

Imagining relevant contexts. Mirinda thought that third-graders should encounter real-world mathematics problems, but had difficulties coming up with examples. The ones she picked were standard ones: sharing food fairly and shopping at a store. Other PSTs had similar difficulties in the interviews, and in particular also chose similar contexts: six out of nine mentioned food, and three mentioned money. The following exchange shows Mirinda proposing a promising context based in students' communities, but then turning it into a static story problem.

> *Mirinda*: For third grade, they are old enough to go to the store and buy snacks and they know how much to pay and if they are getting the right amount of change.
> *Interviewer*: How would you do that in your classroom?
> *Mirinda*: I would say word problems once more. So I would give them a scenario. They go to a store and they buy a certain item, and give them the cost, and ask them how much will the total cost and how much change with they get back.

When asked for an example of a controversial issue, Mirinda shared an example from an assignment completed earlier in the semester, and otherwise responded, "I can't really think of anything."

Perceptions of children and controversy. Mirinda seemed to believe that children cannot understand complex real-issues, stating that "it would be difficult for younger students to understand." Other PSTs in the interviews also stated that children are too young to understand issues or to care about them, even if those children live in circumstances investigated by the real-world problems.

Controversy in the classroom. Mirinda did not see a need for teachers to teach through controversial issues, saying that "it should be optional for teachers to teach it." Of course teaching controversial issues cannot and should not be mandated, but in this statement made by Mirinda we also see a relinquishment of responsibility for teaching for change to a self-selected group of teachers, instead of being distributed to all.

Interest in teaching with controversy. Mirinda also did not feel too much curiosity about teaching through controversial issues. She responded to the question of whether she was interested in learning more about the topic with, "If I find content and issues that go with it, maybe…" This was not the case with other interviewed PSTs, who all expressed interest, if accompanied with doubt, in learning more about controversial issues.

Briana: Eyeing the Line

Briana is a PST from the campus' low-income neighborhood who has lived in different parts of the city and has attended some of the highest- and lowest-performing schools in the region. She wants to teach first grade, and is an average mathematics student. Because of her background she has a greater awareness of neighborhood and community issues than most of her peers. Briana was also concerned about the effect that teaching controversial issues might have on others. In the interview she made references to borders by expressing concerns about "staying on one side of the fence" and making sure not to "cross the line." The four themes emerge in her interview as well.

Imagining relevant contexts. Briana had difficulties coming up with real-world contexts, and her mathematics examples were not particularly rich. While she said she wanted to bring children's lives into mathematics teaching, she was unable to come up with an example beyond counting, for example suggesting that "they could go outside for a homework assignment and they could count how many bikes they see and things like that. However, despite these difficulties, Briana believed that "it's not very hard to incorporate math into real world situations."

Perceptions of children and controversy. Briana believed that children should be aware of the world around them. She had some reservations about their ability to comprehend the issues, but to a lesser extent than Mirinda. For example, when discussing recycling, she noted:

> I think it is good for students and children to know what is going on around them, it may be on the broad spectrum and they may not get the whole picture but it's still important for them to know… [T]hey may not know everything about one thing that you are trying to get the point across, like recycling they don't need to know the pros and cons but they can get the point of, "Oh it's good to recycle, I recycled 10 bottles this week."

We would advocate extending this reasoning to most controversial issues: even if students do not understand it fully yet, there is usually an entry point for discussing it. We should not prevent young children from thinking critically about the world just because they will not be able to understand all its workings.

Controversy in the classroom. Briana wanted to know where to set a limit and how not to go beyond it, because "we don't know where to draw the line, or what's appropriate for what age necessarily." She also briefly mentioned that parent resistance might get in the way of teaching controversial topics to young children. Like other PSTs we interviewed, she was concerned about offending someone by teaching topics some might find inappropriate. This is understandable, and points to the need for MTEs to support PSTs in learning to navigate these complex spaces so that they and their students can learn to both play the game and change it (Gutiérrez, 2007).

Interest in teaching with controversy. Briana was open to and aware of controversial issues, but preferred safer ones. She was interested in the environment, but would pick recycling over climate change, because the latter could "start to get controversial." Similarly, if she talked about the military, it would have to be in a positive light:

> You can't go on the other side of the fence but you have to stay on one side if you talk about that with all the military families, that's where all the controversy comes in, you so you have to be very positive about it, like how many people have family in the military, so let's make them thank you cards.

While this could be viewed as backing away from controversy, it can also be seen as recognition of the local context – criticizing the military is widely unpopular in the area where Briana lives. Briana's concern about not offending people is common, and we as MTEs need to develop tools to help PSTs develop the courage to cross to the other side of this fence, i.e., be critical of and challenge the status quo, and not only help their students succeed academically, but also empower them to challenge inequity and injustice (Gutiérrez, 2007).

Laura: Over the Line, Looking Back

Laura is also White, and wants to teach upper elementary school. Her responses to the interview prompts show deeper thinking than those of her peers, and we have conjectured (Simic-Muller et al., 2015) that this is due to three factors: (a) her mathematical knowledge, as she is getting a middle-level mathematics endorsement; (b) her background, as, like Briana she grew up in the neighborhood surrounding the campus and is aware of its strengths and challenges; and (c) her position on campus as a residence assistant, in which she participated in a variety of social justice programming. We will

compare and contrast Laura's characteristics pertaining to our four stated themes to those of Mirinda and Briana, hoping to understand what sets PSTs like Laura apart from her peers.

Imagining relevant contexts. Laura had no difficulties creating real-world examples for various grade levels. For a real-world context, she gave examples of whole number and percent problems about the numbers of children who speak other languages in order to celebrate bilingual children. For a controversial issue she proposed investigating how many school lunches a favorite celebrity could buy for students with her income, with the purpose of "realizing how do we make it fun but also relate it to our own lives," and noting that "it's not just me compared to them, but our whole society compared to them [which is] easier to look at rather than me compared to you." She claimed that she is "always thinking of things in [her] daily life in math ways," and as she is "pretty confident in the content [she will be] teaching, and the standards [she will be] trying to meet," it is "easier to relate those on the spot to different things and come up with things on the spot."

Perceptions of children and controversy. Laura did not dismiss younger students' ability to deal with real-world, issues, though, like the other PSTs, thought that in-depth conversations are easier to facilitate with older students, partly because younger students do not have as many mathematical tools available to discuss complex issues. She also explicitly discussed children's backgrounds, which she believed should be celebrated, such as the number of languages a child speaks or the number of people living in his household; and she believed that even the basic data collection questions, such as recording how many pets each child has, is a "way to create a community in the classroom and have students learn about other students."

Controversy in the classroom. Laura was confident that she would teach through real-world issues. Although she too had concerns that the administration and student might disapprove of discussions about touchy subjects, she offered strategies for turning these stakeholders into allies. She built on her experience as resident assistant to describe that, when discussing a potentially touchy subject, she would explain to students who might be impacted by it that "it's great for people to be aware … of all these things going on people's lives." Similarly, she said she would "would like to be able to do what I want in my classroom with math" but would ask her supervisor, "I am interested in using these types of situations, how would you suggest, or could you suggest a different way that I bring these up in the classroom?"

Interest in teaching with controversy. Finally, although she did not explicitly say it, her entire interview indicates her openness to learning more about teaching through real-world and controversial issues.

Discussion

The three case studies show a range of beliefs and knowledge related to real-world connections. While additional research is needed to better understand how PSTs' beliefs and knowledge develop, we hypothesize that the cases may represent a possible progression that PSTs first developing familiarity with these ideas may follow.

First, mathematical content knowledge is important for problem posing (Gonzales, 1994), but it is not sufficient. Although Mirinda possessed reasonably strong mathematical knowledge, she was unable to come up with a rich context on the spot. Laura, on the other hand, talked about "always thinking of things in my daily life in math ways," and we believe that the ability to view the world through a mathematical lens, along with content knowledge, can support PSTs' problem-posing skills. A question for our consideration, then, is how to help develop a mathematical lens in all PSTs, especially those who see themselves as bad at math.

Second, if PSTs do not believe that students are capable of engaging with real-world and controversial contexts, then they will not feel compelled to think about these contexts. Doubts about children's interests or abilities repeatedly occur in the interviews, yet we know from research and our

experience that young children do possess the capacity to engage with complex contexts and mathematics (e.g. Turner, Varley Gutiérrez, Simic-Muller, & Diéz-Palomar, 2009; Murphy, 2009; Varley Gutiérrez, 2013). Many MTEs already challenge PSTs' perceptions about children's ability to develop their own strategies, and it is important to challenge their perceptions about children's ability to engage with real-world contexts as well. We need more accessible examples of young children being successful at grappling with complex real-world problems that deal with their lives or the world at large.

Third, PSTs are hesitant to introduce controversial topics into their teaching. With the current climate of excessive teacher evaluation and diminished autonomy, these concerns are far from unfounded. However, Laura's approach, which seeks to turn students and administrators into allies, is a promising one. MTEs need to provide positive examples of communities and schools working together to enact change and to support PSTs in learning to successfully navigate these complicated political spaces.

Finally, as obvious as it may seem, it is essential that MTEs provide PSTs with ample opportunities to learn about real-world mathematics problems and controversial topics. In our survey, though few PSTs readily agree to teach using controversial issues, many more are at least curious about this approach. We will also encounter PSTs like Miranda who do not yet see its point, but may begin to do so through appropriate and relevant examples. We need to better understand pathways to creating curiosity in PSTs who come to us with little prior experience with social justice in order to help them develop interest in teaching mathematics to critically analyze the world.

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PRESERVICE TEACHERS' CONCEPTION OF EFFECTIVE PROBLEM-SOLVING INSTRUCTION AND THEIR PROBLEM SOLVING

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Despite the importance of teachers' conception of effective problem-solving instruction, limited attention is given to this area in the current literature. In this study we examined 96 preservice teachers' (PSTs) views on effective problem-solving instruction and how their conceptions are related and reflected in their views on problem solving and problem solving performance. Analyses of survey responses revealed that our PSTs seem to develop narrow views on effective problem-solving instruction. In addition, we found a positive association between PSTs' conceptions of effective problem-solving instruction and problem solving. However, no such connection exists between PSTs' views problem-solving instruction and their performance.

Keywords: Teacher Beliefs, Problem Solving, Teacher Knowledge, Teacher Education-Preservice

Introduction

Preparing effective teachers of mathematics who promote students' conceptual understanding and problem-solving abilities is one of the most urgent problems facing teacher educators (Morris, Hiebert, & Spitzer, 2009). It is recognized that the quality of problem-solving instruction matters the most for improving students' mathematical abilities (e.g. Stigler & Hiebert, 1999). However, there has not been a clear agreement about what can be counted as effective problem-solving instruction (Krainer, 2005). The interpretation of a problem-solving lesson as 'good (or effective)' or 'bad (or ineffective)' is a value-loaded judgment. In addition, the notion of problem solving has been used with multiple meanings that range from "working rote exercises" to "doing mathematics as a professional" (Lesh & Zawojewski, 2007). Thus what is meant by effective problem-solving instruction is often subject to interpretation in a particular context and ill-defined in the literature, which suggests the importance of clarifying the meaning of effective problem-solving instruction in mathematics education.

Although several researchers focused on students' and teachers' perspectives of mathematics classes includes their meanings of "effective teaching", "good teaching", "good teacher", "good class", or "model class" (Kaur, 2008; Li, 2011; Seah & Wong, 2012; Cai & Wang, 2010), limited attention is given to preservice teachers' (PSTs) conception of effective problem-solving instruction in the current literature in the US context. The purpose of this study is to explore PSTs' conceptions of problem solving and effective problem-solving instruction and to investigate any relationship that might exist among PSTs' conceptions of effective problem-solving instruction, problem-solving, and their problem solving performance. In exploring the relationship between PSTs' conception of effective problem-solving instruction and their problem solving abilities, we specifically focus on fraction topics because it is often reported that not only students but also teachers have difficulties in understanding fractions and fraction operations (NRC, 2004; Son & Crespo, 2009. The research questions that guided this study are: (1) What are the characteristics of PSTs' thinking about effective problem-solving instruction and problem solving?; (2) Is there any relationship among PSTs' conceptions of effective problem-solving instruction, problem solving, and their problem solving performance?; and (3) What are the PSTs' views on what it takes to develop effective problem-solving instruction?

Theoretical perspectives

What constitutes effective problem-solving instruction?
      Problem solving is a powerful vehicle for students' mathematical learning (NCTM, 2000). Schroeder and Lester (1989) identified three types of teaching approaches to problem solving that have been emphasized at different periods of time in mathematics education: (1) teaching *for* problem solving, (2) teaching *about* problem solving, and (3) teaching *through* problem solving. Each of these perspectives offers different affordances. The first approach involves teaching skills or abstract concepts first and then students apply the learned skills or concepts to solve the given problems. The second approach indicates teaching students the process of problem solving or strategies for solving problems explicitly. In the book of *How to Solve it* (1945), George Polya generalized the four steps that can be used regardless of subject matters—(1) identifying a problem, (2) designing a strategy, (3) implementing, and (4) looking back. Teachers explicitly teach the aforementioned four-step process with strategies for problem solving (i.e., approaching methods to a problem). The third perspective includes classroom instruction where students learn mathematical concepts through real contexts and problems, which helps students build meaning for the concepts before moving to abstract concepts (Boaler, 2008; NCTM, 2000, 2014).
      Various criteria can be used in specifying the features of effective problem-solving instruction. Drawn from Stanic and Kilpatrick (1989), we believe that *problem solving as art* should be a goal of effective problem-solving instruction. According to Stanic and Kilpatrick, three different meanings were attributed to the notion of problem solving in mathematics education-- *problem solving as means to a focused end, problem solving as a skill,* and *problem solving as art.* Different from the first and second perspectives where problem solving is viewed as a means to practice skills or as one skill taught in school mathematics, problem solving should be viewed as an act of discovery through creative use of mathematical thinking. Thus, among the three types of teaching approaches by Schroeder and Lester (1989), we consider the third approach--*teaching mathematics through problem solving* as effective problem-solving instruction (NCTM, 2000, 2014).

Research on teachers' conceptions of problem solving and effective instruction
      Prior research has documented that teachers' beliefs and conceptions about the subject and its teaching interact and influence mathematics teachers' planning and delivery of instruction which may impact student achievement (Koehler & Grouws, 1992). However, research directly addressing the issue of 'good (or effective)' classroom instruction from teachers' perspectives is a relatively new endeavor in mathematics education (Cai, Kaiser, Perry, & Wong, 2009). In particular, limited attention is given to the issue of 'good (or effective)' problem-solving instruction from teachers' perspectives in the current literature in the US context.
      Kaur (2008, 2009) carried out a series of studies in Singapore where 8th grade students were asked to describe the qualities of a "good mathematics class" and the "best mathematics teachers". Kaur concluded that "good mathematics teaching in Singapore is student-focused" (2009, p. 346). However, Shimizu (2006), who investigated Japanese students' perceptions on good mathematics lessons, reported that Japanese students consider that a mathematics class is good when there is a "whole class discussion" (2009, p. 316). In general, the research presented herein shows that the views of teachers and students regarding "good teaching" or "effective teaching" vary in relation to multiple factors.
      Using the three meanings of problem-solving by Stanic and Kilpatrick (1989) and the three teaching approaches to problem solving by Schroeder and Lester (1989), we explored PSTs' conceptions of problem solving and effective problem-solving instruction.

Methods

96 PSTs from two different university sites – one from a large northeastern university and the other from a large southwestern in the US – were invited for this study. Participants majored in elementary education and they were either in their sophomore, junior or internship year. A written task was used for the study, which consists of two parts (see Fig. 1).

Part 1: Please answer the following questions in as much detail as possible.
1. When people say problem solving, what does the word "problem solving" mean to you?
2. What do you believe constitutes effective problem-solving instruction?
3. What skills are necessary to create effective problem-solving instruction?
4. How do you believe the skills necessary for teachers to create effective problem-solving instruction develop?

Part 2: Solve the following problems.
1. At both Rivers High School and Mountainview High School, ninth graders either walk or ride the bus to school. 6/7 of the 9th grade students in Rivers High School ride the bus, while 7/8 of the 9th grade students in Mountainview High School ride the bus. If there are 40 9th grade students who walk at Rivers and 25 9th grade students who walk at Mountainview, in which school do more students ride the bus? In which school do a greater fraction of the students ride the bus? Explain your strategies or solutions as much as in detail.
2. For each picture shown below, (i) write a fraction to show what part is shaded. For each picture, (ii) describe in pictures or words how you found that fraction, and why you believe it is the answer.
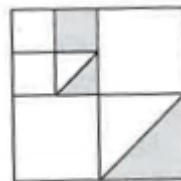
3. Merlyn spends $60 of her paycheck on clothes and then spends 1/3 of her remaining money on food. If she had $90 left after she buys the food, what was the amount of her paycheck? Explain your solution method as much as in detail. You may use representations (e.g., diagrams, rectangles, number line etc.).

Figure 1: Main task of this study.

For the analysis of PSTs' written response to problem solving and effective problem-solving instruction, we used an inductive content analysis approach (Grbich, 2007). We initially organized raw data into an Excel spreadsheet, read all of the responses. PSTs' responses to the notions of problem solving and effective problem-solving instruction were categorized based on themes emerging as researchers read multiple cases. Then we explored the subcategories under each analytical aspect according to the framework (e.g., Table 1). Finally, we interpreted the data quantitatively and qualitatively (Creswell, 1988).

For the problem solving task, we first created a rubric based on correctness of PSTs' responses to each item and their problem solving process and assigned a score for each item. To examine relationship among PSTs' conceptions of problem solving, effective problem-solving instruction, and

their problem solving performance, we ran SPSS statistical program (i.e., chi-squared tests and ANOVAs).

Summary of selected findings

PSTs' conceptions of effective problem-solving instruction and problem solving
To investigate PSTs' conception of effective problem-solving instruction, we reviewed their responses and classified the responses into four aspects based on common themes (see Table 1). Among the four aspects, the first one is the *purpose* aspect of problem-solving instruction (i.e., what is a good problem solving lesson aimed at?). The second aspect is *problem features* (i.e., what is considered as a good problem for problem-solving instruction?) and third one is *problem solving steps* aspect (i.e., what step(s) are/is required for problem-solving lesson?), which involves four steps to solve a problem such as identifying a problem, planning a strategy, carrying out and looking back. The last one is *teaching* aspect (i.e., what instructional strategies or teaching practice are needed for effective problem-solving instruction?).

Table 1: Four aspects of PSTs' conception of a good problem solving lesson and frequencies

Category	Sub-category	# of PSTs	Relation to 3 approaches of teaching PS
1. Purpose aspect (28)	To find a good solution	7	For
	To develop critical/ creative/ logical /reflective thinking (cognitive aspects)	16	Through
	To develop a good understanding of mathematics	5	Through
2. Problem aspect (41)	a. Word problems	2	For/Through
	b. Real-life problems	2	Through
	c. Problems that allow students to use their prior knowledge	5	Through
	d. More practice problems that allow students to apply the same technique	6	For
	a. Problems that require different strategies/multiple solutions	13	Through
	b. Problems that require explanations	0	Through
	c. Problems that require various representations	3	Through
	d. Problems that require creativity	1	Through
	h. Problems that are not overwhelming/not too difficult	9	For
3. Problem solving steps aspects (42)	a. Structuring a lesson based on all four problem solving steps	14	About
	b. Identify problem	12	About
	c. Devise a strategy	3	About
	d. Carry out	2	About
	e. Look back	11	About
4. Teaching aspect (78)	a. Emphasizing different/multiple ways of solving a problem	26	Through
	b. Allowing students to share and discuss their ideas	8	Through
	c. Making sure if students understand the topic	7	Through
	d. Giving examples about how to solve	6	For
	e. Giving definitions	2	For
	f. Providing hands-on manipulatives	1	Through
	g. Giving enough time to work on problems	5	For/Through
	h. Providing a direct and clear direction and structure	15	For/About
	i. Engaging students in solving a problem mentally	6	Through
	j. Lessons that are interesting to students	2	Through

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Note. Majority of PSTs addressed multiple categories. These responses were coded in multiple categories as long as the categories were present in their written responses.

Out of the four aspects, the most popular category is teaching aspect, followed by problem solving steps aspect, problem aspect, and purpose aspect. Among PSTs who mentioned about the teaching aspect, 26 PSTs emphasized that different/multiple ways of solving a problem is important for effective problem-solving instruction. However, interestingly, many PSTs out of the 26 PSTs also considered that it is important to provide a direct and clear direction (15 PSTs) or give examples about how to solve a problem (6 PSTs). This finding suggests that our PSTs perceived the value of multiple solutions, but they believed that they could teach the different/multiple solutions through a *direct instruction* rather than through student-centered discussions. After identifying the four aspects, we collectively considered them to categorize PSTs' conception of effective problem-solving instruction into the three groups by referring to Schroeder and Lester's (1989) identification. Out of 96 participants, 42 participants considered effective problem-solving instruction as teaching *about* problem solving, 23 participants as teaching *through* problem solving, and 31 participants as teaching *for* problem solving. This finding indicates that despite the consistent emphasis on teaching *through* problem solving in current mathematics education, a large portion of our PSTs still did not have a clear view of teaching *through* problem solving.

In a similar way to what we analyzed for PSTs' conception of effective problem-solving instruction, to explore their conception of problem solving, we reviewed their responses and classified the responses into four aspects: problem aspect, process aspect, purpose aspect, and knowledge/skills/ability required. Out of the four aspects, the purpose aspect is the most frequent, followed by process aspect, and problem aspect and only a small number of PSTs considered the aspect of knowledge/skills/ability required for problem solving (Son, Lee, & Arabeyyat, 2015). Based on this analysis, we categorized PSTs' conception of problem solving into three groups by referring to Stanic and Kilpatrick's (1989) identification. Out of 96 PSTs, 55 PSTs considered problem solving as *means* to a focused end, 27 PSTs as a *skill*, and 14 PSTs as *art* of discovery.

Relationship between PSTs' conceptions and their mathematical performance

A chi-squared test showed that there is a positive relationship between PSTs' conception of problem solving and their conception of effective problem-solving instruction, $\chi^2 = 16.888$, $df = 4$, $p = 0.002$. That is, PSTs who perceived problem solving as *means* to a focused end seem to consider effective problem-solving instruction traditionally in that their views on effective instruction were categorized into "teaching *for* problem solving". In addition, the results for ANOVAs revealed that there is a significant difference of mean scores concerning problem solving competence among groups of PSTs who perceived different views on problem solving, $F(2, 75) = 3.292$, $p = .042$. PSTs who perceived problem solving as *art* showed highest mean scores in the problem solving tasks, followed by PSTs who with problem solving as *means* to a focused end (see Table 2).

Table 2: Results from ANOVAs test

		Sum of Squares	df	Mean Square	F	Sig.
1	Between Groups	349.936	2	174.968	3.297	0.042
	Within Groups	3980.0120	75	53.067		
	Total	4329.949	77			
2	Between Groups	25.768	2	12.884	0.216	0.807
	Within Groups	2743.049	46	59.631		
	Total	2768.816	48			

Note: 1 = PSTs' views on problem solving; 2 = PSTs' views on effective problem-solving instruction

However, there was no such connection between PSTs' conception of effective problem-solving instruction and their mathematical performance, $F(2, 46) = 0.216$, $p = .807$. Appendix A presents PSTs' mathematical competence in the problem solving task, focusing on PSTs who perceived problem solving as art.

What knowledge and skills are needed for effective problem-solving instruction?

When the PSTs were asked to indicate types of skills necessary for creating effective problem-solving instruction, our PSTs pointed out not only knowledge and disposition but also skills such as problem solving skills, teaching skills, and lesson design skills (see Table 3). Out of the three big categories, the most popular category is skills, followed by knowledge and dispositions.

Table 3: PSTs' report on knowledge and skills needed for *effective problem-solving instruction*

Knowledge and skills for good lessons	
1. Knowledge	*72*
1.1 Knowledge about content.	22
1.2 General pedagogical knowledge.	6
1.3 Knowledge about problem solving steps.	11
1.4 Knowledge on different solving strategies.	18
1.5 Thinking (critical, organized, creative, thinking)	15
2. Skills	*79*
2.1. Problem solving skills	39
2.1.1. Identifying or understanding questions/problem	18
2.1.2. Breaking problem into easier steps	2
2.1.3 Organizing or logical skill	6
2.1.4 Thinking backward/Reflect or check answers	1
2.1.5 Following the 4 steps (Work on the process behind problem solving)	4
2.1.6 General good problem solving skill (Not specified)	8
2.2. Teaching skills	30
2.2.1 Unpacking knowledge/Step by Step direction	3
2.2.2 Explaining/ Articulating	10
2.2.3 Answering students' diverse questions	3
2.2.4 Attending to students' thinking or work (Noticing skill)	6
2.2.5 Engaging/Motivating students	4
2.2.6 General teaching skill (Not specified)	4
2.3. Lesson design skills	10
2.3.1. Lesson planning	4
2.3.2 Creating a good problem	6
3. Attitude and disposition	*50*
3.1 Patient	15
3.2 Open-mind	12
3.3 Creative	18
3.4 Collaborative	2
3.5 Efforts or working hard	5

When we further explored PSTs' perception of what it takes to develop effective problem-solving instruction, ten categories emerged shown in Table 4. A large portion of the PSTs considered that necessary skills for effective problem-solving instruction are developed through teaching experience (31) or problem solving itself (44).

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Table 4: Resources for developing skills for good problem solving lessons

Category	Total
1. By practicing problem solving or solving problems to be a good problem solver through mastering problem solving skills	44
2. Experience or trial and error (e.g., teaching and making lesson plans)	31
3. Developing dispositions (e.g., creativity, open-minded, compassionate toward kids)	12
4. Time	11
5. By learning knowledge or skills (e.g., different problem solving methods)	10
6. By understanding students' work or working with students	6
7. Teacher education program (e.g., by taking classes)	5
8. Working with other teachers	4
9. Finding resources (e.g., standard, book, videos)	3
10. By observing experienced teachers, experts, or mentor teachers	1

Discussion and Implications

This study contributes to the current literature on problem solving and the knowledge base of teacher education. In particular, this study has implication for teacher educators working to design mathematics education courses for PSTs, as well as for researchers interested in furthering understanding of teachers' knowledge, beliefs, and problem solving strategies. The findings of this study suggest that teacher educators need to find a better way to help PSTs perceive problem solving as *art* and effective problem-solving instruction as teaching mathematics *through* problem solving. One approach would be: Have PSTs experience three different perspectives of teaching mathematics and compare affordances and limitations of each approach. Then teacher educators need to give PSTs more opportunities to experience teaching *through* problem solving in their mathematics methods courses where PSTs engage in mathematical modes of thought by analyzing and interpreting the problems (Son, 2013; 2016). Future studies need to be done with different research tools and in multiple contexts, possibly using interviews or observations to provide more detailed explanations for teachers' responses. Furthermore, intervention studies that experiment with these suggestions are needed to find a better way to support PSTs' conceptions regarding problem solving, problem solving lessons, and their problem solving abilities.

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IMPROVING PRE-SERVICE TEACHERS' NOTICING WHILE LEARNING TO LAUNCH

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Effectively launching a task involves surfacing and addressing misconceptions so that students can make progress on the task. Launching a task is supported by teachers' noticing (interpreting and responding to students' thinking). We investigated the degree to which an intervention supported improvements in pre-service secondary teachers' (PSTs') abilities to notice when launching a rich proportional reasoning task. Through the use of representations of practice and an intervention consisting of opportunities to make sense of and discuss multiple choice options for interpreting and responding to students' thinking, we analyzed whether PSTs improved their abilities to notice. After the intervention, PSTs improved in responding to a student's misconception when PSTs concurrently exhibited expertise with interpreting students' thinking.

Keywords: Teacher Education-Preservice, Instructional Activities and Practices

Effectively introducing or "launching" rich mathematical tasks is an important teaching skill (Jackson, Garrison, Wilson, Gibbons, & Shahan, 2013; Stein & Lane, 1996). When launching rich tasks, effective teachers support students as they make sense of the context. Effective teachers also clarify the problem to be solved and surface and address misconceptions that obscure core mathematical issues.

Jacobs, Lamb, and Philipp (2010) have identified attending, interpreting and deciding how to respond as three aspects of teacher noticing. To launch a task well, teachers need to *attend* to students' thinking, *interpret* students' thinking, and then *decide how to respond* to students' thinking in ways that support students' sense-making without reducing the cognitive load of the task. In this paper, we report on our efforts using animated representations of teaching in pedagogy courses to improve pre-service teachers' (PSTs') abilities to interpret and respond to student thinking while launching a rich mathematical task.

Perspectives on Teacher Learning

When confronted with complex phenomena, experts rely on *schema* to make sense of situations and inform decision-making (Bransford, Brown, & Cocking, 2000). Given the complexity of noticing, we hypothesize that developing sophisticated schema would enable PSTs to interpret and respond to student thinking during a launch. Their interpretations and responses would then support students' engagement with the task.

There is evidence that PSTs possess naïve schema that interfere with their efforts to engage students in productive mathematical work. When interpreting student thinking, many teachers have two categories: students that "get it" and students that do not (Otero, 2006). Additionally, PSTs often assume that students who perform procedures correctly have conceptual understanding (Bartell, Webel, Bowen, & Dyson, 2013). These schema may make it difficult for PSTs to identify and engage students' prior understandings, an essential element of effective launches in which students make sense of rich problems. Therefore, one of our goals for PSTs' learning was to expand their schema for interpreting student thinking in ways that would lead to improved launches. In particular, we hoped that PSTs would become able to: (a) identify important student misconceptions and (b) differentiate between procedural and conceptual understanding.

When responding to student thinking, PSTs also may create challenges for students based on naïve schema. Given the persistent prevalence of IRE (initiate, respond, evaluate) discourse patterns in American math classrooms (Franke, Kazemi, & Battey, 2007), PSTs might be pre-disposed to

respond to student thinking dualistically by correcting incorrect responses and praising correct ones. This precludes opportunities for reasoning and elevates correct answers over deeper understanding (Stein, Grover, & Henningsen, 1996). To promote students' opportunities to reason about mathematics, especially in working through misconceptions that could interfere with productive mathematical work, we wanted to expand PSTs' schema for responding to student thinking so that they would: (a) leverage misconceptions to engage the class in reasoning about important mathematics and (b) assert the value of understanding why a solution makes sense when confronted with a procedural response.

The Process of Teacher Learning

"Practice-based" teacher education situates teacher learning in the actual work of teaching (Ball & Cohen, 1999). However, teaching is complex (Lampert, 2001), and novices struggle to make sense of big ideas when confronted with multiple dynamics of real classroom situations. Representations of teaching that retain some of the complexity of practice, while also providing opportunities for novices to focus on specific elements of practice are a valuable tool for teacher educators as they struggle to situate learning in practice while simultaneously focusing on core ideas (Herbst, Chazan, Chen, Chieu, & Weiss, 2011). *LessonSketch* (http://www.lessonsketch.org) is a web-based platform that allows for creating cartoon storyboards of classroom interactions and embedding them in interactive assignments for PSTs. This platform can present complex situations, but focuses PSTs on specific elements of those situations, thus shaping their attention.

Research Questions

We created a *LessonSketch* experience to introduce and expand PSTs' schema for interpreting and responding to typical examples of student thinking in the context of a rich task on proportional reasoning. As a part of the experience, we presented multiple-choice options for how to interpret and respond to students' thinking. We hypothesized that by working to make sense of possibly new categories for interpreting and responding in the context of potentially realistic classroom interactions during a lesson, PSTs could expand their own schema, which could improve their abilities to effectively interpret and respond to student thinking.

Our investigation was guided by these research questions: (a) After completing a task designed to expand schema for interpreting and responding to student thinking, in what ways, if at all, do PSTs show improvement in interpreting and responding to student thinking during the launch of a complex task? (b) In what ways, if at all, do these improvements reflect the options introduced in the multiple-choice task designed to expand PST's schema?

Context and Methods

This study took place in secondary mathematics methods courses in undergraduate teacher preparation programs at two different Mid-Atlantic universities. Both of these courses were connected to field experiences, shared an emphasis on proportional reasoning, and took place near the end of the PSTs' teacher education coursework. Both courses also focused on pedagogical strategies that support students with developing conceptual understanding of important mathematics through problem-solving, including planning and enacting effective lesson launches.

To engage PSTs in thinking through the launch of a rich task, we designed a *LessonSketch* experience which featured the initial reactions of ten different students to the following task:

> At the hardware store they sell 30 pound bags of sand for 6 dollars. At the lumberyard they sell 50 pound bags of sand for 9 dollars. Where should I buy the sand? Which store has the better deal?

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The experience also indicated that the learning goal of the lesson featuring this task was to understand the quantitative relationships in the problem and understand why scale-factor or unit rate strategies would support finding the solution.

We designed two different versions of this *LessonSketch* experience. One version was administered twice and served as a pre- and post-assessment. In this version PSTs worked individually, writing an interpretation of each student's thinking, and a description of what they (the PST) would plan to do in response.

The second version of the experience was the multiple-choice intervention, designed to expand PSTs' schema for interpreting and responding to student thinking. The task and specific examples of student thinking remained the same. For each instance of student thinking, participants were given a series of choices that represented the schema we were trying to introduce about (a) interpreting and (b) responding to students' thinking. For example, choices for interpreting, or "what can you tell about this student's thinking," included (among others):

- This student is thinking about the mathematics of the lesson in a way that will lead to a correct solution
- This student has a misconception that will get in the way of them creating a correct solution
- This student is working to remember and/or apply a procedure without any evidence of understanding the underlying mathematical relationships

Choices for responding ("What would you plan to do?") included (among others):

- Facilitate a discussion during the launch in which students respond to this student's idea; students do most of the talking
- Briefly explain or clarify to the whole class during the launch. You, the teacher, do most of the talking.
- Explain or clarify to individual student during the launch.

PSTs completed this second version in pairs. We hypothesized that the opportunity to discuss with partners would support PSTs in making sense of the multiple-choice options.

Data Analysis

This analysis focuses on PSTs' interpretations and responses to four of the ten items. Two items (1 & 6) represented univariate thinking: a student attended to only one of the quantities in the problem (i.e. only the sand or only the price) rather than a relationship between the amount of sand and the price (Harel, Behr, Lesh, & Post, 1994). The other two questions (2 & 10) represented procedural thinking: the student talked about a rule for solving the problem without evidence of deeper conceptual understanding.

We created codes for determining whether PSTs' answers were at the levels of novice, emerging expert, or expert. These codes were developed through an iterative process that involved identifying key elements of novice and expert answers, coding answers independently, and meeting to revise disagreements or address questions. For the items involving misconceptions (1 & 6), we developed the following criteria for novice, emerging expert and expert.

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Table 1: Codes for Student Misconception Items

	Novice (n)	Emerging Expert (em)	Expert (ex)
Interpret	• PST does not identify paying attention to one quantity as the problem	• PST uses vague language to identify that student has misconception involving attending to both quantities.	• PST clearly identifies that student has attended to only one quantity
Respond	• PST explains or tells student about relationship between quantities • PST tells student what to do • Solves the problem during the launch	• Asks questions that promote reasoning, directed to single student • Asks questions to the whole class, but does not promote reasoning or discussion	• Initiates a discussion with whole class that promotes reasoning about the relationship between the two quantities

For the items involving procedural student thinking (2 & 10), we developed the following criteria for novice, emerging and expert.

Table 2: Codes for Procedural Thinking Items

	Novice (n)	Emerging Expert (em)	Expert (ex)
Interpret	• Assumes conceptual understanding • Assumes procedure will produce correct answer • No mention of understanding • Non-specific descriptions of using prior knowledge	• Assumes that student does not understand • Aware of possible lack of understanding but assumes correct response *Or* • Qualifying language around prior knowledge	• Aware that student may, or may not understand concepts *and* • Aware that student may not get correct answer
Respond	• Amplifies procedure as method without any prompt for sense-making • No redirection to sense-making • Solves the problem during the launch	• Pushes sense-making with individual student • Discourages using formula • Changes task to forbid or discourage formula use	• Clarifies to the whole group that any solution strategy is acceptable as long as you can explain why your strategy works and makes sense

To develop a shared understanding of our refined codes, we each coded a subset of PSTs and met to resolve disagreements. We then coded the rest of the data. As we identified answers that were challenging to code, we conferred and reached agreements.

To determine whether PSTs' answers improved, we looked at whether or not individual PSTs' answers changed from pre- to post-assessment. For each of the items, we classified PSTs as same (no change), improve or decline. Types of improvements were: from novice to expert, from novice to emerging expert, and from emerging expert to expert. Finally, we tabulated the total results for each item to identify overall trends.

Results

Student Misconceptions: PSTs' Interpretations and Responses

For items 1 and 6, designed to assess PSTs' interpretations of and responses to a specific student misconception.

Item 1 depicts a student saying, "He should just buy it at the lumberyard. He gets a whole lot more sand there."

Item 6 depicts a student saying, "Hardware store; it's only six dollars, not nine."

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We hypothesized that there would be consistency among PSTs' answers for these two items due to the similarity of the design of the items.

Table 3: Changes (Pre-Post) - Interpreting and Responding to a Misconception

Item 1: Interpretation	Item 6: Interpretation	Item 1: Response	Item 6: Response
Same (19):	**Same (22):**	**Same (13):**	**Same (13):**
17 (expert-expert)	20 (ex-ex)	2 (ex-ex)	1 (ex-ex)
2 (emerging-emerging)	0 (em-em)	8 (em-em)	7 (em-em)
0 (novice-novice)	2 (n-n)	3 (n-n)	5 (n-n)
Improve (8):	**Improve (4):**	**Improve (15):**	**Improve (16):**
5 (emerging-expert)	4 (em-ex)	5 (em-ex)	5 (em-ex)
0 (novice-emerging)	0 (n-em)	6 (n-em)	7 (n-em)
3 (novice-expert)	0 (n-ex)	4 (n-ex)	4 (n-ex)
Decline (3):	**Decline (5):**	**Decline (3):**	**Decline (2):**
1 (expert-emerging)	2 (ex-em)	1 (ex-em)	1 (ex-em)
2 (emerging-novice)	1 (em-n)	2 (em-n)	1 (em-n)
0 (expert-novice)	2 (ex-n)	0 (ex-n)	0 (ex-n)

Most PSTs were able to clearly identify the misconception. For interpretations on item 1, 83% (25 out of 30) clearly identified the misconception on the post-assessment, noticing that the student engaged in univariate reasoning. Of these, 57% (17 out of 30) did so on the pre-test as well. 27% (8 out of 30) showed improvement from pre to post. Similarly for item 6, 77% (24 out of 31) clearly identified the misconception students were displaying on the post-assessment. Of these 64% (20 of 31) had done so on the pre-assessment; 13% (4 out of 31) showed improvement from pre to post. (Note: The number of total responses differs because some PSTs did not answer every item.)

Approximately half of the PSTs showed improvement in their response to student misconceptions during the launch. PSTs improved their responses in the post-assessment by engaging the whole class in discussion and / or discussing reasoning about the misconception. 48% of PSTs (15 of 31) showed improvement in their responses on item 1. 52% of PSTs (16 of 31) showed improved responses on item 6. Out of these 31 improvements, 24 responses involved facilitating a discussion with the whole class. A typical example in the post-assessment was this PST's answer:

> Before asking for questions, I could pose the question that if you get more sand at the lumberyard then why not just buy it at the lumberyard? Then let the groups of students turn and talk about that before opening it up to the class.

However, only a subset of the PSTs with improved responses newly included addressing the whole class about a misconception during the launch in their post-assessment answer. Eight PSTs, representing twelve instances of improvement, did not mention addressing the whole class at all during their pre-assessment, yet they did mention discussing the misconception with the whole class during the post test. These improved responses reflected an option from the multiple choice assessment: facilitate a discussion.

The rest of the PSTs' responses improved in ways less aligned with the options in the multiple choice assessment. Six PSTs, representing 11 instances of improvement, responded with whole group discussions in both pre- and post-assessments. The nature of the discussion differed, however; in the pre-assessment, PSTs discussed correct solutions or clarified the context, and they discussed a misconception with the whole class in the post-assessment. Other PSTs (six, representing 8 instances) also improved by focusing on addressing the misconception in the post-assessment, but their post-responses still involved addressing an individual rather than the whole class.

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Students' Procedural Thinking: PSTs' Interpretations and Responses

Two items (2 and 10) assessed PSTs' interpretations of and responses to students' procedural thinking (see Figure 1).
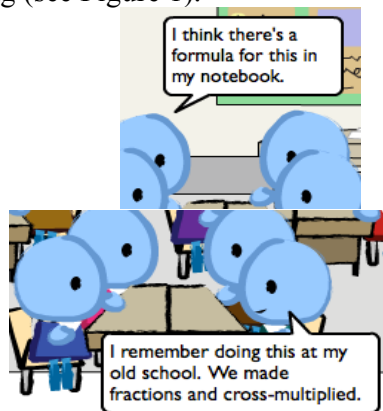
Item 2
Item 10
Figure 1. Procedural Student Thinking.

These items did not provide evidence of whether or not students had conceptual understanding. We hypothesized that PSTs would be more likely to interpret procedural thinking in item 10 incorrectly (i.e. as evidence of conceptual understanding) because that item contained more specific details about students' thinking.

Table 4: Changes (Pre-Post) - Interpreting and Responding to Procedural Thinking

Item 2: Interpretation	Item 10: Interpretation	Item 2: Response	Item 10: Response
Same (24):	**Same (23):**	**Same (19):**	**Same (23):**
0 (expert-expert)	0 (ex-ex)	1 (ex-ex)	0 (ex-ex)
16 (emerging-emerging)	5 (em-em)	10 (em-em)	7 (em-em)
8 (novice-novice)	18 (n-n)	8 (n-n)	16 (n-n)
Improve (5):	**Improve (6):**	**Improve (3):**	**Improve (4):**
2 (emerging-expert)	3 (em-ex)	0 (em-ex)	1 (Em-ex)
3 (novice-emerging)	2 (n-em)	2 (n-em)	2 (N-em)
0 (novice-expert)	1 (n-ex)	1 (n-Ex)	1 (N-Ex)
Decline (2):	**Decline (2):**	**Decline (9):**	**Decline (4):**
1 (expert-emerging)	0 (ex-em)	1 (ex-em)	0 (ex-em)
1 (emerging-novice)	0 (em-n)	8 (em-n)	4 (Em-n)
0 (expert-novice)	2 (ex-n)	0 (ex-n)	0 (Ex-N)

Most PSTs did not change their interpretations of students' procedural thinking or their responses to students' procedural thinking. For interpretations on item 2, 77% (24 out of 31) of the participants' answers did not change from pre- to post-assessment (16 were at emerging expertise and 8 were at novice). For responses on item 2, 61% (19 out of 31) of the participants' answers did not change (10 remained at emerging expertise, 8 were at novice and 1 at expert). For interpretations on item 10, 74% (23 out of 31) of the participants did not change (5 remained at emerging experts, 18 at novice and none at expert). For responses on item 10, 74% (23 out of 31) of the participants did not change (7 were at emerging expertise, 16 were at novice and none were at expert).

However, there are some important differences in the overall interpretations for items 2 and 10 that indicate that PSTs' conceptualizations of procedural thinking may be multi-dimensional and nuanced. In particular, on item 2 the majority of PSTs identified problems with procedural

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understanding by the post-test (20 out of 30 were emerging and 2 out of 30 were expert). In contrast 20 out of 31 PSTs failed to identify any problem with the procedural approach taken by the student in Item 10. As an example, for an interpretation for item 10 on the post-assessment, a PST wrote, "The student is aware of a more advanced problem solving method that they learned previously." This would be a novice interpretation because the PST interpreted the use of cross multiplying as "advanced" without problematizing whether or not there was evidence that the student understood the quantitative relationships in the problem. Also, for a response on item 10, a PST wrote, "I would tell the student to try to solve the problem using what he remembers about cross multiplying. I would then walk away and come back minutes later." This is a novice response because the teacher does not push the individual student (or the class) to reason about why this strategy makes sense.

These results did not reflect the influence of multiple choice options on PSTs' adjustments of either their interpretations or their responses on the post-assessment. We expected that introducing PSTs to the following option for interpreting students' thinking would result in expanding their schema to move toward stronger interpretations of students' procedural thinking: "The student is working to remember and/or apply a procedure without any evidence of understanding the underlying mathematical relationships." But PSTs did not change their answers to reflect this choice. By the post-assessment, our PSTs still did not recognize that students' procedural thinking is not enough information to determine whether the student conceptually understands the quantitative relationships in the task during the launch.

Discussion

Overall, these results illustrate the following: (a) PSTs with expertise in interpreting students' misconceptions can improve their capabilities with responding to students' thinking during a launch, possibly after being introduced to and having opportunities to discuss new options for responding. (b) PSTs without expertise in interpreting students' procedural thinking may not be likely to improve interpreting or responding after being introduced to and discussing multiple choice options. Thus, the multiple choice options appeared to provide some support with expanding PSTs' schema for responding to a student misconception when PSTs had expertise in interpreting students' thinking.

Improvements in PSTs' responses to a student's misconception (items 1 and 6) are quite pronounced. One potential explanation is that PSTs' expertise in interpreting students' thinking on these items supported stronger responses to students' thinking, as their interpretations to these items were strong. It appeared that, for these PSTs, better understanding of student thinking and how it is connected to the important mathematics supported them with making instructional decisions. Recognizing that students are demonstrating a misconception may motivate PSTs to respond more actively and prompt them to look for possible responses.

A subset of the PSTs that improved responses to a student's misconception seemed to mirror one of the categories that we used in the multiple choice intervention. It may be that the motivation that PSTs had to respond to a student misconception made them more receptive to the categories introduced in the intervention, and more likely to remember and refer to them later, in the post-assessment.

The possible role of interpretation in motivating improvement in pedagogical responses can also be seen in items 2 and 10. Very few PSTs gave expert interpretations of students' procedural thinking, and none of them had expert interpretations on the pre-test. Because PSTs failed to identify the full problematic nature of procedural thinking in the context of this launch, they may have been less motivated to look for new ways of improving their responses. This may explain why there was so little improvement in PSTs' responses for these items.

Additionally, these PSTs performed worse on item 10 than item 2, likely because item 10 was more detailed and specific about students' procedural thinking. This points to important elements of procedural understanding. One element of understanding procedures consists of understanding why

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the procedure works, what important mathematical concepts lay beneath it. The continued tendency to ascribe conceptual understanding to the student in item 10 indicates that the PSTs in this study struggled with this idea. A second element of procedural understanding entails recognizing when specific procedures are appropriate, including identifying and remembering specific problem types and larger mathematical relationships. The fact that so many PSTs identified the student in item 10 as having an understanding of the mathematics suggests that they can see this element of procedural fluency (knowing which procedure is appropriate to use in a particular situation) and that they conflate such knowledge with conceptual understanding.

Acknowledgments

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ELEMENTARY MATHEMATICS TEACHER EDUCATOR'S LEARNING THROUGH LESSON STUDY: A CROSS-INSTITUTIONAL STUDY

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The purpose of this paper is to present how mathematics teacher educators at five different U.S. institutions of higher education refined and informed their own practice through the process of lesson study. Although much used at the K-12 level to improve teaching practice, lesson study is not commonly used in higher education. The mathematics teacher educators designed and implemented a common lesson to develop noticing of children's mathematical thinking in pre-service teachers and gained insight into their own practice through lesson study methodology. Qualitative analysis of each lesson iteration including field notes, discussions and reflections on the lesson and pre-service teachers' work, transcription of lessons, and meeting notes were analyzed to identify processes and themes.

Keywords: Teacher Education-Preservice, Instructional Activities and Practices

Introduction

This study was designed to improve the teaching and learning within a mathematics methods or content course for elementary education majors. Using data from the planning and teaching of one common lesson, mathematics teacher educators (MTEs) at five institutions investigated their own teaching and instructional practices through an iterative process of lesson study. As part of the lesson study, the MTEs researched how their instruction during the focus lesson affected what their pre-service teachers (PSTs) noticed when analyzing elementary student work samples and how PSTs made whole class instructional decisions based on those samples. In participating in this iterative cycle of lesson improvement, the MTEs sought to answer the following research question: How do MTEs from different institutions understand, inform, and improve their own teaching through a lesson study?

Theoretical Framework

Lesson study is a process of investigating instruction with the goal of instructional improvement. The iterative lesson study process is generally organized into a cycle of four steps:

- (a) study the curriculum and formulate goals,
- (b) plan,
- (c) conduct research on the lesson, and
- (d) reflect (Lewis, 2002; Lewis & Hurd, 2011).

Since arriving from Japan in the 1990s, lesson study has primarily been used to investigate instruction at the K-12 level. However, Cerbin and Kopp (2006) adapted the cycle slightly for higher education. In later years, Kamen et al. (2011), all MTEs, used lesson study to investigate how to improve their teaching in mathematics methods course so that PSTs could see the value in understanding and encouraging students' invented strategies. Through six iterations they learned better processes for engagement and encouraged more diverse thinking. In their final iteration they

concluded that PSTs were "pushed to actually study the mathematical thinking of others through a deliberate focus on analyzing their peers' mathematical thinking" (p. 173).

In their theoretical framing of how lesson study produces instructional improvement, Lewis, Perry, and Hurd (2009) built upon the traditional four lesson study components by theorizing three ways in which lesson study creates change: (a) teachers' knowledge and beliefs, (b) teachers' professional community, and (c) teaching-learning resources. This framework for understanding potential changes guided our analysis.

Connection to the PMENA 2016 Conference Theme

By conducting this lesson study in higher education and in geographically different institutions around the U.S., this study pushes the border of traditional lesson study work. The idea for this study emerged out of a common goal to understand how to support PSTs to make the leap from planning instructional next steps for one student based on each student's work to planning out instructional next steps based on a set of student work for a whole class.

Methods

Location and Context

This lesson study occurred over two semesters in five different institutions of higher education, located in diverse locations – the West Coast, Midwest, South and Northeast US, with four of the five being public. The institutions varied in size with enrollment ranging from 3,600 to 36,000 and from 6 to 30 students per class. PSTs at four institutions were enrolled in a four year Elementary Education program and at one institution were enrolled in a post-baccalaureate Multiple Subject (elementary) Credential program. The MTEs taught the first mathematics methods or content course in these programs. Similar to the nation's current enrollment in elementary teacher education programs, most students were white and female.

Research Design

Lesson study methodology began with the MTEs investigating student characteristics and long term learning goals for PSTs. A decision was made to focus on the PSTs' "noticing" of children's mathematical thinking with an emphasis on making instructional "next step" decisions (Jacobs, Lamb, & Philipp, 2010).

The researchers started the second phase of planning a lesson to support and facilitate the intended learning goals by developing a research-based lesson that was taught at one of the locations. The group reviewed the video of the lesson and reflected on various aspects of teaching using evidence from the lesson. The lesson was revised and then taught again by another member of the research team. The process of revising and teaching the lesson was done for five iterations (after each MTE conducted the lesson). The result was a final, polished lesson.

The data from various sources in the form of lesson plan iterations, explicit meeting notes, videotapes of the taught lesson, reflections, and discussions on PSTs' work was analyzed in stages. With each iteration of the lesson, the researchers met to analyze the video to determine what best supported PSTs in making thoughtful instructional "next steps." These included lesson strengths and areas for improvement: helpful questions, missed opportunities, teacher moves, etc.

Lesson Content of the Lesson Study

The designed lesson focused on multiplication solution strategies and involved an analysis of the *Mr. Harris and the Band Concert* case scenario, an NCTM publication in the *Principles to Actions* Professional Learning Toolkit (NCTM, n.d.), which also included six students' solution strategies. To understand the range of student solution strategies (direct modeling-> counting strategies->derived facts), MTEs presented videos of students solving multiplication problems in various ways. Then the

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PSTs solved the Band Concert problem individually in multiple ways and discussed their solutions with their table. Next, the PSTs looked at the six samples of student work and analyzed them using the noticing framework by: attending to the math, interpreting the solution strategies, and determining a next step for each individual student. Finally, after a whole-class discussion based on the individual students, the PSTs were asked to determine an instructional next step for the whole class.

Results – Initial Findings

When investigating lesson study, the results include both a description of the process of lesson refinement and areas for potential change identified by Lewis, Perry, and Hurd (2009): (a) teachers' knowledge and beliefs, (b) teachers' professional community, and (c) teaching-learning resources.

Lesson Study Refinement Process

Using lesson study at the university level with MTEs located in five different geographic regions required refinement of the traditional lesson study process. For example, the MTEs could not be physically present to view each other's lessons. This resulted in the need to refine the process to use video and online collaborative tools to both watch and analyze the data. For example, the MTEs used video recording to capture their lesson and uploaded the video to a secure cloud server for all to watch later. While watching, each contributed to a transcription and analysis of the video using Google documents. This allowed for the collaborative discussion and analysis required to refine the lesson for the next iteration.

Areas of Change

Under each category below, an exemplar of a change brought about by lesson study is explained in detail.

Mathematics teacher educators' knowledge and beliefs. After the first lesson iteration in which the PSTs did not study the actual noticing framework of attending, interpreting, and deciding, the MTEs realized the need for PSTs to be explicitly exposed to the components of noticing and thus decided to have the PSTs read Thomas et al. (2014), a practitioner article on noticing. After reading the article, the PSTs were better equipped to notice and interpret what students were doing in the work samples and consider the instructional next steps. As MTEs, we learned that being explicit and providing details of a framework assisted them in applying the framework.

Mathematics teacher educators' professional community. One struggle experienced by all MTEs during the lesson study was the lack of time devoted to the concluding discussion and wrap up to ensure that the goal of the lesson, to plan whole-group instructional next step decisions, was addressed. As early career MTEs, we recognized a common difficulty of concluding our lessons and spent time discussing general ideas and techniques with each other. Though the lesson analysis process, the MTEs developed a community of practice in which they not only discussed improvements to the lesson, but also discussed and refined their teaching practices in general.

Teaching-learning resources. The result of the lesson study was the creation of a refined lesson and tasks for PSTs that used the topic of multiplication and student work to support them to move from analyzing student thinking to planning the "next step" for one student and eventually for a whole class. With each iteration and planning, the MTEs developed strategies and questions that ensured PSTs engaged in whole-group instructional next step decisions. These strategies and questions served a teaching and learning resources for the lesson.

Discussion

The process of lesson study strengthened the practice and community of MTEs. It speaks to the need to create more ways for MTEs to meet and engage with each other across diverse sites in

substantive ways. There need to be additional ways for mathematics teacher educators to find common goals and answer common questions about mathematics education. Working across sites as new assistant professors of mathematics education, the researchers could identify their weaknesses and developed a safe space to not only discuss them, but to work together to try new methods/strategies to overcome them. The outcome was a refined lesson that each MTE can more confidently use with his/her students in the future.

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QUESTIONING THE BOUNDARIES: KNOWLEDGE INTEGRATION IN METHODS AND CONTENT COURSES

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Ball (2000) noted, "subject matter and pedagogy have been peculiarly and persistently divided in the conceptualization and curriculum of teacher education" (p. 241). In this study, we investigated the opportunities secondary pre-service teachers (PSTs) had to integrate their knowledge of mathematics, learners, and pedagogy in a methods and content course taught in tandem. We describe the frequency of opportunities that PSTs had to integrate the various combinations of teacher knowledge and the instructional moves associated with this integration. Recommendations for pre-service teacher education and directions for further research will be provided.

Keywords: Teacher Knowledge, Teacher Education-Preservice

Secondary mathematics teachers must develop a deep and connected understanding of mathematics (M), pedagogy (P), and learners (L) in order to interpret student thinking, choose instructional activities, and plan effective lessons. Unfortunately, pre-service secondary mathematics teachers often perceive a chasm between their education and mathematics courses, creating a divide between their knowledge of mathematics and pedagogy. Ball (2000) succinctly summarized this challenge:

> "Subject matter and pedagogy have been peculiarly and persistently divided in the conceptualization and curriculum of teacher education and learning to teach. This fragmentation of practice leaves teachers on their own with the challenge of integrating subject matter knowledge and pedagogy in the contexts of their work. Yet, being able to do this is fundamental to engaging in the core tasks of teaching, and it is critical to being able to teach all students well" (p. 241).

In addition, "Research that has investigated the development of mathematical knowledge for teaching has shown this process to be less additive (e.g., learn content, then learn to teach it) and more iterative." (Steele & Hillen, 2012, p. 53-54). Preparing future teachers to integrate these different knowledge sets for the purpose of instruction has the potential to improve the quality of mathematics instruction.

To address this challenge, we investigated the opportunities for the integration of M, L, and P in a methods and a content course for secondary pre-service teachers (PSTs). This paper will describe some preliminary findings for the following research questions:

1. What distribution of knowledge types (M, P, and L), and opportunities for potential knowledge integration, were observed in a methods and content course for secondary PSTs?
2. What instructional moves appeared to promote knowledge integration in the methods and content course for secondary PSTs?

Theoretical Foundation

Hiebert, Gallimore, and Stigler (2002) stated that teachers do not separate knowledge types as researchers do, but weave their knowledge together around the problems of practice. With this in mind, we define *knowledge integration* as the use of multiple knowledge types in order to reflect on classroom occurrences or to make instructional decisions. Bishop and Whitefield (1972) proposed that teacher decisions are made using a framework or schema. The main operation of a schema is to

store knowledge through a network of connected pieces of knowledge called "elements" (Marshall, 1995, p. 43). The more connections that exist within a schema, the more useful the schema will be. Hence, the process of knowledge integration leads to the development of a more connected schema, resulting in more informed instructional decisions.

Our investigation of teachers' use of multiple knowledge types was inspired by Shulman's (1987) work on teacher knowledge. Our definitions are as follows:

- *Knowledge of Mathematics (M)*: knowledge regarding the mathematical concept(s) related to the content under investigation. Includes the connections and relationships among ideas, the way(s) and mean(s) of justifying and proving these ideas, and conversations focused on mathematics and reasoning about the mathematical topic.
- *Knowledge of Pedagogy (P)* – knowledge regarding the tasks, curriculum, instructional goal(s), or questions used to further the lesson. Includes comments centered on the lesson implementation or decision-making regarding the flow of the lesson.
- *Knowledge of Learners (L)* – knowledge regarding student thinking. Includes observed as well as anticipated student thinking, conversations about student characteristics, habits, or misunderstandings.

Our definitions of knowledge of mathematics, learners, and pedagogy are related to other models of teacher knowledge, such as mathematical knowledge for teaching (MKT) (Hill, Ball, & Shilling, 2008) or pedagogical content knowledge (Shulman, 1987). However, the definitions in this study are broader in scope, as we are focused more on the *interactions* among these knowledge sets (i.e. knowledge integration). Our study seeks to uncover how PSTs connect their knowledge and if instructional moves exist that promote knowledge integration among PSTs.

Methods

To answer our research questions, extensive field notes were collected for every session of a senior secondary methods and content course of an undergraduate teacher preparation program in the Midwest. A single researcher collected all field notes. The content course explored high school topics from an advanced perspective. The content and methods courses were taught in tandem, with the same set of students, in hopes of further promoting knowledge integration. This arrangement allowed the methods course instructor to use the mathematics of the content course as a foundation for pedagogical discussions. Similarly, the content instructor could highlight and model pedagogical ideas from the methods course.

Using data reduction methodology (Miles & Huberman, 1994) the field notes were initially read to identify individual episodes--sections of the transcript centered on a single idea. Following the identification of episodes, the constant comparative method (Glaser & Strauss, 1967) was used to develop definitions for statements about mathematics (M), learners (L), and pedagogy (P) (see definitions above). These definitions were then used to code all episodes. If multiple forms of knowledge were used during a single episode, we defined this to be an example of potential knowledge integration. For instance, if an episode contained both M and L statements, we identified this as an opportunity for PSTs to integrate their knowledge of M and L. We note that evidence of multiple forms of knowledge in an episode does not ensure knowledge integration, only that the prospect of knowledge integration was present. To answer the first research question we tallied the distribution of all knowledge combinations for both the methods and content course.

Our next step in analysis was to investigate the instructional moves that appeared to promote knowledge integration in the content and methods course. For the purpose of this paper, we focus on the episodes where evidence of M, L, and P were present. We applied open coding to develop categories for these different instructional moves.

Findings

From the 15 weeks of field notes, we identified 405 different episodes in the content course and 273 episodes in the methods course. Table 1 displays the number of episodes that contained evidence of the different combinations of knowledge. For instance, the 26 in the last column for the content course indicates there were 26 episodes in the content course where the knowledge of mathematics, learners, and pedagogy were exhibited.

Table 1: Combinations of Knowledge Used in Episodes from Both Courses

Course	M	L	P	M-L	M-P	L-P	M-L-P
Content	265	4	18	41	32	19	26
Methods	4	10	92	1	17	135	14

In the content course, a majority of the episodes contained only statements about mathematics. However, of the 405 total episodes, there were 118 episodes that contained multiple knowledge types, which constitutes 29% of all episodes. Although knowledge integration was not the norm in this particular class, the data reveal the potential for knowledge integration in this content course for teachers.

A majority of the episodes in the methods course, 95%, contained statements about pedagogy. However, unlike the content course, 61% of the episodes in the methods course contained evidence of multiple forms of knowledge. There were 149 episodes, 55% of the total, which contained both L and P; however, far fewer of the integrated episodes contained evidence of the knowledge of mathematics.

How can we better promote knowledge integration in our content course? How can we better integrate mathematical knowledge into our methods course? If we can answer these questions and uncover the instructional moves associated with knowledge integration, we can help our future teachers develop the robust schema required to make important instructional decisions.

To uncover the instructional moves associated with knowledge integration (Research Question 2) we first examined each episode that contained M, L, and P. Through an open coding process, we have tentatively identified four common instructional moves associated with the MLP episodes. Table 2 provides this initial categorization.

Table 2: Categorization of Episodes with M, L, and P

Category	Episodes in Content Course	Episodes in Methods Course
Reflecting on Instruction: The teacher educator reflects, or has the PSTs reflect, on the mathematical instruction in the Content Course. The conversation is focused on the pedagogical moves associated with this shared experience.	9	5
Development of a Pedagogical Product: The teacher educator asks the PSTs to develop a pedagogical product. (e.g., excerpt of lesson plan, assessment, task, lesson objective, etc.)	8	5
Focus on Student Thinking: The teacher education asks the PSTs to consider how a high school student would think about a particular mathematics topic.	8	0
Evaluation/Critique of Pedagogical Products: The teacher educator asked the PSTs to evaluate a pedagogical product. (e.g., excerpt of lesson plan, assessment, task, etc.)	0	4

To note, we only included instructional moves that occurred in at least four episodes. In addition, the instructional moves identified in this study were those used by these instructors; there could be additional instructional moves, not used by the instructors, which promote MLP integration. The data shows two instructional moves common to both courses (Reflection on Instruction & Development of Pedagogical Products) and two instructional moves used in only one of the courses (Focus on Student Thinking & Evaluation of Pedagogical Products).

Discussion and Future Direction

The initial categorization of instructional moves is based on the 40 MLP episodes. As we continue to expand our coding to the remaining 638 episodes, our categorization will continue to be refined and expanded. Once this categorization is finalized, we hope to have a better understanding of the instructional moves associated with the various knowledge combinations. It is also important to identify instructional moves that are not associated with knowledge integration. For example, the content course instructor incorporated short microteaching opportunities into his class with the hopes of encouraging knowledge integration. Unfortunately, there was only one episode of microteaching coded as an MLP episode.

This raises several important questions. Can we adjust current instructional moves to help bring out integration? Are their critical features of instructional moves that encourage knowledge integration? What other aspects of teacher preparation promote knowledge integration? How does this influence teaching practice? In the future, we hope to continue developing and revising innovative instructional methods that promote knowledge integration, with the hopes of providing mathematics teacher educators strategies that stimulate a robust schema of knowledge.

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COLLABORATION FOR RICH TASKS: DEVELOPING PRE-SERVICE TEACHERS ABILITY TO CONSTRUCT LESSON PLANS WITH MASTER TEACHERS

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Pre-service mathematics teachers collaborated with Master Teachers to experience rich problem solving, design of rich tasks, and authentic implementation of rich tasks in high school classrooms. The pre-service teachers made strides in their own problem solving abilities and made improvements in their construction of rich tasks. There were limitations in their ability and willingness to shift their thinking to design open-ended tasks that promote reasoning and problem solving.

Keywords: Teacher Education - Preservice, Teacher Knowledge, High School Education

Introduction

The *Principles to Action* (NCTM, 2014) call for teachers to "Implement tasks that promote reasoning and problem solving" and to "work collaboratively with colleagues to plan instruction." Beginning methods classes can be challenging environments in which to achieve these goals. Too often, pre-service teachers' own experiences in mathematics classes - both at the school and college level - do not include engagement with rich tasks that promote problem solving (Ball, 1990). Furthermore, teaching methodologies advocated by methods instructors in teacher preparation programs, particularly around teaching with open-ended, differentiated tasks, are often not readily observed in actual classroom settings (Zeichner, 2010). Although pre-service teachers can work collaboratively, creating rich tasks in methods courses in university settings, they often don't see the fruits of that labor because such planning is done in the abstract and not for implementation in an actual classroom. This creates a disconnect between the experience of the methods class and the reality of teaching both in field experiences and upon entering the profession (Allsopp et al. 2006; Meagher, Özgün-Koca & Edwards, 2011).

To address this disconnect, a team of three methods instructors and six practicing teachers collaborated to construct a set of experiences for Preservice Mathematics Teachers (PSMT). Specifically, the experiences required PSMTs to complete the following cycle six times:

1. Six Rich Problem Sets (RPS) constructed by six practicing classroom Master Teachers (MT), reflecting content used in authentic school classrooms.
3. In teams, adapt the Problem Sets, constructing (a) differentiated versions of tasks, (b) scoring rubrics, and (c) online video tutorials that secondary school students will use in the practicing teachers' classrooms as they solve the tasks.
4. Observe a video of the differentiated tasks being implemented in a high school classroom taught by the partnering Master Teacher who constructed the initial RPS.
5. Provide significant written feedback to students on their worksheets in an effort to strengthen and extend students' mathematical understandings; return the assessed work to the high school teacher for distribution to students.

The initial stages of the process were designed to assist teacher candidates in making the shift from being a "doer" to being a teacher of mathematics (Meagher, Edwards & Özgün-Koca, 2013). The later stages of the process gave teacher candidates authentic experiences in terms of the preparation of tasks, the enactment of tasks and assessing/giving feedback to students.

The research is guided by and designed to answer the research question:

What effect does collaboration with a cadre of master teachers (MTs) over the course of a 15-

week semester have on Preservice Mathematics Teachers' (PSMT) attitude to tasks and lesson plans that promote reasoning and problem solving (Allsopp et al. 2006; Zeichner, 2010)?

Literature review and relationship to research

Brown and Borko (1992) argue that there are three important issues in the process of "learning to teach": (a) the influence of the content knowledge, (b) novice's learning pedagogical content knowledge, and (c) difficulties in acquiring pedagogical reasoning skills. Furthermore, they assert that "one of the most difficult aspects of learning to teach is making the transition from a personal orientation to a discipline to thinking about how to organize and represent the content of that discipline to facilitate student understanding" (p. 221).

Pre-service teachers early in their program typically develop lesson plans that are never taught in a classroom - typically in isolation - creating a disconnect between planning, implementation, and assessment of student learning (Allsopp et al. 2006; Meagher, Ozgun Koca & Edwards, 2011). Developing communities of practice (Wenger, 1999) and lesson study groups (Fernandez, 2002) can help candidates and practicing teachers adopt a more research-based focus in their lesson planning and develop a shared repertoire of communal resources which transcend individual contributions. Furthermore, providing candidates with opportunities to experience their lessons taught by Master Teachers in authentic classroom settings increases motivation for lesson writing. Preservice teachers demonstrate a trajectory of learning about lesson plans in a cycle of designing a lesson to be taught by a Master Teacher and reflecting critically on the implementation of that lesson (Meagher, Özgün-Koca & Edwards, 2011).

Methods and Methodologies

The participants (n=18) in this study were candidates enrolled in the second 15-week course of a year-long methods sequence designed for prospective secondary mathematics teachers.

Completing the Rich Problem Sets

Prior to the beginning of the second methods course, the methods instructor and 6 Master Teachers worked collaboratively over a 3-week period to design a series of 6 distinct Rich Problem Set (RPS) experiences for candidates. A crucial aspect of the process was that the RPSs were designed to become progressively more open-ended as the semester proceeded. In completing the RPSs, the candidates were to experience how the content of the high-school curriculum can be framed with rich tasks - rich in content; illustrating connections among various mathematical areas and real-world application; with differentiation for different learners, requiring conceptual understanding, procedural fluency and problem solving skills.

Data Collection and Analysis

The data collected consisted of

1. PSMTs responses to the RPSs and the materials developed for the lesson plan;
2. Video of the lesson plans implemented in school classrooms by the Master Teacher;
3. high school student written work from the lesson;
4. field notes from the class meetings in the university;
5. course evaluations.

Analysis of activities, lesson plans constructed during each RPLTF, and pre- and post-survey, provides a comprehensive picture of preservice teacher development *across* the 15-week study period. Moreover, since these artifacts were constructed collaboratively using Google Docs, modifications that preservice teachers made prior to and following feedback provided by the partnering classroom teacher were tracked, giving insight into the thinking of preservice teachers'

development *within* the development of teaching artifacts for a particular RPLTF. To address the research question the qualitative data was analysed for attitudinal positions exhibited by the PSMTs with regard to tasks and lesson plans that promote reasoning and problem solving.

Results

The first 3 tasks required PSMTs to generalize numerical patterns using symbolic algebra and had one correct answer, for example, in one task PSMTs determined the total number of triangles that can be constructed on a 5 pin by 5 pin geoboard and in another they had to construct an algebraic expression describing the number of unit cubes required to construct a "Skeleton Tower" *n* units tall.

Task 4 was markedly different from the first 3 tasks. First of all, RPS 4 required candidates to write code in Matlab, a multi-paradigm numerical computing environment with none of the PSMTs enrolled in the methods course having had any previous programming experience. Secondly, and unlike previous assignments, RPS 4 required the PSMTs to have a say in the *construction of their own task*. PSMTs were asked to conduct their own research, identify an iterative approach for approximating pi, and implement the approach using the Matlab programming language. Candidates were provided with extra time to explore a variety of Matlab tutorials and supports. Given these differences, it is not surprising that problem set scores dipped from an average of 90.6% to 81.9% from RPS 3 to RPS 4. RPS 5 and RPS 6 continue in a similar vein, each with no one "correct answer" or "correct solution" method.

In RPS 5 PSMTs set up their own test to determine whether or not the shuffle feature on an iPod is random (or not). Teacher candidates were required to construct their own "randomness" criteria and apply these criteria to various playlist data. (Ziegler & Garfield, 2013). RPS 6 was similarly open-ended with no one correct approach or solution.

Although their performance on the RPSs diminished as the tasks became more open-ended, as the PSMTs became more accustomed to the openness of the tasks as the semester progressed, their scores on the RPLTF artifacts and lesson plan materials again improved. That said, there is a similar split apparent in lesson plan scores between the first 3 plans and the final 3 plans. It was clear that the PSMTs struggled to provide students with learning opportunities that included ambiguity and no clear, "correct" solution strategy. When candidates struggled to solve tasks themselves, as "students," they were also less comfortable constructing similar lessons for others. At the beginning of the semester, a sizable portion of teacher candidates noted that their enjoyment of mathematics was due to the "fact that in mathematics there is always one right answer." As one student noted in an early assignment, "1+1=2. I can count on that . . . it's comforting to have something we all can count on . . . something that is actually TRUE" [Artifact, RPLTF 1]. This fairly typical need for "right answers" and "a correct solution method" continued to be important for PSMTs throughout the experience, although we note an increasing comfort with complexity reflected in their scores on the lesson plans. That said, in some cases as tasks became more open-ended and more ambiguous, PSMTs became less comfortable and more resistant to the ideas brought forward in the methods class. In general, as work in the course became more complex - and arguably more closely aligned with CCSSM and NCTM recommendations - teacher candidates became more disconnected from the course goals.

Disconnect between "real" classrooms and RPLTF experience

Teaching methodologies advocated by methods instructors in teacher preparation programs, particularly around teaching with open-ended, differentiated tasks, are often not readily observed in actual classroom settings (Zeichner, 2010). Using qualitative codes "realistic" and "observed by me" the field notes and course evaluations were analysed. The RPLTF experiences were specifically designed to allow these PSMTs to see open-ended, rich, differentiated tasks implemented in actual high-school classes - and yet the PSMTs still struggled to make this connection. Many of the teacher candidates indicated that they were frustrated because, as one student expressed it, "this isn't the

work that high school math teachers do" [Field notes, class meeting 9]. One PSMT stated directly, "the teachers in our field placements [last semester] never did this kind of work with their students" [Field notes, class meeting 14]. The PSMTs had field placements the previous semester where they spent a considerable amount of time in high-school classrooms observing and occasionally teaching. The quotes above indicate the power of the field experiences over PSMTs approach to teaching and, in particular, their approach to tasks and lesson plans that promote reasoning and problem solving. The PSMTs observed the implementation of their lesson plans by a master teacher in an actual classroom but their field experiences did not provide enough models of mathematics classrooms where students were required to come up with their own solution techniques or grapple with the possibility that a task may have multiple correct answers or multiple solution strategies. And so many of the PSMTs thought that the work of the class was "unrealistic," despite being *initiated* by classroom teachers and enacted as part of their work in secondary school classrooms with actual students.

Conclusion

The intervention described above was designed to provide pre-service mathematics teachers with experiences in rich problem solving, experiences in designing rich tasks for implementation, and experiences in how those tasks can be authentically implemented in high school classrooms. We observed the pre-service teachers make strides in their own problem solving abilities as well as make improvements in their construction of problem-solving tasks. However, there were limitations in their ability and willingness to shift their thinking to design truly open-ended tasks that promote reasoning and problem solving with the influence of their field placements in more traditional classrooms proving to be very powerful.

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DEVELOPING QUANTITATIVE REASONING IN WORD PROBLEMS

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Quantitative reasoning is the heart of word problem solving; it requires fluency and understanding with units and the quantities involved. We incorporated a supplementary word problem solving program in a regular arithmetic course for pre-service teachers. In the pre-test most of our pre-service teachers could not clearly identify the qualitative characteristics of the quantities. The supplementary program significantly improved students' performance in their abilities to define measurable attributes in simple word problems.

Keywords: Algebra and Algebraic Thinking, Problem Solving, Teacher Education-Preservice

One of the standards for mathematical practice in the *Common Core State Standards for Mathematics* (CCSSM) (National Governors Association for Best Practices, Council of Chief State School Officers [NGA & CCSSO], 2010) promotes abstract and quantitative reasoning. The CCSSM standards advocate for students to have the ability to *decontextualize word problems* by creating models that contain abstract symbols. Creating such models requires a conceptual understanding of the measurable attributes (*quantities*) and their units.

Researchers (Weber et al., 2014) identified the importance of quantitative reasoning in students' abilities to create mathematical models of word problems. However, students need to comprehend a problem's text before creating the models. A group of researchers (Sulentic-Dowell et al., 2007) found that the ability of teachers to meaningfully discuss word problems with their students depends on their own literacy. Particularly, non-active readers, instead of helping to analyze the text of the problems, point only to key words.

Through informal classroom observations, we found our pre-service teachers, when discussing a problem's text, do not necessarily use scientifically correct language and define measurable attributes with their corresponding units. Hence, we undertook this research to examine the level of pre-service teachers' abilities to discuss word problems' texts while clearly identifying attributes, and the possibility of increasing their abilities by providing specially-designed instruction.

Theoretical Framework

Our theoretical framework is based on Thompson's (1988) interpretation of quantitative reasoning—reasoning about quantity as a *measurable quality of something* and thinking about *a magnitude of a quantity as the quantity's measure in some units* (p. 164). For example, in the sentence, *Tom spent 3 hours doing his homework*, a qualitative attribute is *time*, and a unit of measure is *hour*. In Tom's sentence, the attribute (*time*) is not specified but can be deducted from the text. Thompson's analysis (2011) covers various types of attributes, including complex attributes that one can face in economics or high school physics classes (p. 38), but in our research, we narrowed the list of attributes we discuss with students to talking about a number of objects, amount of money, length-type characteristics (depth, width, distance, height, and length itself), volume, weight (in meaning of *mass*), temperature, and time.

As we observed in our classes, when attributes are not specified, students are not immediately aware what attributes are mentioned in a problem. Besides difficulties with assigning attributes, understanding word problems can be quite challenging, due to the use of word-problem specific vocabulary (Fuchs et al., 2015). To help pre-service teachers develop their quantitative reasoning, we applied the teaching method referred to as the 'dimension variations' (Marton & Pang, 2006) approach. This approach is based on teaching one learning dimension (one type of problem) at a time

while variations are applied. In literature connected with word problem solving, this approach is referred to as a *schemata*. A schemata subdivides one-step word problems into several groups (Jitendra, 2002; Jitendra et al., 2015). We used this schemata to teach students to use scientifically precise language when describing values in *Compare*, *Combine*, and *Change* addition and subtraction word problems. Variations were introduced by incorporating word problems with specified and non-specified attributes, varying problems' logic, etc.

Methodology

After a pretest, scientific language instruction combined with visual representations of the quantitative relations was introduced as a supplementary instruction to an arithmetic course for future elementary teachers in a southwestern research university. The lead author taught the first of two sections of the arithmetic course and implemented the supplementary program along with the standard course textbook (Beckmann, 2013) for future elementary teachers. The supplementary program used a workbook which was designed by the second author and was used successfully with children (ages 6 to 14) in a remediation/enrichment learning center in the Midwest to help students develop quantitative reasoning.

Participants

Thirty-three students enrolled in an arithmetic course for elementary pre-service teachers participated in this study. Seventeen students were in the treatment group and sixteen were in the control group. Both groups were taught the same standard course content. Additionally, the treatment group was given the supplementary workbook on word problem solving. The workbook was self-explanatory and provided a brief explanation of theory as well as some examples and exercises. Students were asked to read the text in the workbook carefully and analyze examples before starting their assignments. Instructor spent 5 minutes of each class time to discuss a concept after students read the text themselves. Students were taught how to rephrase word problems by explaining the attributes the number in the word problem is measuring, to use symbolic representations for word problems, to visualize quantitative relations using illustrations, and to identify units in the word problems.

Data Collection

Students were assigned homework from the supplementary workbook for 12 weeks of a 15-week course. Data were collected from students' written work on a pre-test questionnaire, workbook assignments, quizzes, a mid-term, and a final exam (post-test) from the treatment group. Only post-test data were collected from the control group. One example of the explanation for a post-test question on measurable attributes is shown in Figure 1.

Value-Analysis – an explanation of the meaning of each known and unknown value in a problem.

In value analysis, identify the names of the properties that the values describe.

> Examples of Givens and Unknown Value
> 5 is the number of apples Nina has.
> 6 degrees is the temperature in New York.
> ? is the number of cookies.

Figure 1. Post-test question—explanation on measurable attribute.

Data Analysis

For this preliminary study we analyzed pre- and post-test data following an open and axial coding method (Strauss & Corbin, 1998). We first coded each question in terms of the precision and clarity

on defining the involved attributes and units (Table 1). Then, we recorded students' written responses for each task against each measurable attribute and discussed each student's precision and clarity on each task.

Table 1: Code for precision and clarity in naming attributes

No defining or naming an attribute (N/A)	Low Precision (Low)	High Precision (High)
Repeating text words that do not point to attributes: *how much, size, objects, amount of* without mentioning the attribute.	Describing attribute instead of naming it: *6 is how long; 6 is how much it weighs…*	Naming attributes: Length, weight, volume, and measuring units

Results

We found that about 18% (3 out of 17) of the students in the treatment group precisely identified the measurable attribute in word problems with non-specified attribute "length" in the pre-test. Others have either repeated the keywords from the text or used *how long*, *how much*, *size*, or *amount of ribbon*. Figure 2 demonstrates an example of a student's written response.

After Teresa cut off 6 feet of a ribbon, the ribbon became 10 feet long. How long was Teresa's ribbon to start with?

6 feet is _What Teresa cut off from a ribbon that was 16ft long._

Figure 2. A student's work on the length attribute in the pre-test.

Twelve out of 17 students in the treatment group named the *weight* attribute in the pre-test. We believe it is related to the similarity between the attribute "weight" and the verb "weighs" used in the problem. On the post-test the verb "to weigh" was not used in a weight-related problem. Nevertheless, 88% (15 of 17) of the students from the treatment group named *weight* and *volume* attribute (Figure 3) correctly while none of the control group identified the *weight* attribute (see an example in Figure 4). Table 2 shows the increase in the number of students' precision in naming the attributes from the pre- and post-tests in the treatment group.

Table 2: Comparison of pre-test and post-test results on naming the attribute task

Measurable Attributes	Weight			Length			Volume		
Precision and Clarity	N/A	Low	High	N/A	Low	High	N/A	Low	High
Control Group (Post-test)	16	0	0	13	1	2	8*	-	-
Treatment Group (Pre-test)	5	0	12	13	1	3	-	-	-
Treatment Group (Post-test)	2	0	15	2	0	15	2	0	15
The data above show the number of students who fall under the three categories. *Only 8 completed the problem.									

We found knowledge of the words *length*, *weight*, *time*, and *volume* did not help students to fluently identify the attributes. Their work on supplementary materials reflected gradual progress toward defining attributes in 1–3 step word problems. Students' assignments reflected how the students started to develop a habit of specifying measuring units in all of their calculations.

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Figure 3. A student's work on the volume attribute in the post-test from treatment group.

Figure 4. A student's work on the weight attribute in the post-test from the control group.

Conclusion and Discussion

Pre-test results for the treatment group as well as post-test results for the control group showed most of the students do not name attributes to the quantities mentioned in word problems unless the attributes are specified in the text. It is true for the students before and after taking a standard arithmetic course. However, using supplementary materials based on the schemata, which uses scientific language to describe givens, considerably increased students' abilities to identify measuring attributes in word problems and promoted the habit of naming these attributes and measuring units when presenting solutions to the problems.

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PRESERVICE MATHEMATICS TEACHERS' CONCEPTIONS OF INTENSIVE QUANTITIES COMMONLY USED TO MODEL GLOBAL WARMING

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Here, I report preliminary findings concerning preservice mathematics teachers' (PSTs) conceptions of three intensive quantities commonly used to model global warming: concentration, energy density, and energy flux density. Three PSTs completed four mathematical tasks in two individual, task-based interviews. Analysis revealed that: (a) PSTs' conceptions of these quantities varied greatly depending on the type of quantity (Type 1 vs. Type 2), (b) confusing temperature with energy hindered PSTs' ability to conceive of energy density as ratio as measure, and (c) PSTs conceptions of energy flux density involved two types of per-one associations. The second type seemed to conduce to the development of ratio as measure conception for energy flux density. However, this was an effortful process for PSTs.

Keywords: Cognition, Modeling, Rational Numbers, Teacher Education-Preservice

Introduction

In recent years, there have been several calls to include global warming in school and college instruction (McKeown & Hopkins, 2010; UNESCO, 2012). Global warming provides a motivating context to study important scientific and mathematical concepts. For example, introductory mathematical models for global warming—for which the mathematics can be accessible to high-school students—make use of covariant physical quantities, many of which are defined through the division of other physical quantities, such as energy, area, and volume. However, research has demonstrated that students and future teachers can have persistent difficulties understanding division in terms of quantities and with comprehending and mathematically expressing co-variation between quantities (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Greer, 1992; Nunes, Desli, & Bell, 2003; Thompson, 1994). By examining how secondary preservice teachers (PSTs) learn about introductory mathematical models for global warming, my research attempts to answer two overarching questions: (a) What are PSTs' conceptions of quantities commonly used to model global warming? and (b) How do PSTs conceptualize covariant relationships between those quantities, relationships that link carbon dioxide pollution to global warming? Here, I report the preliminary findings for the first of these research questions.

Quantifying Global Warming

Earth's climate system is powered by the sun and there is a continuous flow of energy between the sun, the planet's surface, and the atmosphere. The planet's surface is warmed by the sun's energy. As the surface heats up, it radiates (infrared) energy to the atmosphere. The majority of this energy is absorbed by *greenhouse gases* (GHG) such as water vapor (H_2O), carbon dioxide (CO_2), and methane (CH_4). The atmosphere re-radiates part of the absorbed energy back to the surface. This continuous energy exchange between the surface and the atmosphere is known as the *greenhouse effect* and influences the planet's average surface temperature. Thus, quantifying changes in the flow of energy and the abundance of GHG in the atmosphere is required to accurately model climate and global warming. My research focuses on PSTs' conceptions of three quantities commonly used to model global warming: *concentration*, *energy density*, *energy flux density*. Concentration is the ratio between the volume of a gas and the total volume of the mixture in which the gas is contained, usually measured in the same units of volume like m^3/m^3 (volume concentration) or in parts per million by volume or ppmv (ppmv concentration). Abundance of GHG in the atmosphere is

measured in terms of concentration. Energy density is the ratio between the energy that an object radiates, absorbs, or reflects and the object's surface area, usually measured in Joules per square meter (J/m^2). Energy flux density is the time rate of energy density, usually measured in Joules per square meter per second ($J/m^2/s$). Flows of energy are measured in terms of either energy density or energy flux density.

Conceptual Framework

Concentration, energy density, and energy flux density are *intensive quantities* (Nunes et al., 2003). Unlike extensive quantities such as length, which are measured in the same measure space (e.g., length is measured in units of length such as foot, meter, or mile), intensive quantities are defined through division and express an invariant multiplicative relationship between two quantities that may or may not involve two measure spaces (e.g. speed is measured in units of mile/hour or meter/second). There are two types of intensive quantities (Nunes et al., 2003): Type 1 has constituent quantities that form a whole and is usually interpreted as a fraction. Type 2 has constituent quantities whose measures remain separate. Here, the intensive quantity is interpreted as a ratio and not as a fraction. Concentration is an example of Type 1, while energy density and energy flux density are examples of Type 2.

I developed a conceptual framework to characterize PSTs' conceptions of concentration, energy density, and energy flux density. The framework builds on Johnson's (2015) work and admits two levels of meaning for intensive quantities: *Ratio Level* and *Rate Level*. The levels are consistent with Thompson's (1994) distinction between *ratio*–a multiplicative comparison between particular values of two quantities–and *rate* – reflectively abstracted constant ratio. The framework describes four conceptions of ratio which, in order of increasing sophistication, include ratio as: *identical groups*, *particular per-one*, *abstract per-one*, and *measure*. Ratio Level includes the first two conceptions, while Rate Level includes the latter two conceptions.

Ratio as identical groups involves an association between particular values of two quantities (Simon, 2006). For instance, consider a diving tank containing 650 liters (L) of air of which 130 L are O_2. Here, the concentration of O_2 is conceived as an association between 130 L of O_2 and 650 L of air, or, as a simplified version, 13 L of O_2 for every 65 L of air. The framework includes two types of ratio as per-one (Simon & Placa, 2012): *particular* and *abstract*. *Ratio as particular per-one* involves associating a value of a quantity with one unit of another quantity for a <u>particular</u> situation. For instance, consider 4,200 J of energy being absorbed by a surface area of 12 m^2. Here, energy density is conceived as an association between 350 J and 1 m^2 for the <u>particular</u> case in which 4,200 J are shared equally among 12 m^2. Although *ratio as abstract per-one* also involves a per-one association, the ratio is conceived as an invariant multiplicative relationship between quantities that <u>can vary</u>. For example, an energy density of 350 J/m^2 is still conceived as an association between 350 J and 1 m^2. However, energy and surface area are conceived as varying quantities that maintain the 350 J per 1 m^2 association constant. Finally, *ratio as measure* no longer involves associating the values of two quantities. Rather, a ratio is conceived as a quantity in its own right, a quantity that measures the *strength* or *intensity* of an invariant multiplicative relationship between two quantities (Simon and Placa, 2012). For instance, a concentration of O_2 of 0.21 is conceived as a measure of the level of *oxygenation* in the air one breathes, or an energy density of 350 J/m^2 is conceived as a measure of the *intensity of the radiation* being absorbed by a surface.

Methods

Three secondary PSTs–hereafter known as Pam, Kris, and Jodi (pseudonyms)–enrolled in a mathematics education program at a large Southeastern university participated in my study. Each PST completed four mathematics tasks in the course of two individual, task-based interviews (Goldin, 2000). Task 1 required PSTs to use volume concentration to compare the level of

oxygenation in the air of two diving tanks. Task 2 required PSTs to determine ppmv concentration to compare the level of carbon dioxide (CO_2) in the air of two diving tanks. The next two tasks involved an experiment in which two devices are radiating energy in the form of heat towards two metallic sheets. Task 3 required PSTs to use energy density to determine which sheet will be hotter at the end of the experiment. In Task 4, PSTs needed to use energy flux density to determine which sheet's temperature is rising faster. These tasks were not directly related to global warming since their purpose was only to assess PSTs' conceptions of these intensive quantities commonly used to model this phenomenon. All interviews followed a *semi-structured* approach and the interview videos and transcripts were analyzed through Framework Analysis (FA) method (Ward, Furber, Tierney, & Swallow, 2013).

Preliminary Findings

PSTs did not show signs of difficulties when working on tasks involving concentration. All three PSTs conceived of volume concentration as *ratio as measure*. PSTs conceived of concentration as a measure of how "concentrated" a particular gas is in the air of a tank. Pam compared concentration to preparing lemonade to illustrate what concentration measures. Pam said that if you have more lemonade powder than water, the lemonade would taste "more lemony" because the lemonade power would be more "concentrated." Here, Pam said the drink's taste represents how concentrated oxygen (O_2) is in the tank. In addition, all three PSTs used a single value–in the form of percentage or decimal–to indicate the magnitude of a particular concentration. Kris interpreted a concentration of O_2 of 0.19 as "out of 850,000 cm^3 of air in tank B, 19% is oxygen." These PSTs also conceived of concentration as an invariant multiplicative relationship between two quantities that can vary (Rate Level). Jodi explained that a concentration of 0.21 "could be created by two different numbers as a ratio of each other … then you'll still get the 0.21 proportion of O_2 to air." Lastly, all three PSTs recognized volume concentration and ppmv concentration as two measures of the same attribute.

Unlike concentration, PSTs faced more difficulties when working on tasks involving energy density and energy flux density. All three PSTs had initial difficulties deciding on how to approach the task involving energy density. These PSTs started by calculating the ratio between energy and surface area, but they had difficulties recognizing why this ratio was important to complete the task. Initially, all three PSTs showed evidence of *ratio as particular per-one* for energy density, associating a particular energy to 1 m^2. PSTs constructed this association in the particular context of the situation described in the task. Pam interpreted energy density as "how many Joules are in 1 m^2." PSTs were able to reach Rate Level and conceived of energy density as *ratio as abstract per-one* by the end of the interview. However, before this happened, PSTs had to distinguish energy from temperature and then connect energy density to temperature. PSTs tended to confuse temperature with energy and had difficulties realizing that temperature involves a comparison between energy and size–larger objects need more energy than smaller objects to reach a certain temperature. Jodi compared a swimming pool to a cup of water being heated up by the sun, so "if you take the temperature of them at maybe midday, then the cup of water would be hotter than the swimming pool. So, size probably does have a role in how hot an object is." Once PSTs realized this, they could connect energy density to temperature since both compare energy to size.

In general, all three PSTs experienced problems conceiving of energy flux density as a single measure of how fast a sheet's temperature was increasing. All three PSTs needed help conceiving of energy flux density as a measure of this attribute. Pam, however, was not able to grasp this concept. All three PSTs conceived of energy flux density as two different types of per-one association. Pam constructed a *double per-one* between extensive quantities, associating energy with 1 m^2 and 1 second. Kris and Jodi associated an intensive quantity with one unit of an extensive quantity. Kris associated energy density with 1 second, while Jodi associated the rate at which energy is radiated

with 1 m². Unlike Pam's association, these PSTs' associations conduced to conceiving of energy flux density as the rate at which a sheet's temperature rises.

Discussion

PSTs' conceptions of intensive quantities commonly used to model global warming varied greatly depending on the type of quantity (Type 1 vs. Type 2). They faced increasing difficulties developing a ratio as measure conception as they moved from Type 1 (concentration) to Type 2 (energy density and energy flux density). Additionally, PSTs usually confused temperature with energy, which hinder their ability to conceive of energy density as ratio as measure. This is an important conceptual obstacle since connecting energy density and temperature is central for global warming. Finally, conceptions of energy flux density involved two types of per-one associations: a *double per-one* association of extensive quantities (energy per second per m²) and an association between an intensive quantity and one unit of an extensive quantity (energy density per second). The second type seemed to facilitate the development of ratio as measure conception for energy flux density. However, this was an effortful process for PSTs.

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ELEMENTARY PRESERVICE TEACHERS' UNDERSTANDING OF EQUITABLE MATHEMATICS TEACHING

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This goal of this study is to investigate how preservice elementary teachers describe equity and equitable mathematics teaching on a brief survey before and after taking an equity focused elementary mathematics methods course. At the beginning and end of this course, the preservice teachers completed a brief survey of questions that asked them to define equity and equitable teaching and give examples of how they would teach in equitable ways. Results indicate that students' pre and post responses changed in meaningful ways before and after the course. The definitions and examples were coded into three different categories: equity as academic success, equity as management of the classroom, and equity as teachers' relationships with students. Findings from this study support the hypothesis that more sophisticated or advanced ways of thinking can be developed through an equity focused methods course.

Keywords: Equity and Diversity, Teacher Education-Preservice

Elementary teachers often have direct and sustained interactions with the students in their classrooms and, as such, strongly influence how much students learn, what they learn, and how they feel about being learners of mathematics. In this way, improving teachers' knowledge through teacher preparation is therefore essential to teacher effectiveness and student achievement. Teachers need specialized training that will prepare them to provide ambitious and equitable instruction, especially in the elementary mathematics classroom. Elementary mathematics methods courses are sites to develop these specialized skills in preservice teachers. It is important to evaluate the effectiveness of the teacher preparation in order to maintain a high quality program that will produce elementary teachers that can teach mathematics in equitable ways.

Review of the Literature

Equity in Mathematics Education

Equity is a term used to describe a way in which all students can be academically successful in mathematics. NCTM (2008) published a position statement that stated equity is "high expectations, respect, understanding, and a strong support for all students" (p. 5). Additionally, Esmonde (2009) describes equity in mathematics as "a fair distribution of opportunities to learn or opportunities to participate" (p. 1010). New teachers often enter their teacher preparation programs with a professed "love of children", but are often unaware of their own bias towards the students who culturally, socially, and linguistically differ from themselves. These unexamined bias and assumptions may hinder teachers from enacting equitable teaching practices that would support all students in developing skills needed for mathematical proficiency (Downey & Cobb, 2007).

Methods as a Site for Developing Equitable Teaching Practices for Mathematics

The quality of mathematics instruction, and ultimately student learning, is directly related to teachers' knowledge and skill (Darling-Hammond, 1999). Mathematics methods courses are places for preservice teachers to develop new ways of thinking about teaching and learning (Ball, 1990), and they are places to challenge deficit views of students' abilities to learn ambitious mathematics as well as develop preservice teachers' abilities to teach ambitious mathematics content. Many mathematics teacher educators use their methods courses to shape preservice teachers' views of

students and how to teach those students. Interweaving multicultural education throughout a teacher preparation program helps elementary preservice teachers develop the necessary equitable teaching strategies in a nonthreatening environment (Chisholm, 1994). However, there is little in the literature that examines how preservice elementary teachers make sense of equity and equitable teaching when a mathematics methods class is designed to attend to equitable mathematics teaching.

Methodology

This study explores how preservice teachers conceptualize equity and equitable teaching before and after an equity focused elementary mathematics methods course by analyzing their pre and post response to a brief survey. Specifically, the research question is: In what ways do elementary preservice teachers define equity and equitable teaching before and after an elementary mathematics methods course that is focused on equitable mathematics teaching?

Sample

The participants in this study were classified as seniors in an elementary teacher preparation program at a large research-intensive university in a southern state. This university is a minority serving, Hispanic serving institution in a large urban city. Graduates from this teacher preparation program typically teach in local urban school districts. Given their anticipated first teaching assignments in urban classrooms, it is especially important that graduates of this program are prepared to work with culturally, ethnically, and linguistically diverse students. The sample for this study includes forty-three participants, only one of which was male while the rest were female. The participants were also ethnically and linguistically diverse with Spanish being the second dominate language spoken.

Mathematics Methods Course

Participants completed an elementary mathematics methods course that focused on equitable and ambitious mathematics instruction during the Fall 2014 semester. It was the first of two mathematics methods courses in the teacher preparation program, and the content focus of the course was number and operations. Equitable teaching practices were infused throughout the methods course. Some of these practices include setting up and managing group work in ways that distribute opportunities for multiple students to participate, using multiple strategies for posing questions about content, choosing and using sensitive and appropriate examples for use with diverse students, and maintaining high expectations for all students. The participants also read case studies and engaged in conversations about issues of race and language and the impact they can have on student learning. In conjunction with the methods course, the participants were also enrolled in full time student teaching. As a part of the required work for this course, pre-services teachers completed a variety of projects where they practiced the equitable teaching practices learned in the methods course and reflected on their teaching using evidence of student participation and work.

Data Collection

The instrument used in this study is a one-page equity survey asking the preservice teachers to write a definition of equity, write a definition of equitable teaching, and describe three ways in which they would teach in equitable ways during student teaching. Participants completed the equity survey twice during their methods course. Question prompts on the pre survey and post survey were identical in order to track on the difference in understanding of equity and equitable teaching by the preservice teachers' before and after the completion of the equity mathematics methods course. The pre survey was given on the first day of class as the first activity for the preservice teachers to complete, and the post survey was administered as the first activity on the last day of class.

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

Data Analysis

The responses to the equity survey gave the researcher a holistic understanding of how the participants conceive equity and equitable mathematics teaching. The pre and post responses for each participant were randomly coded so it could be analyzed without bias of knowing if the survey represents pre or post data. Emergent coding techniques were used to derive "understanding of equity" categories from the data. There were three rounds of coding for the analysis that consisted of descriptive coding and pattern coding. A group lens was used during the analyses to look at the percentage of responses that attended to the same "understanding of equity" categories for the pre and post surveys. This analysis gives insight into how the majority of participants understood equity and equitable mathematics teaching before and after the methods class. It will also show if there are any trends in the shift from one category to another over time.

Results

Three categories emerged from the analysis of the participants' responses and each consists of subcategories that contain specific terms or ideas expressed by the participants. The three categories are equity as academic success, equity as management of the classroom, and equity as teacher's relationships with students. Table 1 shows the percentages of each category and its subcategory from the pre survey to the post survey.

Table 1: Understanding of Equity Categories

Category	Subcategory	Percentage of Pre-Survey Responses Attending to the Subcategory	Percentage of Post-Survey Responses Attending to the Subcategory
Equity as Academic Success	Instruction	44.19%	90.70%
	Opportunity	39.53%	48.84%
Equity as Management of the Classroom	Participation	37.21%	51.16%
	Students' Environment	11.63%	51.16%
	Discipline	30.23%	6.98%
Equity as Teacher's Relationships with Students	Treatment of Students	88.37%	39.53%
	Expectations	11.63%	25.58%

Equity as Academic Success

The "equity as academic success" category contains participant responses that attended to how he or she will teach so that all students can be academically successful or how equity is related to the opportunity for all students to be academically successful. This category combined the two subcategories instruction and opportunity. The instruction subcategory contains responses related to particular teaching techniques, student learning, and ways in which the teacher will help students achieve academic goals. The opportunity subcategory consists of responses that relate equity to students' having an equal or the same opportunity to achieve academic success.

Equity as Management of the Classroom

The "equity as management of the classroom" category consists of participants' responses that attended to how students participant in the classroom, how students will feel in the classroom environment, and how discipline will be handled. Three subcategories were grouped into this one category: participation, students' environment, and discipline. The subcategory participation consists

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of responses that relate equity to students' participating in the daily classroom routines or participating in the lesson that is taught. The students' environment subcategory describes equity as being a safe, supportive, or status-free environment for the students. The responses that received this particular code described ways in which students would interact with their whole environment and not just with the teacher. Finally, the subcategory of discipline described equity and equitable teaching as holding all students accountable to the same rules, receiving the same consequences, or looking at the cause of a behavior instead of just the student.

Equity as Teacher's Relationships with Students

The "equity as teacher's relationships with students" category contains responses that describe equity in relation to how the teacher will treat students or what the teacher expects from students. The two subcategories of treatment of students and expectations were grouped together because they only describe the relationship between the teacher and the student. The subcategory treatment of students consists of responses that relate equity to how the teacher will interact with students or treat the students. The expectations subcategory includes responses about having high expectations for students or expecting the same behavior or academic success from all students.

Discussion

The responses on the pre survey compared to the post survey showed a shift in understanding of equitable teaching. The pre survey responses revealed that pre-service teachers started the course with broad examples of equitable teaching and moved to more specific examples that they would use during a mathematics lesson. In the post surveys, the preservice teachers described equitable practices in much more specific terms revealing an increased knowledge about the complexities of equitable teaching as promoting individual academic success, managing students in ways that position them as smart and capable instead deficient, and creating relationships with students to communicate high expectations.

Conclusion

Results from this study indicate that an understanding of equity and equitable teaching can be developed through an equity focused mathematics methods course. Analysis of pre and post survey results indicate that there was a shift in the types of definitions and examples from ones focused on treating all students the same to more specific equitable teaching strategies. Additional research needs to be done to expand the sample size and scope of the study to more closely approximate causal inferences between what was learned in the course and students' pre and post survey responses. However, the implications for mathematics teacher education is that equitable teaching is a concept that needs to be embedded into a mathematics methods course so that preservice teachers can develop their understanding of equitable teaching practices in the context of mathematics.

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SHIFTING PARADIGMS FOR PRE-SERVICE TEACHERS' INTERNSHIP EXPERIENCES: CO-TEACHING AS A MODEL FOR RELATIONSHIPS

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This paper explores co-teaching as a new paradigm for secondary mathematics internships. We share an overall vision for an internship grounded in co-teaching. We describe specific co-teaching strategies from the literature and discuss our initial findings regarding the use of co-teaching during our secondary mathematics students' full-time internships. We discuss future directions for research on co-teaching and co-planning.

Keywords: Teacher Education-Preservice, Instructional Activities and Practices, High School Education

Pre-service teachers' classroom internships are widely acknowledged as one of the most influential, and potentially problematic, components of teacher preparation programs. In addition to providing students with quality mathematics instruction, there are many other tasks that must be accomplished during the internship. Interns and mentor teachers must negotiate their roles in both the classroom and in planning. Interns, with assistance from their mentors, must make the transition from student to teacher, establish links between their theoretical understanding of pedagogy and the practical applications of that knowledge, and learn about the many non-instructional components of teaching. Mentors must assist interns with all of these tasks while maintaining quality classroom instruction and juggling time commitments to provide interns with adequate access to their expertise. Given the complexity of the internship experience, models with potential to make the experience more productive are worth exploring. In this study we investigate co-teaching as a new paradigm for how interns and clinical teachers work together.

Theoretical Framework

Our work with co-teaching during pre-service teachers' internship experiences is grounded in Lave's (1991) construct of situated learning. As interns go out into the field, their learning moves from a predominately academic experience to an apprenticeship in a community of practice. In such a setting the working relationship between intern and mentor teacher becomes a major determining factor in the intern's ability to participate productively and collaboratively in the practice of classroom teaching. In our work we consider ways to expand traditional visions of this working relationship between intern and mentor, envisioning mentor and intern as collaborators in classroom instruction.

Co-Teaching

Co-teaching involves more than just new models for how mentors and interns communicate and work together. It involves challenging the borders of what it means to be a mentor and intern. In too many internship settings, interns begin as mere observers in the classroom with no responsibility and little identity in the eyes of classroom students. Interns then begin to take on more of the classroom leadership, often abruptly, as the mentor relinquishes classes to the intern. Although excellent internship settings may provide the kind of progressive, scaffolded learning experience that helps

interns develop the skills for classroom teaching, even these settings often keep the intern mired in the identity of a student in the eyes of classroom students and, more troubling, in their own eyes (Valencia, Martin, Place, & Grossman, 2009).

In looking at other models for the intern-mentor relationship, we looked to co-teaching models found outside mathematics education literature (e.g., Bacharach, Heck, & Dahlberg, 2010; Murawski and Spencer, 2011). A synopsis of these strategies, as well as benefits and concerns for each, are shown in Table 1. These models, adapted to a mathematics education setting, envision a far more collaborative relationship, which may help interns develop their self-identity as teachers and provide some focused strategies to facilitate communication and collaboration in this relationship.

Table 1: Co-Teaching Strategies
(Adapted from Bacharach, Heck, and Dahlberg, 2010; Murawski and Spencer, 2011)

Strategy	Definition	Benefits	Concerns
One Teach, One Observe	One teacher leads instruction, while the other teacher gathers specific information.	Extra set of eyes; provides data about instruction or student learning; easy to implement.	Easy to become a habit; must agree in advance what is to be observed.
One Teach, One Assist	One teacher works with the whole class, while the other assists individual students or groups of students.	Provides assistance to individual students; easy to implement; may provide a "voice" to share student concerns.	Too easy to become a habit and for one teacher to always feel like an assistant; changing roles is essential.
Station Teaching	Students divided into three or more groups; students rotate through multiple stations; teachers facilitate individual stations or circulate among stations.	Smaller groups are better for instruction, assessment, and class management; allows for differentiation, movement, and hands-on activity.	Teachers may need to use space differently; class management and transition needs to be structured; independent stations need to be carefully planned.
Parallel Teaching	Each takes half the class. Groups may be doing the same or different content in the same or different ways. Groups do not switch during lesson.	Smaller groups better for instruction, assessment, and class management; teachers have their own groups; interns teach same lesson/mirror teacher.	Teachers need to be willing to use their space differently; both teachers need to plan for their group; class management needs to be structured.
Alternative Teaching	One teacher works with large group of students, other teacher works with smaller group (re-teaching, pre-teaching, or enrichment).	Good for smaller and more specific group work; good for addressing IEP/504 goals; teachers can plan separately.	DO NOT always pull the same kids; need place for group to meet; watch noise levels; plan how to integrate group back into class.
Team Teaching	Both teachers presenting. This may take the form of debates, modeling information, compare/contrast, or role-playing.	Demonstrates parity and collaboration between teachers; good for modeling; fun for role-playing.	Takes willingness to "share the stage"; both need to feel comfortable in front of the class.

Our Study

Educational researchers have worked to broaden the vision of co-teaching strategies beyond their roots in special education (e.g., Bacharach et al., 2010; Murawski and Spencer, 2011). Our work is focused on seeking ways that these strategies might be implemented in the unique setting of secondary mathematics classrooms. We began our work by collaborating with colleagues about math-specific examples of the strategies and used these in training sessions with our interns and mentors. These have been updated and expanded over the last three years with examples from our interns and mentor teachers. In our study we have been investigating how interns use co-teaching strategies in their placements, interns' perceptions of co-teaching as an internship paradigm, and mentor teachers' perceptions of co-teaching. We have also collected data on how interns and mentors co-plan in preparation for co-teaching. We have collected data on 33 interns and their mentor teachers over the past three years. This data includes surveys, informal observations, focus groups, and audio recordings of co-planning sessions. We are still in the process of collecting and analyzing data but initial findings are very promising.

Findings

Our ongoing analysis of data indicates that both understanding of co-teaching as a paradigm shift and the use of particular co-teaching strategies are perceived as helpful by interns and their mentor teachers. Our analysis also indicates that, over the three years of this study, engagement in co-teaching has increased and mentor teachers' resistance to trying it has decreased. For example, two years ago, interns reported using specific co-teaching strategies an average of 3-4 times per month. This semester at least half of our interns are framing all of their classroom experiences in terms of co-teaching.

We have also seen specific instances in which co-teaching served to expand all participants' vision of what secondary mathematics instruction might look like. For example, at one high school, there were two mentor/intern teams teaching the same course during different class periods. The interns were interested in trying the station teaching strategy. The interns co-planned four stations designed to help students review for an upcoming test on angle relationships and triangle congruence. When the interns began to think about implementation of these stations they realized that the lesson would likely be more effective with more than two teachers. Since the intern/mentor pairs had different planning periods, they convinced their mentor teachers to "donate" their planning period for a day so that all four teachers would be available to implement the stations in both classrooms.

Several positive outcomes should be noted from this example. First, the interns conceptualized the lesson during a co-planning session, presented the idea to their mentor teachers as part of their professional learning community, and took the lead on creating all materials for instruction. Second, both of their mentor teachers were willing to allow their interns to try a co-teaching model that neither mentor teacher had previously implemented. They were also willing to use their planning period to co-teach with a colleague and her intern, also a first for them. One of the mentor teachers shared that, despite prior non-content specific co-teaching training, before this experience she could not envision implementing stations with her high school mathematics classroom. After this lesson, she commented that this experience was one of the most interactive and engaging learning activities for not only her students, but also her colleague and their interns. Further, she could see herself continuing to use stations in the future. In this example, co-teaching not only helped to provide an excellent learning experience for the students, it expanded the mentor teachers' view of the type of teaching and collaboration possible in a high school classroom.

Discussion

As described in the case above, initial findings are promising. The two observations noted above point to the potential of co-teaching to engender confidence and leadership within interns, thus

helping them view themselves as teachers rather than students. This is a marked contrast to the challenges faced by interns as described by Valencia et al. (2009). Furthermore, in this example co-teaching allowed the interns to introduce novel instructional practices within their mentor teachers' classrooms. In this sense, co-teaching has the potential to serve as another tool for mathematics teacher educators to empower pre-service teachers and to encourage student-oriented instructional practices in high school mathematics classrooms.

While our initial results are encouraging, we need more research on the effect of co-teaching as mentors, interns, and university supervisors learn more about how to implement these strategies in high school mathematics classrooms. One critical component that also needs careful consideration and study is the role of co-planning, including strategies on how to co-plan, as a support for quality mathematics instruction, especially in a co-teaching setting. There is wide agreement in the literature (e.g., Bryant & Land, 1998; Bryant Davis, Dieker, Pearl, & Kirkpatrick, 2012; Murawski, 2012) that co-planning is essential to co-teaching but very little practical guidance exists about ways to engage in co-planning. We are currently working on developing and testing co-planning strategies to support co-teaching and intern development.

Acknowledgments

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"IT'S WHAT SOCIETY THINKS": EXPLORING IDENTITY NARRATIVES IN LEARNING TO NOTICE CHILDREN'S MATHEMATICAL PRACTICES

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Attention to mathematics teacher noticing is growing in our mathematics education communities. For this study, I intended to examine prospective K-8 teacher noticing across multiple contexts outside of the classroom (e.g. science museum, nature walk); however, stories about identities were particularly salient through our conversations and reflections in learning to notice children's mathematical practices. In this paper I explore how identity narratives (e.g. gender, race, age) interact with developing teacher noticing. I present preliminary findings that suggest the importance of teachers' identity stories in their mathematics teacher noticing development.

Keywords: Teacher Education-Preservice, Equity and Diversity, Gender, Informal Education

Much of the literature on mathematics teacher noticing research primarily focuses on supporting prospective teachers' noticing through various kinds of activities, such as analyzing classroom artifacts (e.g. Goldsmith & Seago, 2011) and video recordings (e.g. Starr & Strickland, 2008), to say something about children's mathematical thinking in classroom contexts. Part of supporting mathematics teacher learning, though, involves understanding how teachers make meaning of their professional practice across multiple learning spaces and experiences. One challenge for mathematics teacher education involves understanding whether and how prospective teachers engage in interrogating dominant limiting narratives about what it means to demonstrate mathematical competencies (e.g. "I'm not a math person") through their professional learning. Identity narratives (i.e. who, in terms of multiple identity markers such as race or gender, is readily seen as a "math person") are also embedded in these discussions.

The purpose of this study is to better understand how prospective K-8 teachers make connections between identity narratives and their learning to notice children's mathematical practices. Specifically, I investigate the following research questions: (1) What identity narratives do a small group of prospective teachers and a university researcher invoke through their participation in a working group focused on learning to notice children's mathematical practices? and (2) In what ways do prospective teachers connect these narratives to mathematics teacher noticing? This research extends work of scholars investigating mathematics teacher noticing to better understand the nuances and complexities of this practice.

Related Literature

In conceptualizing ways of knowing mathematics, Nasir, Hand and Taylor (2008) argued that mathematics knowledge is fundamentally linked to cultural practices. Many scholars have also highlighted this argument through their studies of out-of-school mathematics practices (e.g. Carraher, Carraher, & Schliemann, 1985; Nasir, 2002; Taylor, 2009). Guided by scholarly work on learning as participation (Lave & Wenger, 1991), I view *mathematics* in terms of participation in the practices and discourses involved in *mathematical experiences*, which must include aspects of mathematics and mathematical practices, such as people engaging with quantities, patterns, and spatial reasoning. Mathematics, in this view, is not restricted to a formalized body of knowledge. Instead, this view validates multiple ways of meaning making and practices involving mathematical ideas. Mathematical experiences include the intersections of mathematics and any combination of disciplines and disciplinary ideas. By broadening what constitutes mathematics, I directly challenge limiting perspectives about what counts as competencies and engagement with mathematics.

Noticing Children's Mathematical Thinking

Teacher noticing, as a professional practice, involves more than just the act of observing behavior; it requires specialized knowledge and intentional decision-making that is specific to teaching. Jacobs, Lamb, and Philipp (2010) define this construct as *professional noticing of children's mathematical thinking,* which involves three interrelated skills: attending to, interpreting, and deciding how to respond to children's mathematical thinking.

Narratives as Storytelling

My explicit attention to identity narratives stems from humans making meaning of the world, shaped by experiences, values, and emotions. As Goldsmith and Seago (2011) eloquently state:

> Teachers' interpretations of classroom artifacts are influenced by the very ways they think about mathematics and mathematics teaching and learning. Teachers view video or read a student's worksheet through the lens of their own knowledge, beliefs, and experiences; this lens shapes their very perception of the artifacts themselves (Heid, Blume, Zbiek, & Edwards, 1999). (p. 170)

I consider identity narratives important in mathematics teacher education because these narratives contribute to shaping the lens through which we see the world. Drawing on Shah (2013), I view narratives as storytelling. Specifically for mathematics education, these narratives are stories that teachers tell about children, about themselves, and about mathematics.

Method

This research uses an exploratory case study design to investigate a case of prospective teachers learning to notice children's mathematical practices across multiple contexts outside the classroom. Through discourse analysis of multiple sources of generated data (i.e. written reflections, audio-recordings of meetings and site visits, individual interviews) over a period of time, this research contributes to the literature on mathematics teacher noticing by providing a collection of prospective teachers' stories about their learning.

Research Setting and Participants

The research setting takes place in meetings and site visits of a working group created by six prospective K-8 teachers and myself over a seven-month period. I pursue ethnographic traditions in this work because my active participation supports me in getting close to the teachers across the multiple contexts in which they are learning to notice children's mathematical practices. I draw on ethnographic participation that provides "access to the fluidity of others' lives and enhances sensitivity to interaction and process" (Emerson, Fretz, & Shaw, 2011, p. 3). I am fully aware of my role as both a group member and as a researcher, which is influenced by my positionality as a middle class, Latinx woman who was generally successful in traditional mathematics classroom spaces and is a visibly young beginning teacher educator. One participant (Liv) is a female Asian international student from Indonesia, and five participants (Ann, Beth, Elise, Kate, and Lilli) are female White Americans. All names are pseudonyms.

Preliminary Findings

In our group, we identified multiple narratives about being a "math person" that suggest the power and status mathematics holds within the United States and in our own lived experiences. Furthermore, these narratives point to the importance of identities in mathematics spaces and how we see ourselves as future educators. In this section, I provide examples of identity narratives we invoked and our perspectives on connecting identity narratives to teacher noticing.

Invoking Gender, Race, and Age Narratives

Each member of the working group recognized identity narratives in her written reflections and verbal group discussion contributions. Two narratives that repeatedly appeared in our discourse involved mathematics and gender (e.g. *Boys are better than girls at math; math is for boys*), and mathematics and race (e.g. *Asians are good at math and science*). For example, Elise wrote about wanting to teach math in ways that were applicable to her future female students' lives because "math is often viewed as a man's subject or profession." She also voiced her frustration that some people viewed mathematics teaching as a "waste of that math talent" and "math people" are often encouraged to pursue engineering instead of teaching.

Prospective teachers also invoked narratives about age, and in particular, being young women in mathematics spaces through our teacher noticing activities. In the following, two prospective teachers and the author debriefed an interaction between Ann, Lynette, and a White male volunteer at a science museum site visit where we engaged in our noticing.

Kate: …so much of what he says is kind of condescending. Did he know you were elementary educators at this point [in the interaction]?

Lynette: At this point he did.

Kate: Yeah, and part of it—I don't know if this is applicable, but is he older?

Lynette: Yeah.

Kate: So I don't know if age is—just treating you like, *oh, you're just these young people*. And I feel just how he talks down on everything, it definitely seems a little gendered. A little ageist, maybe. Like, something's going on here. You would think that most people would be welcoming, but he was kind of the whole time not really treating you like [what you were doing] was anything important, which isn't really good at all.

Ann: Well, I definitely felt the whole ageist thing when he realized that I was an undergrad, and at that part where I said, *No, we're observing together*…He kind of looked at me like, *well, you're just her guinea pig, you're just here for her*, and it's like, no, we're here. We're working together. I don't know. He just really rubbed me the wrong way.

Lynette: So did you feel like this interaction was gendered also for you?

Ann: Yeah. Yeah. I felt like it was very much him talking down to us. He's like, *Well, I'm a male physicist and you're just elementary ed, teaching little kids. You don't know anything*… After all that time that he kept going on about what he knows and then he kept bringing up himself how he didn't know [elementary mathematics] things, but then never started acting like—he always acted like he knew better than us, even saying, *Well, I didn't even know that*. Yeah, because you don't do elementary.

This example is one experience in which gender, age, and elementary mathematics content intersect in how these prospective teachers experienced identities in their world and through their learning to notice children's mathematical practices. This positioning speaks to marginalization as *women* in traditionally male-dominated spaces, as *young people* who are devalued as not having enough expertise, and as *elementary mathematics educators* who are devalued in favor of people who work with more 'advanced' and 'difficult' mathematics content.

Perspectives on Connecting Identity Narratives to Teacher Noticing

Overall, prospective teachers in this working group noted the value of recognizing identity narratives in their own developing professional practice, and teacher noticing in particular. Although prospective teachers identified these narratives, they also seemed to dismiss some of these narratives by drawing on their lived experiences. Some teachers also intentionally distanced themselves from these identity narratives. I asked the group why we think these narratives exist at all, and Beth referred to identity narratives as "what society thinks" without any acknowledgement of being part of

society and potentially contributing to what society thinks. Lilli also added that "it's something *they* (emphasis added) use to blame: 'Oh well, you're better than me because you're a boy!'" Later on in the conversation, though, Lilli commented that our attention to narratives is important for our teacher noticing because:

> …it's also for us to be aware that it's not—let's say if a student is struggling in math, there are so many factors. You know, you can't just assume that they don't understand the material. I mean, they could have a mental block that's like, *I'm a girl so I'm not going to be as good.*

Subsequently, Liv pointed to feelings of discomfort by saying, "I'm starting to fear that if this is how people in America, I guess, view Asian students… and the moment us Asian students come into your classroom and you think, 'Oh, you're going to need less help from me.'" Multiple of the White female prospective teachers pointed to this comment as important for their reflections.

Implications and Discussion

This work strives to highlight the complexities in developing mathematics teacher noticing—teachers may notice children's mathematics both inside *and* outside the classroom, and additionally, teachers' own stories about identities and mathematics shape what it is they attend to and interpret in their noticing. In my ongoing investigation, I seek to examine how these stories relate to our teacher noticing over time as we gain expertise. I also want to further explore the relationships between societal and individual narratives in our noticing development.

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PRESERVICE ELEMENTAY TEACHERS' UNDERSTANDING OF MATHEMATICAL DEFINITION OF THE AREA OF A RECTANGLE

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Research indicated that U.S. students had a weak understanding of spatial measurement concepts and they tended to take measurement concepts as procedures rather than meaningful knowledge. Research also found undergraduates' confusion about mathematical definitions and theorems. The formula to calculate the area of a rectangle tends to be regarded as a definition rather than as a deduced result. This study investigated preservice elementary teachers' understanding of definition of the area of a rectangle and their understanding of mathematical definitions in general. I found that most preservice elementary teachers did not know the right way to define the area of a rectangle and they tended to focus on external features when producing and evaluating mathematical definitions. Teachers also demonstrated a lack of understanding of the roles mathematical definitions play in the axiomatic system.

Keywords: Measurement, Teacher Education-Preservice

Definitions play a central role in the discipline of mathematics (e.g., Zaslavsky and Shir, 2005) and in the teaching and learning of mathematics (e.g., Tall and Vinner, 1981). In spite of the importance of mathematical definitions, the notion of definition develops through students' experience in learning specific mathematical concepts and usually is not discussed explicitly during mathematics instruction. Therefore, Wilson asserted (1990) that "Although we frequently use definitions, we rarely focus on the nature of definitions" (p. 33).

Understanding the nature of mathematical definitions is important for elementary teachers. As argued by Ball, Thames, and Phelps (2008), knowledge of mathematical definitions is an important component of Subject Matter Knowledge. *Common Core State Standards for Mathematics* (CCSSM) (National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010) says "Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments" in the practice of "Construct viable arguments and critique the reasoning of others" (NGA & CCSSO, 2010, p. 6). Elementary teachers need to understand how to use definitions to build mathematical arguments in order to help their future students make sense of mathematics. For instance, elementary teachers need to know how the area of a rectangle equal to length times width could be explained from the meaning of area measurement.

Definitions are different from theorems because mathematical definitions are conventional knowledge and cannot be proved (Kobiela & Lehrer, 2015). Educational researchers have reported students' confusion about definitions and theorems. For instance, Dickerson and Pitman (2012) found that college students thought defining was a way to get around difficult proofs. This misconception raises a need to study if preservice elementary teachers are able to articulate key features about mathematical definitions as opposed to theorems.

In this study, I investigated preservice elementary teachers' understanding of definition of area because elementary students' deficiencies in conceptual understanding of spatial measurement was widely reported in the literature (e.g., Kamii & Kysh, 2006). Measurement tend to be taken as meaningless procedures rather than meaningful mathematical knowledge. I focused on preservice elementary teachers' thinking about definition of the area of a rectangle because the formula length times width could be easily taken as a definition rather than a deduced result. I then explored how

preservice elementary teachers think about mathematical definitions in general based on their understanding about this specific definition. Specifically, I aimed to answer the following research questions:

1. How do preservice elementary teachers understand mathematical definition of the area of a rectangle?
2. How do preservice elementary teachers think about mathematical definitions in general, as manifested through their understanding of mathematical definition of the area of a rectangle?

Methods

Participants

Twenty-four preservice elementary teachers (PETs) who were at the end of their teacher education program participated in this study at a Midwest university. At this university, PETs are required to choose one of the content areas (e.g., Mathematics, Language Arts) as their teaching major. PETs whose teaching major is not mathematics need to take two mathematics courses for elementary teachers (Number and operations & Geometry and Measurement). PETs whose teaching major is mathematics are required to take additional advanced mathematics courses. This context offered me an opportunity to investigate if the amount of mathematical training PETs received correlates to their understanding of mathematical definitions. Among the 24 participants, half are mathematics teaching majors (PETs - M) and the other half are non-mathematics teaching majors (PETs - N). I used M1 - M12 to represent PETs - M and N1 – N12 to represent PETs - N.

Data Collection

The source of data were semi-structured interviews from a larger study. This paper reports the findings to two interview questions. The first interview question (Q1) asks PETs to write a mathematical definition of the area of a rectangle. The second interview question (Q2) happened later in the interview. It asks PETs to determine if the statement "The area of a triangle with base b and height h is equal to ½ bh" is a mathematical definition. Immediately after both questions, PETs were asked to explain their thinking. I audiotaped and transcribed each interview.

Data Analysis

An analytical framework was built to categorize PETs' written definitions into four categories: (a) the amount of space inside the rectangle, (b) length times width (or its symbolic representation l x w), (c) <u>both</u> the amount of space inside the rectangle <u>and</u> length times width (or its symbolic representation), and (d) the amount of space inside the rectangle <u>or</u> length times width. The coding not only considered PETs' written responses but also their verbal explanations. I then investigated the aspects PETs attended to while constructing and evaluating definitions to explore how PETs think about mathematical definitions in general.

Results

PETs' Construction of Mathematical Definition of the Area of a Rectangle

Table 1 gives the distribution of PETs who produced one of the four statements as the definition of the area of a rectangle in Q1. Around 20% of PETs indicated that the amount of space inside the rectangle was the definition of the area of a rectangle. Two PETs - M and one PETs - N discussed the dichotomy - what it means vs. how to calculate it - as a way to distinguish a definition vs. a non-definition. One PET - M indicated that the amount of space was more like a definition because the formula can be discovered through the more basic statement "the amount of space inside the

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rectangle". However, M9 mentioned this idea in a teaching setting instead of in a mathematics setting.

Table 1: Types of Statements PETs Wrote as Definitions of the Area of a Rectangle

Type of statement	Total	PSTs - M	PSTs - N
(a) Amount of space inside the rectangle	5	3	2
(b) Length times width	11	8	3
(c) Amount of space inside the rectangle and length times width	7	1	6
(d) Amount of space inside the rectangle or length times width	1	0	1

Almost 50% of PETs indicated that the formula was the definition of the area of a rectangle. M1 regarded the formula as the definition because "it is more specific to a rectangle." Another PET M3 indicated that the reason why she thought the formula was the definition was that the formula was more clear and straightforward. She said "I feel like that [the space contained within the perimeters] wasn't like clear enough." Another two PETs - N indicated that they thought the formula was the definition because in their mind the definitions should help calculate the concept. This is the opposite opinion to the previous two PETs - M and one PET - N who used the dichotomy - what it means vs. how to calculate it - as a way to distinguish a definition vs. a non-definitional statement.

Almost 30% of PETs indicated that both the amount of space inside the rectangle and the formula together formed the definition. Three of them expressed that definitions should include both what the concept is and how to find or use it. M7 mentioned that the formula can be discovered by the students, but this knowledge did not affect her decision to include both amount of space and formula in the mathematical definition. Another PET, N9 indicated that definitions were audience dependent. Thus in order to meet the needs of all audience, both the amount of space inside the rectangle and the formula should be included in the definition.

Overall, I found that PETs - M tended to choose formula as the definition but PETs - N tended to choose the amount of space inside the rectangle and formula as the definition. In addition, eight PETs used dichotomy, what it means vs. how to calculate it to support their claims. But they differed on how this dichotomy related to mathematical definitions. Some thought how to calculate alone cannot be counted as a definition; others thought it can be counted as a complete or part of a definition. Similarly, two PETs M9 and M7 pointed out that formula can be discovered by students in teaching contexts. But they made opposite arguments on if the formula should be included in the definition based on this observation.

PETs' Evaluation of Whether a Statement About the Area of a Triangle Is a Mathematical Definition

Only four PETs gave correct judgment to Q2, namely they rejected the statement "The area of a triangle with base b and height h is equal to ½ bh." as a mathematical definition of the area of a triangle. Because Q1 and Q2 ask for definitions of similar concepts, my hypothesis was that PETs would give similar responses. However, a comparison between PETs' responses to Q1 and Q2 suggested that only half of PETs gave consistent answers to both questions. Among them, only one PSTs - M gave consistent and correct answers to both questions. She wrote the amount of space inside the rectangles as the definition and also determined that the statement in Q2 was not a definition of the area of a triangle. Other PETs either made mistakes in Q1 or Q2.

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Discussions and Conclusions

Overall, twenty percent of PETs and less than 20% of PETs gave correct responses to two interview questions respectively. Only one PET - M gave correct answers to both interview questions. This result indicated PETs' weak understanding of mathematical definition of the area of a rectangle. When PETs were making judgment about mathematical definitions, they did not consider whether a statement could be deduced from other knowledge as an essential feature. Even though two PETs pointed out that formula can be discovered by students in teaching contexts, no PET has ever mentioned this idea in pure mathematics setting. This phenomenon suggests that PETs' understanding of the difference between mathematical definitions and deduced results is weak. PETs tended to focus on external features such as if the statement is specific to a rectangle, if the statement is clear, straightforward, understandable, and can be discovered by their students. These external features are certainly good features to attend for mathematical definitions, but taking these external features as the most important features of mathematical definition is inadequate. Vinner (1991) pointed out that there is a significant difference between "technical concept" (e.g., mathematical concept) and every day concept. Obviously, the PETs in this study were not aware of the differences and many of them made incorrect judgment because they favor external features, which might be more important to a daily life definition.

Another interesting phenomenon I found is that even if PETs - M received more mathematics instruction in college, their understanding of the area of a rectangle and their understanding of mathematical definitions in general were not better than PETs - N. This implies that taking advanced mathematics courses alone does not automatically improve PETs' understanding of mathematical definitions. PETs' inconsistent responses in Q1 and Q2 further supports my claim that PETs are not clear about the nature of mathematical definitions. Therefore, explicit instructional attention should be given to clarify the nature of mathematical definitions and their roles in mathematics. Also, assessment needs to be aligned with the instruction in order to test PETs' knowledge of mathematical definitions as a meta level concept in addition to testing PETs' knowledge of specific mathematical concepts (e.g., fractions).

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PRE-SERVICE TEACHERS' STRATEGIES FOR COMPARING FRACTIONS

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The teaching and/or learning of fractions have been a challenge for students, pre-service teachers, and teacher educators. The creation of a sequenced set of applets developed to model the approach advocated in the Common Core Standards provided the opportunity to examine their impact on pre-service teacher learning of fractions. In this study, we examined how pre-service teacher content and pedagogical knowledge compared when learning from a technology approach using the applets versus a hands-on manipulative approach. This paper reports on two groups of pre-service teachers' ability to compare fractions based on using these approaches in their pre-service teacher methodology courses.

Keywords: Teacher Education – Preservice, Technology

Fractions in the Common Core Standards

Students' struggles with fractions have been well documented in the research literature. As an example, on the 2004, National Assessment of Educational Progress (NAEP), 50% of 8th graders could not order three fractions from least to greatest, and 70% of 17-year-olds could not write 0.029 as a fraction (Siegler et al., 2010, p. 6). Because of these concerns, the *Common Core State Standards for Mathematics* (CCSSM) (National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010) promotes a more coherent learning sequence with an emphasis on unit fractions and fractions as numbers on the number line. The changes are supported by a 2010 U.S. Department of Education's Institute of Education Sciences report on effective K-8 fraction instruction and the 2008 study by the National Mathematics Advisory Panel. The authors of the IES study conclude: "A high percentage of U.S. Students lack conceptual understanding of fractions, even after studying fractions for several years" (Siegler et al., 2010, p. 6).

If future teachers are to engage their students in a coherent learning sequence about fractions, they need to have the required knowledge to help students develop a conceptual understanding. However, the limited knowledge of pre-service and in-service teachers' own understanding of fractions is a concern (Ball, 1990; Ma, 1999). Additionally, mathematics teacher educators consider this topic to be one of the most challenging in their work with pre- and in-service teachers (Tirosh, 2000; Borko et al., 1992). In this paper, we examine the effectiveness of a technology-leveraged approach compared to a non-technology approach in the development of pre-service teacher content and pedagogical knowledge with respect to fractions. The technology-leveraged approach used a carefully sequenced set of interactive documents (applets) focused on fractions and designed to adhere to the didactical approach advocated by the CCSSM, particularly an emphasis fraction as a number and using numbers lines as a central representational tool. By "interactive" we mean the document environment allows the learner to "deliberately take a mathematical action, observing the consequences, and reflecting on the mathematical implications of the consequences" (Conference Board of Mathematical Sciences [CBMS], 2012, p. 34) without requiring significant time to learn to use the technology. The design of the interactive documents has been guided by the Action-Consequence Principle (Dick & Burrill, 2009) in that the technologies should allow students to: 1) take deliberate, purposeful and mathematically meaningful actions; and 2) provide immediate, visual and mathematically meaningful consequences. The research question investigated whether or not the use of interactive documents might provide a significantly different impact on pre-service teacher

fraction content and pedagogical knowledge, both in terms of quantitative performance on measures of fraction understanding, and in the qualitative nature of the explanations and models used when teaching fraction concepts and operations.

Developing Strategies for Comparing Fractions

In this section we examine the one aspect of the results from a larger study regarding the use of a technology-leveraged approach involving the interactive documents described above to develop understandings of key fraction concepts with pre-service teachers (PSTs). In particular, we compared the results of using the interactive documents with a non-technology approach using more traditional manipulatives, such as Cuisenaire rods, fraction strips, and diagrams, e.g. area models, to develop the core fraction concepts of unit fraction, comparing fractions, and operations.

Participants and Setting

The participants in the study include 52 PSTs in an elementary methods course for pre-service teachers. The methods course was the third in a sequence for elementary teaching majors and focused on mathematical content and pedagogy appropriate for K–4 classrooms. PSTs enrolled in the course were in the second or third year of college. The PSTs were in three different classes, which are called T1, T2 and NT, of size 10, 17 and 25 respectively. The PSTs enrolled in the T1 and T2 sections were all female. The NT section consisted of 20 females and 5 males. PSTs in T1 and T2 met for 75 minutes twice a week and the NT class met for 50 minutes three times a week.

Methodology

T1 and T2 classes engaged in 9 lessons on fractions using the applets and supporting materials modified for pre-service teachers. They did not directly engage in any activities comparing fractions. The NT class engaged in 14 lessons using non-technology techniques, one explicitly addressed comparison strategies, such as benchmarking, common numerator, distance to/from and common denominator.

Data Collection & Analysis

Pre-service teachers (PSTs) were given a pre-assessment on their fraction content and pedagogical knowledge at the beginning of the course. The content questions consisted of selected NAEP questions and mathematics education faculty members developed the pedagogy questions. At the end of the course, eleven questions embedded in the final exam served as the post-assessment. The questions were aligned to the pre-assessment content and pedagogy topics.

PSTs completed the pre-assessment during the third week of the course in a regularly scheduled class meeting. The pre-assessment consisted of thirteen questions covering content and pedagogy, and PSTs were given 30 minutes to answer any questions they could. PSTs completed the post-assessment during a common final exam time, which occurred the day after the final class meeting for the NT class and five days after the final class meeting for the T1 and T2 classes.

Strategies for Comparing Fractions

In the larger study, the quantitative performance on measures of fraction understanding on aligned questions on the final exam showed a slight, but not significant positive difference for PSTs using the interactive document approach. In terms of the qualitative nature of the explanations and models, the PSTs in the technology-based classes were more likely to explain their thinking, used a larger variety of strategies, and more often used the number line as a model. As an example, we examine in detail the qualitative differences in PST explanations on one question regarding comparing fractions.

The PSTs in the technology courses were not explicitly taught strategies for comparing fractions. In several activities, PSTs in the T1 and T2 sections had to use a number line or area model to

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describe why two fractions were either equivalent or not equivalent. In the non-technology class, PSTs engaged in one lesson where they learned strategies for comparing fractions other than common denominators. PSTs in the NT section engaged in specific problems to highlight the strategies: benchmark, distance to/from, greater number/larger size, and common numerators. These strategies were reviewed in a subsequent lesson and were part of estimation explanations prior to completing a fraction operation problem.

On both assessments PSTs were asked to determine which fraction was larger for two problems. The two problems were A) $\frac{15}{17}$, _____ , $\frac{15}{27}$ and B) $\frac{5}{12}$, _____, $\frac{7}{9}$. While very few PSTs answered these questions incorrectly, the explanations for how they determine which fraction varied. For problem A on the pre-assessment, 30% of all PSTs used a common denominator approach. On the post-assessment, no PSTs used the common denominator approach. In the non-technology class, 20 of the 25 PSTs used a common numerator approach. In the technology classes, PSTs answers varied among many different strategies, which are summarized in Table 1.

Table 1: Fraction Strategies Used by Pre-Service Teachers on Post-Assessment

Fraction Strategies	Technology courses (n = 27)		Non-technology course (n=25)	
	Problem A	Problem B	Problem A	Problem B
Common numerator	8	0	20	0
Distance to/from	1	0	2	3
Area	1	1	0	0
Number line	7	8	2	3
Benchmark on a number line	2	4	0	0
Benchmark	3	7	0	10
Number of copies	4	3	0	0
Common denominator	0	3	0	6
Greater number/larger size	0	1	1	3

The most prevalent strategies used to solve problem A in the T1 and T2 courses were common numerator and explaining the location of the two fractions on the number line. In their explanations comparing these fractions, four PSTs reasoned the number of copies of the unit fraction was the same but the size of the unit fraction was different.

For problem B, 68% of all PSTs used a common denominator approach on the pre-assessment. On the post-assessment, 3 of 27 (11%) students in the technology classes and 6 of 25 (24%) PSTs in the non-technology class used common denominators. The most used strategy in the non-technology class was the benchmark strategy that identified both fractions on either side of one-half. In the technology classes, the most frequent strategy was a number line followed by two different variations of benchmarking strategies. For the varied benchmarking strategies, seven of the PSTs stated the fractions were on either side of one-half, and four PSTs drew the fractions out on a number and indicated where one-half was located or noted that one fraction was closer to 1 while the other was closer to zero. The three PSTs who used the number of copies strategy explained their reasoning by describing that there were more copies of larger pieces in the ninths than the number of copies in smaller pieces of twelfths.

When explaining their comparison strategy, the PSTs in the non-technology class typically named a strategy, such as distance to/from, while most PSTs in the technology classes wrote out their thinking, such as two copies of seventeenths is closer to one than twelve copies twenty-sevenths.

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PSTs in the technology classes also used a larger variety of strategies for the same problems than the PSTs in the non-technology class. Based on these results, it appears using comparison ideas within activities rather than learning the strategies as tools separately benefits PST's ability to explain their approach to comparing fractions and allows them to apply the ideas flexibly.

Conclusion

If PSTs' pedagogical knowledge is influenced by their experiences in pedagogy courses, the tools used to build their understanding could have an impact on their future instructional choices as in-service teachers. Since the current standards and prior research call for an emphasis on the use of number line, the technology-based approach seems to have promise for helping future teachers understand the benefits of activities that use number lines to explain fraction concepts and operations. As the comparing fraction question from the study showed, PSTs in the technology courses developed their own strategies and explanations for comparing fractions, used a larger variety of strategies and more often used numbers lines when explaining their thinking. In contrast, PSTs that explicitly learned strategies used a prescribed strategy as their explanation for comparing fractions problems. Much of the literature cautions that teachers may use ineffective model and diagrams. However, with the mathematical practices standards call for strategic use of technology tools, the opportunity to learn through the CBMS (2012) recommendation of action, consequence, and reflection may be an approach that translates to deeper knowledge for teaching fractions and pedagogical change in the classroom.

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PRESERVICE TEACHERS USE OF MATHEMATICAL EXPLANATIONS

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Research has shown that preservice teachers struggle to create and use meaningful mathematical explanations and often fall back on procedural, or step-by-step explanations. In this study, we examined how preservice teachers explained a mathematics problem by engaging them in a multi-task activity in which they recorded their own mathematical explanations using the Educreations app, listened to and watched recorded mathematical explanations, and discussed what they considered as important elements of mathematical explanations. The results indicate that the preservice teachers improved their use of language and mathematical terminology in their mathematical explanations.

Keywords: Middle School Education, Teacher Education - Preservice

The Standards for Mathematical Practice set forth by the Common Core State Standards (CCSSI, 2010) clearly call on students to explain their thinking by communicating their reasoning and using mathematically precise language and symbols. This creates a need for mathematics teacher educators to support preservice teachers in developing the practice of communicating precisely when explaining mathematics. Research has shown that preservice teachers (PSTs) struggle to create and use meaningful mathematical explanations and often fall back on procedural, or step-by-step explanations (Ball, 1990; Borko et al., 1992). These same teachers will play an important role in modeling mathematical thinking, processes and practices of mathematics for students. Therefore, it is important to support future teachers in investigating and evaluating mathematical explanations, both their own and those composed by others. The purpose of this study was to examine how PSTs explained a mathematics problem. More specifically, the research question underlying this study is: In what ways does participating in a classroom intervention focused on problematic mathematical explanations support middle level prospective mathematics teachers on providing productive mathematical explanations?

Theoretical Perspectives

While several studies in the field of mathematics have focused on developing students' vocabulary fluency and the importance of vocabulary in learning mathematics (see for example Downing, Earles-Vollrath, Lee, & Herner-Patnode, 2007), there is a dearth of research, if any, that focuses on prospective mathematics teachers use of mathematics vocabulary and mathematical language in their explanations. Furthermore, where verbal descriptions and examples fall short, visual representations can illuminate important mathematical messages. Lesh, Post, and Behr (1987) argue representations, which are essential elements of mathematical explanations, can be spoken, written, symbolic, and/or pictorial. Visual representations can help students make sense of mathematical problems and help members of the classroom mathematical community understand each other's thinking (National Council of Teachers of Mathematics [NCTM], 2014; Taylor & Dyer, 2014). We contend visual representations can be used as tools for thinking that "serve to 'unpack' the structure of the problem and lay the foundation for its solution" (Diezman & English, 2001, p. 77). In addition, we strive to help our PSTs use all forms of representations in their explanations in ways that are both precise and illustrative of the underlying mathematical meanings. We characterize representations that meet both of these goals as *productive representations*.

Methods

We designed a set of classroom activities with the dual goals of providing students with opportunities to reconceptualize mathematical explanations and modeling the types of conversations they might have in their own classrooms around mathematical explanations. We contend that students are in the best position to determine which features of an argument communicate the most meaning to them. Thus, we engaged our PSTs in activities that allowed them to construct a shared vision for mathematical explanations. A total of 26 students, 11 males and 15 females, (> 95% were Caucasian) from two different universities in the south-central region of the United States enrolled in middle level mathematics methods courses at their respective institutions participated in the study.

Data – Mathematical Explanation Task

We engaged our PSTs in a multi-task activity in which they recorded their own mathematical explanations using the Educreations app ([www.educreations.com)](www.educreations.com), listened to and watched recorded mathematical explanations, and discussed what they considered as important elements of mathematical explanations.

Task 1. Students were given a mathematical task (e.g., A circle has a radius of 5 inches. What radius will double the area of the circle?) with the following prompt:

> One of your friends is struggling with the given problem. There may be problems like this on their upcoming test. How would you explain to them so that they understand it and will do well on the test?

Students used an Educreations app, which allows users to simultaneously record voice and write on a virtual whiteboard, to record their explanations. The links to all of students' explanations were collected.

Task 2. Students watched the mathematical explanation of solving for x: $4x - -3 = 15$ that was pre-recorded on Educreations as a class. The recording deliberately focused on common problems PSTs make when giving mathematical explanations. As the class listened to the verbal mathematical explanation, they simultaneously viewed the written and symbolic representations. The following is the transcript of the mathematical explanation with the representation.

> I would begin by rewriting the problem. So, I would rewrite four x minus minus three equals fifteen. Then, I would do keep change change. Next. I will move the three over so that all of the numbers are on the same side.

$$4x + {}^+3 = 15$$
$$4x + 3 = 15 - 3$$

> From this point, I would divide both sides by four canceling out the four on the left side. Then, we have x equals twelve fourths. Finally, we will reduce down the twelve over four and get x equals three.

Students were asked to reflect on what they noticed about the explanation. Specifically, what parts of the explanation promoted understanding, what parts would hinder students' understanding, and how they would explain the problem.

Task 3. Students viewed and discussed important components of their initial mathematical explanations in their small groups. Through this iterative process and whole-group discussion of

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important aspects of various explanations, a list of essential elements for a general mathematical explanation was developed (see Figure 1).

Students at University 1	Students at University 2
1. Clarity – clear language, correct terms, appropriate pace, clear writing 2. Completeness – show every step, don't make assumptions, full explanations, anticipate the "why" 3. Direction – states the goal	1. Correct terminology 2. Accurate and relevant visual representations 3. Provide detail to explain "why" 4. Establish a clear goal

Figure 1. Essential elements of mathematical explanations.

Task 4. After creating the final list (see Figure 1), the essential elements of mathematical explanations were used as criteria to assess students' explanations. Students were asked to record new explanations for their original mathematical task, and score their new explanations using the class-created rubric. The students then compared their first explanation with the new one in a written reflection.

Data Analysis

In order to analyze the data, we collectively watched a PST's initial mathematical explanation on Educreation and coded the representation (i.e., verbal, symbolic, and pictorial) used during the explanation. As we analyzed the explanation, we noticed the PSTs would use a representation in their explanation as an organizational tool. For example, the PST would underline different aspects of the problem or circle parts of the problem for added clarity in the explanation. So, we added a code for organization. We each coded the PSTs' initial and post explanations. During this analysis, we wrote theoretical memos (Grbich, 2007) related to the PSTs' understanding of the mathematical content, vocabulary/language use, and the visual representation used within each explanation. We then discussed the codes and theoretical memos. All discrepancies were discussed and a consensus on the codes and theoretical memos was obtained. After a consensus was reached, we identified themes that emerged across the data.

Results

We found the visual representations the PSTs used in their initial mathematical explanations were less productive (i.e., detracting from the problems conceptual meaning). For example, Dominique was asked to find the radius that would double the area of a circle with radius 5 cm. His explanation began with a sketch of a circle with radius 5.

To solve the problem, Dominique first found the area of the original circle using the area formula for a circle. He then doubled the area and solved for the radius. Although Dominique took the time to create a visual representation, he did not use or refer to it as he explained the problem. Sometimes students have become accustomed to hearing the phrase "draw a picture," they do it without thinking about how the representation will help them understand the mathematical concept (NCTM, 2014). It is possible that Dominique felt compelled to include a visual representation for the sake of having one. Dominique's less productive visual representation is not necessarily harmful, but using it to explain the problem poses a challenge because it did not help the student identify a solution path for the problem.

We also found the PSTs' improved their use of language and mathematical terminology in their mathematical explanations. For example, Kelley was asked to explain $\frac{8}{14} \div \frac{2}{7}$. In her initial

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explanation she showed two approaches for dividing fractions: common denominator method and multiplying by the reciprocal. When she explained the division she used the phrase "those can cancel out." Kelley provided more background knowledge of defining the numerator and denominator in relation to the problem in her second explanation. Unlike the initial explanation where she divided only the numerator by two she clearly articulated that both the numerator and denominator had to be divided by two in her post explanation. Kelley did not say the denominators "cancel" out as in the first explanation, but she clearly demonstrated and explained how seven divided by seven is one. In addition, she explained the meaning of the reciprocal in the post explanation. Kelley was cognizant of her use of precise mathematical language in her post explanation.

Discussion

Our PSTs were able to consider their own ideas and assumptions, both conscious and unconscious, about what constitutes a productive mathematical explanation. We relied on Watson and Mason's (2005) characterization of non-examples as a way to "demonstrate boundaries or necessary conditions of a concept" (p. 65). In much the same way that vague or incorrect mathematical definitions can prevent a student from understanding a mathematical concept, a non-productive visual representation may lead a student to draw faulty conclusions or simply fail to help the student make any meaningful progress towards these goals. In this study, the PSTs were able to reflect on both their verbal and written explanations, which in many cases included the visual representations they used.

Activities like the ones discussed can serve as a foundation for creating mathematics classrooms focused on the kinds of meaningful mathematical discourse described in Principles to Action (NCTM, 2014). Specifically, classrooms in which "students are active members of the discourse community as they explain their reasoning and consider the mathematical explanations and strategies of their classmates" (NCTM, 2014, p. 35). Based on this, we contend PSTs should understand the purpose of mathematical explanations and why they are important.

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UNDERSTANDING AND TRANSFORMING PRE-SERVICE TEACHERS' PROPORTIONAL REASONING

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Proportional reasoning is important to the field of mathematics education because it lies at the border between the transition from additive reasoning in the elementary school and multiplicative reasoning needed for more advanced mathematics. This research reports on findings from a qualitative study which examined pre-service teachers' (PST) proportional reasoning. In particular, this study highlights two misconceptions about proportional reasoning that were prevalent among PSTs and focuses on specific ways to transform these assumptions. Using Mezirow's transformative learning theory, tasks and targeted questioning were used to elicit disorienting dilemmas that allowed PSTs to engage in critical reflection about their proportional reasoning. Ultimately, this led the PSTs to transform their misconceptions and to understand proportional relationships in new ways as they made shifts in their understanding.

Keywords: Teacher Knowledge, Rational Numbers, Learning Theory, Teacher Education-Preservice

Proportional reasoning is important to the field of mathematics education because it lies at the border of transitioning from additive reasoning in the elementary school to multiplicative reasoning needed for more advanced mathematics. Lesh, Post, and Behr (1988) describe the importance of proportional reasoning, saying that it is "widely recognized as a capability which ushers in a significant conceptual shift from concrete operational levels of thought to formal operational levels of thought" (p. 101). This shift in understanding can lead to advanced mathematical thinking.

Pre-service teachers (PSTs) enter college with previous understandings of mathematics and mathematical concepts. Often PSTs have many deep rooted misconceptions about the multiplicative relationships involved in proportional reasoning and struggle with solving problems that involve these concepts. The question becomes: how can we help PSTs transform their understanding of proportional reasoning, so they might develop the necessary specialized content knowledge for teaching mathematics? This report focuses on how particular tasks and specific discourse moves, providing reflection on their conceptions, ultimately led to a transformation of their proportional reasoning.

Theoretical Tools: Transformative Learning Theory

According to transformative learning theory, our mission as mathematics educators is to provide opportunities for our PSTs to become aware of their prior assumptions in order to allow them to transform and deepen their knowledge. Simply presenting PSTs with information about proportional reasoning is not enough to help them alter their understanding. What is required is a transformation of their existing knowledge. Transformative learning theory attempts to first establish and clarify a learner's prior assumptions and then transform these assumptions (Mezirow, 1991). Mezirow (1991) focuses on three key components: 1) "we transform our frame of reference through critical reflection on assumptions" (p. 7), 2) that "rational discourse through communicative learning" is key (p. 78), and 3) that this reflection often takes place within the context of problem solving. Transformative learning theory suggests that in order for PSTs to make shifts in their understandings about proportional reasoning, they need to become aware of their misconceptions and then have these assumptions challenged in ways that encourage reflection about proportional reasoning. This study provides evidence that PSTs frequently made shifts in their strategies to solve proportional problems after their reasoning was perturbed through targeted discourse with the interviewer.

Methodology

Twenty-five PSTs were selected for this study at the beginning of their first mathematics methods courses at a large research university. A nine problem questionnaire was used in order to ascertain each PST's current level of understanding about proportional reasoning. (See Johnson, 2013 for more details on questionnaire). The questionnaires were coded and participants were divided into five distinct groups based on the analysis of their responses. Eleven individual interview schedules were created in order to challenge the PST misconceptions about proportional reasoning, the interviews were implemented, videotaped, transcribed and annotated. Individual interview data was coded and analyzed to create descriptions of the nature of the participants' understanding of proportional reasoning. A group of trained graduate students also coded the data and these codes were then discussed and revised to provide a higher degree of validity and reliability (Johnson, 2013). This report discusses how disorienting dilemmas and targeted questioning were used to transform these misconceptions.

Providing Opportunities for Pre-service Teachers to Overcome Challenges of Proportional Reasoning through Transformative Learning Theory

Mezirow (1991) suggests that problem solving situations often present disorienting dilemmas for learners. For this study tasks were designed to address distinct aspects of PSTs' misconceptions about proportional reasoning that surfaced from the questionnaire. Two of these previous assumptions include: (1) reasoning quantitatively (Thompson, 1994), (2) recognizing ratios as measurement (Schwartz, 1988). I will illustrate how these particular tasks and targeted questioning challenged PSTs' prior misconceptions which led them to transform their understandings of proportional reasoning.

Disorienting dilemma: quantitative reasoning versus computation. PSTs in this study struggled with proportional reasoning situations that involved the distinction between quantitative reasoning and computation. Quantitative reasoning is making sense of the relationship among measureable attributes of objects in a situation (Thompson, 1994) while computation is the result of an arithmetic operation to evaluate quantities. In general, reasoning about quantitative situations involves conceiving of circumstances in terms of quantities by constructing networks of quantitative relationships. For example, PTS often will set up proportions and not understand what the ratios represent in the context of the situation.

The Lemon/Lime task was used to challenge the PST's misconception of quantitative reasoning and computation (see figure 1). In this task participants were asked to compare two different lemon/lime mixtures (3 lemon:2 lime and 4 lemon:3 lime) **without doing ANY calculations** but by representing the mixtures with unifix cubes. The request to not use calculations posed a high degree of difficulty for most of the PST interviewed, because it forced them to reason quantitatively and conceptually not computationally.

> *Jen and Alice are making lemonade. Jen mixes 3 cups of lemon juice with 2 cups of lime juice. Alice mixes 4 cups of lemon juice with 3 cups of lime juice. Whose mixture is more lemonier? Justify your solution without using calculations. You may rearrange the cubes in such a way that they demonstrate which one's more lemony tasting or that they are the same. (Heinz [2000, p. 70])*

Figure 1. Lemon/Lime problem.

Emma is a typical example of how many of the PSTs reasoned quantitatively about the mixture (without the use of calculations). Initially, she used an additive relationship claiming that there was "one more cup of lemon in each mixture so the mixtures were the same." However, when she was allowed to utilize calculations she created a multiplicative relationship (i.e. $3/2 = 1.5$ and $4/3 = 1.333$)

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by dividing the quantities in order to compare the decimal representations of the mixtures. This interpretation of the relationship caused her to reevaluate the mixtures and determine that the 3 lemon:2 lime mixture had more lemon taste than the 4 lemon:3 lime mixture.

Targeted questions to provide reflection. Emma was then asked to reflect on "why her answer with the calculation was different from the conclusion she made with the representation?" The fact that her initial conclusion and the calculations did not match provided a disorienting dilemma for Emma. She claimed that "because the concentration of lemon juice was stronger in the smaller volume (mixture with 5 total cups), then the larger volume (mixture with 7 total cups)" one of the mixtures would have more lemon taste. Emma exemplifies the misconceptions many of the PSTs had when only allowed to reason quantitatively about a situation. This illustrates how the lemon/lime problem posed a disorienting dilemma—multiple conflicting justifications that relate back to the original task— that provided opportunities for Emma to make sense of contradictory responses and allowed reflection on her previous assumptions and transform her reasoning. It is important that we allow PSTs to take the time to reflect and discuss these disorienting dilemmas in order to transform their understandings.

Disorienting Dilemma: Ratio as measurement (reasoning with intensive quantities). Many PSTs in this study had difficulty reasoning with ratios as measurements. Ratio as a measurement might be thought of in terms of intensive quantity. Schwartz (1988) describes quantities that can be counted or measured (e.g., distance or length) as extensive and quantities that form a composed unit between two quantities as a ratio (e.g., speed) as intensive. This distinction explains how ratios as measurements are intensive quantities. The main challenge with intensive quantities was for PSTs to overcome the "more A implies more B" comparison between the two extensive quantities involved in the relationship is a ratio not a difference. PSTs were challenged have when reasoning with intensive quantities or ratios as measures. The Housing problem (see Figure 2) was used to elicit ratio as a measurement in terms of squareness of non-square rectangular lots and intensive quantity that relates *width* and *length*. Additive relationships are often used with linear measurements in mathematics; but this problem in contrast is multiplicative, ratios of linear measurements are necessary to determine which lot is most square. This problem provides specific challenges to PSTs' initial inclination to use additive reasoning due to the presence of linear measurements given for the dimensions of the housing lots.

> *A new housing subdivision offers lots of 3 different sizes: 185 feet by 245 feet; 75 feet by 114 feet; 455 feet by 508 feet. If you were to view these lots from above, which would appear most square? Which would be least square? Explain your answer. (Adapted from Heinz [2000, p. 150])*

Figure 2. Housing problem.

For example, Eve initially claimed that "the lot that is the most square is the one with the smallest difference between length and width." She then calculated the differences between the lengths and widths of each housing lot and reported that "I would say that this one [*circles 75x114 drawing*] would be closer to a square because they are closer in distance."

Targeted questions to provide reflection. Eve was then asked to reflect on her strategy of utilizing the difference between length and width by applying it to a new scenario. In this case she was to determine whether two new lots, one that is 100' x 200' and one that is 300' x 400', is more square. The facts that her initial strategy when applied to this second task caused a disorienting dilemma for Eve since she would have to conclude that the two lots were the same squareness. After some thought, Eve claimed that,

> this [*traces 100 foot side*] length has to be half of this width, whereas this length [*traces 400 foot side*] is NOT half the width [*traces 300 foot side*] it's MORE than half, so it's going to make it appear more square.

This reasoning represents a multiplicative comparison between the length and the width as a ratio whereas Eve's previous reasoning utilized an additive comparison of differences. I then asked Eve to discuss on her response to the first prompt with the three housing lots (figure 2). After some reflection she applied this multiplicative concept of comparing the length and width as a ratio to a benchmark of half to the previously presented lots. Eve says,

> this one [*points to 75x114*] seems more like this one [*points to100x200*] because 75 is more like half of this [*points to 114*] versus 508 compared to 455 [*points to 508 then 455*] is not half so it's gonna look more square over here [*points to 300x400*], which would be like this again [*points to the 300x400*], this one [*points to 185x245*] is kind of in the middle 245 they are not really half.

In this example we see Eve transform her previous misconception of using additive reasoning to solve the problem and revise her thinking that the 455' x 508' lot would be the most square since it has the largest ratio, which involved proportional reasoning. This further illustrates how targeted questioning, discourse and reflection permitted Eve to transform her previous assumptions about ratio as measure.

Conclusion

If proportional reasoning represents an important border between elementary mathematics and higher level mathematics then PSTs need to obtain a deeper understanding of ratios and proportions in order to provide their students with richer opportunities in their future classrooms. This study illustrates how Transformative Learning Theory can be used as a guide to transform PSTs' previous assumptions regarding proportional reasoning. As such, PSTs need to first be made aware of their misconceptions about ratio and proportion and then have these assumptions challenged through questioning and discourse in their university experiences.

By helping these PSTs become aware of their misconceptions about proportional reasoning through these disorienting dilemmas (i.e., thought-provoking problem solving tasks) and providing opportunities for them to engage in rational discourse about these tasks, the PSTs were able to think about proportional problems in new ways and make shifts in their proportional reasoning. This knowledge can be used by mathematics teacher educators to develop courses for both elementary and secondary PSTs that can transform and extend their understandings of ratio and proportion and enhance their proportional reasoning so that future teachers can ultimately improve their students' learning at this important border between higher and lower mathematics.

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PRESERVICE TEACHERS' FRACTIONAL KNOWLEDGE: UNDERSTANDING THE DISTINCT ROLES OF FRACTIONS

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Drawing from task-based interviews, classroom observation, and participants' homework, the present study examines ten middle grades preservice teachers' understanding of the role of fractions as operators, with an eye toward exploring how fractional reasoning is constructed. The results point to the construction of the reversible distributive partitioning scheme as a requisite for understanding fractions as operators. Further discussion will suggest that school curricula and teacher education programs may need to be adjusted to reflect more current understandings of both early childhood cognitive development and future teachers' fractional knowledge.

Keywords: Cognition, Teacher Education-Preservice, Teacher Knowledge

Writing about four decades ago, Kieran (1976) remarked that "most school curriculum materials simply treat rational numbers as objects of computation. Hence, children and adolescents miss many of the important interpretations of rational numbers" (p. 102). Since then, substantial research has further emphasized that there are multiple interpretations—or "subconstructs"—of fractions (e.g., Behr, Khoury, Harel, Post, & Lesh, 1997), and numerous books have been written reflecting this point of view in teaching rational numbers (e.g., Clarke, Fisher, Marks, & Ross, 2010). The current study concerns itself primarily with the role of fractions as "operators," referring to fractions that link two quantities of the same kind or of different kinds (see Vergnaud, 1988).

Students are normally taught fractions from third to fifth grade, before reaching the expected age range identified by Lovell (1972) for developing the proportionality schema (i.e., ages 12-14). Two subsequent studies on adolescents' comprehension of fractions as operators (Kieren, 1976) add some empirical weight to Lovell's (1972) earlier findings, showing a strong relationship between age level and scores that is consistent with this suggested age range. In that regard, Kieren (1976) suggest an important relationship between students' development of the proportionality schema and their comprehension of fractions as operators.

Post, Behr, and Lesh (1982) noted that understanding proportional reasoning is only possible once students have reached Piaget's "formal operational stage," which generally happens in the fifth grade. In that case, many students – perhaps a majority – are not ready to learn the operator role of fractions at the time when fractional foundations are being laid in school curricula. However, in the middle grades, if students cannot understand fractions both as quantities and operators, this can be a barrier to their comprehension of later topics in mathematics.

In light of these considerations, this study investigates preservice teachers' understanding of fractions as operators through task-based clinical interviews designed to reveal any areas of struggle, in order to further analyze the nature of these issues—particularly in regard to what they can reveal about how fractional reasoning is constructed—and to discuss possible solutions.

Theoretical Framework

Steffe (2003) conducted a study that can serve as an exemplar of the radical constructivist approach that forms an essential basis of my epistemological and analytical framework. Steffe (2003) adopted as part of his analytical framework the notion that both social interaction and an individual's action are crucial in learning. He regarded social interaction as a means of "generating situation" for the construction of students' cognitive schemes, and learning as a product of the "auto-regulation" of constructs within the individual's understanding. The emphasis on the individual's self-regulation

and auto-regulation is in line with von Glasersfeld basic principles of radical constructivism (1990, p. 22). What we perceive as reality is thus regarded as a construct within the mind of each individual.

Hackenberg and Lee (2015) showed that only MC3 (able to coordinate three levels of units) students could use fractions as multipliers in the context of writing equations. Unit coordination is related to the ability to construct certain schemes, such as distributive sharing and reversible distributive partitioning (Steffe & Olive, 2010). To illustrate, students who can share two identical bars among three people have constructed the "distributive sharing" scheme. To carry out this operation, they would partition each of the two bars into three parts and pull out one part from each bar, understanding that those two parts make up 2/6 of the original two bars. However, unless they had also constructed the "reversible distributive partitioning scheme," they would not understand that 1/3 of two bars is identical to the two individual thirds of each bar. Thus, students who have constructed the reversible distributive partitioning scheme understand that taking one-third of each of two objects (not necessarily identical) is equivalent to taking one-third of both objects together.

While I make use of a constructivist theoretical framework, my conceptual framework includes both the constructivist notions of fractional schemes and operations and Vergnaud's (1988) explanation of fractions as quantities and operators. For the purposes of the current study, it seems appropriate to make use of a qualitative case study design that will allow for a context-rich and focused investigation using task-based clinical interviews and participant observation.

Research Design

The present research is designed to address the following questions: How do middle grades preservice teachers reason with fractions as operators?

1. What can their reasoning with fractions as operators reveal about their construction of fractional knowledge?
2. What is the nature of their struggles in constructing fractions as operators, and what are some possible solutions to these issues in their understanding?

Building on the concept of the clinical interview developed by Piaget (1951), the "constructivist teaching experiment" (see Steffe & Thompson, 2000, p. 285; Steffe, 2003) is a method by which the researcher's ongoing analysis can interact with data in a fluid way. According to a description by Steffe & Ulrich (2014), "[a] teacher/researcher, through reviewing the records of one or more earlier teaching episodes, may formulate hypotheses to be tested in the next episode. . ." (p. 104). While this research tracks student' learning trajectories across successive teaching episodes, the pedagogical interventions themselves are supplied by the participants' course instructor, rather than directly by the interviewer. The method employed in the current research, therefore, may be regarded as a modified teaching experiment method; rather than one person acting as the "teacher- researcher" in "teaching episodes" (Steffe & Thompson, 2000), I conduct interviews as the "researcher," while the course instructor acts as the "teacher." The interviews themselves—without the expressly pedagogical element characteristic of the teaching episodes of a teaching experiment—closely resemble "task-based clinical interviews" (Goldin, 2000).

This study makes use of a multiple case study design, examining the fractional knowledge of preservice middle grades math teachers in a Math Education department at a major public university located in the Southeastern United States. The primary data source is a series of clinical interviews conducted over the course of two semesters using ten volunteer participants from a math content course. More specifically, I make use of data from task-based clinical interviews, as well as from classroom observation notes and homework submissions.

From among the initial volunteers in the math content course, I divide the participants into groups according to whether or not they have constructed the distributive sharing scheme. These

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selections are based on analysis of an initial round of interviews, classroom observation notes and homework submissions.

Results and Significance

All clinical interviews were recorded on video, transcribed verbatim, coded and annotated for emerging themes, and analyzed. At the beginning of the interview all of participants were unable to make composite units for fractions acting as operators, which means that they had not constructed the reversible distributive partitioning scheme. In fraction multiplication and fraction equation problems, two of the participants either could not produce the drawn model of certain problems, or they could not provide a coherent interpretation of their own drawings. All participants showed signs of confusion as a result of incorrectly applying procedural knowledge. Moreover, the current progress of my data analysis indicates that the construction of the distributive sharing scheme is necessary but not sufficient to ensure a student's conceptual understanding of fractions as operators.

During each interview, variations of fraction multiplication problems were given, and students were asked to design word problems and represent the problem situations visually. For one of the problems, 4/5 times 1/3, several students drew a rectangle and partitioning it into 3 rows and 4 columns (or vice versa). They shaded 1 row and 4 columns and identified the 4 parts in the overlapping region as their answer. One of the word problems a student designed for this problem was: "You have a recipe that calls for one-third cups of flour, and you want to make four fifths of the recipe, so how much flower do you need to use?" When I asked her to show one cup and one recipe, she identified the largest box as both one cup and one recipe. This indicates that she used both fractions as extensive quantities (instead of 4/5 as the operator to show the relationship between the quantity 1/3 and the product 4/15). This shows her limitations in representing two different units in one drawing and in using one fraction as on operator.

Another identifiable limitation was in making composite units. Before classroom instruction, when asked to show 7/3 of two unit bars, two students responded that they must find one-third of one bar and then repeat it 7 times. Thus, by partitioning both unit bars into three parts and repeat-ing one of these parts seven times, in fact they showed 7/3 of just one bar. Their difficulty was in understanding the two bars together as one composite unit (treating it as the "whole" for 7/3).

After the completion of the unit on fractions in the participants' math course, students already knew that the correct answer to the problem (finding 7/3 of two unit bars) could be given as 14 parts, each being 1/3 the size of one unit bar. However, in explaining the answer with a drawing, three out of ten students showed limitations in partitioning and iterating with composite units. All students gave the answer as 14/3 of one bar, but three students still answered that they needed to understand 7/3 as equivalent to 14/6 to determine the result relative to two bars. If they could form one composite unit out of the two bars, they were able to find 1/3 of the two bars by partitioning each bar into thirds and taking 1/3 from each bar, after which they could also make a composite unit of these two parts and iterate it 7 times to make 7/3 of the two unit bars. In that case, the students do not need to consider 1/3 of one bar as 1/6 of two bars. This shows the limitations of these three students' reasoning with iterable composite units, corresponding to limitations in using fractions as operators.

Conclusion

In fraction multiplication contexts, the multiplicand represents an extensive or intensive quantity with one unit as a whole. However, the multiplier shows the relationship between the multiplicand and the product of multiplication. Thus the whole of the multiplier becomes the multiplicand, and students have to be able to use the multiplicand as one composite unit to perform the operation. This requires students' construction of the distributive sharing scheme.

These preliminary findings, although limited in scope, may direct further research that can cast additional light on the construction of fractional reasoning, both in preservice teachers and in primary

school students. Considering the foundational importance of being able to understand fractions as operators, in the very least we need to ensure that preservice teachers have a strong grasp of this fundamental fractional concept in order to teach middle grades students effectively. Moreover, if further research lends additional support to the notion that primary school is generally too early for students to understand the operator role of fractions, school curricula may need to be adjusted to reflect this.

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LEARNING FOR THE TEST: GROWING UP IN THE ERA OF NCLB

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I report on an interview study conducted with a prospective teacher who began school in 2001, the year the No Child Left Behind Act was passed. Using radical constructivism, I seek to understand how the prospective teacher's experience as a learner in mathematics classrooms characterized by standardized test preparation shaped her understanding of mathematics and the possibilities for teaching mathematics in the U.S. I found the prospective teacher does not see a need to understand why in the mathematics classroom. Also she believes accountability pressures tied to standardized tests limit an elementary teacher's ability to teach comfortably in a way that is best for students.

Keywords: Teacher Beliefs, Policy Matters, Teacher Education-Preservice

In the United States, the era of No Child Left Behind (NCLB, 2001) has come to an end with the passage of the Every Student Succeeds Act (ESSA, 2015). NCLB was criticized because of the unintended effects of high-stakes testing (Lambdin & Walcott, 2007). Accountability pressures contributed to emerging school cultures characterized by mathematics instruction which emphasized preparation for end of the year standardized tests (Darling-Hammond, 2010). It is unclear if this trend will continue. Although the era of NCLB has come to an end, the impact of its implementation likely has consequences for the future of mathematics education. Many prospective teachers now enrolled in teacher education programs have spent their entire school lives as students under NCLB. How might a prospective teacher who was a learner during the NCLB era understand mathematics and her role as a teacher of mathematics?

As a doctoral student in mathematics education, I teach an undergraduate mathematics course designed for prospective elementary teachers. A nineteen year old student in the course, Jessie, began kindergarten in 2001, the year that NCLB was passed. Jessie expressed strong opinions about what she perceived to be the United States educational system and the possibility for students to learn mathematics within it. I decided to request interviews with Jessie so that I could better understand her perspective--a perspective that has been molded by her experience as a student in the era of NCLB.

Theoretical Framework
I enjoy radical constructivism and believe it provides a useful theory of learning to view the influence of NCLB on Jessie's perspective. Von Glasersfeld (1995) articulated two basic principles of this theory. The first is that "knowledge is not passively received but built up by the cognizing subject" (p. 18); and the second is that "the function of cognition is adaptive and serves the organization of the experiential world, not the discovery of ontological reality" (p. 18). Jessie's perspective has been directly influenced by her powerful experiences as a mathematics student in the NCLB era. Her constructed perspective is viable given her experience as a learner in mathematics classrooms dominated by test preparation. My goal in this study is to provide the reader with insight into Jessie's constructed reality, and to understand the ways in which her experience as a student during the NCLB era may have influenced her perspective.

Methodology
The data for this study consists of two hour-long semi-structured interviews, the second conducted one week after the first, both audio-taped and transcribed. The purpose of the first interview was to get a broad sense of Jessie's views regarding education, mathematics, and her role as a future teacher of mathematics. The purpose of the second interview was to probe Jessie to

provide greater explanation and detail illuminating and clarifying the ideas that were expressed in the first. I also desired to understand *why* she might have the perspective that she did. I sought to employ the qualitative techniques of open-ended questioning, empathic neutrality, and rich description as discussed by Patton (2015). I wrote open-ended questions and listened attentively to Jessie, probing her to say more about her ideas. I strove to be empathetic and neutral, making it known that I was interested in understanding her ideas, and that my purpose was not to judge. I engaged in data analysis with the goal of writing a rich description of Jessie's perspective and to understand how her perspective may have been influenced by her educational experiences. To this end, I developed a concept map of the ideas she expressed in the interviews. The creation of this map involved listening to interviews and reviewing transcripts, noting important quotes in the transcripts, and assigning codes to each quote. These codes were designed to summarize or capture the meaning expressed in the quote (Saldaña, 2012). I found that the map did not provide a simple translation to written results. I re-read the transcripts noting the important quotes and created a more general open code for each one so I could better organize the results section according to themes.

Results

Jessie's experience as a student of mathematics, especially in middle and high school, consisted of memorizing formulas and procedures in order to pass tests.

> I couldn't tell you even one thing I learned in high school math, at all… Instead of the teachers actually teaching the stuff, most of them honestly would just be like, "Okay here's the study guide with the exact questions and answers that's gonna be on the test. Go home and memorize it." How am I learning by you giving me the test and telling me to memorize answers?

Jessie always received good grades in math, but she "hated math" because "it just felt pointless." She noted she did not need to understand the why behind mathematics, and was angry when other students asked why during mathematics class.

> I remember in high school we would have those students who would be like "Why is this?" and I would get so mad. I would be like, "Shut up and just learn how to do it"... Because we just had to pass the test you know? It just seemed a waste of time to figure out why because, why wasn't going to be on the test. You just memorized the formula and how to do it.

Jessie did not see the value in asking why questions about mathematics. She believes the pressure for students to perform well on standardized tests precludes the possibility of teaching students to engage in proof or understanding why in the elementary mathematics classroom.

> I wasn't really saying, "No they shouldn't do proofs," and "no I don't think that they can do proofs."… My point was that as with the school system today in today's society we can't just teach however we want to teach… I have to go by what the standards, what the government says that I should teach. And that doesn't leave a lot of wiggle room… Just because a kid can explain why fractions are the way that they are doesn't mean they can solve a fraction problem… There's a big difference between what I think and what's realistic in our school system. Because it's not that kids shouldn't understand why, it's just the fact that the kids don't have to understand why, and kids knowing why isn't going to help them pass the test; and if they don't pass the stupid test then I don't keep my job.

Jessie was never required to understand why in the mathematics classroom, and so her perception is that the state standards must not require children to understand why. To meet the standards, she believes should merely ensure her students can perform procedures similar to those that would be required on a standardized examination. To devote time to understanding why would detract from

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classroom time needed to prepare for the standardized test (and possibly cause her to lose her job as a teacher).

The education system (as Jessie perceives it) is not the way she wants it to be or how she believes it should be. She wants it "how it used to be" when teachers "were assessed based on how good of a teacher they were and not how well their kids tested." One of the influences in Jessie's life is her grandmother, a retired kindergarten teacher. Of her grandmother she said:

> That's why she retired. Cause she didn't like it anymore. She said it wasn't fun anymore. And she said it was too much stress on her and too much stress on the kids, and the kids weren't really having fun anymore. Cause she didn't have time to do any fun stuff, cause you're having to make sure they can pass the test… You basically put on paper, you try to make yourself look as good on paper as possible. But I don't know. How you look on paper doesn't make you a good teacher.

I asked Jessie how she felt about the *Common Core State Standards for Mathematics* (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010), and without hesitation she said, "I hate it." Jessie equates the standards with testing and accountability. She believes there are too many standards, and does not believe the standards are relevant to real life. Much of what she believes about the standards are based on her own experience as a student, preparing to meet previous state standards on end of the year examination. As none of what she learned in school was relevant, she has no reason to believe the new standards would be relevant.

> It tells them what they have to teach, and basically like their kids have to know that stuff. And it's a lot of stuff, and not very much time. So they have to like cram all this stuff down their kids' throats so the kid can pass the test… I feel like most of the stuff is not even important… Like I said, like in math like, all of these formulas… I guess it's not as bad at the elementary school level. It's awful in like high school and stuff. But, one, the kids are never going to remember all this information, because I couldn't tell you one thing I learned in high school honestly... I just feel like there's too much pressure on the teachers and that leads to too much pressure on the kids.

For Jessie, meeting the standards means preparing for the standardized test. The test puts pressure on the teachers and this prohibits the teacher from teaching in a comfortable way. Jessie said it would be best if "you could teach however you feel as a teacher that you are most comfortable and how you think that your students would learn best." But, because of time constraints and assessment pressures tied to tests, "comfortable" teaching must be replaced with preparing students to pass the test to meet the standards.

Concluding Discussion

Consistent with reports from states in which narrow-high stakes tests have been used for teacher accountability (Darling-Hammond, 2010), Jessie described her mathematical experience as plagued by mathematics instruction in which the goal was to memorize formulas and procedures in preparation for a test. Jessie's mathematics teachers taught *to* the test, and Jessie has consistently learned *for* the test, and this has affected her beliefs about mathematics, standards, and what is required for an elementary teacher to be successful and maintain a job.

Jessie believes that the "educational system" prohibits good teaching. Jessie, who wants to teach kindergarten, desires the freedom to teach children in a way that she is most comfortable with and that she believes is best for the students. But she notes time and accountability pressures are barriers to good teaching, and good teaching conflicts with preparing students for standardized tests. The National Council of Teachers of Mathematics (NCTM) advocates for mathematics teaching that incorporates student discussion, reasoning, and justification (NCTM, 2014). Jessie's case points to a

possible tension of advocating for instructional reforms that, while being advertised as preparing students to meet career and college readiness standards, appear (to an individual like Jessie) contrary to such a purpose.

Jessie's experience as a learner of mathematics in the NCLB era had a profound effect on her expectations for what mathematics is and how it can be taught. Will student experiences of mathematics fundamentally change with the passage of the Every Student Succeeds Act (ESSA, 2015)? Reflecting on the educational climate, Jessie reflected, "It makes me not even want to teach honestly, but, whatever. Maybe I'll change the education system." Despite a desire that the system change, Jessie does not know how such a change could take place. As educators and scholars, we should consider what can be done so that prospective teachers have a chance to develop their imagination so that they see themselves as agents, capable of improving education.

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NOTICING AND WONDERING: SUPPORTING MENTORING CONVERSATIONS ABOUT MATH TEACHING

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One way that mentors support prospective teachers' (PSTs') development within field placements is through conversations about classroom teaching. However, conversations based on merely descriptive teaching observations tend to be more evaluative in nature and lack educative opportunities for PSTs to reflect or inquire about the practice of teaching (Borko, Jacobs, Eiteljorg, & Pittman, 2008; Clarke, Triggs, & Nielsen, 2014; van Es, 2011). A short orientation session was implemented with three mentor-intern pairs that offered Smith's (2009) noticing and wondering language as a way to structure mentoring conversations that maintain descriptive and interpretive analytic stances. Analysis of pre- and post-conversations offer examples to illustrate how mentor-intern pairs adopt noticing and wondering language in mentoring conversations, and the ways in which this language structure supports interpretive mentoring conversations.

Keyword: Teacher Education-Preservice, Teacher Education-Inservice/Professional Development, Research Methods

Purpose of the Study

Prospective teachers (PSTs) have consistently identified clinical field experiences, such as student teaching, to be the most important experience in their teacher preparation, however the support mentors provide varies significantly based on the variety of ways that mentors engage with PSTs (Wilson, Floden, & Ferrini-Mundy, 2002). These variances have led to a greater problem of inconsistency in the quality of mentoring across schools and classrooms that PSTs experience and are attempting to learn from during field placements. Clarke, Triggs, and Nielsen's (2014) mentoring literature review reveals that PSTs varied experiences with their mentors may be because many mentors "lack specific preparation to enable high quality and developmentally appropriate support for student teachers" (p. 191). In an effort to address this need and assist mentors and PSTs in communicating across borders of teaching expertise, I created a short orientation session to offer Smith's (2009) *noticing and wondering language* for structuring mentoring conversations to mentor-intern pairs. The purpose of this initial study was to explore the ways in which mentor-intern pairs adopt noticing and wondering language and the ways in which noticing and wondering language supports mentors and PSTs to adopt an interpretive stance when discussing mathematical teaching practices.

Theoretical Framework

One way that mentors support PSTs' development within field placements is through conversations about classroom mathematics teaching, but mentoring conversations are not in and on themselves educative. Conversations that are descriptive and interpretive frame talk around investigating teaching, which is the way "[t]eachers learn to teach by treating teaching as an object of study - by trying to improve teaching by studying carefully what works and what doesn't" (Stigler & Hiebert, 2009, p. 36). These interpretive conversations are more valuable in field placements because they produce more educative opportunities for PSTs to reflect or inquire about the practice of teaching compared to only descriptive conversations or evaluative conversations which close educative opportunities (Borko, Jacobs, Eiteljorg, & Pittman, 2008; Clarke et al., 2014; van Es, 2011).

Feiman-Nemser (1998) supports that mentoring skills, particularly those focused on talking about teaching in productive ways, can be developed through professional development sessions and/or involvement in collaborative learning opportunities. I intentionally selected Smith's (2009) *noticing and wondering language* to share with mentor-intern pairs because it seemed like a structure that mentors could easily adopt and apply to mentoring conversations without a large time investment to learn this new mentoring practice. This language structure helps ground conversations in descriptive observations and then uses an interpretive stance to inquire into important teaching moments. Noticing and wondering language also supports the effort to improve a PST's mathematical teaching practice by studying teaching (Hiebert & Morris, 2012; Stigler & Hiebert, 2009).

Methods

The study took place with three mentor-intern pair volunteers from a secondary teacher preparation program at a large public university in the Midwest. Interns were PSTs in their final year of a five-year teacher preparation program. Mentors were veteran teachers that had mentored between five and fourteen PSTs. Mentor-intern pairs' content areas were mathematics and social studies, but for the purposes of this study I analyzed the mentoring conversations about a mathematics teaching episode.

Mentor-intern pairs completed a pre screener activity, participated in a short orientation session, submitted two sample conversations from the classroom, and completed a post screener activity. The pre- and post-screener activities were identical and asked each mentor-intern pair to watch two video episodes (Beginning Teacher Support & Assessment, 2012) of teaching together, and after each episode to have a conversation about the teaching they had just watched. These conversations were referred to as the mentor-intern pairs' pre- and post-conversations, because they occurred before and after the short-orientation session where mentor-intern pairs were introduced to noticing and wondering language. Mentor-intern pairs audio recorded and electronically submitted all their conversations for the study.

Out of the three mentor-intern pairs, I investigated the pre- and post-conversations about a sixth grade mathematics video from the two mentor-intern pairs (Pair 1 and Pair 2) who demonstrated no evidence of noticing and wondering language in their pre-conversations prior to the short orientation session. To illuminate how noticing and wondering language influenced the stance of mentoring conversations, each speaker turn (Johnstone, 2008) in the pre- and post-conversations was identified and coded as having a descriptive, evaluative, or interpretive analytic stance (van Es, 2011). The percentage of turns in each stance category was calculated for each conversation, and used to compare the stance of pre- and post-conversations. To explore the adoption of noticing and wondering language, the frequency of the words *notice* and *wonder* in the pre- and post-conversation transcripts were totaled. The stance of speaker turns that included *notice* and/or *wonder* were further analyzed for patterns about how mentor-intern pairs were adopting the language structure. Overall, 145 speaker turns from the four mentoring conversations served as the data analyzed for this study.

Results

In response to exploring the ways in which mentor-intern pairs adopted noticing and wondering language, the results indicate that mentor-intern pairs more frequently used the words *notice* and *wonder* within post-conversations than pre-conversations. Both pairs went from having pre-conversations that lacked any intentional form of noticing and wondering language (contained *notice* or *wonder* once), to post-conversations that contained *notice* six times and *wonder* three times for Pair 1, and *notice* four times for Pair 2. These slight increases in frequency suggest that mentor-intern pairs were attempting to use the language, and that the short orientation session provided mentors and interns with the knowledge and scaffolding to begin adopting the language structure. Mentor-intern pairs also implemented *notice* more frequently than *wonder*, possibly because it simply

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describes an observation versus extending the thinking about an observation. This may also demonstrate part of a learning curve that mentor-intern pairs experienced as they tried to merge this new structure within their current mentoring conversations.

In response to investigating the ways in which noticing and wondering language supports mentors and PSTs to adopt an interpretive stance when discussing mathematical teaching practices, the results indicated that the mentor-intern pairs' pre-conversations contained a majority of evaluative turns (Pair 1, 67%; Pair 2, 68%), which lack educative opportunities for PSTs to reflect or inquire about the practice of teaching (Borko et al., 2008; Clarke et al., 2014; van Es, 2011). While the post-conversations also contained a high percentage of evaluative turns (Pair 1, 53%; Pair 2, 77%), Pair 2's post-conversations demonstrated an increase in the amount of interpretive turns that do encourage reflection and inquiry about teaching (8% to 33%) and Pair 1's percentage of interpretive turns remained consistent (13% to 11%). A more nuanced investigation of the 14 turns that incorporated some form of *notice* and/or *wonder* revealed that *notice* was in all three stance categories, while *wonder* was only coded as interpretive. Even with the limited number of examples, these turns illuminate the importance of *wondering* in productive mentoring conversations.

The following set of interpretive turns provides an example of the noticing and wondering language structure being utilized in a mathematics mentoring conversation. It showcases how the mentor opens with *noticing* something he finds interesting about the teaching episode, and the intern acknowledges this as a shared observation. Then, the mentor continues his inquiry about the time inconsistency by using follow-up *wondering* turns about what prompted the teacher's decision.

> *Mentor:* Now, it's interesting, the previous video we commented on, I noticed that his, 30 seconds we're gonna do this, but then he pushed it. She said in a couple minutes we'll be playing a bingo game and then at some point they say 12 minutes into the lesson, they still hadn't gotten there. Somehow that feels, his sense of time was 30 seconds well we're going to take a short period of time.
>
> *Intern:* Yeah
>
> *Mentor:* Um hers, I think, I wonder if she actually believed was gonna be a couple minutes, but then her preparation and the technology issues and whatever delayed that, and so I wonder if it was intended to take that long, to get to that point.
>
> *Intern:* Or if she found formatively they're not as ready as I thought they were.
>
> *Mentor:* Right. Was it a planning thing? Was she adjusting things on the fly?

These productive conversational moments of unpacking and making sense of observations tended to follow interpretive turns more than evaluative or descriptive turns. Overall, an interpretive stance is supported by the combination of *noticing* and *wondering* in mentoring conversations, and thus encourages improving teaching practices through inquiry, reflection, and analysis of teaching (Hiebert & Morris, 2012; Stigler & Hiebert, 2009).

Discussion

This study provides beginning support for Smith's (2009) proposed noticing and wondering language. In particular, this study demonstrates how noticing and wondering language can be used to communicate across borders of teaching expertise to support interpretive discussions about teaching episodes that have the potential to be educative for interns. It also provides research tools for studying this language structure and how it has the potential to make a difference in mentoring talk. The research tools designed for this study included: the mentoring conversation prompt of the pre/post screener activity, audio recordings of mentoring conversations, and the analytic stance framework (van Es, 2011) for coding turns. Together these tools allow researchers to examine a small enough grain size within mentoring conversations to identify initial evidence to warrant further study of how this particular language structure can make an impact on the quality of mentor talk.

Specifically, it is the few captured moments when mentor-intern pairs utilized both *notice* and *wonder* that demonstrate the potential for using this language to structure conversations that investigate and make sense of shared teaching moments that have the potential to improve PSTs' mathematics teaching practices (Hiebert & Morris, 2012; Stigler & Hiebert, 2009), and thus warrant future research.

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REASONING ABOUT TELLING IN REHEARSALS OF DISCUSSIONS: CONSIDERING WHAT, WHEN, AND HOW TO "TELL"?

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Telling is often set in opposition to student-centered approaches to teaching. However, current research supports a more nuanced approach that considers what, when, and how one might tell. Using a practice-based approach, we investigate the ways in which rehearsals of leading discussions offer opportunities for secondary pre-service teachers to learn about telling. Through analyzing video of 13 rehearsals with seven pre-service secondary teachers, we found that learning opportunities fell into three categories: unlearning a reliance on telling, reasoning about the role of telling, and learning strategies for telling. This study demonstrates the potential of rehearsals to support novices to connect ideas about telling to the work of leading discussions.

Keywords: Classroom Discourse, Instructional Activities and Practices, Teacher Education-Preservice

Research on mathematics instruction in the US has repeatedly documented that many US mathematics teachers use telling and lecture as predominant ways to teach (e.g., Stigler & Hiebert, 2009). Simultaneously, over the past decades, there have been many efforts to shift instruction away from the teacher as "teller" toward instruction that supports collective sense-making, a form of instruction shown to better support student learning (e.g., Boaler & Staples, 2008). As teachers work to move their practice away from a reliance on telling, they may come to think that they are not supposed to tell anything (Chazan & Ball, 1999). Scholars highlight the need to move beyond dichotomies that set telling in opposition to student-centered approaches; instead, they call for an account of telling that considers *what* might be told, *when*, and *how*. There is general agreement that socially constructed knowledge, such as particular terminology or notation, needs to be told as students are unlikely to re-invent these in ways that align with canonical notation (e.g., Hiebert et al., 1997). Lobato, Clarke, and Ellis (2005) argue for a reformulation of telling as "the set of teaching actions that serve the function of stimulating students' mathematical thoughts via the introduction of new ideas into a classroom conversation" (p. 101). They focus on the function of teachers' actions, the actions' conceptual (over procedural) content, and the relationship to other actions over time. This highlights that teachers also must decide *when* to make new contributions. Timing matters in what students are able to learn from telling, with evidence supporting telling after experience with the content (e.g., Schwartz & Bransford, 1998). Telling might also involve naming or otherwise making explicit something mathematical that students had constructed (Selling, 2016) or co-constructing an explanation with students (Leinhardt & Steele, 2005). This suggests that, in addition to unlearning a reliance on telling (Philipp, 1995), teachers need opportunities to learn about this nuanced reformulation of telling.

One place to examine dilemmas of telling is in the work of leading discussions (Chapin, O'Connor, & Anderson, 2013). A discussion is sometimes viewed as antithetical to telling, as one of its goals is to support students in collective sense-making; however, teachers may make contributions or tell in crucial moments in ways that actually deepen the discussion (Lobato et al., 2005) or help steer toward the point (Sleep, 2012). Alternatively, when challenged to respond to student thinking, teachers may fall back on telling if they are unsure how to respond or handle an error. We focus on ways to support novice teachers in learning about telling in the context of leading discussions.

Theoretical Framing

We build on a practice-based approach to teacher education. One strategy for supporting novice teacher learning of core instructional practices is through the cycle of enactment and investigation (Lampert et al., 2013). Novices work on a practice, such as leading discussion, through learning about the practice, rehearsing it in the methods class, enacting it with students, and analyzing and reflecting on their enactments. The cycle is built around instructional activities (IAs) that bound complex practice in order to support novice learning. Rehearsals represent a key set of learning opportunities as teacher educators can provide in the moment feedback around the core practice. Efforts to work on telling in teacher education, such as working with pre-service teachers around instructional explanations (Charalambous, Hill, & Ball, 2011), have highlighted that telling productively is challenging for new teachers. As novices learn to lead different types of discussions (Kazemi & Hintz, 2014), they must reason about what, when, and how to tell within these discussions in a way that stimulates student understanding. We investigate the ways in which rehearsals of leading whole class discussions offer opportunities for novices to learn about telling and its potential roles in classroom discourse.

Methods

We investigate this question in the context of a mathematics methods course for secondary teachers that we designed and co-taught. The two-semester course was organized around a set of core practices, including leading discussions. The novice teachers were concurrently in field placements in urban secondary classrooms (See Baldinger, Selling, & Virmani, 2016 for more details about context). The novices participated in two cycles of enactment and investigation around leading discussions. The first cycle (weeks 4-7, fall) focused on discussions in which students share, compare, and connect strategies for solving problems (Lampert et al., 2013). The second cycle (weeks 6-10, spring), focused on discussions in which students work to clarify and define mathematical ideas (Kazemi & Hintz, 2014). The data sources include video of all rehearsals, debriefs at the end of each rehearsal, and a final debrief after all rehearsals.

Table 1: Data Sources for the Two Cycles of Enactment and Investigation

Instructional Activity	Rehearsal Data	Debrief Data
Strategy Sharing: Number Talks - Fall Semester	6 novice teacher discussion rehearsals	6 rehearsal debriefs 1 final debrief
Defining and Clarifying Mathematical Ideas:Sorting Task (Baldinger et al., 2016) - Spring	7 novice teacher discussion rehearsals	7 rehearsal debriefs 1 final debrief

The video data were analyzed in two phrases. First, we developed detailed content logs (Derry et al., 2010) of all rehearsals and debriefs. We then identified all interactions that explicitly or implicitly addressed "telling". *Explicit opportunities* were interactions among teacher educators (TEs) and novices in which telling was brought to the collective attention of the group (e.g., a TE coached about an alternative to telling, a novice reflected on what he/she learned about telling). *Implicit opportunities* were instances when a rehearsing novice made a contribution to a discussion but it was not highlighted in the moment. This paper focuses on the explicit opportunities. Next, we coded the explicit opportunities inductively to characterize the different types of opportunities to learn about telling (e.g., unlearning a reliance on telling). Lastly, all opportunities were recoded with the final codes.

Findings

Both cycles of enactment and investigation offered explicit opportunities to learn about telling in mathematics discussions. These arose in both rehearsals and debriefs and were distributed across different rehearsals. The opportunities fell into three distinct but related categories: opportunities to unlearn a reliance on telling, opportunities to reason about telling, and opportunities to learn strategies for telling. We use vignettes to illustrate each category.

Unlearning a Reliance on Telling

Novices had opportunities to recognize and problematize a reliance on telling in discussions. One example of this occurred in Victor's (all names are pseudonyms) rehearsal of a strategy sharing discussion around a multi-digit addition problem. During Victor's rehearsal, one TE, playing a student, asked why Raul had subtracted in his strategy. Victor immediately responded with a lengthy explanation. After he finished, the TE interrupted as a coach saying, "You just did a beautiful job of explaining to us but is there any way you could have gotten one of us to do the work instead?" The TE asked him to go back and replay the episode, at which point Victor asked, "would anyone like to help [TE] out and explain why Raul decided to take 2 away from 17?" In his debrief, Victor reflected that he had learned about "when to throw it back to the class and when to lead the discussion myself". Another novice, Julia, commented on this moment in the general debrief, saying she had "learned the value of having students restate other students' answers instead of explaining it myself". This highlights how rehearsal afforded a moment for a novice to experience his tendency to tell in discussions, to problematize it as potentially taking over the work from students, and to discuss potential alternatives.

Reasoning about the Role of Telling

Both cycles offered opportunities to reason about the role of telling in discussions and when to tell. One example occurred during a sorting task rehearsal about quadrilaterals. The first two cards (a rectangle and a parallelogram) surfaced disagreement among the "students" around whether a shape had to have right angles in order to be a quadrilateral. One TE, playing a coach, suggested moving the conversation to sharing non-examples of quadrilaterals. At this point, Carl, one of the "students", paused the rehearsal to ask about how and when in the discussion one might address or resolve this disagreement. This opened up an opportunity for discussion around telling or otherwise resolving this definitional issue. One TE contributed that it would be productive to address this ambiguity, if it persisted, after the class had more opportunities to grapple with and argue about what makes a quadrilateral. The other TE suggested a strategy for how to record the disagreement to show that the issue had still not been resolved, while not directly addressing the disagreement in the moment. This illustrates how the common problem of practice of responding to errors and disagreement that surfaced in rehearsal offered opportunities to consider when one might tell socially constructed knowledge (e.g., definitions).

Strategies for Telling

There were also opportunities to discuss strategies for making mathematical contributions to a discussion. For example, in Raul's rehearsal of a strategy sharing discussion, only a few different strategies were shared. During his debrief, Raul reflected on being surprised that so few had emerged. In response, one TE suggested contributing a new strategy by framing it as a strategy used by a student from another class and asking the students to comment on or interpret the strategy. The other TE described how this technique allowed the teacher to introduce a new idea while avoiding positioning the teacher as the mathematical authority in this moment. This illustrates how a common problem of practice that arose in the rehearsal helped construct a moment to surface and reason about strategies for telling.

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Discussion

These vignettes illustrate how rehearsals of whole class discussion in secondary methods classes can provide numerous opportunities to explicitly address the complex issue of telling. Novices not only had opportunities to unlearn a reliance on telling, they also had opportunities to reason about the role of telling and develop strategies for productive telling. This set of learning opportunities is not intended to be a comprehensive list of what might be learned through rehearsal; however, this study demonstrates the potential of rehearsals to offer opportunities to address telling in a nuanced way by situating work on telling inside a discussion. The novices were able to connect ideas about telling to the problems of practice they experienced while facilitating classroom discourse. This study extends prior research on rehearsals (Lampert et al., 2013) and on preparing teachers to tell (e.g., Charalambous et al., 2011) to highlight the value of addressing telling within work on other instructional practices. Future research might examine how particular discussion structures might afford different learning opportunities around telling in rehearsals. Additionally, how might explicitly addressing telling in rehearsals influence the ways in which novice teachers tell (or do not) when leading discussions in K-12 classrooms? Finally, in this study we saw that TE moves were often integral to surfacing issues of telling. Future research could TE moves that create and capitalize on problems of practice around telling.

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EXAMINING PRESERVICE TEACHER TASK MODIFICATIONS FOR ENGLISH LANGUAGE LEARNERS

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This study examined how elementary preservice teachers modified cognitively demanding (Stein et al., 2009) mathematics tasks for use with an English language learner during a four-week field experience. In the examination of the preservice teachers' lesson plans, written reflections, and video recorded interviews, we identified three categories of modifications made to accommodate their students: context, presentation, and language. The findings suggest that while the preservice teachers increasingly attended to their students' individual needs, they had limited strategies on which to draw. Implications address the need for teacher educators to address the use of tasks with English language learners.

Keywords: Teacher Education-Preservice, Equity and Diversity, Elementary School Education

Introduction and Framework

English language learners (ELLs) may possess unique experiences in mathematics; schools ask them to adapt culturally to American mathematical practices while also developing academic and everyday language in English. Despite the increase of students whose first language is not English, U.S. teacher preparation programs may not yet adequately address effective strategies for drawing on ELLs' cultural and linguistic resources in conjunction with rigorous mathematics instruction. In fact, it has been found that less than one third of teachers have received training in effective teaching strategies for ELLs (Ballantyne, Sanderman, & Levy, 2009).

This paper examines how preservice teachers (PSTs) modified mathematics tasks to work with ELLs. The modifications teachers make to tasks are important to examine because "the tasks in which students engage [in] provide the contexts in which they learn" (Henningsen & Stein, 1997, p. 525). Moreover, modification to mathematics tasks may lead to the creation of learning contexts more appropriate for ELLs. In this paper, *tasks* refer to an activity or set of activities designed to develop a particular mathematical idea (Henningsen & Stein, 1997). Today, there is growing emphasis on engaging students, including ELLs, deeply in worthwhile mathematics tasks that feature opportunities for high cognitive demand (Stein, Smith, Henningsen, & Silver, 2009).

As we examined the PSTs' task modifications, we drew on a situated-sociocultural perspective (Moschkovich, 2002) to describe how students learn mathematics. This perspective acknowledges that mathematics learning cannot be disentangled from language. Furthermore, it frames ELLs' language as a resource from which to build academic and everyday language. From this perspective we sought to identify the ways in which the PSTs supported ELLs in their language and mathematical learning as they drew on ELLs' linguistic resources. More specifically, we focused our examination on how the PSTs set up mathematics tasks (Stein, Grover, & Henningsen, 1996) as they attempted to increase mathematical and linguistic access. Thus, we sought to answer the following research question: *What modifications do elementary PSTs make to cognitively demanding mathematics tasks in order to accommodate ELLs?*

Methods

The data for this study came from a larger study that investigated four undergraduate juniors in an elementary education program (Fiona, Kimberly, Morgan, and Hannah) who worked one-on-one with an elementary ELL student in a four-week field experience. Prior to the study, each PST had assisted teachers in a classroom setting, worked one-on-one with students in classroom settings on

mathematical tasks, and received formal training on mathematics instruction, curriculum, and assessment as part of her program. That said, the PSTs had limited prior experiences working with and learning about ELLs specifically and were eager to gain additional experience. All four ELLs (Jin, Kyeong-Tae, Hwa-Young, & Ho-Min) in the study were native Korean speakers.

For the first three meetings, the research team provided the PSTs with a cognitively demanding task. These tasks featured opportunities for the ELLs to engage deeply in mathematical practices such as constructing viable arguments, critiquing the reasoning of others, and looking for and expressing regularity in repeated reasoning (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Each week, the PSTs were asked to write a lesson plan to implement their task and were permitted to modify or adapt the task to meet their student's needs. Data included video recordings of each meeting, video recordings of pre and post interviews, the PSTs' lesson plans, and the PSTs' written reflections.

In this study the research team employed an inductive approach (Thomas, 2006) to examine and analyze the modifications PSTs made to the provided tasks. This involved open coding (Strauss & Corbin, 1990) the tasks, lesson plans, and interview transcripts for each PST to identify and agree upon initial themes related to task modifications. We then used axial coding (Strauss & Corbin, 1990) to further refine these themes as we analyzed across all the data.

Findings

The PSTs modified the task context, presentation, and language. These modifications ranged from bolding a single word to significantly altering the task's purpose. The PSTs often made these modifications in response to knowledge gained in prior meetings with their ELL, with each PST waiting until after the first meeting to introduce any modifications.

Context

Mathematics tasks that draw on students' life experiences can simultaneously support language and mathematical learning (Barwell, 2003). Furthermore, such tasks can allow students to build meaning for mathematical contexts (Wager, 2012). However, the connection between culture and mathematical context is not immediately intuitive (Aguirre et al., 2012).

The PSTs in this study waited to alter task contexts until the third week. This task originally asked students to examine and critique the statements of two students, Jeff and Sara, about rounding numbers of baseball stadium seats. The task required the ELLs to discover that Jeff and Sara had rounded to different place values and their different perspectives were both valid. A portion of the task and one PST's modification is shown in Figure 1.

Given Task	Modified Task
Baseball stadiums have different numbers of seats. **Giants'** stadium **in San Francisco** has 41,915 seats and **Nationals'** stadium **in Washington** has 41,888 seats. **Padres'** stadium **in San Diego** has 42,445 seats. Compare these **statements** from two students:	**Soccer** stadiums have different numbers of seats. **U.S.A.s'** stadium has 41,915 seats and **Brazils'** stadium has 41,888 seats. **England's'** stadium has 42,445 seats. Compare these **sentences** from two students:

Figure 1. Fiona's modified task context.

After learning in the previous session that her student played soccer in P.E. class, Fiona changed the task's original context from baseball to soccer stadiums as seen in Figure 1. In her pre-interview she stated, "I changed it because I thought since he liked it, like soccer, playing it would connect more to him." Morgan also changed this task, asking her student to round the number of parts in three Transformer robots. She explained, "I was nervous because it was about the – the Giants'

stadium in San Francisco and the Nationals' stadium in Washington. I wasn't sure if he'd know what those were." Similarly, Hannah converted the numbers of stadium seats to numbers of library books; however, she did not consider her student's interests. Instead, Hannah chose this context to be more accessible, assuming that her student could envision his own school library, which contained thousands of books. Notably, the final PST, Kimberly, did not modify the context of any task and no PST changed the mathematical context (e.g., changing numbers or representations used in the tasks).

Presentation

Three of the PSTs, Fiona, Morgan, and Hannah, modified tasks' presentation in all three weeks by adding emphasis, restructuring, and appending visual aids. Visual supports such as these can aid learning by highlighting important concepts or ideas (Coleman & Goldenberg, 2010). For example, Morgan employed heavy emphasis (i.e., color-coding, bolding, underlining, etc.) in the second week's task, a section of which is shown in Figure 2. Morgan's emphases also matched accompanying, color-coded pictures of space creatures for her student's reference.

Both Morgan and Hannah restructured tasks into several parts, providing ample whitespace and blank lines for student work to make the task more manageable. Drawing on her experiences learning Spanish in high school, Morgan posited, "it's easier to see when it's broken into sentences, kind of separated from each other." Fiona anticipated her student being unfamiliar with the vocabulary of space creatures in the task in Figure 2 and countered this by providing images and cutouts of space creatures to her student. The cutouts also served as a representation her student used in reasoning mathematically. Other uses of visuals by the PSTs included showing images of stadiums on a laptop or pictures within the written task itself.

3. If you have to include **at least** <u>one</u> **2-eyed space creature,** <u>one</u> **3-eyed space creature,** <u>**AND**</u> <u>one</u> **4-eyed space creature,** how many space creatures of each kind are needed to make a group with <u>**24**</u> total eyes? If it is possible, list **all** possible pairs and explain your strategy. If it is impossible, explain why.

Figure 2. Morgan's modified task presentation.

Language

While ELLs' language learning is enhanced through an emphasis on mathematical meaning and processes (Moschkovich, 2012), the PSTs focused their linguistic modifications on single words and short phrases. For example, Kimberly attempted to challenge her high-achieving student by replacing '24' with 'twenty-four' and 'combinations' with 'explanations,' stating, "I'm wondering if this is going prove to be more of a challenge because he's not going to be able to just look at the numbers and figure out oh I have to do this." In contrast, Morgan sought to avoid such challenges, replacing the phrase "compare the statements of the two students" to "look at what the two students say" to relieve her student from speaking in week three.

Although the PSTs modified task language, they did so inattentively. For example, Kimberly and Morgan introduced changes that significantly altered the task instructions; while week two's instructions asked students to "list all possible combinations" for space creatures to have a total of 24 eyes, Morgan asked for "all possible *pairs* [emphasis added]" (Figure 2). Fiona and Hannah also included a number of typos and the inappropriate use of mathematics vocabulary, such as confusing the "tenths place" with the "tens place" in a rounding extension.

Conclusion

Throughout the field experience, all four PSTs attempted to modify the tasks in a variety of ways with differing levels of success. Thus, PSTs seem to utilize certain general strategies to improve linguistic access for ELLs. However, the PSTs struggled to modify tasks while also maintaining the

mathematical rigor, often conflating mathematical and linguistic supports. The ELLs were often relieved of requirements to express their reasoning or thinking and to critique arguments. The PSTs' difficulty maintaining cognitive demand was similar to findings in studies on inservice teachers conducted by Henningsen and Stein (1997).

The PSTs' hesitation to modify the tasks before meeting their student in conjunction with statements they made in interviews indicate their modifications were based more on anecdotal evidence gained from their ELLs rather than previous knowledge of how to implement tasks with ELLs. It was evident that the PSTs were not adequately prepared to accommodate linguistically diverse students through the modification of existing tasks or draw on their linguistic or cultural resources (Moschkovich, 2002). These findings suggest that PSTs would benefit from increased support in their programs to prepare them to effectively modify and use tasks for ELLs while drawing on these resources. Thus, further research is needed to investigate how teacher education programs can best prepare PSTs for future interactions with ELLs.

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SUPPORTING PRESERVICE TEACHERS' USE OF CONNECTIONS AND TECHNOLOGY IN ALGEBRA TEACHING AND LEARNING

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Secondary mathematics teacher education programs across the United States opportunities for preservice teachers to make connections and encounter technology to varying degrees, despite recommendations from the Conference Board of the Mathematical Sciences. We explore the research question: What opportunities do secondary mathematics teacher preparation programs provide for preservice teachers to learn about connections and to encounter technologies in learning algebra and learning to teach algebra? We present patterns and examples of a wide range of opportunities that instructors provided related to algebraic connections and use of technology, based on data collected from five teacher education programs across the U.S.

Keywords: Algebra and Algebraic Thinking, Teacher Education-Preservice, High School Education

Algebra is valued as a foundational subject in mathematics, and thus plays a prominent role in mathematics education reform efforts. In the United States, preparing future secondary mathematics teachers to teach algebra has gained importance as, in response to algebra-for-all initiatives, more states include algebra as a high school graduation requirement (Teuscher, Dingman, Nevels, & Reys, 2008). Not only are more secondary mathematics teachers teaching algebra in their first professional position, but these new teachers are also expected to teach algebra to a more diverse population of students than ever before (Stein, Kaufman, Sherman, & Hillen, 2011). Hence it is imperative to study the strategies and experiences that teaching programs use to prepare preservice teachers (PSTs) for teaching algebra to this diverse population and to support PSTs' development of a deep understanding of algebra. Furthermore, standards for both secondary mathematics content and teacher preparation have emphasized the importance of developing PSTs' abilities to make connections and to use appropriate educational technologies in their own mathematical learning and in their future mathematics teaching. Particularly with respect to PSTs' mathematics courses, *Mathematics Education of Teachers II (MET II)* (Conference Board of Mathematical Sciences [CBMS], 2012) recommended that instructors of mathematics courses support PSTs in "forming connections" (p. 56) and that experiences with technology "should be integrated across the entire spectrum of undergraduate mathematics" (pp. 56-57). This study explores PSTs' opportunities to expand their knowledge of algebra through connections and the use of technology, and to learn how to use both to support the teaching and learning of algebra. We explore the following research question: "What opportunities do secondary mathematics teacher preparation programs provide for PSTs to learn about connections and to encounter tools and technologies in learning algebra and learning to teach algebra?"

Perspectives

The importance of developing PSTs' abilities to see mathematics as a complex, connected system that also intertwines through other non-mathematical disciplines, as well as a way to make sense of mathematics in the real world, is recommended by standards developed by teacher preparation program accreditation agencies (e.g., Council of Chief State School Officers [CCSSO], 1995;

National Council of Teachers of Mathematics [NCTM], 2012). The National Board for Professional Teaching Standards (NBPTS) (2010) emphasized that PSTs must think about mathematics as a "whole fabric" as they make connections among mathematical topics. To support this view of mathematics, PSTs need to make connections within algebra, and between algebra and other mathematical fields, while linking algebra with real-world situations. PSTs should prepare to teach using "rich mathematical learning experiences" and provide their future students with opportunities to "make connections among mathematics, other content areas, everyday life, and the workplace" (NCTM, 2012, p. X). Further, PSTs should prepare to support their future learners in reflecting "on prior content knowledge, link[ing] new concepts to familiar concepts, and mak[ing] connections to learners' experiences" (CCSSO, 1995, p. X).

In addition, teachers must prepare to critically evaluate and strategically use technology in mathematics teaching and learning (CBMS, 2012; CCSSO, 1995; NCTM, 2012). *MET II* emphasized the importance of PSTs' preparation for using a variety of technologies, including problem-solving tools and tools for exploring mathematical concepts (CBMS, 2012). Teacher preparation standards have emphasized the importance of PSTs' own learning of mathematics using technologies as both a "practical expedient" as well as to enhance learning (CBMS, 2001).

For this paper, we focus particularly on opportunities provided by teacher education programs to support PSTs in making algebraic connections and using technology to learn and teach algebra. Making connections in the service of algebra teaching and learning includes making connections within algebra (WA), between algebra and other mathematical fields (OM), between algebra and other non-mathematical fields (NM), and between ideas in advanced algebra and school algebra (AS). Encounters with technology in the service of algebra teaching and learning include using and learning about a variety of algebra-appropriate technology, engaging in rich and technology-integrated algebra tasks and problem-solving, as well as thinking critically about the use of technology to support algebra teaching and learning.

Method

This study is part of *Preparing to Teach Algebra (PTA)*, a mixed-methods study exploring opportunities provided by secondary mathematics teacher preparation programs to PSTs to learn algebra and to learn to teach algebra. *PTA* included a national survey of secondary mathematics teacher preparation programs and case studies of five universities. The study described here is a qualitative analysis of the five case studies focusing on opportunities provided to PSTs to encounter technology and to make connections in learning algebra and learning to teach algebra.

PTA purposefully chose teacher preparation programs at five universities to represent diversity in location, demographics, and Carnegie classifications. We refer to the universities as Great Lakes University, Midwestern Research, Midwestern Urban, Southeastern Research, and West Coast Urban Universities. We conducted approximately 10 course instructor interviews, collecting relevant instructional materials, and two focus groups of 3-4 PSTs at each site. At West Coast Urban, we conducted only three instructor interviews because the program is post-baccalaureate. With the assistance of a site coordinator, we selected courses based on potential opportunities to learn about algebra or teaching algebra. We included mathematics, mathematics for teachers, mathematics education, and general education courses. Among other interview questions, we asked instructors about technology use in their course and about how they supported PSTs in making algebraic connections. Similarly, we asked PSTs about opportunities to use technology and make algebraic connections in required courses.

Prior to this more focused study, the *PTA* team coded all data for algebraic content. For this study, two researchers individually coded data sources based on the four major types of connections (i.e., WA, OM, NM, AS) and met to reconcile their coding. We developed summary documents for each university, including tables of opportunity counts with relevant quotations from courses. We

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analyzed quotations to document types of reported opportunities (e.g., algebraic topics, specific activities, and/or experiences to help PSTs learn to teach connections). For tools and technology, two researchers similarly considered data sources for types of technology, details of how technologies were used to support algebra learning or teaching, and the rationale (if any) given by the instructor its use. We are developing summaries for each university, including types of technology used and how it was used. We are analyzing reported experiences according to whether they seem to use technology as a "practical expedient," to "advance learning," or to provide opportunities for PSTs to think critically about the choice and use of technologies by engaging with potential affordances or limitations (CBMS, 2001).

Preliminary Findings

For the purpose of this proposal, and due to space limitations, we present examples of *types* of experiences provided to PSTs by two of the five programs (Great Lakes and Midwestern Urban) in Abstract Algebra, Linear Algebra, and Secondary Mathematics Methods.

Connections

Great Lakes University. The Linear Algebra instructor reported opportunities related to all four types of connections: discussing the meaning of solving an equation through distributivity and associativity (WA), probability through Markov chains (OM), population dynamics through modeling (NM), and connections between solving systems of linear equations and college algebra (AS). The Abstract Algebra instructor provided examples of all types except NM: emphasizing common structures and themes behind different number systems, discussing connections between ring isomorphisms and graph morphisms in Discrete Mathematics course, and discussing relationships between high school level division algorithm and machinery in the division algorithm. The Secondary Mathematics Methods instructor described two types of connections (WA and NM), but focused on how PSTs made connections, specifically through creating lesson plans and participating in reading workshops.

Midwestern Urban University. Instructors of the three courses made different types of connections: the Linear Algebra and Secondary Mathematics Methods instructors reported opportunities related to three types of connections (excluding AS). The Abstract Algebra instructor described opportunities related to three types of connections, excluding NM. The Linear Algebra instructor mentioned that PSTs studied how to solve systems of equations, connected them to topics in the course, and learned how technology could best assist them. The Abstract Algebra instructor reported PSTs' opportunities to learn about abstract proofs related to college algebra and the usefulness of number theory and set theory. The Secondary Mathematics Methods instructor provided a specific activity (border problem) where PSTs discussed meanings of the variable in context and generalized the situation using both words and symbols, which provided them opportunities to connect different representations and geometry

Technology

Great Lakes University. The Abstract Algebra instructor reported using little technology in his course, but described experiences using instructional technologies to facilitate communication: students collaboratively developed class agendas using Google Docs. Use of technology was extensive in Linear Algebra, as students used Maple and Java applets in weekly computer lab activities. In speaking of computer lab activities, the instructor used phrases like "discover the concept" and "develop intuitive understanding." The Secondary Mathematics Methods instructor described activities supporting PSTs in thinking critically about technology use; for example, asking "which five problems you want to use the calculator on and why."

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Midwestern Urban University. The Abstract Algebra instructor reported little use of technology, but said he hoped he would have the time to incorporate more in the future. He described using Wolfram Demonstrations to "explore the symmetries of the equilateral triangle and the square." The Linear Algebra instructor reported heavy use of the TI84 calculator in his course, mainly to perform operations on matrices so that they can spend more time on what the matrix is used for and what conclusions can be drawn. The Secondary Mathematics Methods course instructor incorporated a technology workshop on using graphing calculators for algebra, and required PSTs to include one lesson in their unit plan that used this technology

Discussion

Preliminary results showed different types of opportunities related to making algebraic connections and encountering technology in learning and teaching algebra. Instructors reported a wide range of opportunities related to algebraic connections, including: lists of topics and ways that they made connections (e.g., Linear Algebra at Great Lakes) and specific activities that engaged PSTs to make connections (e.g., Secondary Mathematics Methods at Midwestern Urban). Instructors also described different ways of using technology in algebra teaching and learning: as a practical expedient for focusing on the meaning of mathematical objects instead of on operations (e.g., Linear Algebra at Midwestern Urban), to enhance learning to discover concepts and develop their understandings (e.g., Linear Algebra at Great Lakes), and to think critically about technology use (e.g., Secondary Mathematics at Great Lakes).

In our presentation, we will share patterns we have seen among the major types of connections and uses of technology across all five programs. We will additionally provide detailed examples of how instructors provided PSTs with opportunities to make algebraic connections and use technology as recommended by policy documents (e.g., CCSSO, 1995; NCTM, 2012), along with examples that other educators can use to enhance their own programs.

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DEVELOPING PRESERVICE TEACHERS' NOTICING OF PRODUCTIVE STRUGGLE

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This study examines how two important aspects of classroom instruction, teacher noticing and productive struggle, can be introduced to and developed in preservice teachers (PSTs). We examined PSTs in a 2-week professional development experience associated with a summer math camp for elementary and middle school students. Findings suggest that an awareness of productive struggle can be developed through extending the borders of teacher noticing to include this construct. 13 PSTs were immersed in a setting that allowed them to observe teaching, interact with students, and reflect on both their actions and the classroom setting. The PSTs reflected on and discussed the development of their own observations of student thinking and the role productive struggle can play in both teaching and learning mathematics.

Keywords: Teacher Education-Preservice, Instructional Activities and Practices

Introduction

Schoenfeld (2011) writes that *professional teacher noticing* matters and urges the mathematics education community to examine the development of this construct and its transferability to various settings and contexts. More recently, *Principles to Actions* (National Council of Teachers of Mathematics [NCTM], 2014) outlined eight Mathematics Teaching Practices that includes *support productive struggle in learning mathematics*. This practice encourages the use of, "effective teaching and supports to engage in productive struggle as they (students) grapple with mathematical ideas and relationships" (p.10). *Productive struggle* can support mathematical understanding and self-confidence in all students (Hiebert & Grouws, 2007).

The constructs of *teacher noticing* and *productive struggle* play important roles in the teaching and learning of mathematics (Warshauer, 2015a; Jacobs, Lamb, & Philipp, 2010). Research suggests that *teacher noticing* can be developed (Miller, 2011; Warshauer, Strickland, Namakshi, Hickman, & Bhattacharyya, 2015) and mathematics educators encourage providing opportunities for students to engage in struggle as an integral part of doing mathematics (Warshauer, 2015b; Hiebert & Grouws, 2007).

The purpose of this study is to examine how these aspects of classroom instruction can be introduced to and developed in preservice teachers (PSTs) and thereby stretching the borders of the constructs of *teacher noticing* and *productive struggle*. Our study examines PSTs in a 2-week summer math camp professional development (PD) experience. The seminar portion of the PD included written and oral reflections using the *teacher noticing* framework proposed by Jacobs, Lamb, & Philipp (2010). In addition, the seminar included discussion of content and pedagogy of the morning math camp classes.

The purpose of our focus on *teacher noticing* was to examine how to promote and support *productive struggle* as students engaged in mathematical tasks. By *productive struggle*, we mean what occurs when, "students expend effort in order to make sense of mathematics, to figure something out that is not immediately apparent" (Hiebert & Grouws, 2007, p. 387). Research suggests that engaging students in tasks that promote struggle may contribute to strengthening

students' disposition towards engaging in doing mathematics and foster persistence in students' efforts to understand and make sense of challenging mathematics (Warshauer, 2015a; Hiebert & Stigler, 2004). However, it is critical to understand how to prepare PSTs to address student struggle in a productive manner.

Studies suggest that teaching does not readily provide opportunities for students to struggle productively, and that teachers do not see the importance of *productive struggle* in learning mathematics (Hiebert & Stigler, 2004). Finding ways to expose PSTs to what *productive struggle* looks like through the lens of *teacher noticing* can introduce the PSTs to the value of these two important aspects of teaching practices through actual experiences.

In this paper, we report on the nature and development of *professional teacher noticing* among the PSTs in a professional development opportunity that promoted *productive struggle* in doing mathematics. The research questions for this study are:

1. What do PSTs attend to in the context of the morning camp observations and instructions? How do the PSTs interpret what they observe?
2. When PSTs observe students' *productive struggle* in a math camp setting, what is the nature of their *teacher noticing*?
3. How does the PSTs' *teacher noticing* of *productive struggle* develop over the course of the two-week camp experience?

This study seeks to provide insight into what PSTs notice in general and in particular what they notice about *productive struggle* in the context of students' learning of mathematics.

Methodology

We use qualitative methods (Corbin & Strauss, 2008) to open-code what the PSTs noticed in their classroom setting and describe how they appear to develop their noticing skills, particularly as they pertain to observations of *productive struggle* during the math camp professional development experience. The seminar was designed to expose the PSTs to the research base for the practices they would observe in the classrooms.

Context

Setting: The study took place in a two-week summer mathematics camp for elementary and middle school students associated with a university in the southern part of the United States. Each PST was assigned to a master teacher teaching one of five levels offered at the camp. They assisted the master teacher in the classroom for 4 hours each morning. After lunch, the PSTs attended a 2-hour seminar facilitated by two mathematics faculty members of the university who authored curriculum implemented in the camp. The seminar began with a daily written reflection followed by a discussion of the PSTs morning observations with an eye towards using the *teacher noticing* framework and observing *productive struggle* in the classroom. The seminar component of the PD fostered among the PSTs an understanding of the constructs of *teacher noticing* and *productive struggle.* PSTs also received training on using the Jacobs et al. (2010) *teacher noticing* framework to observe aspects of *productive struggle.*

Participants: The participants in the study came from a larger group of undergraduates, not all PSTs, who served as assistants in the camp. There were 13 PSTs: 3 had previously participated as assistants; 6 were just beginning their teacher preparation courses; and 4 were ready to begin their student teaching.

Data Collection: The data gathered included PSTs daily written reflections completed at the start of each seminar session; facilitators' field notes of daily reflection discussion during the seminar; pre and post surveys that included questions about PSTs understanding of *teacher noticing* and

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productive struggle and their possible value in teaching and learning mathematics; 3 questionnaires every 3rd day targeting PSTs observation of the nature of student struggles and their resolutions; and interview transcripts of the PSTs conducted at the end of the camp.

Data Analysis: In order to answer our three research questions, we began with open-coding (Corbin & Strauss, 2008) of the reflections, surveys, and interviews for aspects of *teacher noticing* while superimposing the Jacobs, Lamb, and Philipp framework (2011). To establish inter-rater reliability of our coding, 15% of the PST reflections were examined by the researchers and reached 90% agreement on our coding. We continue our analysis and remain open to other codes that may arise in the PSTs observations of *productive struggle* and *teacher noticing*.

Findings

We report preliminary findings for each of our research questions.

Research Question 1: Based primarily on the daily reflections, PSTs noticing initially focused on attending to the routines of the classroom and the management of students by teachers. For example, PST A wrote, "We tested and started with number lines/greater then/less than". Regarding the management, PST A wrote, "One of our students threw a fit in class today because she did not get the color car she wanted. The teacher insisted that she could not switch cars or she would have to do it for everyone." However, over the two weeks, the PSTs began to notice teacher moves such as asking a series of questions instead of telling the students answers and a push for precise use of vocabulary. PSTs observed how students were encouraged to share ideas about the concepts from their experiences as in this account by PST B, "When the class was revisiting negative numbers, a student said that negative numbers don't exist. So the class talked about if they existed in real life or not." PSTs took notice of teacher's use of visual models and representations and physical objects such as blocks and number lines on the floor. One PST C reports, "Mrs. Fox really took a lot of time to work in front of the class and explain how to do the car model while subtracting w/ positives and negatives". PSTs were able to observe multiple ways students were thinking. Others noticed different levels at which students were engaging with the mathematical concepts as PST E noted, "Although some kids are much more experienced, they still get involved by teaching the other students (and they enjoy it!)".

Research Question 2: PSTs often attributed student struggles to lack of background knowledge or vocabulary as PST F wrote: "This seemed to be a vocabulary struggle or interpretation on what was being asked." The PSTs found that by having students reread the instructions or if the PST or teacher rephrased the question, many of the students were able to move forward with the task as noted by PST G: "It may have been the instructions because once I helped and restated the instructions, then they would be okay and begin (the) task." As the camp progressed, PSTs found questioning students to be an effective response to determine the source of the students' struggle. For example PST C states "… then I asked her what she needed to do next and … asked her if it was positive or negative. She then said negative and I asked her which way she had to face if it was negative … then she told me and we look at the sign in the problem and I asked her which way we would drive. And she figured that out". While PSTs observed that probing questions supported the struggle productively, they also found alternate means of addressing the task such as use of visual models or examples as PST A noted "I made him go to the number line with me so we could pretend to be cars. We walked out the problem."

Research Question 3: The PSTs noticing of productive struggle over the course of the 2-week camp experience seemed to have developed particularly in the two PSTs who were preparing for student teaching. Many PSTs had not considered *productive struggle* as a construct as noted by PST H, "I had never heard the term 'productive struggle'. However, after the afternoon classes, I searched for and noticed 'productive struggle'." Most expressed that incorporating this construct is an important part of learning mathematics as PST J noted: "Productive struggle in math for a child looks

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like them building and discovering the pathways to connect concepts of mathematics." By the final survey, PSTs shared PST K belief that focus of their teaching should be on supporting the students to figure things out and asking why things work rather than seeking immediate answers to the questions," Instead of asking 'do you understand?', ask 'can you explain that'?". They also valued questioning students to better understand the students' thinking which integrated the *teacher noticing* along with support of *productive struggle* as noted by PST K "Instead of giving hints, I'd ask more questions".

Conclusion

This study explored PSTs' *teacher noticing* and using this to support *productive struggle.* Findings suggest that an awareness of *productive struggle* can be developed through extending the borders of *teacher noticing* to include this construct. PSTs were immersed in a setting that allows them to observe teaching, interact with students, and reflect on both their actions and the classroom setting in an intense professional development experience. The PSTs reflected and discussed the development of their own observations of student thinking and the role *productive struggle* plays in both teaching and learning mathematics. This type of professional development introduces PSTs to the practice of recognizing and responding to student struggles in a productive manner.

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MATHEMATICAL VISUALIZATION SUPPORTED BY ALTERNATING PROBLEM POSING AND PROBLEM SOLVING

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Research about mathematical visualization supported by problem posing is rare. This study investigated visualization during an alternating problem-posing and problem-solving process. The findings indicate that problem posing and problem solving inspired prospective elementary teachers' visualization; the visualization in turn supported their problem-posing and problem-solving performance; and visualization has specific features at different stages of that process.

Keywords: Teacher Education-Preservice, Teacher Knowledge, Problem Solving

Introduction

Researchers use the word visualization to refer to the process of constructing and using geometrical or graphical representations of mathematical concepts, principles or problems that are beneficial in doing mathematics (Presmeg, 1997b). Effective visualization helps improve teaching and learning efficiency (Duval, 1999; Guzman, 2002). Problem solving is considered a complex activity that demands cognitive thinking rather than simple recall of facts and procedures. Lesh and Zawojewski (2007) proposed an even broader definition of problem solving that emphasized problem solving as a process of interpreting a situation, during which a problem solver is usually required to refine clusters of mathematical concepts from various topics within and beyond mathematics. Problem posing is another cognitively demanding activity that asks learners to either formulate new problems based on a given situation or restructure a given problem (Kilpatrick, 1987).

Researchers have found that problem posing is closely related to problem solving (Silver & Cai, 1996). Even so, there are still many unanswered questions about the relatedness between problem posing and problem solving (Silver, 2013). Research about visualization in problem solving is limited (Presmeg, 2006), while research about visualization in problem posing is more rare. This study aimed to explore US and Chinese prospective elementary teachers' visualization performance during problem posing and problem solving. The following questions guided this study: (1) What are the patterns of visualization during US and Chinese prospective elementary teachers' alternating problem-posing and problem-solving process? (2) What are specific features of visualization during problem posing prior to problem solving, during problem solving, and during problem posing after problem solving?

Methodology

This study is part of a larger research project that was conducted among 87 first-and second-year prospective elementary teachers. Thirty-two participants were from three universities in the Northeastern part of the US and the remaining participants were from three universities in China. No participants had formal experience in problem posing before. By adapting the Active Learning Framework developed by Ellerton (2013) as well as the four cognitive problem-posing processes proposed by Christou, Mousoulides, Pittalis, Pitta-Pantazi and Sriraman (2005), this study designed a developmentally appropriate way of engaging participants in alternating problem-posing and problem-solving activities. Those four processes include: (1) Translating process for posing problems based on given graphs, diagrams or tables; (2) Comprehending process for posing problems based on given operations; (3) Editing process for posing problems according to provided information such as a real-life situation but without any other restriction; and (4) Selecting process for posing problems

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that are appropriate to specific given answers. Figure 1 showed the two sets of tasks that we had administered.

For each set of tasks, we first asked the participants to individually pose a problem for the Translating process on a separate sheet of paper, no story was given. We then asked them to pose problems for the other three processes on another sheet of paper according to a real-life story, which corresponded to the previously provided figures. Next we asked them to solve a problem, and finally to pose two more problems on a fourth sheet of paper. All 85 participants completed the first set of tasks. We then selected 43 of them according to the criterion that the majority of their posed problems (usually more than 50%) during the first task administration were solvable mathematical problems to complete the second set of tasks.

Results and Discussion

Answer Research Question 1

To answer the first research question, we examined the mathematical concepts, the associations among those concepts, and creative ideas about visualization demonstrated in participants' posed problems and their problem-solving solutions.

Concepts contained in posed problems. The participants used a large number of mathematical concepts for posing new problems during both task administrations. They commonly used the concepts of area, perimeter/circumference, ratio, percentage, proportion, inscribed circle/square, circumscribed square, as well as some concepts about three-dimensional objects. These findings demonstrate prospective elementary teachers' ability and skills in mastering mathematical concepts when visualizing geometrical objects. These findings also show that problem posing with visual images provides learners opportunities to apply mathematical concepts that they have learned.

Associations of concepts contained in posed problems and problem-solving solutions. We found that the participants made interesting associations among concepts that they used. For example, one US participant posed a problem asking about how the change in length and width of a rectangle would impact its change in area. For problem solving during the first task administration, twenty-five participants provided at least two different solutions. For the second set of tasks, more than half of the participants utilized at least one of the following associations in their posed problems or problem-solving solutions: (1) the radius of a circle was half of its circumscribing square's side length; (2) the diameter of a circle was the diagonal of its inscribed square; and (3) the Pythagorean Theorem played a significant role of connecting neighboring circles and squares. These findings show that problem posing and problem solving encouraged the participants' use of visualization; the associations discovered by participants through visualization, in turn, supported their problem-posing and problem-solving performance.

Some creative ideas. A large number of participants posed problems involving comparison ideas and these problems usually involved more than one operation. The ideas such as how many times a small rectangle fit into a big rectangle were also commonly used. Those problems involved a division with fractions as well as a division with a remainder, which were documented as challenging topics (Rizvi, 2004). Some participant even developed route problems with given speed or time, engineering problems about water injection with different rates, and so on. All these ideas indicate that prospective elementary teachers possess a huge imagination space when posing problems by visualizing geometrical figures.
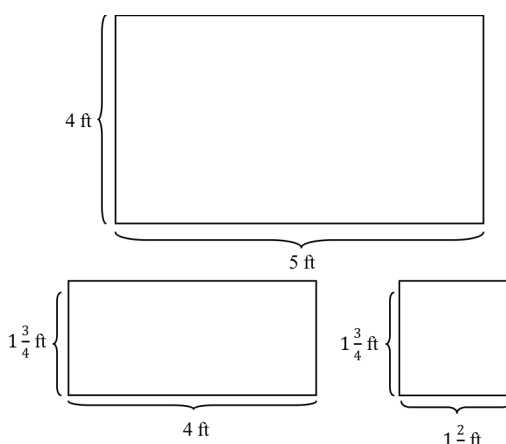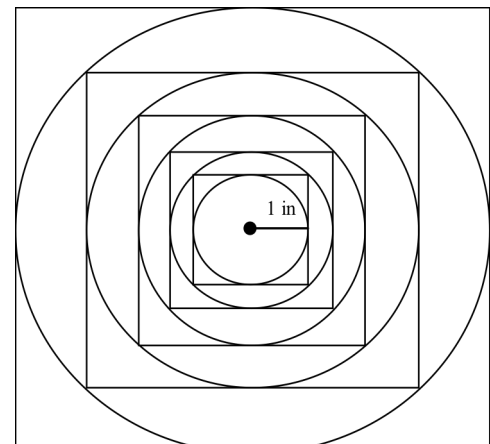
First Set of Tasks	**Second Set of Tasks**

First Set of Tasks

Given the following figures:

Task I: Problem posing [40 min]

Translating: Write a mathematical story problem according to the given figures.

<u>**Here is a real life story:**</u>

Diana bought a piece of cloth 4 feet wide and 5 feet long. It cost $16. She cut off a piece that was $1\frac{3}{4}$ feet wide and 4 feet long to make a scarf. Her sister saw Diana's cloth and really liked the material. She asked for a piece that was $1\frac{3}{4}$ feet wide and $1\frac{2}{3}$ feet long to also make a scarf.

Comprehending: Write an appropriate mathematical story problem representing the following calculation: $1\frac{3}{4} \times (4 + 1\frac{2}{3})$, according to the given situation and figures.

Editing: Write a mathematical story problem according to the given story, which should be different from the problems you posed before.

Selecting: Write an appropriate mathematical story problem according to the given story so that the answer to your problem is $\frac{121}{12}$ square feet.

Task II: Problem solving [20 min]

Solve this problem: In this story, if Diana's sister wanted to pay Diana, how much should Diana charge her?

Task III: Further problem posing [20 min]

Write two more mathematical story problems that are different from all problems you posed or solved previously according to the same story.

Second Set of Tasks

Given the following figure:

Task I: Problem posing [40 min]

Translating: Write a mathematical story problem according to the given figure.

<u>**Here is a real life story:**</u>

A factory that specializes in producing dinnerware decides to produce a set of five bowls in different diameter sizes. The sizes are constrained using the following model diagram and the radius of the smallest bowl is 1 inch.

Comprehending: Write an appropriate mathematical story problem representing the following calculation: $\sqrt{1^2 + 1^2}$, according to the given situation and figure.

Editing: Write a mathematical story problem according to the given story, which should be different from the problems you posed before.

Selecting: Write an appropriate mathematical story problem according to the given story so that the answer to your problem is $2\sqrt{2}$.

Task II: Problem solving [20 min]

Solve this problem: Given that the radius of the smallest circle is 1 inch, what is the ratio of the area of the largest circle to the area of the smallest circle?

Task III: Further problem posing [20 min]

Write two more mathematical story problems that are different from all problems you posed or solved previously according to the same story.

Figure 1. Two sets of tasks administered for this study.

Answer Research Question 2

For answering the second research question, we looked into the features of visualization throughout the entire process. During the problem-posing process prior to problem solving, the majority of participants had a difficult time in posing problems. This was partially because none of

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them had formal problem-posing experience before. Many participants posed one mathematical problem focusing on only part of the given figures (e.g., used only one rectangle during the first task administration instead of incorporating all three given rectangles). The problem-solving process after problem posing provided the participants opportunities to justify whether they accurately applied the mathematical concepts as well as the associations among those concepts in the posed problems, and whether those ideas could serve as aids for solving the given problem. During the problem-posing process after problem solving, the participants usually posed better problems, namely, problems that involved more computational steps or different creative ideas, compared to the problems they posed prior to problem solving. In this case, the participants' visualization ability was developed because many aspects, familiar or unfamiliar, were now associated with given figures.

Conclusion and Implications

This study focused on examining prospective elementary teachers' visualization during an alternating problem-posing and problem-solving process. We found that they used a large number of mathematical concepts as well as multiple associations among those concepts when trying to interpret the given figures. There were many creative ideas involved in their posed problems. We further found that visualization during different stages of problem-posing and problem-solving process had different features. Existing studies have documented that students are reluctant to engage in visual processing and reasoning (Dreyfus & Eisenberg, 1990). There have been no studies investigating visualization in the problem-posing area. Therefore, the role of problem posing in developing students' visualization remains an active question. This study serves as a first example of documenting the interplay of problem posing and visualization, specifically when problem-solving activities are alternately involved. The exploration about specific aspects and/or skills of the use of visualization during problem posing needs to continue.

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CONCEPTUAL AND PROCEDURAL UNDERSTANDING: PROSPECTIVE TEACHERS' INTERPRETATIONS AND APPLICATIONS

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The preparation of prospective secondary mathematics teachers often revolves around working to improve knowledge of mathematics for teaching and understanding the conceptual development and trajectories of mathematics. "Effective teaching of mathematics builds fluency with procedures on a foundation of conceptual understanding so that students, over time, become skillful in using procedures flexibly as they solve contextual and mathematical problems" (NCTM, 2014, p. 42). Prospective teachers need to be prepared to teach concepts along with procedures (Ball, Thames, & Phelps, 2008) particularly with the implementation of the Common Core State Standards (CCSS). In their own experience as a learner of mathematics, however, many prospective teachers come with procedural understandings of mathematics and many struggle to understand the underlying concepts and why those procedures work. Challenging prospective teachers to examine their own understandings of mathematical concepts and their preconceived ideas of good mathematics instruction becomes an important aspect of mathematics teacher preparation.

In this study, prospective secondary mathematics teachers were asked to read *Principles to Actions'* section on Conceptual Understanding and Procedural Fluency (NCTM, 2014). Having individually defined conceptual and procedural understanding in their own words, they were asked to apply those understandings to determine how a student might solve a percentage problem with a conceptual and with a procedural understanding. Prospective teachers' definitions and student solutions were examined to answer the question: In what ways do prospective secondary mathematics teachers define conceptual understanding and procedural understanding and subsequently apply those definitions to solve a percent problem?

Using the prospective teachers' own definitions of these two terms, the researchers compared the definitions with how each prospective teacher distinguished between the types of understandings when applied to the given percent problem. Data showed some disconnect between definitions and applications. Additionally, responses of prospective teachers to the percentage problem could have been either conceptual or procedural based upon varying aspects of student solutions.

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COMIC STRIPS AS A TOOL FOR MAKING SENSE OF INEQUITABLE TEACHING PRACTICES IN MATHEMATICS EDUCATION

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Introduction

This poster presentation will provide an opportunity for mathematics teacher educators to engage with ideas around teacher candidates' sense making of supporting diverse populations of mathematics learners. More specifically, it will discuss the use of student created comic strips as an alternate way (moving beyond the borders) for teacher candidates to demonstrate their understanding of inequitable teaching practices in mathematics education. Comic strips have supported students with autism in learning conversation skills (Gray, 1994), helped English language learners develop foundational literacy skills (Arif & Hashim, 2008) and brought to light myths and inaccurate conceptions about teaching (Ayers & Alexander-Tanner, 2010). Pushing even further beyond borders of alternative ways to allow teacher candidates to present their thinking in math methods courses, I will share my analysis of the text features of these comic strips to answer the question: What text features do teacher candidates use in self created comic strips to make sense of inequitable practices in mathematics education?

Methods and Analysis

As a form of culturally relevant pedagogy (Ladson-Billings, 1995), 13 special education teacher candidates were given choices about some of the learning experiences in which they would participate during their math methods course. One of the choice assignments in the "Math in the Community" component of the course was to create a comic strip dealing with a specific math issue related to equity, diversity, ability and/or social justice. Three students from this cohort chose to create comic strips. The text features of the comic strips were first coded for specific text features such as narrative captions, action words and quotations. This category also included images, which was defined as a visual that might contain words but does not use any features of explicit communication such as speech or thought bubbles. The comic strips were also coded for content. Content included state testing, student interest, individual learning abilities, and teaching for understanding. A checklist matrix (Miles, Huberman & Saldana, 2013) was used to uncover patterns and themes. Preliminary findings show that the teacher candidates' use of images supported evidence of a deeper level of understanding than their use of other text features.

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A CROSS-INSTITUTIONAL STUDY ON PRESERVICE TEACHERS DECIDING "NEXT STEPS" THROUGH NOTICING CHILDREN'S MATHEMATICAL THINKING

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Keywords: Teacher Education-Preservice, Teacher Knowledge, Elementary School Education

Teaching is complex with teachers making whole-class decisions based on what they notice about individual students' thinking. van Es and Sherin (2008) state that teachers can improve their noticing by changing what they notice, how they reason about student work, and how they make decisions. In the case of preservice teachers (PSTs), noticing students' mathematical thinking can be particularly difficult. The poster examines elementary PSTs' professional noticing of student written work (Jacobs, Lamb, & Philipp, 2010). The focus of the study was fostering PSTs' ability to provide instructional "next steps" for a whole class based on analysis of multiple student work samples.

Elementary PSTs enrolled in a mathematics methods or content course at five different institutions in the U.S. participated in the study. The researchers planned and taught the same multiplication lesson at their respective institution that consisted of watching children's videos, analyzing a case study with written student work, and whole-class discussions on "next steps." Qualitative multiple case-study analysis was used to analyze the data, with each institution taken as a separate case.

Preliminary results indicated that providing the whole-class case study assisted PSTs in gaining insight into (a) attending and interpreting students' mathematical thinking such as identifying multiplication strategies like repeated addition and derived facts and (b) the sophistication of these strategies and how they can be used to move students to a higher level of thinking.

However, contrary to our prediction, the PSTs struggled with providing specific "next steps," such as creating specific tasks or a question to pose to move an individual student forward in their knowledge of multiplication. It was also difficult for the PSTs to use individual "next step" ideas to make whole class instructional decisions. While the process of noticing and eventually deciding the "next steps" for a whole class was difficult, there were noticeable differences seen in their responses when talking about individual student work samples and whole class needs.

This study provided a means to begin to help PSTs make evidence-based shifts from describing general strategies for individual students to deciphering key characteristics that describe important mathematical understandings for a whole class.

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MATHEMATICS AS (DOUBLE) GATEKEEPER, STUDENT AS BORDERCROSSER: A CASE STUDY

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Keywords: Equity and Diversity, Mathematical Knowledge for Teaching, Policy Matters, Teacher Education-Preservice

In this paper, we consider "unjust uses of mathematics" (p. 15) for one student at a large public university in the southeastern United States (Stinson, 2004). We focus on the oppressive and withholding functions of one particular mathematics assessment as Jamesha attempted to cross the border into teacher education.

This case study traces the story of an undergraduate middle grades preservice teacher who was provisionally accepted into an urban teacher residency program for her final practicum and student teaching year, and then subsequently denied access to the program due to low scores on a mathematics entrance exam. Though Jamesha's areas of concentration were language arts and history, she was required to pass a program admission assessment in mathematics that required knowledge of advanced mathematics in order to gain official entrance into the middle grades program and move forward in the urban residency program. After two failed attempts to pass the math exam, Jamesha studied alongside peers and professors and worked through test prep materials she and others had purchased. She took the test three more times and still did not receive scores high enough to "pass" the exam. Jamesha was removed from the residency program and denied admittance into the teacher education program. She was forced to change majors and in the following semester, she failed all of her classes outside of the education department. Jamesha remained determined to become a teacher; she decided to take the assessment a 6th time and reenter the education degree program.

In our consideration of this case, we utilize a poststructuralist lens (Foucault, 1981/2000) to think about mathematics as "gatekeeper" (Stinson, 2004) as well as mathematics and 'teacher' subject position construction, complication, and negotiation (Britzman, 2003; Davies, 2003). We pose and consider the following questions: How does mathematics assessment function as a border to teacher education for non-mathematics teachers? How is a student's subjectivity constructed through assessment? "How does school mathematics as gatekeeper function? Where is school mathematics as gatekeeper to be found?" (Stinson, 2004, p.16).

The poster will map some of the ways that the mathematics program admission assessment functions as a gatekeeper for Jamesha. We will also consider how Jamesha's subjectivity as teacher and mathematician was (re)constructed by the assessment, as well as how her subject position as a student in a large university affected her navigation and negotiation of the structures in place in the teacher education program and subsequently in her alternate degree program.

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WHY DON'T YOU RUN HAPPILY WITH YOUR MATHEMATICS TEXTBOOK IN HAND AT UNIVERSITY ANY LONGER?

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Keywords: Affect, Emotion, Beliefs, and Attitudes, Post-Secondary Education, Reasoning and Proof

For many, crossing borders from one country to another can be quite a challenge. For at least 100 years the move from school to university is described as an equal challenge in the mathematics and mathematics education communities. In 1908, Felix Klein (1849–1925), with a special interest in secondary-level teacher training, characterized this transition from school to university as the first part of a "double discontinuity" (Klein, 1908/1932, p. 1). Recent studies show that what Klein described in 1908 is still a major problem today. Hefendehl-Hebeker and her colleagues (2010) noted that the transition problem might be connected to some kind of "abstraction shock" – meaning that pre-service teachers face a major epistemological obstacle from school to university regarding the nature of mathematics.

Our research began with the following hypothesis, which was informed by an approach based on the conception of belief systems, subjective domains of experience, and "Theory theory" (Gopnik & Meltzoff, 1997), and which we use for explaining epistemological difficulties of students struggling with new contents or situation.

The change from empirical-concrete to formal-abstract belief systems of mathematics constitutes a crucial obstacle for the transition from school to university. On epistemological grounds, similar changes regarding different natures of mathematics can be described for the history of mathematics. The explicit analysis of the historical genesis provides support for students dealing with their individual transition processes.

To test this hypothesis, we conducted research on an intensive, three-day seminar that we designed and implemented in spring 2015. We present a description of the seminar, including the readings and historical materials used with participants (see also Witzke et al., forthcoming). We then discuss the case of Inga (pseudonym), and, using excerpts from the data provide a summary to describe her experience with the transition between school and university mathematics.

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EXPERIENCES AND INFLUENCES OF MATHEMATICAL AUTONOMY

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Keywords: Teacher Education-Preservice, Affect, Emotion, Beliefs, and Attitudes

Teachers at all levels are increasingly expected to foster high quality mathematical thinking in their classes. Their students should be making sense of problems, generating solution strategies, and considering and creating representations of those strategies. This type of thinking can provide students opportunities to be more autonomous, or self-directed, in their learning, pursuing their own solution strategies and ideas to go beyond following prescribed processes given by teachers or textbooks. Many teachers were never given the opportunity to engage with mathematics in these newer ways when they were students. This poster will facilitate discussion of future K-8 teachers' experiences of autonomy in math class and how we can prepare these future teachers to foster students' confidence in their own abilities to reason mathematically.

Future teachers' experiences of mathematical autonomy will be articulated from a qualitative case study situated within in a math content course. Specifically, characterization of the course, based on observations, field notes, and video, revealed that the future teachers were expected to (a) discuss mathematics with others, including suggesting solution strategies, considering others' perspectives or answering questions, (b) construct their own solution strategies, (c) be intellectually courageous by asking questions when confused, suggesting potential ideas, and being willing to make mistakes, and (d) to let their thinking develop over time, understanding that problems sometimes require perseverance or revision of thinking. In order to carefully attend to the future teachers' affective experiences in this case study (as opposed to asking generally about feelings or emotions), I leveraged the constructs of competence (feeling capable), relatedness (social belonging), and autonomy (self-direction) from self-determination theory (Ryan & Deci, 2000). My goal was to particularly note how future teachers' affective experiences related directly to the cognitive and participatory demands of a classroom focused on promoting student engagement, and these constructs lent themselves to such connections.

As stated above, this poster will focus on the future teachers' experiences of autonomy, but it will situate these results within the larger study, expanding on the conceptual framing, data collection, and analysis briefly touched on here. The primary goal will be to highlight both the future K-8 teachers' prior experiences in their K-12 years and their experiences in the course studied, as analyzed from autobiographies and interviews, and to focus particularly on how their experiences of mathematical autonomy helped them articulate the type of classroom they hope to create for their future students. Perhaps not surprisingly, many future teachers described lacking feelings of autonomy in their prior math courses, recalling frustration with little freedom for or support of student generated ideas. However, there was a sense of autonomy in this classroom, that not only allowed, but encouraged students to see themselves as mathematical authorities. At the end of the course, when envisioning their future classrooms, each future teacher emphasized their desire to support student autonomy as it had been supported for them in this course.

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HELPING PRESERVICE TEACHERS MAKE SENSE OF WHOLE NUMBER CONCEPTS AND OPERATIONS: WORKING IN BASE 8

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How to improve teacher education programs to allow preservice teachers (PSTs) make sense of mathematics for depth, is a topic debated among educational researchers, especially in mathematics education. PSTs and inservice teachers (ISTs) need to be prepared to teach concepts and to understand how and why students come up with different strategies and whether those strategies will work all the time. Teaching whole number concepts and operations (WNCO) in a base other than 10 can challenge PSTs and ISTs to think about mathematics in ways similar to an elementary student. Previous studies have reported PSTs and ISTs struggle when using base systems other than base 10 in an education class or a professional development workshop just like their elementary students might struggle with base 10 (Andreasen, 2006; McClain, 2003; Yackel, Underwood, & Elias, 2007; Zazkis & Zazkis, 2011).

At one university, PSTs work in base 8 for an entire 4 week unit to gain a deeper understanding of WNCO. While learning in base 8, PSTs have to make sense of mathematics strategies in a deeper level than if they were learning about them in base 10. For example, if students are asked to add 7+6 in base 10, they will say 13. When asked "how do you know?" their most likely response is that they just know. By asking the same question in base 8, PSTs make sense of regrouping in a way that is similar to how beginning learners make sense of regrouping in base 10. This can address an important barrier in supporting PSTs and ISTs to understand WNCO in ways that are important for teaching.

The purpose of this research was to use qualitative data from interviews and observations to attempt to answer the question: In what ways do PSTs perceive learning WNCO in base 8? Interviews were conducted to ascertain how PSTs may or may not have changed their views on learning in base 8. Data from interviews of four PSTs indicated they initially struggled with learning WNCO using base 8 during the four week unit. In the beginning, PSTs were frustrated with learning in base 8 and did not understand why they needed to learn this way. However, as the unit continued, researchers found that PSTs were more accepting and realized the importance of learning in base 8. Most understood that although they will not be teaching their elementary students in base 8, learning in this base system had benefits they did not initially realize, for example understanding WNCO in depth.

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LEARNING TO FACILITATE GROUPWORK THROUGH COMPLEX INSTRUCTION

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Keywords: Teacher Education-Preservice, Instructional Activities and Practices

Complex Instruction (CI) is a form of collaborative learning, grounded in research on students' thinking and learning that focuses on equitable groupwork collaboration (Cohen & Lotan, 2014; Featherstone, Crespo, Jilk, Oslund, Parks, & Wood, 2011). CI research posits that how a student is perceived by their peers in the group with respect to their capability to make meaningful contributions (that is, issues of status) effects whether the group will listen to that individual. If a student is seen as low status, they are seen as unable to contribute to the intellectual work of the group, regardless of how well the student understands mathematics. In contrast, a high status student will be allowed to take over a group, even if the student touts erroneous mathematical ideas. The purpose of our study was to implement three research-based pedagogical strategies grounded in CI—*establishing groupwork norms*, *designing groupworthy tasks* and *setting group goals*—that would provide access and opportunity towards equalizing status and broaden participation by all preservice teachers (PSTs) in an elementary mathematics content course.

Student reflections and post course evaluations were collected from 36 elementary PSTs across two course sections. Preliminary results indicate that CI reforms had a positive impact on PSTs' perspectives about learning mathematics through equitable collaboration with their peers. Below are some exemplar responses that support these findings.

> At the beginning of class, I was dreading working in a group. I believed, "I am smart enough to do this on my own." I have learned that using others can help because you learn the way others see a problem.

> The groupwork was the best part of the course because we got to teach one another and think about different ways to solve a problem. We were smarter together than alone. Before taking this course, I was not confident about begin able to do math. I've never really been very good at it. After this class working in groups, I see myself as capable of doing math now.

Many undergraduate elementary education majors enter their content courses with little confidence in themselves as mathematical learners. Through CI, we've seen a shift in our PTS' view that they are capable of doing mathematics and that taking risks and exploring beyond ones' comfort zone can lead to deep mathematical learning and understanding.

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SUPPORTING SECONDARY MATHEMATICS TEACHER PREPARATION: A COLLABORATIVE TETRAD MODEL

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During student teaching, pre-service teachers (PSTs) are expected to put into practice the integration of content and pedagogy under the mentorship of knowledgeable others in a classroom setting. This integration is traditionally accomplished through a triad model of student teaching supervision which involves daily interaction with a mentoring in-service teacher (IST) coupled with periodic visits by a university supervisor, typically a mathematics educator. Although mathematicians are responsible for significant portions of secondary teacher preparation prior to the student teaching, they are noticeably absent from the triad for mentoring and supporting PSTs during their field placement. This results in an exacerbation of the separation of PSTs' mathematical knowledge development from their pedagogical knowledge development, and from their K-12 classroom-based experiences. Importantly, this separation of mathematical and pedagogical knowledge is experienced not only by the PST, but also by university mathematics educators and mathematicians. Faculty members' vision of PSTs' needs and experiences are restricted, and cross-disciplinary knowledge development is inhibited. In general, jurisdiction over teacher preparation is divided across separate and distinct academic units, and administrative and cultural barriers impede communication and joint work. Further, mathematicians, and in some cases mathematics educators, are not directly involved in working with student teachers, or with recent graduates. Thus, they do not see the ways in which the courses they teach do, or do not, influence how their students teach mathematics.

In this paper, I propose expanding the collaborative discourse community from a triad to tetrad model through the inclusion of a mathematician. The tetrad model brings together PSTs, ISTs, mathematics educators, and mathematicians represented visually as the vertices of a tetrahedron. The tetrad model is a creation of a fourth space, visually the interior of the tetrahedron. This space is not typically reflected in current secondary teacher preparation efforts. Sociocultural learning theory helps us see PSTs' development for teaching secondary mathematics as collaborative, complex, and contextual (Fernandez & Erbilgin, 2009; Thompson, Beneteau, Kersaint, & Bleiler, 2012). From this perspective, we can examine the entire system, in this case the tetrad collaborations, as the unit of analysis. Through careful study of tetrad interactions, one can investigate (a) the knowledge and perspectives that members bring to the tetrad collaborations, (b) the interactions and negotiations of knowledge and practice, and (c) the feedback mechanisms that directly influence mathematicians' and mathematics educators' teaching practices in their university secondary mathematics teacher preparation courses.

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INTERCONNECTED INSTRUCTIONAL AND MATHEMATICAL PRACTICES ENACTED WHILE AMBITIOUSLY 'GOING OVER A PROBLEM'

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To create opportunities for teacher candidates (TCs) to develop pedagogical skill that engages all students in authentic mathematics we have designed an instructional activity (Lampert & Graziani, 2009), *Going Over a Problem* (GOP) (Elliott, Aaron, & Maluangnont, 2015). GOP embeds two learning goals; opportunities for TCs to build skill with instructional practices and opportunities for secondary students to engage with mathematical practices that are critical to sense making on mathematical problems. Our research question is: How does GOP, as enacted by each TC, support or constrain *both* TC and student learning?

We used GOP in cycles of investigation and enactment (McDonald, Kazemi, & Kavanagh, 2013) with eight TCs in a secondary methods course. We coded video attending to TCs' use of discourse moves, mathematical representations, questioning strategies, eliciting and responding to students' reasoning (NCTM, 2014) and students' coordination of ratio quantities (Lobato, Ellis, & Zbiek, 2010) and argumentation (Lannin, Elliott, & Ellis, 2011). We found that TCs' use of instructional practices created opportunities for students to engage in key mathematical practices. Conversely, students' arguments created opportunities for TCs to develop skill with key instructional practices of interpreting and responding to student contributions.

The second finding, that TCs opportunities to learn are supported by interacting with students' emergent thinking may not be surprising, but it is instrumental in understanding pedagogies of investigation and enactment. TCs' enactments of GOP allowed them to confront the improvisational and responsive aspects of teaching which they may not encounter in teaching situations where they can rely on familiar instructional practices such as IRF questioning (Mehan, 1979). We claim that the ambitious practices deployed in GOP expands the types of student contributions that TCs elicit from students and to which they are required to respond in the moment.

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ABRIENDO FRONTERAS: HOW PRE-SERVICE TEACHERS LEARN TO TEACH PRODUCTIVE STRUGGLE TO ALL STUDENTS, INCLUDING ELLS

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Introduction and Purpose of the Study

Productive struggle is a necessary component of learning mathematics with meaning and refers to students grappling to make sense of problems (Hiebert & Grouws, 2007). Murrey (2008) advises teachers to scaffold ELLs' language if necessary, but in doing so, teachers should strive not to reduce the mathematical rigor of problems for ELLs.

Methods

The context of this descriptive multiple case study was an elementary mathematics methods course of 22 PSTs with field placements, in which I was the instructor. I analyzed the following data: open response surveys (pre-and post), homework reflections, lesson plans, university supervisors' reports, host teachers' reports, and semi-structured interviews of PSTs, university supervisors, and host teachers.

Findings and Implications

Multiple data sources suggested that all eight PSTs facilitated their students to connect with the mathematics and to provide access for students. However, only three PSTs showed signs of keeping the cognitive load high for their students in the field, which I refer to as productive struggle.
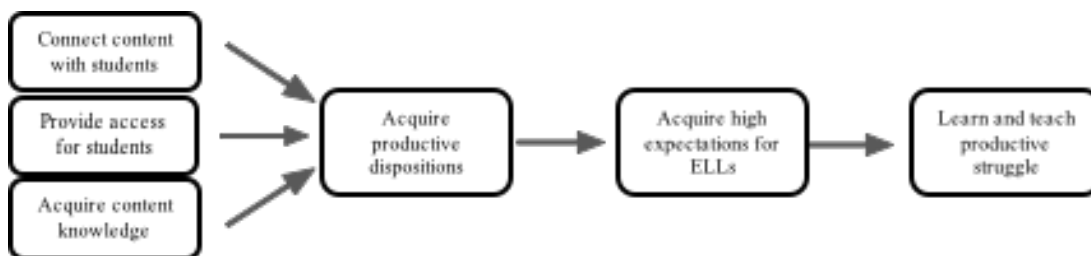
Figure 1. Trajectory for PSTs to engage students in productive struggle.

The trajectory represents a possible order for learning how to engage all students, including ELLs, in productive struggle. PSTs may need to learn how to connect and provide students' access to mathematics and acquire strong content knowledge themselves before developing a productive disposition, acquiring high expectations for students, and teaching productive struggle.

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INFLUENCES AND MOTIVATIONS IN CHOOSING TO BECOME A MATHEMATICS TEACHER OR NOT

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Keywords: Affect, Emotion, Beliefs, and Attitudes, Teacher Education-Preservice, Policy Matters

Mathematics education is a field that has suffered and continues to suffer teacher shortages, particularly in relation to schools that serve low-income students (USDOE, 2012). In meeting the shortage challenge, it is valuable to research and understand the influences on and motivations of individuals, including those typically underrepresented in the field of teaching (e.g., Hispanic, Black/African American), for becoming mathematics teachers or not. Reasons for pursuing teacher education (not specifically mathematics) are inter-related to an individual's goals for the future (Schutz, Crowder, & White, 2001). According to Schutz et al. (2001), "goals are subjective representations of what one would like to happen and what one would like to avoid in the future" (p. 229). In their study, Schutz et al. (2001) found four main sources that were predominant in their participants' goal history of becoming a teacher: (a) family influences, (b) teacher influences, (c) peer-influences, and (d) teaching experience. Most studies on motivations to become a teacher (i.e., Kyriacou & Coulthard, 2000) suggest that reasons for becoming a teacher fall into three categories: intrinsic, extrinsic, and altruistic.

Survey data was collected at a minority serving institution from undergraduate students (62% Hispanic/Latino, 18% African American/Black, 9% Asian, 9% White non-Hispanic,2% Other) engaged in early experiences trialing teaching mathematics as a career option. Initial data analysis revealed that undergraduates continuing to pursue becoming a mathematics teacher reported more positive views of teaching among their parents/guardians, family, and close friends than those not becoming teachers; thus, aligned with Schutz et al. (2001). With respect to extrinsic motivations for becoming mathematics teachers, being respected by family and the community, and being seen as a good mathematics teacher by family and teachers/instructors were rated more highly by those choosing to pursue becoming a mathematics teacher than those that did not. For intrinsic motivations, seeing themselves as teachers, enjoying teaching mathematics to others, and working with people rather than objects were rated more highly by those choosing to pursue teaching in contrast to those that did not. Finally, with respect to intrinsic motivations, making a difference for people, helping people, and helping solve societal problems were rated more highly for undergraduates continuing to pursue mathematics teaching than those that did not. Findings suggest directions for recruiting or influencing students interested in mathematics related fields (at the undergraduate or secondary school levels) to pursue teaching mathematics as a career option.

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TEACHING PRE-SERVICE TEACHERS TO DECONTEXTUALIZE WORD PROBLEMS

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One of the main goals of the *Common Core State Standards for Mathematics* (CCSSM) (National Governors Association for Best Practices, Council of Chief State School Officers [NGA & CCSSO], 2010) is to promote quantitative and algebraic reasoning (Thomson, 2011). One of the aspects of algebraic reasoning includes an ability to decontextualize word problems by presenting them in a form of equations that characterize the essence of word problems in symbolic form, while defining the meaning of each symbol. However, we observed pre-service teachers' difficulties with decontextualizing word problems when presenting them in a form of equations and defining parameters.

To meet our students' needs, we incorporated a supplementary instruction (12 weeks) in a regular 15-week arithmetic course for pre-service teachers and thoroughly monitored teachers' progress with creating equations, identifying measuring units, and defining parameters.

Seventeen pre-service teachers from a western research university participated in this study. They read self-explanatory instructions, created equations, and designed visual models as tools for decontextualizing simple arithmetic problems. When teaching the supplementary program, the instructor constantly monitored students' mistakes found in their written work (students' workbooks) and regularly spent five minutes in class discussing uncovered misconceptions. For this preliminary study, we analyzed the mid-term exam, quizzes, and final exam written data using an open and axial coding method (Strauss & Corbin, 1998).

We found students commonly misinterpret symbols used in equations and identify them as equivalent to objects, units, or words—consistent with other studies (Booth, 1988; MacGregor & Stacey, 1997). In the current study, students demonstrated difficulties with defining symbols they used when presenting 1–2-step arithmetic word problems in symbolic form. Students' work reflected (1) misconceptions in regards to the symbols' meaning and (2) misconceptions regarding the defining attributes represented by the values. Supplementary instruction in word problems solving allowed us to significantly improve pre-service teachers' precision in defining parameters. In the future, we plan to deepen the precision of our studies by redesigning the pre-test and adding a comparison with a control group—no supplementary instruction.

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INTEGRATING CHILDREN'S LITERATURE IN MATH: FOSTERING PRE-SERVICE TEACHERS' UNDERSTANDINGS AND CONNECTIONS

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Interdisciplinary teaching is growing but despite numerous children's books available, pre-service teachers (PSTs) still are unaware of various strategies of integrating children's literature in mathematics. The knowledge about integrating children's literature needs to be built in teacher education programs (Cotti & Schiro, 2004; Ruiz, Thornton & Cuero, 2010; Ward, 2005) to help them develop conceptual models of integration for future teaching decisions. This research is a step towards providing experiences to PSTs for fostering their understanding on integrating children's literature in an elementary math classroom. This study probes, enhances, and aims to transforms PSTs perceptions about integration. The study examines attitudes and perceptions of PSTs and provides the mathematics community with a model of integration that can have significant impact in the teacher preparation program. The intervention aims to present PSTs with a model that can overcome cultural and language borders in their classroom along with borders in their own thinking about using children's literature.

Methodology

Quantitative multiple case study analysis was used to analyze 31 PSTs attitudes and perceptions about integrating children's literature over two semesters in a math methods course at a public university in north Texas. The intervention was an integration model in which PSTs rotated through several stations engaged in math activities with children's literature, designed ways to use children's literature to teach math, and participated in whole class discussions about children's literature. Data in the form of semi-structured pre- and post-interviews, class discussions, small group discussions, PSTs reflections and work, and lesson plans were analyzed through open and axial coding. Triangulation, constant comparison, and member checking were used to identify emerging themes and commonalities.

Results

Results reveal not only lack of awareness and understanding of integrating children's literature but also hesitancy and fear related to the pedagogy of integration. Results also show that the exposure to the integration model not only led to an increase in awareness, preparedness, and understanding but also translated to interdisciplinary integration in other disciplines. In addition, PSTs related using children's literature as a means to reach different learners in the classroom. The researchers propose the integration model as a useful and adaptive tool for higher education to engage PSTs in developing flexible attitude towards interdisciplinary integration.

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(RE)DEFINING SMARTNESS: EXPLORING MULTIPLE ABILITIES IN ELEMENTARY AND SECONDARY MATHEMATICS TEACHER EDUCATION

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Realizing equity goals in mathematics education demands effective instructional approaches that embrace student diversity and "go beyond 'good teaching,'" to ensure every student has opportunities to engage successfully with challenging mathematics content (NCTM, 2014, p. 68-69). Complex instruction (CI) offers one approach to questioning borders within classrooms around who is able to do challenging mathematics (Boaler & Staples, 2008). CI emphasizes *multiple ability tasks* that require a range of ways of 'being smart' and are *groupworthy* because they necessitate positive interdependence (Lotan et al., 1994). When students value multiple abilities as important in mathematics, they are more likely to participate eagerly and to encourage others to contribute their unique strengths.

We, two mathematics teacher educators, integrated CI into our mathematics methods courses, one elementary and one secondary, with prospective mathematics teachers (PSTs). We aimed to investigate how PSTs came to understand, enact, and recognize one slice of CI, namely, integration of multiple abilities in mathematics tasks and the recognition of different ways students can demonstrate 'smartness' in mathematics. PSTs in both courses worked in pairs or trios to design multiple ability, groupworthy tasks to learn to identify potential multiple abilities inherent in high-cognitive demand mathematics tasks and to modify these tasks to better promote collaboration and equal participation among students. The PSTs in both courses created lists of multiple abilities inherent in a high-cognitive demand task and then amended this list after revising the same high cognitive demand task to make it groupworthy. Using qualitative textual analysis, we categorized each listed ability as supporting (1) a general mathematics practice, (2) specific mathematical work of the task, or (3) productive collaboration, and compared the PSTs' initial lists with their final lists to see how they differed.

Overall, preliminary findings suggested PSTs identified vague or general abilities for the initial versions of the task, but they identified detailed abilities, grounded more firmly in the specifics of the groupworthy versions, suggesting a deepening of understanding of the different ways students can demonstrate 'smartness' in mathematics. We hope our poster will foster discussion about how changes in pre- and post-multiple abilities lists provide mathematics teacher educators, across elementary and secondary methods, insights into PSTs' understanding of groupworthy as distinct from other "good" mathematics tasks.

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PRESERVICE TEACHER LEARNING TO HELP ELS UNDERSTAND MATHEMATICAL PROBLEMS

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There is a common belief that teachers do not need to differentiate their instruction for English Learners (ELs) and that helping ELs adjust in school culture is not a teacher's responsbility (Walker, Shafer, & Liams, 2004). This belief is due to their lack of proper training to teach ELs, as statistics show only 13% of teachers are adequately prepared to teach ELs while more than 40% of teachers have ELs in their classrooms (NCES, 2002). Responding to the great need for an adequate teacher preparation for EL in mathematics education, this study was designed to investigate preservice teacher (PSTs) learning to help ELs understand mathematics.

The problem space model (Campbell, Adams & Davis, 2007) was adapted to construct a conceptual framework for this study in order to address the following questions: (a) What strategies do middle school PSTs use to help ELs understand cognitive demanding problems? (b) In what ways do middle school PSTs change in using strategies as they worked with ELs and after they have received interventions? Three middle school mathematics PSTs volunteered to participate in this multiple-case study. Participants prepared a one-hour lesson the same week based on the same problem, each designing their own lessons. After each session, the PSTs were interviewed and provided research-based EL interventions (Chval & Chavez, 2011). The collected data (lesson plans, interviews, video, reflections, and surveys) were analyzed using *the constant comparative analysis method* (Fram, 2013). We developed a code manual after coding implemented strategies in our conceptual framework.

Data showed our intervention influenced certain strategies more than others. All of the PSTs did not consider connecting mathematics with students' life and culture before the relevant intervention. Another impact of the intervention manifested in the PSTs' use of visuals (pictures, drawings, or diagrams). They started using various visuals in more math–related ways after the related intervention. Designing a linguistically and conceptually rich lesson in order to make it accessible to ELs is an important skill for teachers of ELs. Hence, PSTs need to develop this skill in their preparation programs and consider the four components of the conceptual framework.

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PRESERVICE TEACHERS' UNDERSTANDING AND USE OF COMPOSITE UNITS

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Research has shown that fractional knowledge plays a significant role in students' algebraic reasoning and whole number multiplicative structure is strongly related to fraction multiplication and division (Hackenberg, 2013). This study investigates how middle grades preservice teachers who show evidence of limitations in coordinating three composite units make sense of fraction multiplication problems involving composite units. I acknowledge a need to construct a model or hypothesis to represent their ways of their thinking (Steffe, 2010), taking into account the processes of assimilation and accommodation (von Glasersfeld, 1995). I will present one hypothesis to account for how preservice teachers who are unable to maintain composite units while partitioning and iterating may use available schemes or procedural knowledge to find correct answers in such contexts. This study draws data from a series of hour-long, one-on-one clinical interviews with ten middle grades preservice teachers attending a mathematics pedagogy course at a university in the Southeastern U.S., analyzing participants' understanding of using composite units in fraction multiplication and measurement division contexts. Interview tasks prompted students to use Java-bars 5 to visually represent their thinking as they partition bars into equal parts, pull out (disembed) parts, and repeat (iterate) them.

Analysis and Findings

Jacky was unable to maintain 3/5 as a composite unit in finding 1⁄4 of 3/5. The way she made sense of 1⁄4 of 3/5 was to create a bar as a unit, break it up into 5 columns, and pull out 3 columns. Afterward, she broke each fifth into 4 parts vertically so that she could pull out 3 parts from each fifth, without recognizing 3/5 as a composite unit (and instead treating it as a separate unit bar). Following this process, she was able to produce the correct result, but when asked to give the answer relative to the original unit bar, she had to go back to the original bar and partition all fifths into fourths to find the total number of parts in order to make a comparison between two whole numbers. Preservice teachers who are unable to maintain fractions as composite units while partitioning or iterating are able to rely instead on other existing schemes or procedural knowledge to make sense of problems involving these operations. If students conceive of partitioning and iterating along a single axis, this requires maintaining a composite unit, but students who are unable to do so may make use of two axes (partitioning both vertically and horizontally), allowing them to produce the correct result by graphically representing their mathematical calculation.

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MODELING CO-TEACHING APPROACHES TO ENHANCE PRESERVICE MATHEMATICS TEACHERS' KNOWLEDGE AND SKILLS FOR TEACHING

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In the United States, teacher educators must decide how they will respond to the increase in academically diverse populations. Likewise, the national reform for mathematics teacher education partnership (MTEP) strongly encourages secondary teacher preparation programs to rethink strategies among others in how they prepare mathematics teachers for their future classrooms. In response, co-teaching, an instructional practice that blends instructors' various teaching strengths, can be offered to provide students with differentiation and teaching support when learning mathematical concepts (Bauwens, Hourcade, & Friend, 1989).

At an urban, southeastern university, two instructors of a methods course, assigned prior to the field experience, introduced and explored with a co-teaching-for-student-teaching (CT4ST) model, which is designed to assist both cooperating teachers and preservice secondary school mathematics (PSSM) teachers as they collaborate in planning, organizing, delivering, assessing, and sharing the physical space of the classroom. The PSSM teachers then worked with their cooperating teachers using the CT4ST model to share teaching responsibilities and better meet students' needs (Bacharach & Heck, 2012).

The participants, the co-instructors and PSSM teachers, reflect on the CT4ST model integrated at both the university classroom experience and the student teaching field experience. The co-instructors as researchers examined the following research question under a constructivism framework (Ertmer & Newby, 2013): *what are the perceptions of the co-instructors and the PSSM teachers experiencing the co-teaching-for-student-teaching model (CT4ST) during the transition from course preparation to student teaching?*

With a graduate research assistant's help in interviewing and preparing the documents for analysis, the researchers coded the participants' reflections and transcribed interviews. Findings revealed that the CT4ST model integrated into both the methods course and student teaching experiences served beneficial. An instructor stated that "each member [PSSM teachers] in the small group setting took advantage of the individual attention of the instructor." Several PSSM teachers' comments confirmed that when the CT4ST model was utilized, the teacher to student ratio decreased and students received more individualized attention. Implications from this study suggest that the CT4ST model is viable to enhance the collaboration among co-instructors, PSSM teachers, and cooperating teachers. Furthermore, the researchers present an action plan to support CT4ST model work to increase teachers' knowledge and skills for teaching mathematics.

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VISUALIZATION TASKS IN MATHEMATICS TEACHER PREPARATION: HOW A MATHEMATICS GAME IMPACTED PRESERVICE TEACHER LEARNING

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Keywords: Teacher Education-Preservice, Technology, Instructional Activities and Practices, Learning Trajectories (or Progressions)

Innovative Visualization Tools for Learning to Teach

Digital mathematics textbooks, instructional materials integrating interactive diagrams, interactive visual examples are increasingly common in mathematics education. For what learning purposes can these innovative visualization tools be employed in mathematics teacher preparation? Exploring this question, the poster presents learning-to-teach visualization tasks based on *Spatial Temporal Mathematics,* an interactive game-based curriculum for teaching K-8 math visually without words (Petersen, 2011).

Theoretical Frame

I approach the course from a cognitive situated perspective. This means that I actively seek to elicit, develop, and assess the mathematical and pedagogical understanding of PTs through cycles of individual, small group, and whole class problem solving and interactive discussion (Simon, 2008). The visualization task is semiotically inspired (Presmeg, 2013) and builds on Bruner's (1961) theory of mathematics learning whereby mathematical abstractions and symbols evolve from carefully sequenced learning tasks whose logic progresses from visual to symbolic.

Activity, Learning Goals, Results

The ST Math Area game task consists of three levels requiring interaction between PTs and the game environment as follows: *Level 1,* given a rectangle subdivided into unit squares, PTs highlight unit squares equal in number to given rectangle's area; *Level 2,* given a rectangle subdivided into unit squares with numerical dimensions given, PTs highlight unit squares equal in number to given rectangle's area; and *Level 3,* given a rectangle and its numerical dimensions, PTs highlight unit squares equal in number to give rectangle's area. For each level PTs (1) write a learning objective*,* (2) describe how level *n+1* moves game player's mathematical thinking forward from level *n,* (3) identify math content taught, and (4) identify math standard taught.

Summary of Findings

Preliminary findings indicate ST Math visualization tasks are especially useful for learning to logically sequence learning tasks and write learning objectives.

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INVESTIGATING PRESERVICE TEACHERS' CONCEPTIONS OF AND METAPHORS FOR PROBLEM-POSING AND THEIR RELATIONSHIP TO PROBLEM-POSING PERFORMANCE

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With increased interest in curricular and pedagogical innovation in mathematics education, mathematics educators have paid growing attention to problem-posing. However, the notion of problem-posing has been used with multiple meanings such as "the generation of new problems," "the re-formulation of the given problem", or "inquiry activity of persons with exceptional creativity" (Kilpatrick 1987; Silver, 1994). To understand how this critical population of future teachers perceive problem-posing and how their perceptions are reflected in their performance, this study investigates PSTs' notions of problem-posing in relation to their generation of a fraction multiplication word problem. The research questions that guided this study were: (1) How do PSTs view problem-posing?; (2) What metaphors do they use in describing problem-posing and why?; and (3) How are their notion of problem-posing reflected in their ability to pose a fraction multiplication problem?

Participants in this study were 96 PSTs in their internship year of elementary teacher preparation programs at two sites, a large northeastern university and a large southwestern university in the US. For the study, participants completed a written task consisting of two parts. In the first part, they were asked to answer three open-ended questions regarding their conception of and metaphor for problem-posing. The second part called for creating a word problem when given a fraction multiplication expression (improper fraction × whole number) and then solving their problem. For the analyses of PSTs' written responses, we used an inductive content analysis approach (Grbich, 2007). For the problem-posing task, the correctness of PSTs' responses was determined and then various strategies used and common misconceptions were explored.

Findings suggest that there was no clear connection between their views of problem posing and their problem posing abilities. However, PSTs who viewed problem-posing as a means of improving problem-solving tended to create a correct math problem in various contexts, which is consistent with prior research (Silver, 1997). The findings of this study suggest that elementary mathematics teacher education programs need to include more problem-posing activities so that PSTs can experience the benefits of problem-posing from a variety of perspectives.

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EXPLORING PROSPECTIVE TEACHERS' FORMATIVE FEEDBACK PRACTICES

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Letter writing activities have been implemented and studied in teacher preparation courses to research the development of teacherly-talk (Crespo, 2002), and problem-posing skills (Crespo & Sinclair, 2008; Norton & Kastberg, 2012). We consider the use of a letter writing activity in a methods class as a means to explore and develop prospective teachers' (PTs') notions for providing effective feedback when corresponding with multiple learners in individual letters. Research indicates that effective feedback, particularly feedback given without a grade (Crooks, 1988), has potential to positively influence learning (Black & Wiliam, 1998; Hattie & Timperley, 2007). Little is known concerning how PTs develop effective feedback practices to promote learning. In this study, feedback provided by PTs during a letter-writing activity between secondary PTs and high school learners, is examined to explore the question: How do the levels of feedback provided to learners by PTs vary across learners and across the letter exchange?

We define feedback as communication to a learner for the purpose of furthering their understanding and which is expected to be taken up by the learner. We adopted Hattie and Timperley's (2007) feedback framework which states effective feedback should answer the questions: "Where am I going? . . . How am I going? . . . and Where to next?" (p. 86). They further categorize feedback as focused on the task, process, self, or self-regulation and discuss the effectiveness of each type.

Data for this study were the letters exchanged between PTs and learners along with reflective papers and other methods class artifacts related to the letter exchange. Three PTs corresponded with 21 learners in an Algebra II course. PTs wrote five letters involving discussion of a problem that could be modeled with a trigonometric function. Feedback in the letters was coded according to the type of feedback provided (task, process, self, or self-regulation) and learner solutions were coded according to correctness, depth of explanation, and progress toward a solution. Corroborating and disconfirming evidence was coded throughout the additional data.

Preliminary analysis revealed that PTs provided both feedback on the task and the process but that PTs became more directive in their feedback as the exchange progressed. Insight into PTs feedback practices has potential to provide teacher educators information from which to build activities that develop feedback practices and attend to PTs existing ideas about feedback.

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LANGUAGE PATTERNS OF PROSPECTIVE SECONDARY MATHEMATICS TEACHERS: A LINGUISTIC PERSPECTIVE

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Mathematics education has taken up researching discourse in mathematics classrooms through linguistic approaches. This research has added to what we know about the variety of skills and language practices teachers need to create collaborative learning environments. Teachers can use interpersonal language as they interact with students to draw attention to specific mathematical content and also to build relationships with their students in the classroom. However, much remains to be understood about the ways student teachers (STs) of secondary mathematics use language to support students' engagement and interactions during shared constructions of mathematical meanings. At an interactional level of discourse analysis, I use Halliday's (1985) theory of systemic functional linguistics (SFL), drawing on a basic premise that language use is purposeful behavior, structured to make meanings through exchanges of words and grammar. For example, particular uses of pronouns can reveal personal involvements in the discourse. This study examines the ways STs *initiate* an interactive, problem-solving dialogue in secondary mathematics classrooms and the ways students *respond.*

Research Design and Findings

I video-recorded 53 conversations between three STs (all at different high schools) and small groups of algebra students working collaboratively to solve mathematical tasks. Initial passes through the data helped identify excerpts of interactional discourse patterns (IDPs) between STs and students. I have defined IDPs as two or more verbal exchanges made by students in response to initiating language moves made by the ST (statements, offers, questions, or commands). In the brief IDP example that follows, Gabe (ST) has asked a group of Algebra 2 students the question "…can *I* rewrite cosine squared in terms of sine?" (Pronouns are italicized.)

> *Barry*: So, then, *we* just replace this with cosine squared of *x* minus one. Right?
> *Ana*: … yeah, cosine squared of *x* minus one… No, *I* don't know. This is so complicated.
> *Barry*: *We* need more numbers and less letters.
> *Gabe*: *I* appreciate that.

Basic speech roles that speakers take on are either *giving* or *demanding*. Gabe initiated this IDP and demanded information by asking a question. The pronoun *I* suggested that Gabe is doing the problem solving. Barry responded with an answer but included himself in the solving process (*we*). Ana aligned her response with Barry, but then hedged by claiming '*I* don't know'. I interpreted Barry's next response to be interpersonal in nature – attempting 'lighten the mood' by implicating that students (*we*) find mathematics to have too many variables. While not agreeing with Barry, Gabe responded that *he* understood Barry's perception that the abstract nature of variables can complicate students' understandings. Examining IDPs can reveal meanings not only related to mathematical content but also to interpersonal relationships, attitudes, and roles.

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UNDERSTANDING PRE-SERVICE TEACHERS' BELIEFS ABOUT MATHEMATICS

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Among education literature, one of the most prolific categories of research is teacher beliefs (Cochran-Smith et al., 2015). Within both the classical and contemporary belief literature, traditional cultural beliefs, also known as unproductive beliefs, are cited as a major limiting border or obstacle in reforming mathematics classrooms toward the National Council of Teachers of Mathematics (NCTM) vision of excellence and equity and toward the actualization of the federal mandate to effectively implement challenging State standards. NCTM's most recent publication Principles to Action (2014) reports, "Dominant cultural beliefs about the teaching and learning of mathematics continue to be obstacles to consistent implementation of effective teaching and learning in mathematics classrooms (Handal 2003; Philipp 2007)" (p. 10).

The purpose of this study was to thoroughly examine pre-service teachers' normative beliefs at the completion of their preparation program, investigating (a) how they compare to discursive claims, (b) how they transformed over time, (c) whose transformed, and (d) potential influencing factors such as content knowledge. Results were intended to help mathematics educators determine (a) how to best address the obstacle of unproductive beliefs, determining where to focus efforts to maximize the potential for transforming the traditional mathematics classroom culture toward the reform-based, innovative culture advocated by NCTM; (b) what traditional conceptions of teaching, learning, and the nature of mathematics are the most deeply rooted and challenging to change; and (c) what novice-teacher challenges can be discussed explicitly during pre-service preparation to assist novice teachers in negotiating the tensions between reform-based and traditional pedagogical practices and conceptions of teaching and learning mathematics. Too, results were intended to help educators better pinpoint who needs greater scaffolding in shifting their beliefs. The study's guiding questions included: What do pre-service teachers believe about teaching mathematics, the nature of mathematics, and their abilities to teach mathematics? and How are these beliefs transformed during their pre-service preparation?

To answer the guiding questions thoroughly, a mixed methods approach was employed. Statistical analysis of three iterations of a Likert-scale survey was conducted. Open-ended survey responses, lesson-plan data, reading responses, responses to reform-based mathematics classroom video clips, archival data were analyzed to identify common themes and compare to the quantitative analysis, and a typology analysis was conducted using all of these data sources, for which individual cases were identified as representative of each "ideal" type of belief transformation/transmission that emerged from the analysis (Ayres & Knafl, 2008). All pre-service teachers (n=85) entering the undergraduate, early-, elementary, and middle-level education programs as juniors in August 2011 were asked to participant in the study.

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PRESERVICE TEACHERS DEVELOPMENT OF AN UNDERSTANDING OF FUNCTION USING DIFFERENT METAPHORS

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Keywords: Technology, Algebra and Algebraic Thinking

Functions are widely regarded as the unifying element of much of secondary mathematics and are a critical base for a mathematical understanding of the STEM disciplines. This is recognized in the Common Core mathematics standards where the study of functions is given its own category, separate from Algebra, in grades 9–12 (National Governors Association Center for Best Practice & Council of Chief State School Officers, 2010). The curricular and logistical obstacles to putting function front and center are challenging enough, but conceptual obstacles have been well documented in the literature (e.g., Arnon et al., 2013) and prove particularly hard to overcome. Rather than constructing their own definition of function based on tasks, students are often presented with a highly theoretical definition, resulting in a disconnect between their concept definition and their concept image (Even, 1990). Thus it is important that teachers have a deep conceptual understanding of functions and understand how to engage students in tasks to develop their own definitions.

The goal of this study was to examine the ways in which preservice teachers (PSTs) definition and understanding of function changed as they engaged in a sequence of technological tasks that utilized different representations of functions. The sequence began by asking PSTs to each write a definition of a function and provide examples and non-examples. Then PSTs engaged in two technological tasks, "Identify Functions" (adapted from Steketee & Scher, 2011) and "Function Machine" (McCulloch, Lee, & Hollebrands, 2015) that utilized different representations and metaphors of functions. Following the task sequence PSTs were again asked to provide a definition of function and reflect on the different metaphors and representations of functions presented in the tasks. Analysis of PSTs' definitions revealed that though they had extensive mathematics backgrounds, very few had a correct concept image or definition of function at the start. The removal of quantitative characteristics (i.e., no numerical, graphical, or equation representations) in the tasks seems to have allowed the PSTs to focus on the relationships as objects, rather than just processes, which proved helpful correcting their concept image and definitions. Further analysis will be conducted on how PSTs' engaged with the "Function Machine" task and how that task impacted their understanding of function.

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CROSSING BORDERS: MOVING PROSPECTIVE MATHEMATICS TEACHERS FROM INSTRUMENTS OF INEQUITY TOWARDS AGENTS OF CHANGE

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Keywords: Equity and Diversity, Teacher Education-Preservice, Teacher Beliefs, Curriculum

Introduction

Mathematics teacher educators (MTEs) are uniquely positioned to support pre-service teachers (PSTs) in pushing against mathematics education's "long history of giving access to students differentially, particularly based on the ideological construction of race" (Battey, 2013, p. 332). Martin (2013) asks "how can we more effectively partner with teachers to uncover and understand the external forces that position them in ways where their teaching and classroom practice are put in service to larger political projects and ideologies?" (p. 328). As critical mathematics educators, we are deeply concerned with the borders within mathematics that limit access to mathematics and deeply divide our students. As such, we ask what is the role of MTEs in developing awareness and agency regarding issues of equity for elementary PSTs?

Project Background

This project began as a collaborative review of the article *"A Possessive Investment in Whiteness": Access to Mathematics* (Battey, 2013). In that article, Battey highlighted the systemic marginalization of students of color in mathematics education by "analyzing national data to calculate the wage-earning differential attributable to differences in mathematics coursework by ethnic/racial groups across three time points: 1982, 1992, and 2004" (p. 332). He exposed the "differential access to mathematics" found within the data is connected with projected economic stratification based on race. We designed a social justice mathematics methods learning module that combined findings from the article and relevant mathematical content and practice standards. The learning module was used with approximately 100 PSTs at two public universities, one in Mississippi and one in New York. Quantitative and qualitative data from engagement in the module were collected and analyzed.

MTEs must understand how to breakdown the borders by developing PST awareness of the inherent power toward academic and economic opportunities that exist within the position of mathematics teacher. While many MTEs may want to teach for social justice, incorporate critical mathematics material in their courses, and encourage conversations about equity in, there may be a lack of time, resources, and support for this work (Gutstein, 2007), not to mention navigating the difficulty of doing this work well (Bartell, 2013). This poster session will provide a necessary space for critical dialogue about doing this work.

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PRESERVICE TEACHERS' MATHEMATICAL KNOWLEDGE FOR TEACHING (MKT) AND KNOWLEDGE OF CONTENT AND STUDENTS (KCS)

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One of the principal challenges facing teacher candidates in mathematics education is making the shift from "student" to "teacher." This transition requires teacher candidates to reorient from thinking about how they solve mathematics problems to engaging with students and their work, understanding student representations, and asking questions to guide students to move their thinking forward.

In order to scaffold this transition, we developed a five-step approach that we call the Mathematics as Teacher Heuristic (MATH). In the MATH heuristic, candidates complete the following five-step process involving increasingly teacher-centric tasks.

Our identification, from an earlier study (Meagher, Edwards, Ozgun-Koca, 2013), of the engagement with student work as a key juncture in the MATH heuristic led us to use the Professional Noticing Framework of Jacobs, Lamb & Philipp (2010) as an analytic tool to examine and understand teacher candidates' developing KCS and MKT as shown in their engagement with the student work.

The twenty-one participants in this study were undergraduate students enrolled in the second course of a year-long methods sequence for prospective secondary mathematics teachers. This course was built upon candidates' initial experiences with planning and assessment activities in the previous semester. The MATH heuristic was employed early in the semester.

The Noticing Framework of Jacobs et al (2010) was a useful analytic tool for us to be able to see the development of candidates' pedagogical knowledge and also to identify the deficiencies, and some successes, in the candidates' deep engagement with the mathematical thinking of the students which is to say deficiencies in their KCS and MKT. No candidate discussed explicitly patterns of misconceptions across the more than 100 student samples and, in our analysis we saw cases where candidates were unable or unwilling to *notice* and *interpret* student thinking. The data shows that the kind of analysis that would show strong MKT does not seem to come naturally to students but their development in this area is vital if they are to become effective teachers. We plan for an adjustment in the prompts for the MATH heuristic with a requirement to describe trends in student learning as, indeed, is asked for in the edTPA.

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EXAMINING POTENTIAL SUPPORTIVE FACTORS IN HELPING PRE-SERVICE TEACHERS SELECT SOLUTION STRATEGIES FOR CLASS DISCUSSIONS

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Keywords: Teacher Education-Preservice, Teacher Knowledge, Classroom Discourse

One goal of mathematics instruction is to support students' learning of concepts (Fennema & Romberg, 1999). A set of five practices (anticipating, monitoring, selecting, sequencing, and connecting) is intended to support teachers in planning for and facilitating class discussions around solution strategies to cognitively challenging tasks in ways that will build up to a conceptual learning goal (Stein, Engle, Smith, & Hughes, 2008). The selecting practice might be especially critical because the mathematics students are exposed to depends on which solution strategies are shared during the class discussion. However, purposefully selecting solution strategies in ways that will promote a conceptual learning goal might be difficult for pre-service teachers (PSTs) because they typically lack understanding to be able to identify how solution strategies relate to mathematical concepts (Peterson & Leatham, 2009) and do not spontaneously unpack learning goals into their underlying concepts (Morris, Hiebert, & Spitzer, 2009).

The purpose of this study was to explore whether and how two factors (content knowledge and the grain size of a learning goal) might support pre-service teachers in selecting solution strategies for class discussions in ways that will promote concepts underlying a learning goal. PSTs (N=61) in an elementary mathematics methods course first completed a content knowledge instrument aimed to assess their content knowledge related to division of fractions and, in a second sitting, completed a selecting task where they were asked to select solution strategies to a division of fractions mathematical task and provide rationales for their decisions. Thirty-one PSTs were randomly assigned to a condition in which the learning goal on the selecting task was unpacked down to its underlying concepts and 30 PSTs were randomly assigned to a condition in which the learning goal was stated in its general form only on the selecting task.

The main finding from this study was that when PSTs possessed higher content knowledge or had selected solution strategies designed to be likely to promote the concepts underlying the learning goal, they were more likely to justify their selecting decisions with reasons related to these concepts. This finding suggests that which solution strategies are selected for a class discussion matters and that teacher educators need to find ways to support PSTs in identifying relationships between solution strategies and the concepts underlying the learning goal.

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ASSOCIATION BETWEEN PRESERVICE TEACHERS' MINDSET AND OBJECTIVIZATION IN MATHEMATICS

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Keywords: Learning Theory, Teacher Education-Preservice, Teacher Knowledge

Purpose

The purpose of the study was to determine if mindset for learning mathematics could be identified in preservice teachers and if it could, what impact did mindset (Dweck, 2006) have on preservice teachers' ability to identify objectives for a lesson.

Research Question

What objectives do preservice teachers with growth or fixed mindset identify in math lessons?

Theoretical Framework

The study uses Uznadze's (1966) levels of mental activity by likening Uznadze's level of set to Dweck's fixed level of thought (2006) and likening objectivization, Uznadze's second level of thinking (1966), to Dweck's growth level of thought (2006). For Uznadze (1966), two planes of reality affect behavior: external reality (the world around us) and verbal reality (the world represented in words); this may be likened to an external domain and an internal domain where each contributes a level of thought and each develops language differently. Important to learning is that teachers recognize both levels of thought to maximize learning experiences.

Methods, Methodology, and Data

 The research was conducted as a case study of four preservice teachers in mathematics methods courses at a university in a predominantly Hispanic community. Data included content knowledge, mindset (Yeager, In Review), narratives describing lesson objectives, and individual interviews. Participants interviewed represented four profiles: high content knowledge, growth mindset, high match to expert objectives; high content knowledge, fixed mindset, high match to expert objectives; low content knowledge, growth mindset, high match to expert objectives; and low content knowledge, fixed mindset, high match to expert objectives.

Results and Significance

Mindset is measureable. Four themes emerged as knowledge needed to be effective teachers: 1) teachers' roles include more than teaching content, 2) knowledge of process standards to engage students, 3) knowledge content to recognize mathematical thinking and clarify misconceptions, and 4) using time effectively. Objectives identified included mostly process standards. Teachers as helpers emerged above teaching any kind of mathematical content or any other purpose. Participants with growth mindset identified expert lessons objectives.

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PRE-SERVICE TEACHERS' DEFINITIONS OF QUADRILATERALS: CHALLENGING ASSUMPTIONS ABOUT TRANSITION FROM HIGH SCHOOL TO TEACHER PREPARATION

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Keywords: Teacher Education-Preservice, Geometry and Geometrical Spatial Thinking, Learning Theory

Many studies in pre-service teacher education assume that novice pre-service teachers (NPSTs) have conceptions of mathematics similar to those held by elementary or high school students. Constructivist theory (Fosnot & Perry, 1996) suggests that teacher preparation programs might best-support NPSTs' learning by first understanding the specific kinds of mathematical ideas NPSTs hold when beginning their studies in a teacher preparation program. However, relatively little research has been done in this area (Thanheiser, Browning, Edson, Kastberg & Lo, 2013). In this study I explore how NPSTs define various types of quadrilaterals reflecting hierarchical or partitional classifications and what implications these analyses have for mathematics teacher educators (MTEs).

Forty-four NPSTs each provided a definition for rectangle, rhombus, parallelogram, trapezoid, and kite. Each definition was coded into one of three categories. (1) *Partitional* definitions described a relationship among shapes in which no quadrilateral type could be considered a sub-category of another (deVilliers, Grovender and Patterson, 2009). (2) *Hierarchical* definitions did include another class of shapes as a sub-category of the shape being defined (deVilliers, Grovender and Patterson, 2009). (3) *"Not functional"* was used to describe definitions which did not provide an accurate partitional or hierarchical description of the shape.

Analyses revealed forty-four partitional definitions (20%), forty-one hierarchical definitions (19%), and the remaining one hundred thirty-five definitions classified as "not functional" (61%). The large percentage of "not functional" definitions shows that many NPSTs need to review basic quadrilateral types. The findings also suggest that although we often assume NPSTs consider shapes only in a partitional classification (Fujita, 2012), nearly half of all functioning definitions provided were hierarchical in nature. This provides a starting point for MTEs when introducing the hierarchical classification. More detailed results for each shape type can also help MTE's when reviewing types or selecting and sequencing tasks involving hierarchy. There is a need for continued examination of what kinds of conceptual mathematical knowledge NPSTs bring as they traverse the border into teacher preparation programs.

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FACILITATING CLASSROOM DISCOURSE: A PROJECT DESIGNED TO HELP PRESERVICE TEACHERS IMPLEMENT HIGH-COGNITIVE DEMAND TASKS

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Introduction

Standard one of the *Standards for Mathematical Practice* states that students should be able to "make sense of problems and persevere in solving them" and mathematically proficient students "can understand the approaches of others to solving complex problems and identify correspondences between different approaches" (National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010, p. 6). Stein and Smith (1998) argue that the ways in which tasks are actually presented and implemented to students have an important impact on what students actually learn from completing those tasks.

Methodology

In working with our preservice teachers, we have found that many of them struggle to present tasks in ways that promote the type of learning advocated by the Common Core. As a result, we created a class project focused on promoting the use of high-cognitive demand mathematics tasks to facilitate classroom discourse during the preservice teachers' field experiences. The purpose of this qualitative study was to explore the circumstances which impacted the successful implementation of this project.

The Setting and Participants

This research study took place at a four-year university in the Rocky Mountain region. The participants were enrolled in a methods for teaching secondary mathematics course. Six of the eight enrolled students in the class agreed to participate in this study. All of the participants were completing their field experience practicums in either a middle or high school setting.

Data Collection and Analysis

All project documentation was collected from participants; including their anticipated student solutions, lesson plan, video-recorded lesson, and lesson reflection. In addition, a semi-structured, simulated recall interview was conducted with participants shortly after teaching their lesson. An open coding process was used to develop themes across all participants.

Results and Discussion

Data analysis suggested that issues relating to (1) lack of supportive classroom norms, (2) lack of host teacher support and buy-in, and (3) conflicting student teacher beliefs impacted the success of the project. The research team believes that the results of this study can inform how mathematics teacher educators can better support pre-service teachers in the field.

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ELEMENTARY PRESERVICE TEACHERS' PERCEPTIONS OF EXPERIENCING A REFORM-BASED NUMBER CONCEPTS COURSE

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Keywords: Teacher Education-Preservice, Number Concepts and Operations

In line with reform efforts in mathematics education, teacher educators are designing mathematics courses that allow preservice teachers to experience reform-based learning with an emphasis on sense making and conceptual understanding (e.g., Thanheiser, 2015). However, unlike the typical learner in school, elementary preservice teachers bring to their elementary education mathematics courses a sense that they already know what they are being asked to learn (e.g., Thanheiser, Philipp, Fasteen, Strand & Mills, 2013). Familiarity with the concepts being taught may create apathy and hinder preservice teachers from fully benefiting from reform-based courses. To better understand the challenge this may pose, this poster explores the research question: What are elementary preservice teachers' (PSTs) perceptions of experiencing reform-based learning for the first time in an elementary number concepts course?

To answer this question, 18 PSTs at a large Mid-western university were interviewed towards the end of their number concepts course, the first mathematics content course in the elementary teacher education program. The interview included questions about their expectations as they came into the course, their perceptions of their competence at the beginning and end of the course, and how their thinking changed about the content in the course. The interviews were transcribed and read for common themes across the interviews. Themes included discomfort with a new way of thinking, surprise at the number of different ways one can do math, and a new understanding of the meaning behind operations. Once the themes were identified, each interview was coded for evidence of those themes. Analysis of the data revealed that going through the reform-based course was an experience that perturbed their understanding of elementary school mathematics and what they thought they knew about it. In addition, despite their initial reluctance, by the end of the course the majority of the PSTs developed an appreciation for learning mathematics with meaning.

The results of this study highlight the importance of supporting PSTs, as they go through the challenging process of transitioning from the routine learning of their own school experiences to learning mathematics with understanding and provides some insight into ways teacher educators can provide that support.

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INTERPRETING "STANDARD" PROPORTION EQUATIONS IN TERMS OF A QUANTITATIVE DEFINITION FOR MULTIPLICATION

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Keywords: Teacher Knowledge, Teacher Education-Preservice, Algebra and Algebraic Thinking, Number Concepts and Operations

The purpose of this study was to examine how 5 future middle grades mathematics teachers interpret proportion equations in terms of quantities, especially when they were prompted to consider a definition of multiplication. In the present study, we focus on "standard" proportion equations of the form $7/4 = Y/X$ and $X/4 = Y/3$ in which there are exactly two numbers and exactly two letters. Although many studies have demonstrated that both students and teachers formulate standard proportion equations even when such an equation is not normatively correct (e.g., Jacobson & Izsák, 2014), we know of no study that has examined whether students or teachers can interpret these equations in exactly two letters when such an equation does apply.

The present study's framework is based on a quantitative definition of multiplication presented by Beckmann and Izsák (2015), so the quantities in the multiplication equation $A \cdot B = C$ can be interpreted as follows: A is the number of equal-size groups, B is the number of units in 1 or in each group, and C is the number of units in A groups. This definition of multiplication leads to two solutions to problems involving "standard" proportion equations using the variable parts perspective and strip diagrams. Data for this paper come from an ongoing project of future middle grades (grades 4-8) mathematics teachers' ecology of multiplicative reasoning. Beckmann taught a cohort of future teachers for two semesters in 2014-2015. In the first semester, the instruction developed the quantitative definition of multiplication. In the second semester, the future teachers reasoned with the variable parts and multiple batches perspectives on proportional relationships (Beckmann & Izsák, 2015) and developed algebraic equations by reasoning about relationships among quantities. Six future teachers were recruited based on their performance on a fractions survey that focused on multiplication and division with fractions. Data for the present study consist of one-on-one cognitive interviews conducted with 5 future teachers at the end of the second semester.

The main finding we report is that the explicit use of the definition of multiplication facilitated future teachers' interpretation of "standard" proportion equations in terms of quantities. Strip diagrams were found to be useful because they support those teachers' thinking about "how many of these are in that" language, which is critical for interpreting these equations.

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ELEMENTARY MATH PROSPECTIVE TEACHERS AND RELATIONAL EQUITY

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Teacher preparation programs hold a responsibility for supporting prospective teachers (PSTs) to engage in practices that support equity in their future teaching. In particular, relational equity, which means more equitable relationships through mutual respect among students in groups (Boaler, 2008), is one dimension of equity that elementary mathematics PSTs need to make sense of as they learn to teach for mathematical understanding.

Greater attention to relational equity holds promise for supporting PSTs to cross borders of status, race, class, etc. to create more equitable classrooms in their future teaching. Strategies for teacher educators to highlight relational equity can emerge from deeper understandings of how PSTs perceive what students experience in small groups in mathematics classrooms, but researchers have not specifically paid attention to how elementary mathematics PSTs perceive students' experiences in small groups regarding relational equity. The research questions of this project are, therefore, how do elementary mathematics PSTs perceive elementary students' experiences in small groups and how do those perceptions connect to relational equity.

This project took place in a mathematics methods course for elementary PSTs in their senior year in a teacher preparation program of a large Midwestern university. Data sources included two semi-structured interviews with 3 senior PSTs and written artifacts from coursework (e.g., math stories, written lesson plans). The first interview investigated what the PSTs noticed about students' experiences in two episodes of students' inequitable participation from existing equity-related research (Bishop, 2013; Langer-Osuna, 2011). The second interview examined what the participants envisioned in lesson plans they created with respect to students' participation in small groups.

According to data analysis using thematic analysis (Glesne, 2016) to identify emerging themes across interviews and written artifacts and to identify possible implications for the PSTs' preparation to support relational equity, the participants noticed status issues to some extent in student interactions in written cases, and it is uncertain that they envisioned deeply the inner workings of small groups in terms of their lesson plans, and their life experiences in mathematics seemed to influence their perceptions of participation in small groups.

Investigating PSTs' perceptions can help teacher educators delve into ways to support PSTs' understanding of relation equity. Teacher educators in elementary methods courses can build on PSTs' perceptions to encourage, challenge, or disrupt PSTs with regard to elementary students' experiences in small groups.

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PROSPECTIVE TEACHERS' NOTICING OF MATHEMATICS IN PLAY

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The importance of play in early childhood classrooms for learning broadly and in mathematics in particular has been recognized widely and for a long time. For example, in their joint statement the National Council of Teachers of Mathematics and the National Association for the Education of Young Children (2002, p. 3) encouraged teachers to provide "ample

time, materials and teacher support for children to engage in play, in a context in which they explore and manipulate mathematical ideas with keen interest." However, despite this and similar recommendations, prospective early childhood teachers have few opportunities to learn about supporting mathematical engagements during play (Parks & Wager, 2015). The purpose of this study is to investigate what prospective elementary teachers at the student teaching stage notice in videos of mathematical play taken in prekindergarten and kindergarten settings. To guide the study, we used the construct of professional noticing (e.g., Jacobs, Lamb, & Phillip, 2010), which provided a framework for investigating how prospective teachers made sense of play in classrooms. Jacobs, Lamb, and Phillip (2010, p. 169) defined professional noticing as an "expertise" of "interrelated skills including (a) attending to children's strategies, (b) interpreting children's understanding, and (c) deciding how to respond on the basis of children's understanding." Early childhood contexts present special challenges for teachers in noticing mathematical engagements because of the unpredictability of play contexts.

Data collection and analysis for this study are ongoing. To date, 30 elementary student teachers have completed an online, open-ended survey that was distributed through their science education methods course. All of the students had previously completed two mathematics education courses at a large public university. The goal of the study is to have a minimum of 50 completed surveys. The survey contains 12 open-ended questions that ask respondents to write about the mathematical ideas they see in videos or photos of children's play. For example, one question asks respondents to view a video of a child folding doll clothes and to name the mathematical ideas that a teacher could highlight in such play. Data analysis will seek to identify patterns through coding and data matrixes (Miles, Huberman & Saldaña, 2014) with the goal of describing what the respondents attended to in the play and their proposed strategies for responding. Informal analysis of the early results shows that respondents more frequently attended to mathematical topics common in middle and upper elementary grade levels (e.g., fractions) than topics central to early childhood (e.g., composing and decomposing.) The analysis will provide insight into the ways in which an elementary preparation program has and has not supported prospective teachers' abilities to notice mathematics in play.

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UNDERSTANDING TEACHER EFFICACY FOR ELEMENTARY SCHOOL PRESERVICE TEACHERS

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In a distinctly mathematics context (e.g., teachers with a great deal of mathematics content-knowledge teaching secondary school mathematics courses) teacher efficacy often increases significantly and shows significant relationships to other meaningful constructs such as teacher concern and teacher orientation. Considering an often limited mathematics content-knowledge background for elementary school preservice teachers (EPTs), what can be said about their teacher efficacy when they consider their classroom teaching of mathematics?

Teacher efficacy is a belief teachers carry about their abilities to perform in a classroom, that is, as a teacher with students. Teacher concern is a nested construct with self-concerns at the centre, surrounded by task-concerns, and finally encompassed by impact-concerns. Teacher orientation is a conceptualization of preservice teachers' attitudes toward their teaching, in this case, the teaching of mathematics. Preservice teachers may express single or combinations of concerns and/or orientations as they reflect on their preservice learning. (See Pyper, 2014, for a more comprehensive explanation of this conceptual framework.)

The data presented here is the quantitative data set from an ongoing multi-year mixed methods study. Data was collected four times for teacher efficacy and teacher concern, and three times for teacher orientation from two different teacher preparation program iterations in the same university. A small twelve-item scale known to accurately measure teacher efficacy, the Teachers' Sense of Efficacy Scale (TSES) (Tschannen-Moran & Woolfolk Hoy, 2001), served as the basis for the questionnaires EPTs completed at each phase of data collection. The online questionnaire also included two short answer questions, a) about the concerns they had about teaching, and b) about the contributions they felt had an important effect on their teacher efficacy (used for exploring orientation). Teacher concern and teacher orientation data were quantitized from short answer responses into ordinal variables. Descriptively, graphs of teacher efficacy, teacher concern, and teacher orientation show distinct and interesting increases and decreases over time suggesting a developmental pattern to EPT learning. Statistical analyses show significant change in the increase of teacher efficacy over time, significant change for teacher concern over time, and significant change of teacher efficacy over teacher concern.

Collectively, these statistical analyses and descriptive graphs allow for a complex interpretation of the development of EPTs' teacher efficacy, and show promising insight into changes over time in a teacher preparation program. Further replications of this study over future years will add to the data pool and potentially refine the results of the changes in teacher efficacy and its related constructs, and improve our understanding of the changes that occur.

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PROBLEM SOLVING OF PROSPECTIVE TEACHERS

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While several studies have focused on students' problem solving strategies (Hmelo-Silver, 2004), pre-service teachers' conception of problem solving as a part of teaching and the curriculum has received inadequate attention. According to Dooren, Verschaffel, and Onghena (2003), although pre-service teachers can utilize various problem solving strategies, their flexibility with multiple strategies is often limited and does not improve during their traineeship. These researchers also show that pre-service teachers prefer algebraic approaches both in solving problems on their own and in evaluating student's work. This lack of variety in heuristics can hinder a teacher's ability to work with students' problem solving strategies.

This study sought to compare problem solving strategies of prospective mathematics teachers across grade bands, looking at elementary, middle, and secondary teachers focusing on similarities and differences observed while they worked on non-routine mathematics problems. Using Schoenfeld's (1985) conception of problem solving, the resources and heuristics used by teachers were investigated. Also, the impact of control on their problem solving was studied.

For the purpose of this study, task-based interviews (Goldin, 2000) were used to elicit the thinking of PTs. Three PTs from each grade band participated in the study comprised of one video-recorded task-based interview. Using Schoenfeld's (1985) model the data were analyzed at an individual problem level first, then broaden to a grade band view in order to compare across prospective teachers who intend to work with different grade bands. The outcomes of the research indicated that resources that were brought to the problems by all nine PTs were fairly consistent despite their varied mathematical backgrounds, raising questions about mathematical preparation. The use of control appears to be based on a PT's strength of resources and confidence in the problem, and the use of heuristics appeared to be subconscious for all PT''s.

While this shows their strength as problems solvers who are able to use various methods to approach a problem, it raises concern about their ability to help students become better problem solvers. As the design of the interviews did not ask the participants to be explicit about their methods of solving the problem, future studies should look to understand the awareness of PTs about their own problem solving.

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ANALYSIS OF PRE-SERVICE TEACHERS' GENERALIZATION AND JUSTIFYING STRATEGIES IN SOLUTIONS TO PATTERN-GENERALIZATION TASKS

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Findings from research with pre-service teachers (PSTs) document that PSTs have difficulties generalizing and justifying pattern generalizations, particularly for far generalization tasks (Hallagan, Rule, & Carlson, 2009). In order to help their future students learn about, analyze, generalize, and justify pattern generalizations, they need substantial learning to overcome their own difficulties in this area.

We analyzed 37 PSTs' written solutions to four figural pattern generalization tasks, video recordings of class discussions, and audio-recordings of problem-based interviews during which the PSTs were asked to solve one pattern generalization task, to answer the following research questions. (1) *What relationships and structural aspects of a figural pattern do PSTs build upon to formulate pattern generalization?* (2) *How do they utilize uncovered relationships and structural aspects of a figural pattern to justify their general rules?*

The results revealed that PSTs generalized by building on the structural aspects of a figural pattern or by using numerical information they collected about the task. With respect to the structural aspects of a figural pattern, PSTs focused on the changing or invariant pattern characteristics. We classified the generalizations that were developed with attention to the figural pattern structure as (a) holistic analysis of change in pattern structure, (b) decompositions of pattern structure into chunks and analysis of change in structural chunks, (c) holistic analysis of invariant and changing aspects of pattern structure, and (d) decompositions of pattern structure into chunks with a focus on their invariant and changing aspects. At the beginning of the semester, 37% of PSTs generalized patterns by drawing on their understanding of figural pattern structure compared to 67% of PSTs who generalized patterns in this manner at the end of the semester. The higher proportion of PSTs who analyzed pattern structure to develop their rules at the end of the semester was statistically significant at the 0.05 level; $z = 2.602$. Across analyses of PSTs' written solutions to all figural pattern tasks, the PSTs' ability to justify their general rules was highly correlated with their understanding of the pattern structure; $r = 0.766$, $p < 0.01$. This study contributes to the ongoing discussion about ways of supporting grades 1-8 PSTs' ability to generalize and justify.

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A COMPARISON OF INSTRUCTOR AND SECONDARY PRESERVICE TEACHER NOTICINGS USING CONCEPT MAPS

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Concept maps are a visual tool which illustrate connections between ideas around a topic (Novak & Gowin, 1984). In mathematics education, it has been shown that concept maps are an effective way to capture student understanding in ways not possible with more traditional forms of assessment (Williams, 1998). Using concept maps as a formative assessment tool in mathematics education has many opportunities for reflection, both on the part of the learner and on the part of the instructor (Kinchin, 2014). Whether students are reflecting on their own understanding and identifying opportunities for new learning, or instructors are assessing student understanding to inform future coursework, the first step is to determine what connections are being made in the content via what is represented in the concept maps. However, it's not clear that the ways in which students read and reflect on their own understanding as represented in concept maps is the same as the way instructors read and reflect on these same tools.

A concept mapping task was given to fifteen mathematics preservice teachers enrolled in a graduate level Algebra content course for teachers in a university in the northeastern United States, one during the first course meeting at the beginning of the semester, and one during the final course meeting at the end of the semester. Additionally, students were asked to reflect on the differences they noticed in their pre and post concept maps, and the course instructor examined on all students' pairs as well. The reflections of the students and the instructors were then compared for similarities and differences in content and theme. Although our previous quantitative analysis found that the concept maps produced by the preservice teachers increased in both size and complexity over the semester, we found that increases in size did not necessarily appear to correspond to an increase in the depth of student understanding. This highlights the tension between the efficiency of quantitative analysis of concept maps and the limitation of using them for summative assessment. Thus, the power of concept maps lie in their use as a formative assessment tool, focusing attention on the depth and quality of the content and connections within the map.

This poster highlights what students in the class noticed when analyzing their concept maps, what the course instructor noticed, and the differences and similarities between them. This poster adds a unique perspective, combining a metacognitive perspective of the preservice teachers with the formative assessment perspective of the course instructor.

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DESCRIPTION AND INITIAL RESULTS OF THE PRESERVICE TEACHERS SEMINAR "ÜBERPRO_WAHRSCHEINLICHKEITSRECHNUNG" – UNDERSTANDING THE SECONDARY-TERTIARY TRANSITION

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Keywords: Affect, Emotion, Beliefs, and Attitudes, Probability, Reasoning and Proof, Teacher Education-Preservice

The secondary-tertiary transition in STEM Education is a well-known, but still unsolved problem (Holdren & et al., 2012). The proposed poster is focused on the question of how we can improve our understanding of the transition that pre-service teachers experience and support them in challenging it. Additionally, the poster presents initial results based on students' research/learning journals. A theoretical framework based on a network of theories (Bauersfeld, 1983; Schoenfeld, 1985; Gopnik & Meltzoff, 1997; Balzer, Moulines, & Sneed, 1987; Burscheid & Struve, 2010) was constructed for specifying the stated question, and which made it possible to formulate the following hypothesis describing the research perspective on the transition process. The change from an empirical-concrete to a rather formal-abstract belief system of mathematics constitutes a crucial obstacle for the transition from school to university. On epistemological grounds, similar changes regarding different natures of mathematics can be described for the history of mathematics. The explicit analysis of the historical genesis provides support for students dealing with their individual transition processes.

To test this hypothesis I developed an intervention seminar, which took place at University of Siegen in winter 2015-2016 (14 weeks/sessions) with approximately 20 participants, who were just about to finish their bachelor's degree. The main sources I used for making the students aware of different views on mathematics were the "Foundations of Probability" by A.N. Kolmogorov (Kolmogorov, 1973) and the "Lectures on Probability" by R.E. von Mises (Mises, 1945), which are prototypical examples of a formal and an empirical view on mathematics. The methodology for analyzing this seminar is the case study research approach by Stake (Stake, 1995), and is based mainly on the participants' own research/learning journals.

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PRE-SERVICE TEACHERS ENGAGED IN TEAM TEACHING AND COLLECTIVE OBSERVATION USING THE MATHEMATICS QUALITY OF INSTRUCTION

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In this study, we examine pre-service teachers' experiences in their hybrid fieldwork experiences in a Professional Development School (PDS), specifically examining the impact that using the Mathematical Quality of Instruction (MQI; Learning Mathematics for Teaching, 2014) instrument has on their collaborative planning and collective observations. In particular, the authors highlight the nature of the professional learning that takes place from the shared experiences, the reflective dialogue, and making teaching public through team teaching. We explored the following research questions:

1. How did the collaborative planning and collective observation using MQI enhance teacher candidates planning and teaching?
2. What were the affordances of a Professional Learning Team model in a Pre-service teacher education methods course?

The recurring themes included: a) Learning from one another and finding instances of Mathematics Quality of Instruction in their peer teaching. Through the co-teaching activity, our teacher candidates had the opportunities to watch their colleague respond to the teaching episode "live". By having the group lessons videotaped and on Edthena, pre-service teachers, in essence, had the ability to "rewind the lesson". In her interview, Katie commented on her colleague's questioning technique; b) Having to negotiate and being challenged during planning with diverse perspectives that improved their overall lesson; c) Supporting one another when faced with uncertainty and having pre-established trust.

The teaching profession has become more collaborative recently with the Professional Learning Community movement (DuFour, DuFour, Eaker, and Many, 2006). DuFour (2004) points out that "educators who are building a professional learning community recognize that they must work together to achieve their collective purpose of learning for all. Therefore, they create structures to promote a collaborative culture" (p. 9). We learned through this team teaching assignment that pre-service teachers also created a professional learning community that allowed for them to recognize that they needed to work together to achieve their collective purpose of teaching and impacting student learning. With many schools adopting PLC as structures for professional learning among peers, teacher education programs and educators working with pre-service teacher education courses should consider incorporating as many collaborative job-embedded activities and experiences for our pre-service teachers. In doing so, we believe our pre-service teachers will be better equipped with the skills and dispositions to work collaboratively in professional learning communities.

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PRE-SERVICE TEACHERS' CULTURAL DIVERSITY KNOWLEDGE BASE AND CURRICULAR ADAPTATIONS

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Keywords: Curriculum, Teacher Education-Preservice, Equity and Diversity

Theoretical Framing and Purpose

Gay (2000; 2002) proposed a culturally responsive teaching model that included what teachers should know about students' culture—their *cultural diversity knowledge base* (CDKB)—and how teachers should use their knowledge of culture to identify or create culturally responsive curriculum that promotes academic excellence for students. We addressed these research questions:

1. What do pre-service teachers (PSTs) report about their CDKB, in general, and in relation to mathematics?
2. How do PSTs attempt to adapt mathematics instructional materials so that they are culturally responsive to African American or Latin@ students?
3. What evidence is there for a relationship between PSTs' CDKB in mathematics and ability to adapt algebraic word problems for culturally relevance?

Methods

We administered an online survey to PSTs at a large Midwestern university who were at least halfway through a mathematics teacher development program, including having completed an algebraic thinking course. PSTs reported demographic characteristics, community preference for teaching, and their number of opportunities to develop their CDKB such as classes, literature, and conversations about race with minority stakeholders. Then, PSTs were asked to adapt an algebraic word problem for cultural relevance to African American and for Latin@ students.

Results

Of the 15 PSTs who opened the survey, only 7 provided responses. All respondents were European American and six were female. There was no clear relationship between PSTs' reported CDKB and their willingness or ability to adapt the word problems for cultural relevance to African American or for Latin@ students. There were only two PSTs who attempted the word problem adaptation tasks. Both Abby, who was the only respondent who reported having at least one conversation with an African American or Latin@ student about the role of race in mathematics education, and Edward, who was the only student who expressed a desire to teach in "a school of need or urban environment," created word problems that were culturally unresponsive, according to Gay's (2002) model. Both PSTs stated or otherwise indicated a colorblind approach in which curriculum is perceived to be culture neutral.

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Chapter 9

Statistics and Probability

Research Reports

Brief Research Reports

Posters

RAZONAMIENTO DE ESTUDIANTES UNIVERSITARIOS SOBRE VARIABILIDAD E INTERVALOS DE CONFIANZA EN UN CONTEXTO INFERENCIAL INFORMAL

UNIVERSITY STUDENTS´ REASONING ON VARIABILITY AND CONFIDENCE INTERVALS IN INFERENTIAL INFORMAL CONTEXT

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En este artículo se presentan resultados de una investigación con una metodología de tipo cualitativo con un grupo de 15 estudiantes universitarios de ciencias sociales, sobre el razonamiento inferencial informal que desarrollaron en un ambiente computacional sobre conceptos que intervienen en los intervalos de confianza. Los resultados señalan que los estudiantes desarrollaron un razonamiento correcto sobre la variabilidad del muestreo y lograron visualizar intervalos razonables de variabilidad en un muestreo repetido, a su vez identificaron relaciones correctas entre el tamaño de muestra y la confiabilidad en la amplitud del intervalo y el margen de error, e identificaron la aleatoriedad de un intervalo de confianza. Sin embargo, tuvieron dificultades para conceptualizar la confiabilidad como el porcentaje de intervalos que capturan al parámetro en un muestreo repetido en condiciones idénticas.

Palabras clave: Análisis de Datos y Estadística, Tecnología, Modelación

Introducción

Hacer inferencias sobre poblaciones e interpretar resultados de estudios estadísticos se ha vuelto parte de la vida profesional y cotidiana de las personas. Un ejemplo concreto muy recurrente son las encuestas de opinión que aparecen casi a diario en los medios de comunicación, en las cuales se reportan estimaciones sobre parámetros de una población, margen de error y confiabilidad, entre otros conceptos. La investigación reporta que los conceptos y el razonamiento que caracteriza a la inferencia estadística son complejos para la mayoría de los estudiantes (Castro Sotos, Vanhoof, Noortgate, & Onghena, 2007), incluso para profesores e investigadores que la aplican en su profesión (Liu & Thompson, 2004).

En el caso particular de la estimación de parámetros mediante intervalos de confianza, la literatura reporta diversas dificultades de comprensión y errores en la interpretación de resultados. Por ejemplo, un error muy persistente consiste en considerar que un intervalo de 95% de confianza indica que existe un 95% de probabilidad de que el parámetro poblacional se encuentre entre los límites del intervalo. Otros errores consisten en no reconocer la aleatoriedad y naturaleza inferencial del intervalo e ignorar el efecto del tamaño de muestra y la confiabilidad en la amplitud del intervalo y el margen de error; creer que en distintas muestras se obtendrá el mismo intervalo, entre otros (Olivo & Batanero, 2007).

Entre las principales causas que se ofrecen como explicación de la complejidad de la inferencia estadística y el razonamiento a partir de muestras, destacan la multiplicidad de conceptos abstractos que se entrelazan en una inferencia (Chance, delMas & Garfield, 2004; Pfannkuch, Wild & Parsonage, 2012); el enfoque formal deductivo a través del cual se ha abordado la enseñanza de la inferencia (Lipson, 2002); y la dificultad para ver las muestras y cálculo de estadísticos como eventos estocásticos, que en un muestreo repetido presentan una distribución que revela información importante para hacer la estimación de un parámetro (Saldanha & Thompson, 2014). Un acuerdo generalizado entre investigadores, orienta a reemplazar o complementar el enfoque formal por un enfoque más conceptual y más accesible que brinde oportunidad a los estudiantes de comprender las grandes ideas que subyacen a la inferencia estadística (Cobb & Moore, 1997; Wild, Pfannkuch &

Reagan, 2011); este enfoque es conocido como *inferencia estadística informal,* y el razonamiento que lo caracteriza como *razonamiento inferencial informal.* Entre sus objetivos está generar comprensión de los conceptos de la inferencia sin depender de los métodos formales basados en la teoría estadística y la probabilidad.

El avance de las tecnologías digitales proporciona grandes posibilidades para generar este cambio de enfoque en el estudio de la inferencia estadística, dado el carácter dinámico, interactividad, múltiples representaciones y capacidad de simulación que caracterizan a algunas tecnologías educativas, lo cual les confiere un potencial cognitivo que permite visualizar e interactuar con las representaciones de los datos, el proceso de muestreo, el cálculo de estadísticos y su distribución muestral; objetos matemáticos complejos a partir de los cuales se construyen los intervalos de confianza, el margen de error y la confiabilidad. En este contexto, nos hemos propuesto analizar el razonamiento inferencial informal que desarrollan estudiantes universitarios de ciencias sociales sobre la variabilidad y los intervalos de confianza en un ambiente computacional como que el provee el software TinkerPlots (Konold & Miller, 2011). En específico, nos interesa investigar si los estudiantes identifican relaciones correctas sobre el muestreo, el efecto de tamaño de muestra y la confiabilidad en los intervalos de confianza, y si logran interpretar correctamente el margen de error y la confiabilidad en una estimación.

Marco conceptual

Una inferencia estadística es una aseveración sobre una población, la cual es generada a partir de una sola muestra y con un nivel explícito de confianza. El razonamiento inferencial informal involucra ideas y relaciones como centralidad, variabilidad, tamaño de muestra y control de sesgo (Rubin, Hammerman & Konold, 2006), y se define como la habilidad para interconectar ideas de distribución, muestreo y centralidad, dentro de un ciclo de razonamiento empírico (Pfannkuch, 2006). Zieffler, Garfield, delMas y Reading (2008) lo definen como la forma en la que los estudiantes usan su conocimiento estadístico informal para hacer argumentos para apoyar inferencias acerca de poblaciones basándose en muestras. Makar, Bakker y Ben-Zvi (2011) identifican una serie de elementos clave interrelacionados que son necesarios para apoyar el razonamiento inferencial informal, como son: el conocimiento estadístico, el conocimiento del contexto del problema, normas y hábitos desarrollados con el tiempo y ambientes de aprendizaje basados en cuestionamientos e investigación.

En el contexto de los intervalos de confianza, Pfannkuch, Wild y Parsonage (2012) proponen una ruta conceptual para desarrollar la idea intervalo de confianza desde una perspectiva informal utilizando técnicas de simulación, y definen una comprensión estocástica de los intervalos de confianza como un proceso que contempla las siguientes etapas:

- Concebir un proceso de muestreo aleatorio como la selección de una cantidad de elementos de una población y el registro de cada dato de los elementos seleccionados, para después calcular un estadístico de la muestra (por ejemplo la media o mediana) y estimar el parámetro de la población.
- Imaginar repetidamente la selección de muestras de un tamaño dado y determinar si el intervalo de confianza calculado de la muestra, "captura" el valor del parámetro.
- Comprender que este proceso producirá una colección de resultados de la forma "captura" o "no captura" el verdadero valor del parámetro.
- Comprender que en el muestreo aleatorio existe variabilidad en los resultados, pero conforme se incrementa el tamaño de la muestra, la distribución de resultados adquiere una forma más estable y centra en el verdadero valor del parámetro.
- La proporción del resultado "captura" en una larga corrida es el nivel de confianza asociado al método.

Metodología

La investigación se llevó a cabo con 15 estudiantes de ciencias sociales que tomaban un curso básico de probabilidad y estadística. Los estudiantes tenían pocos antecedentes matemáticos en la materia, por lo que decidimos enfocar el curso hacia la modelación y simulación de eventos aleatorios y muestreo de poblaciones utilizando el ambiente computacional que proporciona el software TinkerPlots, tomando como referencia contextos reales de estudios de opinión en el área de ciencias sociales publicados por empresas encuestadoras mexicanas. Por cuestiones de espacio en el presente trabajo se analizan y discuten resultados de una de las últimas actividades del curso. Como instrumentos de recolección de información se utilizaron hojas de trabajo para cada actividad, archivos del software y entrevistas con algunos estudiantes.

El software TinkerPlots permite el análisis y visualización de datos en forma dinámica e interactiva, con un gran potencial para la modelación y simulación de eventos aleatorios, como es el caso del muestreo. Para el caso específico de los intervalos de confianza, el software dispone de una herramienta de modelación conocida como "Sampler", en la cual, a través de mecanismos aleatorios (ruletas, urnas, diagramas de barras) los usuarios generan el modelo de una población y sus parámetros; posteriormente extraen una gran cantidad de muestras y visualizan el proceso de muestreo y cálculo de estadísticos conforme éste se desarrolla, para generar la distribución muestral del estadístico en cuestión, en forma gráfica o tabular. Las actividades de enseñanza se diseñaron con el propósito de desarrollar en los estudiantes una concepción estocástica de los intervalos de confianza y desarrollar un razonamiento informal adecuado sobre los conceptos como el muestreo, tamaño de muestra, distribución muestral, confiabilidad y margen de error.

Resultados y discusión

Para el diseño de la actividad y el análisis de los resultados hemos tenido en cuenta las etapas y procesos definidos por Pfannkuch, Wild & Parsonage (2012) para desarrollar una comprensión estocástica de los intervalos de confianza.

Actividad:

El tema de la legalización del consumo de mariguana en México ha generado opiniones contrarias en sectores de la sociedad. La empresa Parametría realizó una encuesta para estimar la opinión de los mexicanos (http://www.parametria.com.mx/carta_parametrica.php?cp=4816). Utilizó una muestra aleatoria de 800 personas mayores de edad y reporta una confiabilidad de 95% y una margen de error de $\pm 3.5\%$ en los resultados.

Tabla 1: Resultados de la encuesta

Opinión sobre la legalización de la marihuana	Porcentaje
A favor de	20%
En contra	**77%**
No sabe aún	3%

Considera los resultados anteriores como los parámetros de la población objetivo, en particular considera la proporción de mexicanos que están en contra de la legalización de la marihuana, esto es, P=0.77.

Población, muestreo y variabilidad muestral

El punto de partida consiste en formular el modelo de la población, para después extraer muestras aleatorias y explorar la relación entre los resultados muestrales con los parámetros de la población. Los estudiantes formularon el modelo con facilidad, como resultado de su experiencia en los temas

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de probabilidad previamente vistos (ver figura 1).
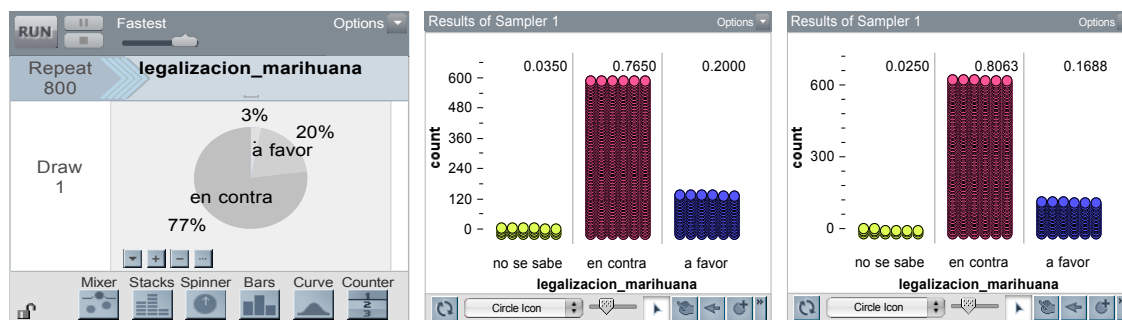
Figura 1. Modelo poblacional y dos muestras aleatorias (n=800) extraídas de la población.

La comparación de los resultados de varias muestras con los valores poblacionales permitió a los estudiantes identificar la variabilidad como una característica intrínseca del muestreo y formular un intervalo intuitivo razonable de resultados esperados. Como evidencia se presentan las respuestas que María José (MJ) y Andrea (A) proporcionaron al investigador (R) en una entrevista:

R: Una vez que construiste el modelo de la población, en la primera muestra de 800 personas [tal como lo hizo la encuestadora] obtuviste un proporción en contra de la legalización de 0.76, ¿te pareció razonable el resultado?

MJ: Sí, me parece que no varía mucho, si tomamos en cuenta que el valor verdadero de la proporción es 0.77.

R: Si repites el muestreo, ¿esperas tener resultados iguales o diferentes?

MJ: Espero resultados diferentes, pero no muy alejados de 0.77.

R: ¿Un intervalo razonable en el cual se esperas los resultados de las muestras?

MJ: Como mínimo 0.74 y como máximo 0.80.

R: ¿Por qué lo consideras así?

MJ: Porque el margen de error que proporciona la encuestadora es del 3.5%, entonces podemos tomar el 0.77 como punto medio y sumar y restar el margen de error.

R: Si en lugar de 800 personas en la encuesta se hubieran utilizado 1500 personas, ¿crees que hubiera resultado el mismo intervalo?

MJ: Los porcentajes se elevarían, el 0.77 quizá sería más grande porque la muestra es más amplia, pero también podría bajar porque se está preguntando a más personas.

R: ¿El intervalo entre 0.74 y 0.80, sería el mismo?

MJ: El margen de error disminuiría.

R: ¿Los resultados que obtuviste en las tres muestras te parecen razonables?

A: Sí, porque el valor verdadero es 0.77, no varían mucho del parámetro.

R: ¿Podrías establecer un intervalo razonable de variación para los resultados muestrales?

A: Del 0.75 al 0.79.

R: ¿En qué te basaste para establecer el intervalo?

A: Consideré que no puede ser un margen de error tan grande, si el parámetro es 0.77.

R: Por ejemplo, sin en lugar de 800 personas se hubieran encuestado 1500, ¿que pasaría con el intervalo?

A: Sería más estrecho.

Las respuestas de Andrea (A) y María José (MJ) muestran que tienen una idea correcta de la variabilidad muestral alrededor del parámetro, y que ésta disminuye conforme se incrementa el tamaño de la muestra. Construyen un razonable intervalo intuitivo de variación de los resultados muestrales esperados. En el caso de María José relaciona el intervalo esperado con el margen de error

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de la encuesta de manera correcta, lo cual significa que tienen un idea de intervalo formado por el estimador y la suma y resta del margen de error. Ante la pregunta sobre incrementar el tamaño de muestra, no tiene claro el efecto que tendría en la estimación, pues le atribuye mayor variabilidad, cuando en realidad muestras más grandes deben parecerse más a la población.

Distribución muestral, confiabilidad y margen de error

TinkerPlots permite visualizar el muestreo como un proceso repetible, calcular el estadístico en cada muestra y acumular los resultados en una tabla que posteriormente puede ser graficada; es decir, genera la distribución muestral para una cierta cantidad de muestras (ver figura 2).
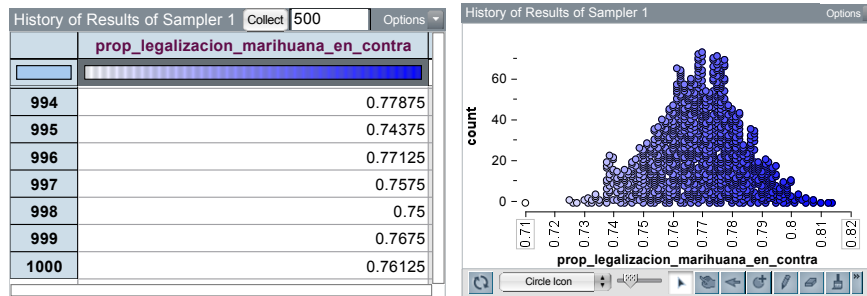
Figura 2. Distribución muestral para 500 muestras de tamaño 800 (P=0.77).

Los estudiantes seleccionaron muestras de tamaño 800 (como lo hizo la empresa encuestadora), y muestras de tamaño 300 (otra posible empresa), con el fin de comparar las distribuciones muestrales y ver el efecto del tamaño de muestra. Se agregaron bandas que sombrean una parte de la distribución muestral y que hacen el papel de intervalos gráficos capturando 90% y 95% de las muestras respectivamente (ver tabla 2).

Tabla 2: Comparación de distribuciones muestrales (n=800 y n=300) y niveles de confiabilidad (90% y 95%)

n	Confiabilidad 90%	Confiabilidad 95%
800		
300		
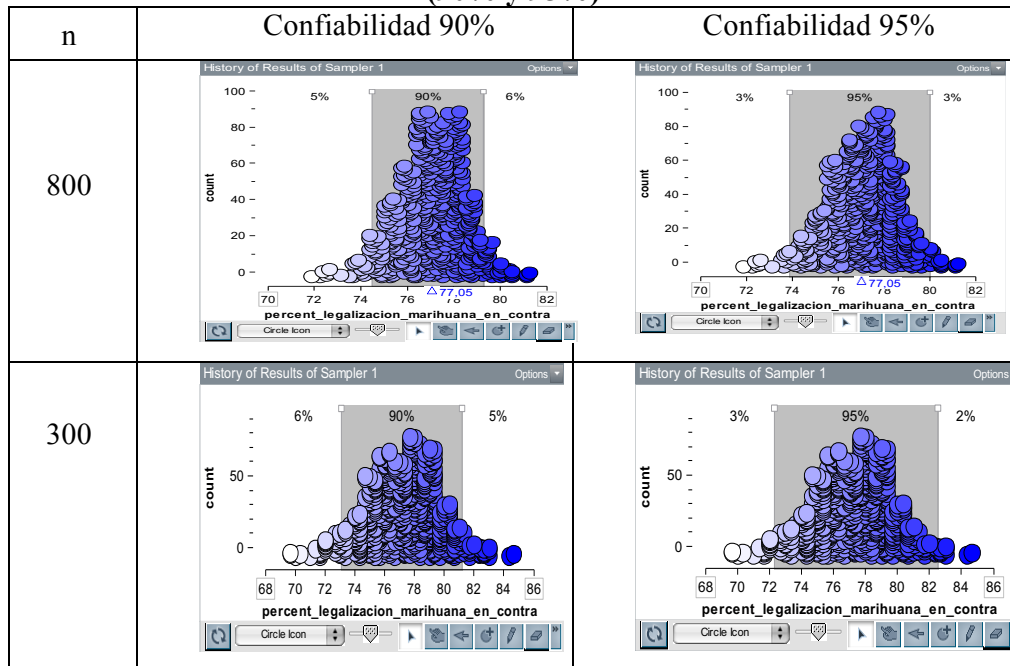
La comparación de distribuciones para cada confiabilidad y tamaño de muestra, permitió a los

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estudiantes identificar algunas relaciones importantes como se muestra en las respuestas que dieron Perla y Katya en la hoja de trabajo, y Andrea (A) y Anaid (AN) en la entrevista:

"Entre más grande es la confiabilidad los intervalos serán más grandes". Perla

"El comportamiento del tamaño de la muestra en relación con la amplitud de los intervalos es a la inversa que con la confiabilidad. La distribución de las muestras de 800 es más angosta que la distribución de muestras tamaño 300". Katya

R: ¿Qué efecto tiene el incrementar la confianza en el ancho del intervalo?
A: Entre menor es la confiabilidad se hace más estrecho el intervalo.
R: ¿Qué ventajas crees que tendría un estudio con una confiabilidad alta?
A: Que cierto porcentaje de la población muy probablemente cae dentro de ese intervalo. Pero si el intervalo es muy amplio puede que no sea muy útil, ya que intervalos grandes son más confiables pero menos precisos.
R: ¿Que se puede hacer para aumentar la precisión?
A: El tamaño de muestra se debe aumentar.
R: ¿Qué nivel de confianza preferirías en un estudio: 90% o 95%?
AN: Es mejor el de 90% porque el margen de error es mas pequeño.
R: ¿Acaso no es mas confiable uno de 95%?
AN: En el de 90 tienes menos posibilidades que caiga dentro y el 95 es más grande y tiene mas posibilidades.
R: ¿Qué pasa con el aumento del tamaño de la muestra en el intervalo?
AN: El ancho del intervalo aumenta al bajar el tamaño de muestra.

Las respuestas de las estudiantes muestran que han identificado correctamente el efecto de la confiabilidad y el tamaño de muestra en el ancho de un intervalo. Sin embargo, el significado de confiabilidad para Andrea es erróneo, al considerar que representa un porcentaje de la población que caerá dentro del intervalo, una concepción muy persistente ya documentada en otros estudios (Olivo & Batanero, 2007).

Otra idea importante que nos propusimos explorar es la aleatoriedad de un intervalo, esto significa de una muestra a otra los limites y el ancho del intervalo pueden cambiar. Para esta parte de la actividad nos propusimos desarrollar en la hoja de cálculo de TinkerPlots los cálculos que se involucran en un intervalo de confianza para una confiabilidad de 90% y 95% respectivamente y repetir la simulación para 500 o 1000 muestras (ver figura 3).

	prop_legaliza...	Error_estandar	margen_de_error	limite_inferior	limite_superior	resultado
478	0.7375	0.0155561	0.0311122	0.706388	0.768612	Cae fuera
479	0.7725	0.0148216	0.0296432	0.742857	0.802143	Cae dentro
480	0.765	0.0149906	0.0299812	0.735019	0.794981	Cae dentro
481	0.77	0.0148787	0.0297574	0.740243	0.799757	Cae dentro
482	0.75375	0.015232	0.030464	0.723286	0.784214	Cae dentro

History of Results of Sampler 1 — Collect 500 — Options

Figura 3. Hoja de cálculo con los elementos de un intervalo de confianza (500 intervalos generados).

A continuación se muestran las respuestas de Andrea (A) y Anaid (AN):

R: ¿Al repetir el muestreo esperas que salga el mismo intervalo?
AN: No, porque las muestras son aleatorias.
R: ¿Varían todos los elementos de un intervalo?
AN: Si porque cada vez que lo corres da un P diferente, aunque cercano.

R: ¿Que relación crees que tiene el 97% de resultados capturados con la confiabilidad? (ver tabla 3).

AN: La confiabilidad es del 95% y ese es muy cercana a 97%, puede incluso ser igual.

R: El 96% de los intervalos que simulaste cayeron dentro y un 4% cayó fuera, ¿tiene alguna relación con la confiabilidad de 95%?

A: Si recuerdo que los resultados que caen fuera el intervalo son lo no contienen al parámetro

R: ¿Eso podría suceder en una encuesta real?

A: Si

R: ¿Lo tomarías como un error de la encuestadora?

A: No, como algo que pasa por el azar, y que sucede con poca frecuencia.

R: ¿En una distribución muestral esos valores donde los ubicarías?

A: En los extremos de la distribución.

R: Viendo estos resultados, ¿qué significa la confiabilidad?

A: Es el porcentaje que un encuestador puede decir que sus muestras contienen al parámetro, que son verdaderas.

Tabla 3: Gráficas con porcentajes de intervalos que capturan y no capturan al parámetro para los niveles de confiabilidad 90% y 95%
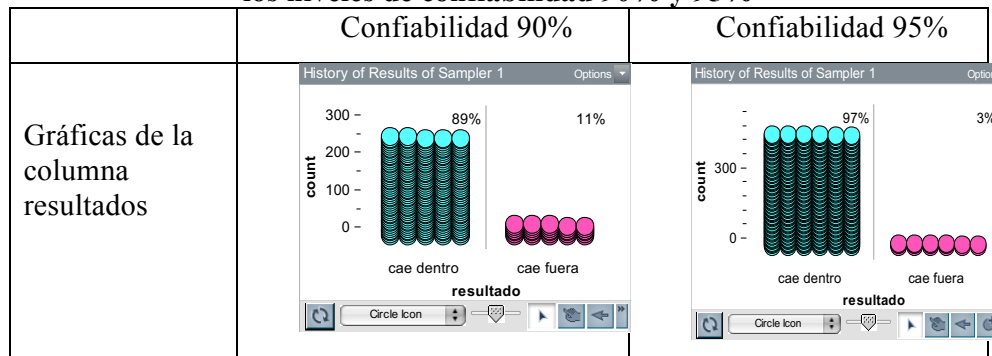
Las respuestas de Anaid y Andrea señalan que tienen claro la aleatoriedad de un intervalo, porque depende de los resultados variables de una muestra. No logran establecer un significado correcto sobre la confiabilidad pese a que visualizan en la gráfica los porcentajes de muestras que capturan y no capturan al parámetro, respectivamente. Sin embargo, cabe resaltar que Andrea está consciente que en una encuesta real se pueden presentar intervalos que no capturan al parámetro, los considera poco frecuentes y los ubica correctamente en las colas de una distribución muestral.

Conclusiones

Los resultados señalan que los estudiantes razonaron correctamente sobre algunos conceptos que integran una comprensión estocástica de los intervalos de confianza definidos por Pfannkuch, Wild & Parsonage (2012), tales como la relación entre el tamaño de muestra y la variabilidad muestral, el efecto del nivel de confiabilidad y el tamaño de muestra en el ancho de un intervalo de confianza. Lograron identificaron intervalos razonables de los resultados esperados en la muestra e identificaron además, el carácter aleatorio de un intervalo, conceptos que se reportan como complejos por investigaciones previas. Sin embargo la confiabilidad resultó ser un concepto muy difícil para todos los estudiantes, y no lograron conceptualizarlo correctamente, aún cuando las actividades enfatizaron en la repetición de muestras para visualizar el porcentaje de intervalos que capturan al parámetro y relacionarlo con la confiabilidad previamente establecida. El ambiente computacional como el que proporciona TinkerPlots en complemento con actividades que promueven la relación explícita entre los conceptos que intervienen en un intervalo de confianza parecen ser adecuados para el diseño de

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trayectorias de aprendizaje que promueven un razonamiento inferencial informal correcto en los estudiantes.

This article presents the results of a qualitative research with a group of 15 university students of social sciences on informal inferential reasoning developed in a computer environment on concepts involved in the confidence intervals. The results indicate that students developed a correct reasoning about sampling variability and visualized reasonable intervals of variability in a repeated sampling, at the same time students identified correct relationships between sample size and confidence level in the width of an interval and margin of error, and identified the randomness of a confidence interval. However, they had difficulties conceptualizing the confidence level as the percentage of intervals that capture the parameter in a sampling repeated under identical conditions.

Keywords: Data Analysis and Statistics, Technology, Modeling

Introduction

Making inferences about populations and interpreting results of statistical studies has become part of people´s professional and daily lives. A concrete and very recurrent example is the opinion polls that appear almost daily in the media, in which estimates of population parameters, margin of error and confidence level, among other concepts are reported. The research reports that the concepts and reasoning that characterizes the statistical inference are complex for most students (Castro Sotos, Vanhoof, Noortgate, & Onghena, 2007), even for teachers and researchers who apply it in their profession (Liu & Thompson, 2004).

In the particular case of parameter estimation by confidence intervals, the literature reports diverse difficulties of understanding and errors in the interpretation of results. For example, a persistent error is to consider that a 95% confidence level indicates a 95% probability that the population parameter is between the limits of the interval. Other errors are not recognizing the randomness and inferential nature of the interval, and ignoring the effect of sample size and the confidence level in the width of the interval and the margin of error; believing that in different samples, the same interval will be obtained, among others (Olivo & Batanero, 2007).

Among the main causes offered as an explanation of the complexity of statistical inference and reasoning from samples, those that stand out are the multiplicity of abstract concepts that are intertwined in an inference (Chance, delMas, & Garfield, 2004; Pfannkuch, Wild, & Parsonage, 2012); deductive formal approach used to teaching inference (Lipson, 2002); and the difficulty to recognize samples and computation of statistics as stochastic events, which in a repeated sampling show a distribution that reveals important information for estimating a parameter (Saldanha & Thompson, 2014). A generalized agreement among researchers, aims to replace or supplement the formal approach with a more conceptual and more accessible approach that provides opportunity for students to understand the great ideas behind statistical inference (Cobb & Moore, 1997; Wild, Pfannkuch, & Reagan, 2011); this approach is known as an informal statistical inference, and reasoning that characterizes, informal inferential reasoning. One of its objectives it is to generate understanding of the inference concepts without relying on formal methods based on statistical theory and probability.

Advances in digital technologies provide great potential to generate this change of approach to the study of statistical inference, given the dynamic properties, interactivity, multiple representations and simulation capabilities that characterize some educational technologies. These technological advances allow for a cognitive potential to visualize and interact with representations of the data, the sampling process, the calculation of statistics measures and their sampling distribution; complex mathematical objects from which confidence intervals, the margin of error and confidence level are

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built. In this context, we intend to analyze the informal inferential reasoning that university students of social sciences develop on the variability and confidence intervals in a computer environment such as that provided by the TinkerPlots software (Konold & Miller, 2011). Specifically, we want to investigate whether students identify correct relations on sampling, the effect of sample size and confidence level in the confidence intervals, and whether or not students correctly interpret the error and confidence level in the estimation of a parameter.

Conceptual framework

A statistical inference, is a statement about a population, which is generated from a single sample and with an explicit confidence level. The informal inferential reasoning involves ideas and relationships as center, variability, sample size, and control of bias (Rubin, Hammerman & Konold, 2006), and is defined as the ability to interconnect ideas of distribution, sampling and centrality, within a cycle of empirical reasoning (Pfannkuch, 2006). Zieffler, Garfield, delMas, and Reading (2008) define it as the way that students use their informal statistical knowledge to make arguments to support inferences about populations based on samples. Makar, Bakker, and Ben-Zvi (2011) identify a number of interrelated key elements that are needed to support the informal inferential reasoning, such as: statistical knowledge, knowledge of the context, rules and habits developed over time and learning environments based on questions and research.

In the context of the confidence intervals, Pfannkuch, Wild, and Parsonage (2012) propose a conceptual pathway to develop the idea of confidence interval from an informal approach using simulation techniques, and define a stochastic understanding of confidence intervals as a process that includes the following steps:

- Conceiving a random sampling process as selecting a number of elements of a population, and recording each data of the selected elements, then calculating a statistics measure (e.g. the mean or median) to estimate the population parameter.
- Imagining repeatedly taking samples of a given size and determinate whether or not the confidence interval calculated for each sample, "covers" the parameter value.
- Understanding that this process will produce a collection of outcomes that would either "cover" or "not cover" the true parameter value.
- Understanding that because of the random selection process there is variability in the outcomes, but as the sample size increases, the distribution of outcomes becomes stable and centered at the true parameter value.
- The long run proportion of ''covers'' is the confidence level associated with the method.

Methodology

The research was conducted with 15 social science students taking an introductory course in probability and statistics. The students had little mathematical background in the subject, therefore we decided to focus the course on the modeling and simulation of random events and population sampling using the computer environment that provides TinkerPlots software, using real contexts of opinion polls in the area of social sciences published by Mexican pollsters. Due to space limitation, in this paper we analyze and discuss results from the last course activities. The data collection instruments used were worksheets for each activity, computer activity files, and interviews with some students.

TinkerPlots software enables analysis and visualization of data in dynamic and interactive way, with great potential for modeling and simulation of random events, such as sampling. For the specific case of confidence intervals, the software has a modeling tool known as "Sampler," which, through random mechanisms (spinners, urns, bar charts), generates the model of a population and its parameters. Following, users select a large amount of samples and visualize the sampling process and

statistics measures calculation (e.g. mean, median) as they develop, to generate the sampling distribution of the statistic in question, in graphical or tabular form. The teaching activities were designed with the purpose of developing in students a stochastic conception of confidence intervals and developing an appropriate informal reasoning on concepts such as sampling, sample size, sampling distribution, confidence level, and margin of error.

Results and Discussion

For the design of the activity and the analysis of the results, we have taken into account the stages and processes defined by Pfannkuch, Wild, and Parsonage (2012) to develop a stochastic understanding of confidence intervals.

Activity

The topic of legalization of marijuana in Mexico has generated conflicting opinions in diverse sectors of society. Parametría Company conducted a survey to estimate the opinion of Mexicans (http://www.parametria.com.mx/carta_parametrica.php?cp=4816). It used a random sample of 800 people and reported a confidence level of 95% and a margin of error of ± 3.5% in the results.

Table 1: Survey results

Opinion on the legalization of marijuana	Percent
Agree	20%
Against	**77%**
Do not know yet	3%

Consider the above results as the parameters of the target population, particularly considering the proportion of Mexicans who are against the legalization of marijuana, that is, $P = 0.77$.

Population, sampling and sampling variability

The starting point consists in formulating the population model with the purpose of taking random samples and exploring the relationship between the sample results with the population parameters. Students formulated the model easily due to their experience on the issues of probability previously studied (see Figure 1).
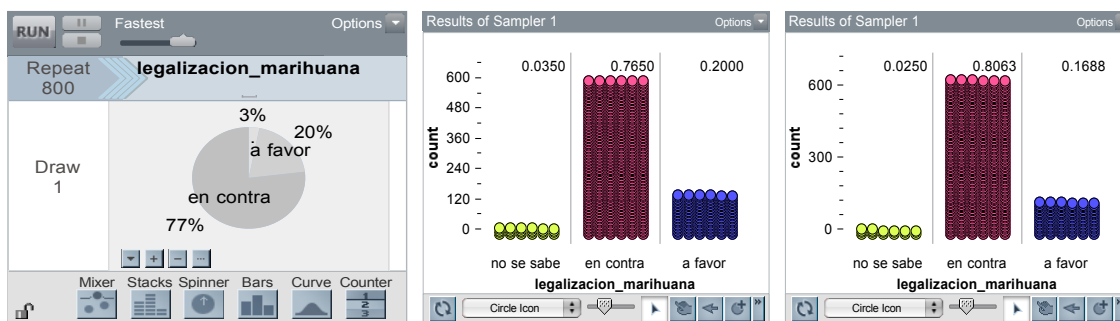
Figure 1. Population model and two random samples (n = 800) drawn from the population.

Comparing the results of several samples with population values, allowed students to identify the variability as an intrinsic feature of the sampling and formulate an intuitive and reasonable interval of expected results. As evidence we present the answers that Maria Jose (MJ) and Andrea (A) provided to the researcher (R) in an interview:

R: Once you built the population model, in the first sample of 800 people [as the pollster did] you obtained a proportion against the legalization of 0.76. Did the result seem reasonable to you?

MJ: Yes, I think that it is does not vary much, if we consider that the true value of the proportion against is 0.77.

R: If the sampling is repeated, do you expect to have the same or different results?

MJ: I expect different results, but not far from 0.77.

R: A reasonable interval in which you expect the results of the samples?

MJ: 0.74 minimum and maximum 0.80.

R: Why do you think so?

MJ: Because the margin of error provided by the pollster is 3.5%, then we can take 0.77 as the midpoint and add and subtract the margin of error.

R: If instead of 800 people surveyed, 1500 people had been used in the survey, do you think that would have been the same interval?

MJ: The percentages would rise, perhaps 0.77 would be bigger because the sample is wider, but it could also lower because more people are been surveyed.

R: The interval between 0.74 and 0.80, would be the same?

MJ: Decrease the margin of error

R: Do the results you obtained in the three samples seem reasonable to you?

A: Yes, because the true value is 0.77, it does not vary much from the parameter.

R: Could you establish a reasonable range of variation for sample results?

A: From 0.75 to 0.79.

R: On what base did you set the interval?

A: I considered that it may not be so great a margin of error, if the parameter is 0.77.

R: For example, if instead of 800 people they had surveyed 1500, what would happen to the interval?

A: It would be narrower.

The responses of Andrea (A) and Maria Jose (MJ) show that they have a correct idea of the sampling variability around the parameter, and it decreases as the sample size increases. They built an intuitive and reasonable interval of variation of the expected sample results. In the case of Maria Jose, she relates the expected interval with the margin of error in the survey correctly, which means they have an idea of interval formed by the estimator, adding and subtracting the margin of error. When asked about increasing the sample size, she is not clear on the effect of the estimate, because she attributed a greater variability, when in fact, larger samples should be closer to the population.

Sampling distribution, reliability and margin of error

TinkerPlots displays the sampling as a repeatable process, calculate a statistics measure in each sample and accumulate the results in a table that one can then graph; that is, it generates the sampling distribution for a certain amount of samples (see Figure 2).
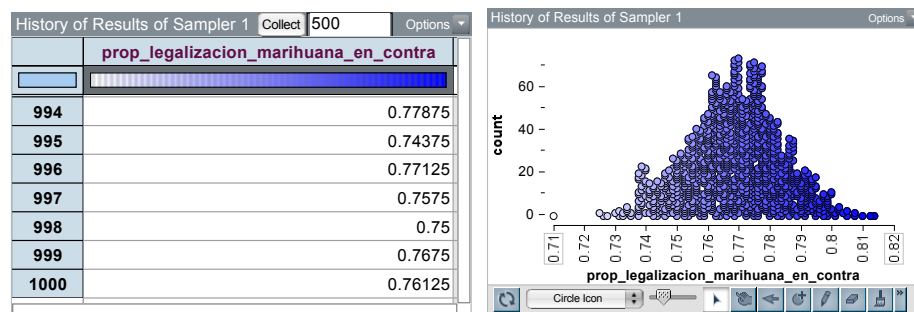
Figure 2. Sampling distribution for 500 samples of size 800 (P = 0.77).

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Students selected samples size of 800 (as the pollster did), and samples size of 300 (another possible company), in order to compare the sampling distributions and see the effect of sample size. Shading bands were added to a portion of the sampling distribution that make the role of graphics intervals capturing 90% and 95% of the samples respectively (see Table 2)

Table 2: Comparison of sampling distributions (n = 800 and n = 300) and confidence levels (90% and 95%)

n	Confidence level 90%	Confidence level 95%
800		
300		

The comparison of distributions for each confidence level and sample size, allowed students to identify some important relationships as shown in the answers given by Perla and Katya in the worksheet, and Andrea (A) and Anaid (AN) in the interview:

"The larger confidence level, the larger intervals will be." Perla

"The behavior of the sample size in relation to the width of the intervals is reversed with respect to the confidence level. The sampling distribution of size 800 is narrower than the sampling distribution of size 300." Katya

R: What is the effect of increasing the confidence level on the width of the interval?
A: The lower is the confidence is, the narrower the interval becomes.
R: What advantages do you think a high confidence level study would have?
A: That certain percentage of the population likely falls within that interval. But if the interval is very wide, it may not be very useful, since large intervals are more reliable but less precise.
R: What can be done to increase the accuracy?
A: We should increase the sample size.
R: What confidence level would you prefer in a study: 90% or 95%?
AN: 90% is better, because the margin of error is smaller.
R: Is it not an interval of 95% more reliable?
AN: At interval of 90% you are less likely to fall within, and 95% is larger and has more possibilities.

R: What happens to the interval when you increase the sample size?
AN: The width of the interval increases by lowering the sample size.

The responses by the students show that they have correctly identified the effect of confidence level and sample size in the width of an interval. However, the meaning of confidence level for Andrea is wrong, considering that it represents a percentage of the population that falls within the interval, a very persistent misconception documented in other studies (e.g. Olivo & Batanero, 2007).

Another important idea that we proposed to explore is the randomness of an interval, this means that the limits and width of the interval may change from one sample to another. For this part of the activity, we decided to develop, in the TinkerPlots' spreadsheet, the calculations involved in a confidence interval for a confidence level of 90% and 95% respectively, and repeat the simulation for 500 or 1000 samples (see Figure 3).

History of Results of Sampler 1						Collect 500	Options ▾
	prop_legaliza...	Error_estandar	margen_de_error	limite_inferior	limite_superior	resultado	
478	0.7375	0.0155561	0.0311122	0.706388	0.768612	Cae fuera	
479	0.7725	0.0148216	0.0296432	0.742857	0.802143	Cae dentro	
480	0.765	0.0149906	0.0299812	0.735019	0.794981	Cae dentro	
481	0.77	0.0148787	0.0297574	0.740243	0.799757	Cae dentro	
482	0.75375	0.015232	0.030464	0.723286	0.784214	Cae dentro	

Figure 3. Spreadsheet with elements of a confidence interval (500 intervals generated).

The answers of Andrea (A) and Anaid (AN) are shown below:

R: When you repeat the sample, do expect the same interval?
AN: No, because the samples are random.
R: Do all the elements of an interval vary?
AN: Yes, because every time you run it, it gives a different P, although it is close.
R: What relationship do you think 97% of captured results has with the confidence level? (See Figure 3).
AN: The confidence level is 95% and that is very close to 97%, it may even be the same.

R: In your simulation, 4% of the intervals did not capture the parameter; Does that have any relationship with the confidence level of 95%?
A: Yes, I remember that the results that fall outside the interval do not contain the parameter.
R: Could that happen in a real survey?
A: Yes
R: Would you take it as a pollster error?
A: No, I would take it as something that happens by chance, and that it happens infrequently.
R: Where would you locate those values in a sampling distribution?
A: At the ends of the distribution.
R: Seeing these results, what does reliability means?
A: It is the percentage that a pollster can say that their samples contain the parameter, which are true.

Table 3: Charts with percentages of intervals that "covers" and "not covers" the parameter for confidence levels 90% and 95%

	Confidence level 90%	Confidence level 95%
Charts of the results column		
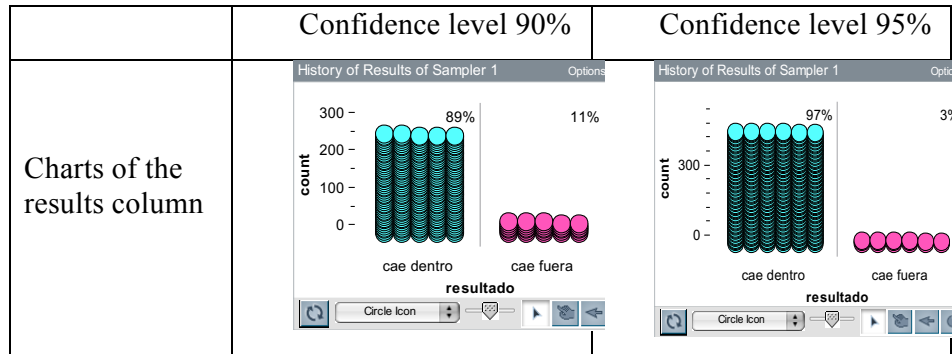
The responses of Anaid and Andrea indicate that they are clear about the randomness of the interval, because it depends on the variable results of a sample. They are unable to establish the correct meaning of the confidence level despite the fact that the graph displays the percentage of samples that cover and do not cover the parameter, respectively. However, it should be noted that Andrea is aware that in a real survey, there may be intervals that do not cover the parameter, she considered them rare and located them correctly in the tails of a sampling distribution.

Conclusions

The results indicate that the students reasoned correctly on some concepts that integrate a stochastic understanding of confidence intervals defined by Pfannkuch, Wild, and Parsonage (2012), such as the relationship between sample size and sample variability, the effect of confidence level and sample size on the width of a confidence interval. The students identified reasonable intervals of expected results in a sample and the random nature of an interval, concepts reported as complex by previous studies. However, confidence level proved to be a very difficult concept for all students, and they failed to conceptualize it correctly, even when activities emphasized repetition of samples to display the percentage of intervals that covers the parameter, and relating it to the previously established confidence level. The computer environment such as that provided by TinkerPlots joint with activities that promote the explicit relationship between the concepts involved in a confidence interval seem to be suitable for the design of learning pathways that promote a correct informal inferential reasoning in the students.

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TOWARDS A FRAMEWORK FOR A CRITICAL STATISTICAL LITERACY IN HIGH SCHOOL MATHEMATICS

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In the spirit of questioning, crossing, and blurring disciplinary borders with(in) mathematics education, this paper proposes a theoretical framework that merges conceptions of critical literacy and statistical literacy for high school mathematics curriculum. I begin by discussing the political nature of education and then situate this work within the goals of education for active citizenship. Next, I provide a detailed background of both critical and statistical literacy from multiple educational research perspectives. Lastly, by synthesizing notions of critical and statistical literacy, I present a theoretical framework for a critical statistical literacy in the context of high school mathematics curriculum.

Keywords: Data Analysis and Statistics, High School Education, Equity and Diversity

Introduction

In modern society, which is drenched in data (Steen, 2001), individuals need to be statistically literate (Gal, 2004). There has been an explosion of data in every facet of life including medicine, economics, education, and public opinion, just to name a few (Ben-Zvi & Garfield, 2008). As a result statistical literacy is becoming a crucial literacy for being a citizen in today's modern societies. As Franklin et al. (2007) states, "every high school graduate should be able to use sound statistical reasoning to intelligently cope with the requirements of citizenship" (p.1). However, it is important for citizenship education that this literacy goes beyond the tradition 'consumption and production' conception of statistical literacy (Ben-Zvi & Garfield, 2004; Franklin et al., 2007; Gal, 2004) to include fostering sociopolitical awareness and critique. Large scale social and environmental issues, such as racism, climate change, refugee crises, and poverty need to be addressed in school curriculum so that students are not only aware of them, but have experiences investigating them (Apple & Beane, 2007). In this way, schools can serve as sites for fostering students/citizens, who can thoughtfully engage issues in their local and global community for the purposes of creating a more just world for tomorrow.

Today in the context of secondary education in the United States, the teaching of statistics and data analysis is situated in the mathematics curriculum (National Council of Teachers of Mathematics [NCTM], 2000; National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010). As data analysis and statistical methods have gained importance in society, they have also gained emphasis in the standards for mathematics instruction in grades 6-12, for example, with the introduction of the *Common Core State Standards for Mathematics* (NGA & CCSSO, 2010). Unfortunately, situated in this context statistics concepts and practices are frequently reduced to mundane computations (Cockcroft, 1982) stripping them of the power they have in investigating issues that are meaningful to students lives as well as their historical and sociopolitical contexts. In the K-12 context there needs to be an emphasis put on teaching statistical concepts and practices consistent with the discipline of statistics, and that will be useful when investigating sociopolitical issues that are prevalent in the world today. To engage in active, critical citizenship, students need a critical statistical literacy.

Before I elaborate on what I mean by a critical statistical literacy I will situate this idea in the broad sociohistorical and political context of US education. I will also draw from multiple discourses

concerning literacy to position this form of literacy, and rationalize its importance in education, and more specifically mathematics education.

The Political Nature of Education and Citizenship

Education is inherently political in that organizations and individuals make decisions on what topics, issues, and concepts are learned, and who can learn them, therefore endorsing certain ideas, views, and individuals, while disadvantaging and excluding others. As Labaree (1997) argues, "the central problems with American education are not pedagogical or organizational or social or cultural in nature but are fundamentally political" (p.40). The problem is rooted in differing, and at times competing, goals of education. Labaree argues that one of the predominant goals of US education is democratic equality. Americans take great pride in living in a democracy, where decisions are made by the people, for the people. Education centered on citizenship benefits society by fostering a literate and educated citizenry who can make critical decisions, and in turn create a stronger democratic society for all. I firmly believe in this goal for education, viewing it as a public good, preparing youths to take up the role of citizens (Apple & Beane, 2007; Giroux, 1989; Labaree, 1997); and it is upon this goal that I situate this paper.

There are many views of what constitutes "good" citizenship, which are culturally, socially, and historically situated. These views can be roughly categorized into three main types of citizenship: personally responsible, participatory, and justice oriented, as described by Westheimer and Kahne (2004). The image of a personally responsible citizen is a hard working individual who follows the law, has good character, and is responsible in his/her community. As indicated by its name, the participatory vision of citizenship is one where citizens are active participants in the government and community. They work within the system to try to improve conditions. The third view of what makes a "good" citizen is the justice oriented citizen. "Good" citizens in this view are ones who question and critique the injustices of societal structures. This type of citizen seeks to find and address the root causes of injustices in society as well as engage in activism, and leverage social movements, to effect systemic change.

There are strengths and weaknesses to each of these views of citizenship. For example, in the personally responsible view of citizenship there is no active participation in government/community affairs, or working to change unjust laws, inequities, etc. This type of citizenship maintains the status quo. The problem with this mentality is that the status quo today includes unjust structures of racism, classism, and sexism that favor certain individuals and disadvantage others (Giroux, 1989; Knoblauch & Brannon, 1993). For example, wealth is becoming increasingly polarized with a small number of individuals holding the majority of the wealth, and an increasingly large proportion of the population sinking into poverty based on hegemonic economic structures (Brown, 2006). I see the ideal view as a blending of participatory and justice-oriented. In my opinion, a good citizen should participate actively in their community and/or government, but should also interrogate the structures at play within their community and government, which produce conditions of injustice, and actively work to change those that (re)produce injustices. Giroux (1989) describes this type of citizenship in his view of education for democracy. One of the key aspects of this view of citizenship that Giroux describes is that,

> It is important to acknowledge that the notion of democracy cannot be grounded in some ahistorical, transcendent notion of truth or authority. Democracy is a "site" of struggle, and as a social practice is informed by competing ideological conceptions of power, politics, and community. This is an important recognition because it helps to redefine the role of the citizen as an active agent in questioning, defining, and shaping one's relationship to the political sphere and the wider society. (p.28-29)

I agree with this view of democracy, as a site of struggle, and with citizens as active agents in shaping the meaning of democracy. In today's modern societies, there are a plurality of different views, values, and ideas, which citizens must be able to negotiate and navigate in daily life. Furthermore, strengthening the bonds between fellow citizens through a common goal of democracy, while appreciating the plurality inherent in our society, is also crucial to being a citizen (Giroux, 1989). I will henceforth refer to this participatory and justice oriented view of citizenship for democracy as *critical citizenship*.

Education based on critical citizenship would include learning both how institutions works, and how to critique and change them, as well as the historical, social and political discourses that shape society and the views of citizens. Students would be provided opportunities to give back to their communities, and encouraged to take action, by forming and leading their own efforts to fight injustices they see in the world around them. An example of this view of citizenship being played out in the context of mathematics education can be seen in Gutstein's (2006) descriptions of his teaching in a Chicago public middle school. Gutstein showed students how to participate in their community and government by taking them to town hall meetings, and also provided them with experiences in using mathematics to investigate injustices related to racist policing practices and gentrification present in their community. It is in this direction, I believe education needs to go, to educate students to function in society, as well as participate in shaping, and improving society for future generations. Critique is an important aspect in this type of education, but as Giroux (1989) points out, "the discourse of democracy also needs a language of possibility, one that combines a strategy of opposition with a strategy for constructing a new social order" (p. 31). It is important for students to have experiences critiquing and analyze discourses and institutional structures, however it is equally important for them to have experiences shaping and creating just practices for the wellbeing of others, in their local and global communities. It is in dialogue between these two types of experiences that schools can help students learn about democracy as a site for struggle, in constant negotiation by citizens who actively participant in creating it. One possible approach that Giroux (1989) describes for making pedagogy more political, for the purposes of educating for a critical citizenry, is to organize pedagogy around critical literacy. Literacy is already a predominant goal of education. Therefore, a critical literacy can be an entry point into school curriculum for the purposes of fostering critical citizens.

I have laid out my view of education, which I see as a public good that should promote critical citizenship for participation in democratic societies. Based on this view of education I will now describe what I mean by a critical statistical literacy, and why it is an important literacy to foster in secondary school classrooms.

Literacy

The theme of literacy has been an educational focus for some time, and has had an ever-changing history. Literacy is commonly defined around the acts of reading and writing (Gee, 2014; United Nations Educational, Scientific, & Cultural Organization [UNESCO], 2005). However the commonalities generally end there. One large difference in scholars' interpretations of literacy is whether the focus should be on the skills necessary to read and write (Scribner, 1984), or on the practices of reading and writing themselves (Perry, 2012). Another aspect of the literature on literacy that varies greatly is the *object* of reading and writing. Much literacy work has focused on the reading and writing of the written languages, in the dominant language of the community in which an individual is a member (Freire & Macedo, 2003; Scribner, 1984; UNESCO, 2005). Overtime, this has expanded to include a multitude of different types of literacy such as financial, quantitative, digital, media, and technological literacies. In this section, I will discuss two types of literacy, statistical and critical, and propose a framework for merging the two types of literacy into a critical statistical literacy. As a note, there is a large research base around quantitative and mathematical

literacies that intersects with statistical literacy. However, I am making a political choice not to draw upon this work, but to instead explicitly draw upon statistical literacy work, which is deeply situate in statistics education and the discipline of statistics, to emphasize its importance, so it is not easily subsumed in the high school mathematics curriculum.

Critical Literacy

Critical literacy draws from a sociocultural definition of literacy in that it is viewed as the practices and abilities associated with being literate. Scholars of critical literacy also foreground the connection between literacy and power (Lankshear & McLaren, 1993). Many discuss literacy as an emancipatory force (Darder, 2014; Freire, 1970; Freire & Macedo, 2003; Giroux, 1989, 1993; Gutstein, 2006), beginning with learning to read the word *and* the world, which can then lead to individuals to being able to write both the word and world—transforming their lived realities through the power of literacy. When democracy is viewed as a site for struggle, through the dialogue of a plurality of views, literacy for critical citizenship needs to include the practices of critiquing and interrogating the discourses and structures in society that reproduce oppressions and injustices. As Gutstein (2006) points out, "U.S. schools socialize students into non-questioning roles, creating and maintaining passive identities so that students do not believe in their own power to shape the world" (p.88). It is in shifting this dominant form of socialization in American schools that the work of critical literacy scholars can speak volumes. By providing students experiences to see how they are situated by social structures, and also how their own schooling is shaped by historical, political, and socially constructed institutions and discourses, can help students create and maintain active citizen identities, where they believe in their own power to influence and shape the world. Without such socialization, how can we ever expect students to become active participants in their communities, to become critical citizens working to negotiate and shape the democracies they live in? As Giroux (1989) describes,

> Critical literacy can provide the theoretical basis for presenting students with the knowledge and skills necessary for them to understand and analyze their own historically constructed voices and experiences as part of a project of self and social empowerment. Central to this view of literacy is an understanding of how knowledge and experience are constructed around particular forms of intellectual, moral, and social regulation within the various relations of power that characterize schools, families, workplaces, the state, and other major public spheres. (p. 34)

Bringing critical literacy into schools can help prepare students to be critical citizens in democratic societies actively participating in the struggle to (re)define meaning and principles of democracy in their world. An important aspect of literacy for critical citizenship includes a statistical literacy, which I will spend some time describing in the next section.

Statistical Literacy

Similar to other types of literacy, statistical literacy can be viewed in terms of reading and writing. The difference is that while literacy is typically considered as reading and writing the written symbol system of the dominant language of a community that an individual is a member of (Freire & Macedo, 2003; Scribner, 1984; UNESCO, 2005), statistical literacy is based on reading and writing a *specialized* symbol system, socially constructed by the discipline of statistics. The practice of reading and writing statistics draws on the dominant language of a social group, but also relies on a specialized symbol system that includes quantities and mathematical symbols, as well as symbols specific to the discipline of statistics. For example, the symbol $\bar{x}$ in statistics represents the arithmetic mean of a sample of data points. However, by itself, the quantity that represents the mean of a data set is relatively meaningless without using it in context, which requires using the dominant language of the individuals which the information is being presented to. For example, the mean

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salary for public school teachers in Massachusetts was $73,908 in the 2013-2014 school year (Massachusetts Department of Elementary and Secondary Education, n.d.). The quantity 73,908 is relatively meaningless until the English language is used to describe and give context and meaning to the quantity.

In the data drenched societies of today, it is crucial for students to be educated and critical consumers of data based arguments (Ben-Zvi & Garfield, 2008; Franklin et al., 2007; Steen, 2001). Citizens today need to be able to make sense of, and critically evaluate, the validity and usefulness of the information presented in statistical arguments, and make informed decisions based on the augments presented. Reading statistical arguments is a key element of statistical literacy (Franklin et al., 2007; Gal, 2004). Gal's (2004) definition of statistical literacy describes this element well shown here:

> (a) people's ability to interpret and critically evaluate statistical information, data-related arguments, or stochastic phenomena, which they may encounter in diverse contexts, and when relevant (b) their ability to discuss or communicate their reactions to such statistical information, such as their understanding of the meaning of the information, their opinions about the implications of this information, or their concerns regarding the acceptability of given conclusions. (p.49)

Reading is only one part of literacy however; there also needs to be a writing component, as the two have a dialogical relationship with one another, just as reading the word and the world operate in dialogue. Writing in statistics involves actively investigating a phenomenon through a statistical investigation, and communicating the results of that investigation to others. The *Guidelines for Assessment and Instruction in Statistics Education* (GAISE) framework (Franklin et al., 2007), outlines four components of the statistical investigative cycle: formulate questions, collect data, analyze data, and interpret results. In drawing from this description of the statistical investigative cycle, writing statistics could be defined as,

> The ability to formulate statistical questions, collect or find data relevant to statistical questions, analyze data using appropriate graphical and numerical methods, and interpret analyzed data addressing the statistical question(s) being investigated.

Combing this description of writing statistics with the earlier description of reading statistics from Gal's (2004) definition of statistical literacy, creates a coherent description of statistical literacy, that includes both the elements of reading and writing, consistent with the broader field of literature around literacy. This also provides a view of statistical literacy that can be merged with the earlier described view of critical literacy, centered on reading and writing.

Critical Statistical Literacy

Now that I have situated the focus of this work, I think it is time I provide a description for what I am calling critical statistical literacy. I see this concept as an intersection of critical and statistical literacies. My goal here is to emphasize the importance of some powerful statistical ideas that are crucial to a critical literacy in democratic education. I have used the word critical first in order to foreground a critical literacy perspective, which is important across all curricula in education, because it is rooted in practices for participating in, critiquing, and (re)shaping structures and discourses in society that are crucial for critical citizenship in society. Next, I have used the word statistics, which is important to emphasize in the high school mathematics context, because it is a distinct discipline (Franklin et al., 2007) whose concepts and practices frequently get subsumed in the school mathematics curriculum, and stripped of their power, reduced to a litany of calculations (Cockcroft, 1982).

Table 1: Framework for a Critical Statistical Literacy

	Read	Write
Statistical Literacy	• Making sense of and critiquing statistical information and data based arguments encountered in diverse contexts. • Discussing or communicating the meaning of statistical information. • Evaluating the source, collection and reporting of statistical information.	• Formulating statistical questions. • Collecting or finding data relevant to answering posed statistical question(s). • Analyzing data using appropriate graphical and numerical methods. • Interpreting analyzed data addressing the statistical question(s) being investigated.
Critical Literacy	• Making sense of symbol systems. • Identifying and interrogating social structures in the world. • Gaining an awareness of sociopolitical issues in society. • Gaining an awareness of the dialectical tensions in society • Understanding one's social location, subjectivity, political context and having a sociohistorical and political knowledge of self and world.	• Creating and communicating one's own meaning through symbol systems. • Actively influencing and shaping structures in society. • Working to alleviate and resolve sociopolitical issues of injustice. • Actively negotiating and navigating dialectical tensions in society. • Communicating one's social location, subjectivity, and political context to others and how it influence and shapes one's meaning making of the world.
Critical Statistical Literacy	• Making sense of language and statistical symbols systems and critiquing statistical information and data based arguments encountered in diverse contexts to gain an awareness of sociopolitical issues in society. • Identifying and interrogating social structures which shape and are reinforced by the data based arguments being considered. • Understanding one's social location, subjectivity, political context and having a sociohistorical and political knowledge of self and understanding how it influences one's interpretation of information. • Evaluating the source, collection and reporting of statistical information and how they are influenced and shaped by the author's social positon and sociopolitical and historical lens.	• Using statistical investigations to communicate statistical information and arguments in an effort to destabilize and reshape structures of injustice for a more just society. • Using statistical investigations to alleviate and resolve sociopolitical issues of injustice • Negotiating societal dialectical tensions when formulating statistical questions, data collection and analysis methods and highlighting such tensions in the results of a statistical investigation. • Communicating one's social location, subjectivity, and political context to others and how it shapes one's meaning making of the world when reporting results of a statistical investigation.

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Beginning with the element of reading, a critical statistical literacy would include: i) making sense of language and statistical symbols systems; ii) critiquing statistical information and data-based arguments encountered in diverse sociopolitical contexts; iii) gaining an awareness of sociopolitical issues and social structures in society; iv) interrogating discourses and social structures that are shaped and reinforced by data based arguments; v) understanding one's own social location, subjectivity, and political context to develop a sociohistorical and political knowledge of self, and how it influences one's interpretations; vi) evaluating the source, collection, and reporting of statistical information, and how they are influenced and shaped by the authors social position and broader discourses, institutions, and historical forces

Moving from reading to writing shifts the focus from being an active and critical consumer of statistical information and arguments to being a producer of statistical information and arguments. Writing within a critical statistical literacy would include: i) using statistical investigations to communicate statistical information and arguments, in order to question and reshape institutions and social structures; ii) using statistical investigations to alleviate and resolve sociopolitical issues of injustice; iii) negotiating and navigating dialectical tensions in society when formulating statistical questions, data collection, and analysis methods, and furthermore explicitly discussing these tensions related to, for example, objective versus subjective truths, or constructed categories versus continuous scales (e.g. gender, race/ethnicity, etc.), in the communication of the results of a statistical investigation; iv) communicating one's social location, subjectivity, and political context to others, and describing how it influences and shapes one's meaning making of the world, when reporting results of a statistical investigation.

A critical statistical literacy based on the elements I have described of reading and writing are summarized in Table 1. This type of literacy is crucial to critical citizenship in today's data centric societies. Statistics is now part of many of the dominant discourses of society, which critical citizens must critically make sense of and evaluate. Furthermore, because of statistics position of power in dominant discourses for providing 'evidence of truth,' it is crucial for critical citizens to be able to use that power to influence, shape, and transform the socially constructed discourses and structures around them in order to create a more just world.

Implications

If schools are to prepare students to tackle large scale social issues, they need to be addressed in school curriculum so that students are both aware of, and have experiences investigating, them (Apple & Beane, 2007). Going forward, I see two major hurdles in fostering a critical statistical literacy in secondary school mathematics curriculum. As I described earlier, critical statistical literacy draws from two main discourses, statistical literacy and critical literacy, neither of which is common in secondary mathematics curriculum. For students to develop sociopolitical awareness and the tools necessary to read and write the world with statistics as critical citizens, they need to have access to powerful statistical ideas. Furthermore, they need to have experiences using those practices and concepts to explore and interrogate structures and injustices in society, to foster sociopolitical awareness, and the practices of critique and active citizenship. I think the following statement by Giroux (1993) concerning critical literacy is quite appropriate for the critical statistical literacy I am proposing as it fits within the theme of questioning borders with(in) mathematics education,

> The pedagogical and ethical practice which I am emphasizing is one that offers opportunities for students to be border crossers; as border crossers, students not only refigure the boundaries of academic subjects in order to engage in new forms of critical inquiry, but they are also offered the opportunities to engage the multiple references that construct different cultural codes, experiences, and histories. (p. 375)

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The critical statistical literacy framework proposed here is by no means complete. There are many questions that need to be considered, such as what does a curriculum that fosters such a literacy look like? How would such a curriculum be enacted in the classroom? How do we train and prepare teachers to teach such a curriculum, especially when situated in high school mathematics where many teachers have had little past experience with statistics (Shaughnessy, 2007)? These are but a few of the many questions that could be posed based on this framework.

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THE DEVELOPMENT OF A PROFESSIONAL STATISTICS TEACHING IDENTITY

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Motivated by the increased statistics expectations for students and their teachers because of the widespread adoption of the Common Core State Standards for Mathematics, this study explores exemplary, in-service statistics teachers' professional identities using a theoretical framework informed by Gee (2000) and communities of practice (Lave & Wenger, 1991; Wenger, 1998). Twelve exemplary, primarily Advanced Placement (AP) Statistics teachers participated in two semi-structured interviews. Findings indicate that these teachers developed teaching identities that cross the borders of mathematics and statistics, though these were not always recognized by administrators, other teachers, and parents. Two major contributing factors to the development of a statistics teaching identity—teachers' specialized knowledge and teacher isolation—are discussed. This work has implications for teacher preparation and professional development.

Keywords: Data Analysis and Statistics, High School Education, Teacher Beliefs, Teacher Education-Inservice/Professional Development

Background

Statistics is a mathematical science – like economics or physics – but is not a type of mathematics (Cobb & Moore, 1997; Moore, 1988; Rossman, Chance, & Medina, 2006; Usiskin, 2014). Still, statistics has been included within the mathematics curriculum at the K-12 level (National Council of Teacher of Mathematics [NCTM], 2000; National Governors Association Center for Best Practices, Council of Chief State School Officers [NGA & CCSSO], 2010) and will likely remain there for years to come (Scheaffer, 2006; Usiskin, 2014). Despite sustained calls for increasing the statistical preparation of teachers (e.g. Conference Board of Mathematical Sciences [CBMS], 2012), the current state of teacher preparation for statistics is regarded as inadequate (Franklin et al., 2015; Shaughnessy, 2007). While more mathematics teachers are being asked to teach statistics because of its inclusion in the Common Core State Standards for Mathematics (CCSSM) (NGA & CCSSO, 2010) in grades 6-12, teachers involved with the Advanced Placement (AP) Statistics program have already overcome challenges associated with an increase in statistics expectations. Therefore, AP Statistics teachers can serve as an analogue for understanding how mathematics teachers overcame challenges they encountered teaching at the border of mathematics and statistics.

Overview of the Study

A key expectation for mathematics teachers is that they develop a professional identity as such (NCTM, 1991). This professional mathematics teacher identity is fostered by teacher preparation programs, on-going professional development, and immersion in the school mathematics culture. However, the benefits of a professional mathematics teacher identity may not be directly applicable to teaching statistics, and there is little extant research on statistics teachers' identities (e.g. González & Zapata-Cardona, 2014; Peters, 2009). This study explores the professional identities of exemplary AP Statistics teachers at the high school level. The primary research question is: "What professional identities do exemplary statistics teachers develop?" Assuming that a statistics teaching identity is found, a secondary question is considered: "What are the major contributing factors to the development of a professional statistics teaching identity?"

Theoretical Framework

There are several theoretical conceptions of identity in the literature. This study draws heavily on two conceptions of identity that form the basis for its theoretical framework: Gee's (1999, 2000) contextual or socially-situated identities and Lave and Wenger's (1991; Wenger, 1998) communities of practice. Other conceptions of identity have also contributed to this theoretical framework (e.g. Juzwik, 2006; Philipp, 2007; Sfard & Prusak, 2005).

Gee's (1999, 2000) conception of identity postulates a core identity that is stable across contexts and contextual identities that are recognized by others and more prone to change over shorter time frames. There are four perspectives for contextual identities described by Gee (2000) (natural, institutional, discursive, and affinity) which are "interrelate[d] in complex and important ways" (p. 101). By employing these four perspectives, identity can be examined in several ways which has implications both for data collection and data analysis. The contribution of contextual identities to the theoretical framework of this study is important because of the multiple roles which in-service teachers have in their careers.

Identity is at the heart of the communities of practice framework. In a community of practice, learning is synonymous with the development of the practices and identity associated with the community in question (Barab & Duffy, 2000; Lave & Wenger, 1991). Identity and practice are further connected through five modes of belonging: negotiated experiences, community membership, learning trajectories, nexus of multi-membership, and the local-global relationship (Roesken, 2011; Wenger, 1998). It is through these modes of belonging that perpetually in-development identities are examined. In this study, the most salient aspects of communities of practice are teachers' learning trajectories and their nexuses of multimembership. The use of both Gee's (1999, 2000) conception of identity and communities of practice (Lave & Wenger, 1991; Wenger, 1998) ensured that the data sources addressed identity in a broad manner.

Methods

Participants

Participants were recruited using *critical case* sampling, a sampling strategy that prioritizes specific, important participants. Patton (2002) characterizes critical cases as those for which the researcher can say "if it doesn't happen there, it won't happen anywhere" (p. 236). In the present study, successful, experienced statistics teachers constitute a critical case because their professional identities (as teachers of statistics) have enabled their success; if the teachers in this study do not develop professional identities that are aligned with teaching statistics then it would not be reasonable to believe that such identities would be developed by other mathematics teachers tasked with teaching statistics.

To identify these critical cases, referrals were solicited from experts in the statistics education community with far-reaching contacts and connections. These experts were asked to identify "statistics teachers that you consider to be exemplary … primarily at the middle and high school levels" in a broad sense: no restrictions were placed on experience, gender, courses taught, etc. From these sources, a list of 24 statistics teachers – all known primarily for their work teaching statistics at the high school level – was compiled. Each of these teachers was emailed and asked if they would like to participate in the study; all 12 who responded affirmatively were included.

Among the 12 participants, there were five women and seven men. The highest level of education was a bachelor's degree for two participants, a master's degree for seven participants, and a doctoral degree for three participants. While most of the participants earned degrees in mathematics or mathematics education (i.e. they had traditional mathematics teaching backgrounds), there were several notable exceptions. One participant's highest education was a master's degree in statistics, and another participant earned a doctorate in statistics. Additionally, one participant earned a

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

doctorate in a science. The participants were primarily mid-career and late-career teachers, having an average of 30 years of teaching experience; the minimum years of teaching experience was 7, the first quartile was 19.5, the median was 32.5, the third quartile was 42, and the maximum was 50. Pseudonyms are used throughout.

Data Source

Because of the rich, multi-faceted nature of identity, semi-structured (formal) interviews were employed for data collection. As recommended by Hatch (2002), the interview protocols were viewed as 'guiding questions' rather than enforcing a rigid structure. In general, two interviews lasting about 60-90 minutes each were conducted with each participant; exceptional circumstances resulted in only one interview being conducted with two of the participants for a total of 22 interviews. Questions on the interview protocol were written by the researcher in consultation with published interview protocols from studies using similar frameworks (e.g. Burton, Boschmans, & Hoelson, 2013; Carlone, Haun-Frank, & Kimmel, 2010; Krzywacki, 2009; Settlage, Southerland, Smith, & Ceglie, 2009). The use of semi-structured interviews allows participants to describe how they recognize themselves and how they perceive themselves to be recognized by other people (Carlone & Johnson, 2007). All interviews were transcribed.

Data Analysis

The data in this study are being analyzed using inductive analysis as conceptualized in Hatch (2002). This type of inductive analysis is characterized by repeatedly re-reading the data to identify and refine domains which are then used to find larger themes (Hatch, 2002). Throughout this process, data examined both to see if it supports the working domains and themes and to see if it runs counter to them (Hatch, 2002). This flexible data analysis framework was chosen because it is consistent with the constructivist epistemology used in this study and because it allows for the use of several data sources; beyond the semi-structured interviews used in this paper, quantitative surveys and a brief biographical sketch were used as data sources.

Results

To answer the first research question, the different contextual identities that participants had with administrators, other teachers, and parents are explored. To answer the second research question, evidence of two major contributing factors to the development of a statistics teaching identity distinct from a mathematics teaching identity are presented, statistics teachers' specialized knowledge and their isolation in schools. These two factors presented are a subset of major contributing factors that have been identified in this study.

The Professional Identities of Statistics Teachers

The participants in this study typically had several professional identities, notably both a professional mathematics teaching identity and a professional statistics teaching identity. In Gee's (2000) conception of identity, one's contextual identities are predicated on being recognized by others in a certain way, i.e. in a given context, others' perceptions of a person are a core part of that person's identity for that context. Several contextual identities are evident in participants' relationships with administrators, other teachers, and parents, and these different contexts are characterized by different trends in how participants are recognized. These results report on participants' perceptions of the perceptions that others have of the participants; ideally, the others (e.g. administrators) would be directly interviewed about their perceptions, but this would substantially increase the scope of this project.

Generally, participants reported that administrators viewed them as either a solely a statistics teacher (four participants) or as both a mathematics teacher and as a statistics teacher (seven

participants); one was viewed as exclusively a mathematics teacher. Administrators tended to view participants in the role of a mathematics teacher for two primary reasons: either they continued to teach mathematics courses or they had become the chair of their school's mathematics department. One participant who was viewed as both by the administration said, "they think of statistics as being math because it lives in the math department here, but some people—who know of my background in statistics—would refer to me as the as a statistics teacher" [Samuel:385-386]. The experience of the administration not differentiating between mathematics and statistics, except in specific circumstances, was common. Some participants reported being viewed exclusively as a statistics teacher by the administration: "I think primarily they view me they saw me as a stat teacher because that was my claim to fame … that was two periods of my six-period day but yeah they would say 'she's the stats teacher'" [Laura:473-475]. Great success teaching statistics tended to be a necessary but not sufficient condition for being recognized by the administration as exclusively a statistics teacher.

Other teachers tended to view the participants as either solely a statistics teacher (four participants) or as both a mathematics teacher and a statistics teacher (eight participants); none were viewed as exclusively a mathematics teacher by other teachers. Patricia described the reason she is viewed as a statistics teacher by other teachers saying:

> a stat teacher to the other teachers, only because statistics is fortunately popping up and all the other courses—not just math but also the science courses, in psychology, economics—so [if] somebody has a statistics question they usually end up in my room [Patricia:246-248]

Similarly, Louie was viewed as a statistics teacher exclusively saying, "even within the department … I think the perception is that I'm a stats guy who happens to be chairing a math department right now" [Louie:351-352]. Other participants were viewed as both by their colleagues, reflecting a recognition by their peers that teaching statistics differs from teaching mathematics.

This recognition as a statistics teacher did not extend to parents of their students, even when restricted just to students enrolled in statistics courses. Three participants were viewed exclusively as statistics teachers by parents, two as both mathematics and statistics teachers, and five were viewed exclusively as mathematics teachers; two participants felt unable to judge parents' perceptions of them. Participants were viewed exclusively as mathematics teachers by parents even when they only taught statistics courses; Paul describes the reason as, "[students] get a math credit for it on their transcript … [parents] don't necessarily think of it as a distinction between math and stats" [Paul:449-452]. This was a common experience for participants. Participants who were viewed as solely a statistics teacher or as both attributed this recognition to widely-known successes in their schools' statistics programs or highly-educated parents. Samuel, who is viewed as both by parents, describes them saying,

> we get a lot of students whose parents are faculty members … you know really highly educated … [they'll ask] 'do you teach t-tests?' … I got a parent asking 'oh, do you do randomization tests?' … they have in their mind what a statistics class [is] [Samuel:256-260].

However, most participants did not encounter parents with well-formed conceptions of a statistic class, and so were viewed more often as mathematics teachers by parents than by administrators or other teachers.

Depending on the context of an interaction (Gee, 2000) with administrators, other teachers, or parents, participants reported having to navigate professional identities at the borders of mathematics and statistics. Every participant was viewed by other teachers—the colleagues who would know them best—as a statistics teacher, and sometimes additionally as a mathematics teacher depending on the specifics of their school. On the other end of the spectrum, parents—who would be expected to be less familiar with participants than teachers or administrators—largely viewed them as

mathematics teachers, reflecting a limited understanding of the differences between mathematics and statistics. Administrators had mixed perceptions of participants, often recognizing them as both mathematics and statistics teachers. A primary reason for this recognition as statistics teachers is that participants had specialized knowledge that was useful for helping others with both teaching and research.

Major Contributing Factors

To explain why participants had different mathematics and statistics teaching identities, major contributing factors were explored. Two such factors are described here, teachers' specialized knowledge and their isolation as statistics teachers within a school.

Specialized Knowledge. Within their schools, participants were able to use their specialized statistical knowledge to support their colleagues' teaching. This support was often in the form of extended collaborations between courses or more focused teaching support to help individual teachers. Environmental science provided opportunities for robust collaboration with colleagues for two participants in this study. Early in his statistics teaching career, Louie co-taught AP Statistics along with AP Environmental Science; this collaboration served both courses by incorporating the collection of real data with a meaningful context which could be analyzed and interpreted appropriately using statistical tools. Similarly, Samuel is in the early-phases of a collaboration with an English teacher at his school to link AP Statistics with an environmental literature course that is being taught. This connection between statistics and environmental science is not isolated to individual schools. While not a local collaboration with an individual teacher as in the cases of Samuel and Louie, Patricia has presented workshops nationally about integrating AP Statistics and AP Environmental Science. Such collaborations with statistics are long-term and are a benefit to both halves of the partnership.

Other collaborations address more immediate statistical teaching needs of teachers. After conversations with the AP Biology teacher at his school about how best to teach statistical topics such as the chi-squared test in response to changes in the curriculum, Bob was invited into the science classroom for several years to teach that material. Additionally, because of the new emphasis on statistics in the AP Biology curriculum, Bob worked closely with that teacher to ensure that he was comfortable teaching statistics. Bob recounts, "I spent almost 4 years working with him to get his stat up to the point where he's comfortable doing that now so I don't have to go in [to his class] anymore" [Bob:325-327]. While Bob's work with the biology teacher spanned several years, its focus was more limited in scope than a partnership between two courses as above. Still, Bob was being consulted for statistical expertise that many mathematics teachers would not have.

Addressing immediate needs of teachers is not limited to helping science teachers. Rebecca has helped algebra teachers with statistics material that is included in the CCSSM such as regression analysis and residuals. This type of assistance can range from offering out-of-class descriptions and help with specific problems to co-teaching lessons and offering question-and-answer sessions in-class for students. Rebecca recounts:

> We co-teach [because] with the students you know they feel they can ask questions that she may or may not be able to answer as in depth, and then I have another teacher that asked me about residuals. We co-taught residuals together for a day. [Rebecca:506-508].

Part of the reason that Rebecca welcomes questions from her colleagues about statistics is that the asking of such questions reveals that they are trying to teach the material rather than skipping it. She says, "Several of them come to me with stats questions, it's in the Common Core units that they have, which is [why] I welcome that because that means that they are teaching the units" [Rebecca:287-288]. In response to curricular changes such as the CCSSM or updates to AP Biology, additional statistics content beyond what these colleagues of the participants have been prepared to

teach are required; statistics teachers are being consulted specifically–rather than mathematics teachers–because of their specialized knowledge and the different roles they serve within a school.

Isolation in Schools. Perhaps unsurprisingly, many of the teachers in this study reported being the only statistics teacher at their school, and also reported that this was also the case for many colleagues they knew. To address this situation, the exemplary statistics teachers sought out a different community from the school mathematics community. For Rebecca, her isolation was unlikely to change in the immediate future:

> I'm the only person at my school that's been to the training for AP Statistics, and I'm probably one of the only ones that really wants to teach it. There's not too many people that really like statistics [Rebecca:13-15]

Patricia began teaching statistics because "when [AP Statistics] first came out nobody else wanted it, so I just kind of fell into it" [Patricia:28]. When colleagues are not prepared and express no desire to teach a course, an isolated teacher can have difficulty even finding another teacher to discuss ideas with. As described by Amber, teachers of mathematics courses generally do not have this problem:

> One of the first challenges was not having anyone to be a soundboard for you. You know, 'What do you think about this? You think this might work?' … like where you have in algebra or precalculus there's going to be multiple teachers–you can work together. Having to work in isolation was really difficult in terms of teaching statistics. [Amber:231-234]

Participants cited not having "that regular opportunity for collaboration" [Paul:407] within their schools and the ensuing challenges as a reason for engaging with the broader statistics education community. Engaging with a professional community was a major contributing factor to the development of the participants' professional statistics teaching identities.

Discussion and Conclusions

The exemplary statistics teachers in this study developed professional identities as both mathematics teachers and statistics teachers. Depending on the specific context of an interaction, one professional identity may be brought to forefront. The effect of this is that the exemplary statistics teachers are continually working at and across the border between mathematics and statistics, even for teachers who only teach statistics classes.

The participants in this study had many years of teaching experience and had grown comfortable operating with these contextual professional identities and were able to use their specialized knowledge to serve their colleagues as a resource. Mathematics teachers who are less experienced with statistics, however, may not be so comfortable. The participants in this study each took several years developing their professional statistics teaching identities.

Improvements should be made to the statistical preparation provided mathematics teachers. Such recommendations are not new (CBMS, 2012; Franklin et al., 2015), but the experience of these exemplary statistics teachers provides further support for these calls. Participants in this study generally reported not feeling prepared to teach statistics when they were early in their careers. None was specifically trained to be a statistics teacher, and developing a statistics teaching identity required engaging with colleagues beyond their local school.

Participants found it critical to be connected with a broader community because they were often so often isolated, being either the only or one of only a few statistics teachers. Many of the participants reported that participation in the AP Statistics Reading or the AP Statistics Teacher Community (formerly the AP Statistics Listserv) provided initial and on-going experiences that helped them become exemplary teachers. Unfortunately, no direct analogues for either the AP Statistics Reading or AP Statistics Teacher Community exist for non-AP Statistics teachers at the 6-

12 level. Addressing teacher isolation within schools by connecting statistics teachers (particularly those new to the subject) with broader communities—either virtual or in-person—may be a viable approach for supporting teachers' transitions and growth. Teachers could be directed to such resources while they are still in their preparation programs.

Mathematics teachers who are being asked to teach more statistics now in grades 6-12 under the CCSSM (NGA & CCSSO, 2010) are likely to require different preparation and professional development than is required for mathematics. However, administrators, other teachers, and parents—groups who could conceivably support new statistics teachers—may not know enough about the disciplinary and instructional differences to recognize that additional, different supports are needed.

Asking mathematics teachers to teach statistics is more than simply asking them to teach new or additional mathematics material. Rather, because of widely-acknowledged disciplinary differences between mathematics and statistics (e.g. Cobb & Moore, 1997; Moore, 1988; Rossman et al., 2006), mathematics teachers are being asked to cross the border between mathematics and statistics. There are many potential benefits for teachers and students that may arise from this arrangement, e.g. new avenues for teaching quantitative literacy (Scheaffer, 2003; Steen, 2001) and changes in mathematics teaching practice. However, the complexities of the identities of these exemplary statistics teachers suggest that purposeful reforms to teacher preparation and professional development must be made and cannot be done in ways that recognize only the professional mathematics teaching identity.

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EXPLORATION OF THE VARIABILITY OF INFORMAL INFERENCES DRAWN FROM SIMULATED EMPIRICAL SAMPLING DISTRIBUTIONS

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Two groups of introductory statistics students (at the secondary and tertiary level) participated in a model exploration activity that explored the dotplot representations of simulated empirical sampling distributions. Students compared and discussed informal inferential claims drawn from the sampling distributions as the amount of simulated data in the sampling distributions increased. The students began to develop a global view of sampling (Pratt & Noss, 2002) by exhibiting an understanding of the long-term randomness of sampling by describing the trend of the growing collection of simulated data and the informal inferences that could be drawn.

Keywords: Data Analysis and Statistics, Modeling

Introduction

Influential documents such as the *Guidelines for Assessment and Instruction in Statistics Education Report* (Franklin et al., 2007) and *The Common Core State Standards for Mathematics* (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) have emphasized the use of simulations in statistics coursework. Technology now allows simulations to be performed nearly instantaneously, thus making accessible new approaches to teaching statistical inference through simulations. Researchers have claimed that technology can allow the teaching of statistics to be visual, dynamic, and interactive with students engaging in experimentation with data rather than computations and can support student learning through the automation of calculations, emphasis on data exploration, visualization of abstract concepts, and use of simulations as a pedagogical tool (Garfield & Ben-Zvi, 2008). In this study, introductory statistics students used *TinkerPlots* (Konold & Miller, 2014) to simulate data, make informal inferential claims from simulated empirical sampling distributions, and explore the effects that increasing amounts of simulated data had on their informal inferences.

Review of Literature

For this study I drew from two bodies of research literature: modeling and informal inferential reasoning. Models are "conceptual systems … that are expressed using external notation systems, and that are used to construct, describe, or explain the behaviors of other system(s)" (Lesh & Doerr, 2003, p. 10). In this study, students participated in a model exploration activity (MXA), which often use computer graphics, diagrams, or animations in order to develop a powerful representation system for the students to make sense out of the situation (Lesh, Cramer, Doerr, Post, & Zawojewski, 2003). The goal of MXAs is to introduce students to conventions that have taken mathematicians and statisticians many years to develop and provide more in-depth understanding of their model.

Central to the MXA in this study was students' informal inferential reasoning, which is the drawing of conclusions from data that extend beyond the data, from the viewing, comparing, and reasoning with distributions of data (Makar & Rubin, 2009). Researchers have claimed that the use of simulation to teach informal inferential reasoning can help students to build a deep understanding of the abstract statistical concepts and suggested that informal inferential reasoning may support the development of students' understanding of formal inferential reasoning (Bakker & Gravemeijer, 2004; Saldanha & Thompson, 2002). Some research has shown that the use simulations can develop students' understanding of the value making data-based inferences from larger collections of samples (Stohl & Tarr, 2002), yet earlier research indicated that while students were proficient at describing

short term randomness, identified as a local view, through observations such as the variations between subsequent samples, they had trouble expressing their understanding of the long term randomness, identified as a global view, such as the overall trend of a growing collection of samples (Pratt & Ross, 2002).

Design and Methodology

This study is a qualitative case study and part of a larger study where four introductory statistics classes at the secondary and tertiary levels participated in a month long instructional unit. I will report the findings from two groups of students (Groups X and Y) while participating in the second activity in the instructional unit, a model exploration activity, during one 75-minute class period. The groups consisted of three to four students with each student at their own computer station with *TinkerPlots* (Konold & Miller, 2014). During the class period I videotaped each group and collected written classwork in order document the students' developing models of sampling and inference. In the results section I will provide a narrative of developments in the students' models.

Prior to this activity, the students participated in a modeling eliciting activity (MEA) that told a brief story of an octopus that predicted eight of eight soccer matches correctly during the 2010 World Cup. Local zookeepers planned to have their octopus predict the winners of eight upcoming basketball games. Groups of students were asked to determine the range of outcomes that they thought were likely for the octopus to correctly predict. Students then participated in a model exploration activity (MXA) that investigated the same context with Ophelia as the MEA, but students now used *TinkerPlots* (Konold & Miller, 2014) to simulate outcomes. I created an environment in *TinkerPlots* that contained a sampler, results, and dotplot window (Figure 1).
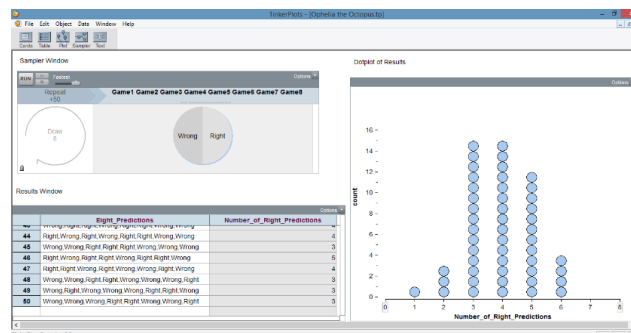
Figure 1. *TinkerPlots* environment for the MXA.

The sampler window in *TinkerPlots* (shown in the upper left) used a spinner with half of the area marked right, the other half wrong. This corresponded to the flipping of one coin to predict the octopus' predictions. The results window (shown in the lower left) showed the outcomes simulated by the spinner in sets of eight predictions. The dotplot window (shown on the right) displayed the number of right predictions in each set of eight. Each student ran the simulation on an individual computer. The activity asked the students to compare their individual dotplots and inferences with their group members' and guided the students to explore how the simulation of an increasing number of samples affected the inferences that they drew from the data. With the dotplot representation of the data, students had the opportunity to visually explore the shapes of the distributions of data simulated by fellow group members, and the changing shape of their own distribution of data as an increasing amount of samples were simulated.

Results

Both groups of students began by simulating 10 samples of Ophelia's predictions and examined their dotplots. A student from group X stated that most of his outcomes were as he expected them to

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be, in the range of four to six correct predictions, but he also generated outcomes of one and two correct predictions. His fellow group member stated that these outcomes "seemed very strange because they are so far off from what the outcomes should have been (4/8, 50%)". This second group member had similar results, with most of his values between three and five correct predictions with additional outcomes of one and seven that he also thought seemed "very unusual because it is so far from 50%". These students did not anticipate that the simulation of a small number of samples would produce what they considered as unlikely results.

When students from group Y each simulated 10 samples and compared their dotplots, one student stated that two of their dotplots:

> … had some form of a bell curve, while one was skewed to the left …They all thought it was going to be a bell curve, but the results didn't comply. They [the dotplots] should be the same because all trials are similar and they are independent and there is a 50/50 chance of it being correct or not.

I interpreted the statement that "results didn't comply" to demonstrate that this student, like those from group X, did not anticipate the variability in sampling. Students in Group Y anticipated the shape of the distribution to be bell curved and both groups of students anticipated aspects of the distribution of data and expected most, or all, data to be in the center of the dotplot.

When focus group X simulated 50 samples, the students noted that each other's dotplots were "very similar and all/most of the dots were around 50% or 4/8", and that each of their dotplots had a "pyramid shape". They concluded that the likely ranges were three to six correct predictions and three to five correct predictions. These inferences were similar, but still not the same.
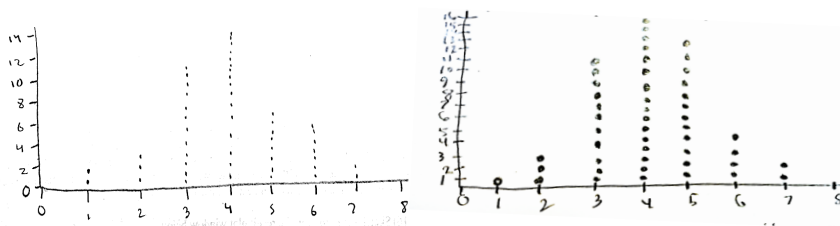
Figure 2. Group X's 50 simulated samples.

After students in group X each simulated 1000 samples, one student noted that they had "identical graphs" and "that all dots / number of dots per # of games correct were very similar" I interpreted this to mean that he was referring to the number of times that each outcome occurred was similar between him and his partner's dotplots. Both students concluded from these dotplots that Ophelia is likely to predict three to six of the eight basketball games correctly.

When group Y simulated 50 and then 1000 samples, the students claimed that each dotplot looked like a bell curve and did not change their predictions for the likely ranges of correct predictions. One student asserted that, "the more trials conducted will help conclude a more accurate prediction". I interpreted that by more accurate, this student meant that as more samples were simulated, the dotplot would look more like the bell curve that the group members had anticipated. Another student from group Y continued to simulate samples and came to the conclusion that at some point, simulating more samples did not change the look of the dotplot. She concluded that the plot was "saturated with data".

Discussion and Conclusions

This study demonstrated the use of *TinkerPlots* (Konold & Miller, 2014) to view dynamic dotplot representations of empirical sampling distributions. As more samples were simulated, the students compared their sampling distributions with those simulated by other students, and observed

that as more samples were simulated, the variability between each sampling distribution decreased. This variability in sampling distributions led students to draw differing informal inferences from the data when simulating a small number of samples, but to draw identical inferences when simulating many samples. Consistent with Stohl and Tarr (2002), these results suggest that students can recognize the importance of simulating large numbers of samples to draw their inferences. I assert that the students began to develop a global view of sampling (Pratt & Ross, 2002) by exhibiting an understanding of the long-term randomness of sampling when describing the trend of the growing collection of simulated samples. The results also indicated that the use of model exploration activities can be used to promote informal inferential reasoning. Future statistics curricula may benefit from the use of modeling and model exploration activities.

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CROSSING THE BOUNDARY: RESULTS FROM THE TEACHER ATTITUDES AND BELIEFS TOWARD STATISTICS SURVEY (TABSS)

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Due to the increased attention given to statistics in the school curriculum over the last several decades, there is a need to study the attitudes and beliefs of secondary mathematics teachers (SMTs) with respect to statistics since research has linked teachers' affect to student learning (Estrada & Batanero, 2008). Because the attitudes, beliefs, and emotions (affect system) of practicing SMTs toward statistics are not exactly known, an inventory of these constructs was conducted via a survey instrument. Most of the survey items were developed in light of the GAISE framework and the CCSSM; hence SMTs' affect toward statistics was described in terms of these documents. Based on responses to the survey items, SMTs indicated confidence in their own ability to master statistical content, belief that students can learn statistical concepts, and some agreement with the goals established by the GAISE framework.

Keywords: Teacher Beliefs, Data Analysis and Statistics, Affect, Emotion, Beliefs, and Attitudes

Over the last several decades, mathematics education researchers have given increased attention to students' and teachers' attitudes and beliefs toward mathematics and statistics, but no work has been done that examines practicing secondary mathematics teachers' (SMTs') attitudes and beliefs towards statistics in light of the *GAISE* framework and the *Common Core State Standards for Mathematics* (*CCSSM*). This study begins to address this gap in the research by analyzing the results from the Teacher Attitude and Beliefs toward Statistics Survey (TABSS), a synthesis of items taken from the Survey of Attitudes Toward Statistics (Schau, 2013), the Statistics Course Attitude Scale (SCAS), and newly developed items reflecting current trends in thinking about K-12 statistics education.

Purpose of the Study

The focus of the study was to investigate and characterize the affect system of SMTs toward statistics via the Teacher Attitudes and Beliefs toward Statistics Survey (TABSS) as the method of data collection. The items on the survey were designed to explore SMTs' affect systems toward statistics relative to their experiences as teachers and students of mathematics and statistics. The purpose of this paper is to filter the data related to attitudes and beliefs in order to describe and clarify the affect system that SMTs, as a group, hold toward statistics. General analysis of the data from the study is highlighted and conclusions are suggested that help to characterize this population of teachers.

The primary goal of the study was to characterize the attitudes and beliefs of practicing SMTs toward statistics based on their responses to the TABSS. Specifically, the *affect system* of practicing SMTs toward statistics was determined by the responses to the items on this survey instrument. Therefore, each of the 10 survey items was analyzed individually for mean responses from the total group of teachers ($n = 141$). The measures of center, together with the variation observed in the data set, were used to characterize the attitudes and beliefs of the population of SMTs based on the responses of this sample of teachers. Hence, the research question was initially explored via the summaries of responses to each of the ten survey items.

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Methods

The SATS-36 served as a basic yet thorough inventory for SMTs' attitudes and beliefs towards statistics, and thus it was a valuable tool for an introductory study. Furthermore, the SATS-36 was the most comprehensive survey available that situated attitudes and beliefs towards statistics in an educational context. While it had not been used with this population of SMTs, it had been validated as a stand-alone instrument for use in similar populations (university students) (Schau, C., Stevens, J., Dauphine, T., & del Vecchio, A., 1995; Schau, 2013) and provided a basic accounting of SMTs' attitudes and beliefs towards statistics. Because the attitudes, beliefs, and emotions of SMTs was expected to be similar to this population (Pierce & Chick, 2008), the existing instrument provided the best basis on which to begin studying this new population. Researchers agree that existing survey instruments designed to measure attitudes and beliefs regarding statistics could be strengthened if they included more items related directly to statistics teaching (Pierce & Chick, 2008). Hence, the TABSS was developed by adding items to the SATS-36 and SCAS in order to address specific concerns related to the *CCSSM* and teaching statistics in light of the *GAISE* document's emphasis on recent statistics education reform efforts.

The teacher response data was collected by using an online survey system which emailed the TABSS to a random sample of SMTs from a Midwestern state. The survey was initially emailed to 502 teachers. Of the 502 who were emailed the survey, 92 filled out the survey giving a response rate of 18.3%. Based on this response rate, the survey was emailed to an additional random sample of 276. Of the 276 who were emailed the survey, 49 filled out the survey giving a response rate of 17.8% for the second mailing. Thus, the overall response rate was 18.1%. The results for the ten survey items are found in Table 1.

Table 1: Summary Statistics for All SMTs (*n* = 141), Arranged in Ascending Order

Survey Item	Mean	Std. Dev.
[a] There are only a few methods that are useful for teaching statistical concepts to students.	3.79	2.12
Understanding probability and statistics is becoming increasingly important in our society, and may become as essential as being able to add and subtract.	3.84	2.33
A student's statistical reasoning should be assessed based on experience with concepts, not age or grade.	3.85	2.19
[a] I found it difficult to understand statistical concepts.	4.24	2.14
[a] I had no idea of what was going on when I learned topics from statistics.	4.46	2.18
[a] I had trouble understanding statistics because of how I think.	4.48	2.22
I understood statistics equations.	4.52	2.11
[a] Typical students cannot expect to understand statistics; they should just memorize and apply what they have learned mechanically and without understanding.	4.60	2.45
Hands-on, active learning is an important part of learning and understanding statistics.	4.60	2.48
Students should be allowed to use appropriate technology when learning statistics.	4.97	2.55

[a] Items are rescaled positively.

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Discussion of Results

Items with Negative Responses

The SMTs responded to the survey items based on a scale of 1 (strongly disagree) to 4 (neither disagree nor agree) to 7 (strongly agree); hence a higher score on a particular item indicated a more positive response to the item. The *lowest* mean response out of the 10 final items was given for "There are only a few methods that are useful for teaching statistical concepts to students." Thus, the low mean score of 3.79 (low in comparison to the neutral survey response of 4) indicated teachers' slight agreement with the statement. Overall, teachers ($n = 141$) agreed that there are indeed only a few methods that are useful for teaching statistical concepts to students. This item was original to the TABSS and was constructed in response to the *CCSSM*, indicating some agreement among the teachers with the aims of that document.

The next two items, "Understanding probability and statistics is becoming increasingly important in our society, and may become as essential as being able to add and subtract" and "A student's statistical reasoning should be assessed based on experience with concepts, not age or grade" were constructed in response to the *GAISE* document. The results related to these items can be further compared since wording of the items is positive and the teachers' mean responses to these items were close (3.84 and 3.85, respectively). These mean scores were below the neutral response of 4, indicating overall slight disagreement with the items. Thus, the teachers' responses to these items revealed that they did not seem to agree that statistical skills are growing more important within modern society and they generally disagreed with the guideline suggested by the *GAISE* document that statistical skills should be taught based on experience with concepts. Based on the results for the first three survey items in Table 1, the SMTs seemed more in agreement with the principles supported by the *CCSSM* rather than the *GAISE* document.

Items with Positive Responses

For the last seven items in Table 1, the responses were more positive than the first three. The determination of "positive" responses is based on comparisons with the neutral survey response of four, which indicates no agreement or disagreement with each item. This discussion focuses on the three items with the most intensely positive reactions.

Teacher responses to the items were moderately positive, and they are each related to teachers' attitudes toward statistical learning. Two of these three items directly follow from the *GAISE* document, which supports hands-on active learning and technology use in the statistics classroom. Additionally, teachers tend to moderately disagree with the item stating that typical students could not expect to understand statistics. This item was constructed in response to the *CCSSM* document, and stands in contrast to those items related to the *GAISE* document.

Thus, the SMTs as a group are in disagreement with some of the ideas reflected in the *GAISE* document. They *disagree* that "Understanding probability and statistics is becoming increasingly important in our society, and may become as essential as being able to add and subtract" and "A student's statistical reasoning should be assessed based on experience with concepts, not age or grade." They *agree* with "There are only a few methods that are useful for teaching statistical concepts to students," which was written for the TABSS to reflect the *CCSSM* document. Despite these findings, SMTs have a high level of confidence regarding their own learning of statistical topics, indicating general agreement with "I understood statistics equations." These findings seem contradictory. While SMTs feel confident in their own abilities to learn and solve statistical problems, they also indicate general disagreement that understanding probability and statistics is important.

Summary of Findings

Teachers' responses to the survey items provide valuable information about SMTs' general affect systems toward statistics, and the study provides an important first step towards studying teacher affect in light of teacher practice (Zumbrun, 2015). Based on the responses to the survey, the SMTs agree that typical students *can* expect to understand statistics, hands-on active learning is an important part of learning and understanding statistics, and students should be allowed to use technology when learning statistics. They are also in common agreement that as students they *did not* find it difficult to understand statistics concepts, they *did* have an idea of what was going on when they learned statistics, they *did not* have trouble understanding statistics, and they *did* understand statistics equations, in general. These findings signify that SMTs, as a group, are largely confident in their own ability to master statistical topics and believe that their students can also learn statistics. Hands-on, active learning and appropriate technology are also important parts of the learning process. These commonalities provide a strong basis for beginning to form a profile of SMTs' affect toward statistics, which was the fundamental goal of this study.

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CHARACTERISTICS OF SECONDARY STATISTICS TEACHERS

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The importance of developing students' statistical literacy prior to entering post-secondary education has gained increased attention over the past decade (Franklin et al., 2007). However, secondary mathematics teachers have limited statistical background knowledge and feel inadequately prepared to teach such content (Banilower et al., 2013). Further, many schools offer Advanced Placement (AP) Statistics and other college preparatory statistics courses, implying teachers of such courses should have stronger statistics backgrounds. Since students who pass the AP Statistics exam typically receive college credit, non-AP statistics courses provide the largest opportunity to prepare students for introductory college level statistics, where students make minimal gains (delMas et al., 2007). Furthermore, non-AP statistics teachers may experience a wider range of student achievement levels. This implies a need for both deep content and teaching knowledge that benefits a variety of students in developing statistical literacy.

Data from Banilower and colleagues' (2013) national survey provides an opportunity to examine non-AP statistics teacher characteristics. Specifically, the research aim is to capture teacher characteristics regarding their (a) background knowledge, (b) pedagogical practices, and (c) ability to apply this knowledge across various student achievement levels. A secondary analysis on the teachers indicating they taught a non-AP statistics course resulted in some important findings. In particular, a large proportion of teachers reported a wide range of student achievement levels in their courses and not feeling well prepared to teach statistics. Moreover, a notable proportion reported not frequently engaging students in building arguments for solutions and about half reported not frequently engaging students in considering multiple representations.

The findings of this study provide some direction for building professional development for non-AP statistics teachers that focuses on specific types of pedagogical practices that can be used across achievement levels. The border between secondary statistics and collegiate study currently appears to limit success and opportunities for developing statistical literacy. Such borders could be reduced with targeted professional development for teachers of non-AP statistics courses.

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INVESTIGATING SECONDARY STUDENTS' INTEGRATION OF EVERYDAY AND SCIENTIFIC IDEAS OF PROBABILITY (INDEPENDENCE AND DEPENDENCE)

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Keywords: Probability, Learning Theory, High School Education, Instructional Activities and Practices

Rationale and Theory

Lev Vygotsky (1987) asserted that learning occurs via the convergence of *everyday* understandings (real-world/concrete) and *scientific* understandings (school-based/abstract). However, students tend to experience a natural *border* between their everyday ideas and what they learn in school. Using Vygotsky's perspective, I conducted a study to determine ways in which instruction could be purposefully designed to foster the convergence of students' *everyday* and *scientific ideas,* as expressed through their language. I chose the topic of probability because of its importance in K-Post-Secondary education (Common Core State Standards Initiative [CCSSI], 2010; National Council of Teachers of Mathematics [NCTM], 2000) and because students must "recognize and explain" probability in "everyday language and everyday situations" (CCSSI, 2010). More specifically, I chose the concepts of *independence* (events not affecting each of their likelihood of occurrence) and *dependence* (one event affecting the likelihood of another's occurrence).

Methods and Analysis

Four high-school students participated in a four-lesson intervention. To elicit their *everyday ideas*, students were first asked to identify pairs of everyday events so that events in one pair were associated and events in the other pair were not associated. Then they were to explain if the events for each pair were independent or dependent. While receiving initially abstract and gradually contextualized instruction designed to develop their *scientific ideas* of probability, students routinely revisited their original contexts. Participant data were coded using a-priori and grounded categories to determine if the integration of *everyday* and *scientific ideas* transpired throughout the lessons.

Results and Discussion

Analysis revealed that most students' *everyday* and *scientific ideas* of *dependence* achieved integration, yet with some logical errors and misunderstandings of probability. For *independence* however, such integration occurred later in the lesson or not at all. One hypothesis for the greater *removal of borders* for *dependence* is that the term is often used in common language to indicate that one event affects the likelihood of the other (e.g., "that depends"). However, *independence* is often used in a way that indicates a relationship involving authority (e.g., "gaining independence from your parents"). This study contributes to the field's understanding about using instruction to intentionally foster the integration of *everyday* and *scientific ideas*. More research is needed to test this hypothesis and to determine whether scientific underpinnings exist for everyday ideas of other concepts.

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THINKING INSIDE THE BOX: STUDENTS' DIFFICULTIES WITH BOXPLOTS

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Common Core State Standards and other reform documents ask teachers to foster statistical reasoning at all grades (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). To do that, students need to have a profound understanding of statistical representations (Ben-Zvi & Garfield, 2005). Understanding and interpreting statistical representations are crucial as students build and improve their statistical reasoning and conceptual understanding in statistics. Boxplots are statistical representations that are fairly easy to create but complicated to understand and interpret. In this study, our research questions were: What are the strengths and difficulties of middle school students in constructing boxplots? How do middle school students read and interpret information presented in a boxplot?

Based on our interpretation of the literature on boxplots (Bakker, Biehler, & Konold, 2004; Pfannkuch, 2007; and others), we created a conceptual framework having five constructs: a major focus on the five-number summary, boxplots presenting data in an aggregated format, the density display of boxplots, the median as a measure of center, and students' previous knowledge of concepts. We studied 259 students in two middle schools in a large metropolitan area in the Midwestern United States. The teachers in both schools assured us that their students had previously studied boxplots. In order to investigate our research questions, we administered a pre-test and post-test the day before and the day after students completed a boxplot activity that used TI-Nspire calculators. In this paper, we report only what we found in the pre-test results.

As a result of cross-tab analysis on multiple items of the pretest, we found that a correctly constructed boxplot was not a strong predictor of providing correct responses to questions asking for an interpretation of information presented in a boxplot. Using previous knowledge and skills to construct a boxplot or make sense of a boxplot often interferes with this new way of representing and interpreting data. For instance, students learn about the frequency bar and the longer the box, the more data there are. When current previous knowledge and experiences do not provide a strong enough foundation for students to learn, think about, and interpret a new concept, it acts as a barrier preventing students from moving forward. When this happens, we need to think about new ways to introduce the new concept. With boxplots, we can get help from technology, which offers novel ways of visualizing and experiencing statistical representations with an access to dynamically linked multiple representations. Using these novel capabilities of cutting edge technologies, we can create representations that students can both act on and interact with as they learn.

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INVESTIGATING MIDDLE SCHOOL STUDENTS' UNDERSTANDING OF RISK AND SOCIAL JUSTICE IN THE MATHEMATICS CLASSROOM

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Educators and researchers are advocating for the teaching of risk and its exploration within an educational setting. For example, studies focus on teachers' understanding of risk (Levinson, Kent, Pratt, Kapadia, & Yogui, 2012; Pratt et al., 2011), as well as high school students' understanding of risk (Radakovic, 2015). These studies address risk in a personal context (e.g. risk of surgery) and a socio-scientific context (e.g. the safety of nuclear power plants). However, there is a gap in the literature when it comes to teaching risk in a social justice context (i.e. on the exploration of how risk affects various groups and how societal structures play a role in an unequal distribution of risk). The closing of this gap should be top priority, as it is well documented that underrepresented groups are more likely to be exposed to certain risks. The research questions for the study were: 1) How do the middle school students reason about risk? 2) What is the relationship between math and students' understanding of an authentic issue? and 3) What is the role of context in their understanding of risk?

Methods

We conducted semi-structured interviews with 14 seventh graders. The interviews focused on the students' understanding of the concept of the average life expectancy at birth and the factors that may influence it. The interviewer (one of the authors) presented students with average life expectancy data across countries, the US states, and socioeconomic groups. The interviews were one-on-one and were audio recorded, video recorded, and screen-recorded. The interview transcripts and video files were imported into NVivo 10 and coded using open codes. Using the constant comparative method, different themes were then identified. The themes include: student understanding of gender, socioeconomic status, and nationality in relation to the average life expectancy.

Findings and Discussion

This study is an initiating work in the critical pedagogy of risk as it merges two research programs: one in the pedagogy of risk and another in critical mathematics education. The findings show a complex relationship between students' understanding of risk (i.e. risks that may influence the average life expectancy) and their mathematical knowledge. Consistent with Levinson et al. (2012) and Radakovic (2015), understanding of the notion of average life expectancy depended on representations (graphs) as well on student values, experiences, and personal and social commitments (e.g. stance on gun control). The study serves as a blueprint for the further exploration of the critical pedagogy of risk in the classroom.

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INFORMAL REASONING ABOUT VARIABILITY IN SIMPLE BINOMIAL DISTRIBUTION BY HIGH SCHOOL STUDENTS

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In this work, we are interested in observing how high school students reason with or ignore variability when faced with a situation of prediction/uncertainty in which the underlying distribution is the binomial distribution $b(x, 2, \frac{1}{2})$.

The Conceptual Framework for this work is formed by concepts related to probabilistic reasoning and informal reasoning. The first one is the reasoning in which one of the 'big ideas' of probability occurs. These big ideas are randomness, variation, independence and the pair of complementary ideas, predictability and uncertainty, Gal (2005). The informal reasoning is a process in which the student builds a model of the situation, articulating several of its elements and obtaining consequences with the help of common sense and previous knowledge. Informal probabilistic reasoning is informal reasoning that involves some of the big ideas of probability.

The method consisted of four-step study with 37 high school students (15-16-year-old) who had not previously taken a course in statistics and probability. The task of predicting the results of drawing 1000 times the random variable $b(x, 2, \frac{1}{2})$ and giving its distribution, was administered in four steps: 1) Students were asked to respond according their actual knowledge. 2) Students simulated the situation with manipulatives. 3) Students simulated the situation using the software Fathom. 4) Students were asked to respond again the same questions.

As a result, we found that there are two main patterns of responses. 1) Responses that consider the theoretical distribution but without integrating variability of the results, for example: "probability distribution: ¼, ½, ¼. Prediction: 250, 500, 250 [expected frequencies]". 2) Responses that consider empirical distribution where variability is reflected but without consider the theoretical distribution, for example: "probability distribution: 0.23, 0.52, 0.25; prediction 234, 521, 245". Only in one case, in the final step, a student identified the distribution ¼, ½, ¼, and predicted frequencies different of expected frequencies. In closing, some students choose the frequency approach to probability but without considering the underlying distribution. Other students calculate classic probabilities and propose determined expected frequencies without considering variability. Articulating classic and frequency approaches, including the role of variability, requires larger processes of conceptualization.

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UNIVERSITY STATISTICS INSTRUCTORS' BORDER CROSSINGS: FROM LECTURE TO ACTIVE-LEARNING

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Keywords: Post-Secondary Education, Instructional Activities and Practices, Teacher Beliefs

The Modules for Teaching Statistics with Pedagogies using Active Learning (MTStatPAL) project creates and studies modules used to implement active learning lessons in collegiate introductory statistics courses. Modules include online pre/post-class activities for students, instructor materials, and in-class activities focused on active learning. While much is known about the benefits of active learning at the university level (Prince, 2004), less is known about university instructors' implementation of active learning in statistics classrooms. It is no simple matter for instructors to transition from lecture to active learning. Understanding how instructors develop new forms of practice while participating in mathematics education reform efforts requires an understanding of how instructors orient themselves to mathematics, to learning in general, and to mathematical learning in particular (Simon, Tzur, Heinz, Kinzel, & Smith, 2000). This ongoing project uses design experiment methodology (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) to understand how the learning ecology of statistics classrooms is impacted while teachers transition to implement the MTStatPAL modules.

This poster focuses on the question: how do faculty who have taught primarily using classroom lectures adapt their teaching practices to implement the MTStatPAL modules' active learning approach, and what issues arise during the implementation? Instructor interviews and observations of classroom instruction provide data for this investigation. Through a qualitative analysis of the data (open coding, memoing, and axial coding), connections have been observed between the instructors' orienting perspectives (Simon et al., 2000), conceptions of teaching (categorized along a continuum that describes the extent to which instructors have a relational view or a treatment view of teaching (Noddings, 2003)), and the ways instructors implement the modules. In the poster, we display ways in which instructors' conceptions of teaching and orienting perspectives are associated with the instructors' implementation of the MTStatPAL modules. This research addresses an important topic in mathematics education. Specifically, it reveals important issues to which university instructors must attend while crossing the border from lecture to active-learning pedagogies in introductory statistics classes.

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PRELIMINARY GENETIC DECOMPOSITION OF INDEPENDENCE OF EVENTS

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Shaughnessy (2003) indicates that "students have difficulty just sorting out the mathematics of whether events are statistically … independent" (p. 221). Ollerton (2015) attributes this difficulty to a disconnect between an intuitive sense of the term and its mathematically correct definition. This is supported by the fact that the mathematical definition is reliant upon conditional probability, however, independence is often introduced intuitively prior to instruction on conditional probability. As a means of determining the conceptual structures students must construct to eventually link their intuitive understands to formally conceptualized independence, I constructed a preliminary genetic decomposition of independence of events, using APOS theory, based on self-reflection and research on student thinking (Arnon et al., 2014).

Necessary mental constructions to conceptualize independence begin with simple probability. The basis for constructing an action conception of simple probability involves randomness, quantification, and logical disjunction (Piaget & Inhelder, 1951/1975). Within this frame of understanding, students require manipulatives to complete probabilistic tasks and are unlikely to be perturbed by a mismatch of frequentist and classical results, as students are unlikely to understand these two types of problems as equivalent. The process of calculating probability is external and cannot be anticipated mentally. To advance to a process conception, students must interiorize their actions, thereby generalizing the frequentist and classical probability to represent equivalent situations. This allows for probabilistic reasoning without manipulatives, the comparison of probabilities, and the reversibility of the mental structure. With a process conception, students have yet to formalize the law of large numbers, and are likely to be perturbed when frequentist and classical results are unequal. Not until simple probability is encapsulated into an object can students reason about multiple (compound) and nested (conditional) probabilities, as an object conception can support the coordination of combinatorial reasoning schemas with probability. Through this encapsulation, students resolve their perturbations regarding classical and frequentist probabilities and the law of large numbers. Ultimately, constructing independence involves the coordination of two objectified probabilities, and therefore, the construction of a schema conception of probability.

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Chapter 10

Student Learning and Related Factors

Research Reports

Brief Research Reports

Posters

IS THIS MATH? COMMUNITY APPROACHES TO PROBLEM SOLVING IN YUCATEC MAYA MATH CLASSROOMS

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In Yucatec Maya middle schools in the Yucatán, math scores are low and drop out rates are high. Although addressing larger social and economic causes may ameliorate these issues, improving math instruction may be a more immediate, feasible approach. This ethnographic, mixed-methods study explores community approaches to problem-solving relevant to middle school math classrooms. Findings indicate: (1) community members possess specialized, practical mathematical expertise that is overlooked in the research around rural, impoverished students, and (2) formal math instruction misses opportunities to capitalize upon cultural approaches involving autonomy and improvisational expertise. Results are relevant for curriculum reform in the US, México, and beyond.

Keywords: Equity and Diversity, Advanced Mathematical Thinking, Middle School Education, Problem Solving

A Yucatec Maya carpenter remodels a kitchen using no tape measure, yet armed with specialized knowledge of the 3-4-5 right triangle, a length of string, and a makeshift level crafted from plastic pipe and water. Despite this engineering expertise, he does not consider his mathematical knowledge to be legitimate, because it is not a product of formal schooling.

Introduction
This carpenter lives in a typical village in the Yucatán with low national math scores and high drop out and poverty rates. His story illustrates the tension between formal and informal schooling in rural, high-poverty communities in many countries. Today, in México and the U.S., academic achievement of socioeconomically disadvantaged students is a pressing issue, because the majority of students enrolled in public schools in México (INEGI, 2005), and the U.S., are low-income (Suitts, 2015). My study departs from the typical deficit framework as it documents, often unrecognized, expertise of socioeconomically disadvantaged students. Results inform mathematics instruction, because they illuminate undocumented cultural assets that can be capitalized upon in math instruction to teach 21st century skills like innovation and creativity.

Bourdieu states that students arrive to school with a "habitus," a well-established set of dispositions and knowledge imparted by families and communities that may be incongruent with school culture (1986). Several studies build upon Bourdieu's theory to illustrate that redressing tensions between home and school cultures of ethnic/racial minority students improves academic outcomes (Au and Mason, 1981; Lee, 1995; Lipka, 2005). Given that recent research in the U.S. demonstrates that the socioeconomic achievement gap is as salient as the ethnic/racial achievement gap (Carnoy & Rothstein, 2014; Reardon, 2011; Suitts, 2015), my study focuses on impoverished students. While the majority of studies emphasize deficits of low-income students such as high absenteeism and low rates of word recognition, one researcher, Laureau, takes an asset-based approach. She finds that working-class students in the U.S., because of their upbringing, are afforded certain qualities like autonomy that are not readily available to their more affluent peers (2003). Because schools, ultimately determine what constitutes legitimate knowledge and learning (McDermott, 2006; Saxe, 1985), these "assets" are not valued when navigating the U.S. school culture, which is constructed to advantage more affluent students. Like Lareau, I am pioneering asset-based research of economically disadvantaged students. I explore a specific case where two community assets in one Yucatec Maya Village are relevant to math instruction. Several landmark

ethnographic studies demonstrate antagonism between formal schooling and problem-solving approaches of impoverished or indigenous youth (Chavahay & Rogoff, 2002, Nunes, Schliemann, Carraher, 1993; Saxe, 1988). However, my study explores tensions that are especially pertinent to current educational initiatives in both México and the US to teach 21st century skills like innovation and creativity.

Results are relevant to math curriculum in México, because stronger Yucatec Maya ethnic identity and sense of school belonging are linked to positive academic outcomes for indigenous students (Casanova, 2011; Reyes, 2009). If formal schooling could reinforce cultural identities and foster a sense of belonging, then students might simultaneously maintain ties to their cultural heritage while excelling academically. Perhaps, capitalizing upon cultural approaches to problem solving like student autonomy and improvisational expertise could ultimately improve outcomes in the math classroom. Results are also relevant to math instruction in the US and elsewhere because 21st century skills require students to learn innovative and creative problem solving. In math classrooms, this means emphasizes solving open-ended, inquiry-based, and real-life math problems that have multiple entry points, requiring skills of adaptive expertise. This learning approach is linked to improved academic achievement and engagement for minorities (Boaler, 2002), and requires that students exercise autonomy and improvisational expertise rather than simply relying on pre-determined algorithms to solve single-solution problems.

My study is an ethnographic, mixed-methods study documenting how one rural, indigenous community in the Yucatán employs autonomy and an improvisational expertise to solve problems in everyday life. Furthermore, it explores the degree to which local middle school teachers capitalize upon these assets in the math classroom. Through grounded research, my study seeks to contribute to the discussion about: (1) math education of socioeconomically disadvantaged students using an asset-based framework; and (2) how to conceptualize autonomy and improvisational expertise in a manner relevant to implementing the national math curriculum promoting 21st century skills in both the US and México.

Methods

This village is a specific case of a rural, indigenous community that possesses a wealth of community problem-solving expertise, but low math scores. The three middle-school teachers are from the local community, but possess different experience and training. Research questions are:

1. How do community members use math in everyday life?
2. What is considered "legitimate" math knowledge and what traits make someone "good" at math according to community members, students, and teachers?
3. To what extent does math instruction capitalize upon community approaches to problem-solving?

Math In The Community And The Classroom

I define "community approaches to problem solving" as a constellation of culturally imparted mindsets, reasoning, skillsets, or strategies used to solve everyday problems that involve logic, spatial reasoning, navigation, or practical engineering. To explore how community members approach problem solving in everyday life, I conducted over 500 hours of participant observations in the village over a five-month period. In addition, I conducted 14 interviews with community members--including a focal family. I focused on businesses, transportation, community events, and central activities like farming. In addition, I conducted 15 interviews with six insiders to verify findings and adapt methodology. See Table 1 for the complete list of data sources and analyses. To explore the extent to which math instruction incorporated two community approaches to problem solving, I spent over five months at the local middle school. I conducted: 15 classroom observations;

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5-10 informal interviews and one video stimulated recall interview with each of two teachers; six student interviews; observations of two student tasks with 62 students with follow-up interviews with 3 focal students; and 280 student math mindset surveys.

Table 1: Matrix of Research Questions, Data Sources, and Analyses Methods

Question	Data Source	Data Analysis
1.) *How do community members use math in everyday life?*	Field notes from observing 500 hours of community activities over 5 months.	Open code. Write initial memos, then focused coding and integrative memos.
	14 interviews from a range of community members. Snowball sampling used.	Audiotape & transcribe. Open-code, a priori code, or statistically consider items.
	15 hours of Community Advisory interviews with 6 people to verify findings with school and community members.	Audiotape. Transcribe. Open code. Write initial memos, then focused coding and integrative memos.
2.) *What is considered "legitimate" math knowledge and what traits make someone "good" at math according to community members, students, and teachers?*	14 interviews with community members, 5-10 formal and informal interviews with each of 3 teachers, 6 student interviews, 15 classroom observations.	Audiotape and Transcribe (some). Open code. Write initial memos, then focused coding and integrative memos. Some items statistically analyzed (IRA & IRR)
	Audiotape and Transcribe. Open code. Write initial memos, then focused coding and integrative memos.	15 hours of Community Advisory interviews with 6 people to verify findings with school and community members. Some items statistically analyzed (IRA & IRR)
	280 Student surveys (questions 10 and 11)	Descriptive statistics, correlation analysis, subgroup analysis.
3.) *To what extent does math instruction capitalize upon student approaches to problem-solving?*	Series of 5-10, informal interviews with each of 3 teachers.	Open code. Write initial memos, then focused coding and integrative memos.
	One, 1-hour, VSR interview with two of 3 teachers on selected clips from one lesson.	Audiotape, transcribe, open-code for beliefs & reported practices.
	Field notes from observations of 5 lessons from each of 3 teachers.	Open code. Write initial memos, then focused coding and integrative memos.
	1 Videotaped lesson of two teachers.	Transcribe, open-code for observed teacher practices.
	280 student surveys with math mindset questions.	Qualitative and quantitative analysis (IRA & IRR).
	Observations of 2, one-hour student tasks with 62, 9th grade students that draw upon student approaches involving autonomy and improvisational expertise—with follow up interviews with 3 students.	Videotape. Transcribe. Open code. Write initial memos, then focused coding and integrative memos. Attitudinal question on tasks are a priori coded. Descriptive statistics on some items. (IRA & IRR)

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Results

The Disconnect

There is a disconnect between school and community math (Darling, 2016a). In this study, I defined math as any activity done outside of school that involves reasoning, justification, logic, arithmetic, or spatial orientating to solve problems. However, community members considered math as arithmetic one learns in school: counting, adding, dividing, paying. I asked a motorcycle-taxi driver, José, how he used math in everyday life. He responded, "I do not use math in everyday life, because I stopped going to school in the fourth grade." Interviews, student surveys and classroom observations indicated that community members thought someone who was "good" at math was someone who was "rapido" (fast) at doing arithmetic or solving school math problems.

Two Approaches: Autonomy And Improvisational Expertise

I documented two approaches to problem-solving in everyday life in the community; autonomy and improvisational expertise (Darling, 2016a). In essence, they are part of a student's habitus that specifically relates to problem solving. Similar to Rogoff, Paradise, Arauz, Correa-Chávez, & Angelillo (2003), I define autonomy as an independence, a self-directedness that is likely fostered by repeated opportunities to engage in independent play and problem solving without adult supervision. Like Yackel and Cobb, I reject "the conception of autonomy as a context free characteristic of the individual" (1996, p. 473). Instead I define autonomy with respect to students' participation in practices in specific contexts within the community. Lareau describes autonomy as independence among working class U.S. students that is at odds with navigating the school culture (2011). I define autonomy as an independence in terms of problem solving and how it may be at odds with math instruction. Also, I describe how it fuels improvisational expertise. I define improvisational expertise as a culturally imparted skillset or mindset used for solving real-life problems in innovative ways. For example, the mototaxista, José, tells me how he does not use math. However, when I ask him how he improvises in everyday life, without hesitation, he says, "yo calculo la kilometraje" (I calculate my mileage). Using no gas gauges or odometers, he calculates his mileage using an elaborate system based on collected fares, estimated distances between landmarks, and gas tank capacity. I documented six distinct methods used by mototaxistas. This is consistent with Saxe's finding that there is wider diversity of problem-solving methods in communities where there is limited access to schooling, as people tend to derive math meaning from solving real-life problems (1988). Findings indicate that autonomy fuels the improvisational expertise much in the same way that Rogoff, et all found that autonomy informs collaboration among indigenous heritage Mexican children (2003). In my study two children aged 5 and 9, with no money to purchase a kite, engineered one from garbage bags, salvaged sticks and string remnants. Unaccompanied by adults, they flight-tested their kites for an hour at the ocean's edge. They launched it with the wind, against the wind, and from the edge of a rock wall. They scavenged beach materials to add weight to the tail, lengthen the string, or adjust the kite's cross spar. They approached problem solving with autonomy and improvisational expertise, but will math teachers capitalize upon this practical, problem solving expertise when they go to middle school?

Opportunities Untapped In The Classroom?

School math instruction incorporated student autonomy and improvisational expertise in terms of completing tasks, but not in terms of solving math problems (Darling, 2016b). Many students chatted while the teacher was speaking. At 10-minute intervals during class at least 1/3 of the students were "off task". Students completed classroom tasks at the pace and order in the order they chose. Were they disrespectful? No. When an adult entered the room, students rose in unison and chimed, "Buenas tardes Maestro Olegario. " Teachers in this pueblo school accommodated students'

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autonomy and improvisational expertise in the math classroom in one non-math domain. This was similar to Boaler's 2002 study that demonstrated that minority students' math achievement improved in inquiry-based math classes where students were afforded the opportunity to complete tasks at their own pace and in their own ways. Olegario, the oldest teacher in my study said: Teachers provide "libertad" (freedom to act); students improvise while learning and completing tasks; consequently, students learn to act responsibly. In essence, teachers granted student autonomy, and students chose how and when to be "on task." Their autonomy informed their improvisational expertise, and students learned how to act responsibly. Although students were encouraged to be autonomous and improvise in terms of how they completed their tasks, teachers' strict adherence to the national math curriculum limited student opportunities to improvise while solving math problems. This is not surprising, because almost all opportunities to learn math involved single-solution, single-method problem solving.

I observed 62 students doing two math tasks based on real-life problems that invited improvisation. I gave attitudinal questions and did follow-up interviews. These triangulated to indicate that students welcomed opportunities to improvise, but did not like the "math part." The first task was presented in an ethics class to mask the math aspect. It required students to identify social problems in the community and then to address them by designing a community center according to explicit geometric specifications leading to the discovery of the Pythagorean theorem. Interviews indicated that students liked the improvisation, but not the math. The second task was completed in math class and yielded similar results. Students were asked to develop a plan to insure that a motorcycle taxi driver does not run out of gas. They were provided with real-life data in the form of a table. Three students interviewed said that at first they thought the problem was hard because they had to use math, then they said, It was easy when they realized they only had to use "sentido común"(common sense). In conclusion, teachers capitalized upon the cultural approaches of autonomy and improvisational expertise in terms of the completion of classroom tasks, but did not capitalize upon either in terms of solving math problems. Students enjoyed solving open-ended, multiple solution problems that were radically different than the typical math problems completed in class (Darling, 2016b).

Implications

Study results are relevant for alleviating academic achievement gaps in math in México. Incorporating previously overlooked cultural assets of student autonomy and improvisational expertise into the curriculum may improve math outcomes for Yucatec Maya youth, because ethnic identity and school belonging may be nurtured. Also, it is likely that these assets are not unique to this one Yucatec Maya community in México. Findings may be relevant to other impoverished, rural, or indigenous heritage immigrant communities in the U.S. Finally, the constructs of autonomy and improvisational expertise are relevant to math instruction that focuses on 21st century skills like innovation and creativity. Adaptive expertise is emphasized in reform mathematics classrooms, because it shifts the learning focus from algorithmic, procedural knowledge to deep conceptual understanding (Torbeyns, Verschaffel, & Ghesquière, 2006). Rather than teach students adaptive expertise from scratch, perhaps scholars could study how to capitalize upon already existing cultural assets to teach reform mathematics to historically marginalized students. Finally, perhaps we can learn from this Yucatec Maya community to teach other students to approach problem solving with autonomy and improvisational expertise to promote 21st century skills.

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USING LINGUISTICS TO EXAMINE A TUTORING SESSION ABOUT LINEAR FUNCTIONS

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We applied techniques from systemic functional linguistics to examine how a student and a tutor construed meaning related to linear functions during a 1-1 tutoring session. The student and tutor varied in how they discussed rates of change. This difference highlights that there are multiple correct ways to use this term in algebra, although small differences in speakers' use may create the potential for confusion. Additionally, the different ways that the student and tutor spoke about rates of change illuminate how any scenario may be represented by multiple different linear functions. This study has implications for providing teachers in a variety of settings information about how students construct meaning through their discussions.

Keywords: Algebra and Algebraic Thinking, Classroom Discourse, Instructional Activities and Practices

In most school districts in the U.S., Algebra 1 is a required course for students to graduate from high school and go on to post-secondary opportunities. In this way, Algebra 1 can be viewed as a "gatekeeper course"; in other words this course is a border between the work students are required to complete and the opportunities that come beyond it. Many students struggle to learn the foundational concepts of a typical Algebra 1 course, and disparities in student performance are greater along racial and socioeconomic lines (Chazan et al., 2007; Lubienski, 2002). There are efforts in place to provide one-on-one or small group tutoring services outside of students' typical mathematics classes, so that students can receive individualized attention and instruction (Hord, DeJarnette, & Marita, 2015; Hord, Marita, Walsh, Tomaro, & Gordon, in press). We conducted a project in which a group of university pre-service teachers provided tutoring for struggling eighth-grade students in Algebra 1 at an urban, high-needs public school. We sought to understand the features of students' and tutors' interactions that supported students' achievement in the course and conceptual understanding of algebra.

In this study, we focus specifically on one tutoring session between a tutor, whom we call Emily[1], and an eighth-grade student named Tanisha. We pose the question, *what are potential sources of ambiguity in a conversation between a student and a tutor about linear functions?* We use Systemic Functional Linguistics (SFL) (Halliday & Matthiessen, 2014) as an analytical tool to identify potential ambiguity in the pair's discourse. By looking for ambiguities in conversation, we bring up some of the barriers in communication that may surface when experts and novices talk about mathematics. Through a better understanding of how teachers and students construct meaning through discourse, we see opportunity to overcome these barriers and support all students to be successful in and beyond Algebra 1.

Theoretical Framework

We draw on a social semiotic framework to inform this study. This perspective emphasizes the importance of our choices in representation for constructing meaning through social activity (Kress & van Leeuwen, 2006; O'Halloran, 2014). Interactions in mathematics classrooms are multi-semiotic, in that communication between teachers, students, and textbooks requires a variety of representation systems including visual representations, symbolic notation, and gesturing (Alshwaikh, 2011; Arzarello & Edwards, 2005; Chapman, 1993; Dimmel & Herbst, 2015; O'Halloran, 2003, 2005; Radford, 2009). The use of spoken language can be considered one of the

primary means through which academic subjects such as mathematics are taught and learned (Lemke, 1988). In particular, spoken interactions can provide a primary means of support for struggling students in mathematics (Hord et al., in press; Ketterlin-Geller, Yovanoff, & Tindal, 2007; Scheuermann, Deshler, & Schumaker, 2009). For these reasons, we attend to the interactions between a student and her tutor through the lens of how the pair co-constructed meaning related to linear functions through their conversations about a task.

Thematic analysis is a method within the theory of SFL, which focuses on the ways that ideas are connected to one another (Lemke, 1990; see also Chapman, 1993; Herbel-Eisenmann & Otten, 2011; O'Halloran, 2005). The primary assumption guiding thematic analysis is that meaning is not made through individual words alone. Instead, meaning is given to words and phrases through the ways in which they are connected to other words and phrases. In an example provided by Herbel-Eisenmann and Otten (2011), the base of a triangle refers to two different constructs depending on how the term is used. In one context, the *base* of a triangle refers to a tangible part of the triangle, specifically the edge of the triangle that serves as the base. In another context, the *base* of a triangle refers to a measurement of that edge. Neither use of the term is more correct, but the term "base" is given meaning based on how it is used in relation to other words. Although communication is achieved through choices in representation—in this case, choices about the ways in which words and phrases are connected to one another—those choices are not always made explicit at the moment of an interaction (Lemke, 1990). Thematic analysis provides an analytical framework to examine the implicit connections between ideas.

Data and Methods

Setting of the Study

We conducted this study in a large, urban public school serving grades 7–12. All students at this school take Algebra 1 during their eighth-grade year, and students who do not pass the course are required to re-take the course during the summer. In collaboration with the eighth-grade mathematics teacher and the special education teacher, the second author established a tutoring program with the school. Beginning in December 2014, and running through the end of the school year, pre-service teachers went to the school on a weekly basis to work individually or in small groups with struggling students on their current classwork and homework. Students were selected for the tutoring program by the mathematics and special education teachers, who identified students that they expected to benefit from individualized attention.

In this study, we focus on one particular student, Tanisha, and her work with a tutor named Emily. Emily had developed rapport with Tanisha prior to the implementation of the present study. Specifically, Emily was an undergraduate pre-service special education teacher at a local university, and she had worked with Tanisha during a prior semester as part of a field placement focusing on literacy interventions. Tanisha was struggling in Algebra 1 and requested to participate in tutoring. Tanisha and Emily had a positive relationship, and Tanisha often requested to work with Emily. Tanisha was especially talkative around Emily, and she seemed highly motivated to use her tutoring time to make progress on her classwork.

Data Collection

We audio recorded all of the tutoring sessions, in addition to making copies of student work and taking field notes during the sessions. In total, we have records of 7 different tutoring sessions with Tanisha, ranging from March-May of 2015. For this analysis, we focus on one particular session, which took place between Emily and Tanisha in early May, when students were preparing for final exams. The task, included below, presented a scenario in which an individual named Katie was saving money from her part-time summer job.

Task: Each week in the summer Katie earned $95 as a lifeguard. Katie deposited 10% of her earnings into her account. At the start of the summer, Katie had $60 in her account. Write an equation to represent the amount of money Katie had in her bank account after a certain number of weeks.

At this time of the year, students had studied linear functions, and they were accustomed to setting up linear equations of the form *y=mx+b*. Students had previously learned about using linear functions to represent real-world scenarios. The above task served as a practice problem for students to prepare for their final exam. The conversation about the task lasted approximately 2 minutes and 30 seconds. We produced transcripts of the pair's conversation based upon the audio records, field notes, and work samples.

Methods

After transcribing the interaction between Tanisha and Emily, we identified the semantic relations between terms and phrases related to the rate of change. *Semantic relations* (Halliday & Matthiessen, 2014; Lemke, 1990) refer to relationships between words or phrases in a text. For example, when the *base* of a *triangle* refers to a specific part of the given triangle, the semantic relationship between the terms *base* and *triangle* would be that of part/whole (Herbel-Eisenmann & Otten, 2011). The part/whole relationship is used when a particular term refers to an object or item that is part of some larger object (Lemke, 1990).[2] Alternatively, when the *base* of a *triangle* refers to the length of that specific part of the triangle, the term *base* would be related as a measure of the *triangle* (Herbel-Eisenmann & Otten, 2011). By identifying the semantic relations among terms, it is possible to describe how words and phrases are connected to one another through speech and, thus, how meaning is construed in these interactions.

We focus our analysis for this presentation on Tanisha and Emily's use of ideas specifically related to the rate of change of a linear function. To do so, we first identified the key terms and phrases that Tanisha and Emily used in their comments about rates of change. For example, the pair used phrases such as "the amount that Katie earns per week," and "she gets that much money," in addition to phrases such as "rate of change." After noting these key phrases, we described the semantic relations between those words and phrases. Prior work using SFL has identified many of the relations that may exist between mathematical terms (Table 1) (Chapman, 1993; Herbel-Eisenmann & Otten, 2011; Lemke, 1990; O'Halloran, 2005). These relations build upon the identification of semantic relations among participants within and between clauses (e.g., Halliday & Matthiessen, 2014). We include in Table 1 only the semantic relations that are most relevant to this study.

Table 1: A Partial List of Semantic Relations Connecting Mathematical Terms

Relation	Description	Example
Part/Whole	An object or figure is part of a larger figure.	Katie deposits 10% of the $95 she earns.
Identified/Identifier	An object or figure is identified in a specific way.	$95 is the amount that Katie earns per week.
Synonyms	Two terms or phrases are equivalent.	The amount Katie earns is the amount of money Katie gets per week.
Subcategory/Category	A category fits inside a broader category.	Katie's total earnings are one example of varying quantities.

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Analysis and Findings

We present our analysis and findings together, focusing on the conversation Tanisha and Emily had about the task presented above. We include specific excerpts of the interaction where Emily and Tanisha discussed the rate of change of the function. We share parts of the transcript with our analysis, to exemplify our findings related to potential sources of ambiguity in the conversation. Numbers in parentheses indicate turn numbers. We identify speakers by their first initials. We bold parts of the transcript to highlight specific points in the analysis. To begin, Emily posed a question about Katie's earnings.

> *Emily*: So how much does **Katie earn per week?** (17)
> *Tanisha*: **95 dollars.** (18)
> *Emily*: Cool. So she **earns** a certain amount of money per week. So each week she **gets that much money**, right? (19)
> *Tanisha*: So **that's** the **rate of change.** (20)
> *Emily*: Yes, awesome. (21)
> *Tanisha*: So **that's m.** (22)
> *Emily*: Yes, wonderful. (23)

With this section of the transcript we can begin to identify the semantic relations among ideas in Emily and Tanisha's talk. We see that $95 identifies the amount that Katie earns per week, and Katie's earnings are synonymous with how much money Katie gets (lines 17-19). After establishing that, Tanisha made a statement that *that* (i.e., $95, or the amount of money Katie earns) was synonymous with the rate of change, which is denoted by *m* in a typical linear equation (lines 20-22). At this stage in the conversation, *Rate of change* seemed to be synonymous with *Katie's earnings per week*, at least as described by Tanisha in line 20.

After determining the starting value of the linear function to be 60, Emily returned to the issue that Katie would only be depositing 10% of her weekly earnings into her bank account.

> *Emily*: So what are we gonna do with the **10%**? The tricky part… (35)
> *Tanisha*: Probably **put it in for x.** Won't we put, $y=mx+b$, for *m* we're gonna put 95, b we're gonna put 60, and then that *x* **we're gonna put 10.** (36)
> *Emily*: You're right there. You're so close, and it's crazy. Cuz you're so – Okay, so **how much money does she put in her account per week? That's what *m* is gonna equal, each week how much money she puts into her account.** (37)

Following the question about what to do with the 10%, Tanisha suggested that they might substitute 10% for *x* in the equation $y=mx+b$ (lines 35-36). From an outside observer's perspective, Tanisha's response might indicate a complete misunderstanding of linear functions. However, consider the scenario from Tanisha's perspective. She had already determined that Katie's earnings were synonymous with the rate of change of the function, which would be represented by *m*. Following that, the pair had quickly determined the starting value to be $60, which would be represented by *b*. With those assumptions, Tanisha was limited in how she might use the value of 10% in an equation of the form $y=mx+b$. At this point, from Tanisha's perspective, *x* was the only variable that had not yet been assigned a value.

The source of ambiguity in the conversation between Katie and Emily starts to become clear in line 37, when Emily noted, "*m* is gonna equal, each week how much money she puts into her account." This statement seems to be in direct contradiction to the earlier exchange in lines 17-22, when Emily and Tanisha established that *m* would be equal to the amount of money Katie earns per week. To make sense of this contradiction, we consider the possible differences between the semantic relations used by Emily and Tanisha in making sense of the rate of change. For Tanisha, the semantic relation between *Katie's earnings* and *Rate of change* seemed to be a synonym relationship

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(Figure 1). In other words, the rate of change was equivalent to Katie's earnings, which had been identified as the specific value of $95 per week.
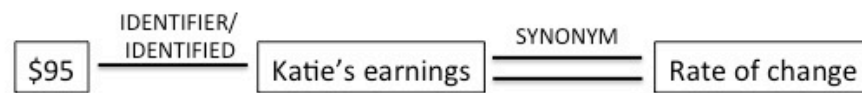
Figure 1. Tanisha's semantic relations between earnings and rate of change.

Once Emily clarified that the value of *m* would be determined by how much money Katie deposited per week, Tanisha needed to redefine the rate of change of the function. They agreed that Katie deposited 10% of her earnings, so Tanisha suggested that perhaps 10% would be the rate of change of the function.

> *Tanisha*: I mean, I meant, put **10%**, right? (42)
> *Emily*: Mm hmm. (43)
> *Tanisha*: So it'll be $y=\textbf{10}x+\textbf{60}$? (44)
> *Emily*: Close. Close. (45)

Here, Tanisha replaced *Katies's earnings* with *Katie's deposits* as synonymous with the rate of change of the function (Figure 2). Tanisha substituted 10 for *m* in the equation $y=mx+b$, as 10% identified specifically what Katie deposited on a weekly basis.
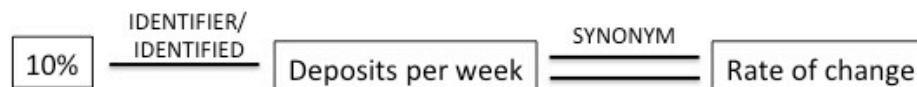
Figure 2. Tanisha's semantic relations when replacing earnings with deposits.

The semantic relation in Figure 2 seemed to replace Tanisha's earlier construction. Because Katie's earnings and deposits were not the same thing, it would be impossible for them to both be synonymous with the rate of change of the function. In order to account for the 10% deposit, Tanisha needed to disregard the previous information about Katie's $95 weekly earnings.

Emily seemed to be using a slightly different semantic construction to describe the relationships between earnings, deposits, rate of change, and the value of *m*. After determining with Tanisha that 10% of $95 is $9.50, Emily made the following comment:

> *Emily*: So **that is our new *m***. Because since she doesn't, since she doesn't deposit all 95 dollars, then that's not our rate of change. Our **rate of change is how much she deposits per week.** (55)

Importantly, in turn 55, Emily noted that the value of 9.5 represented a new *m*. To suggest that Katie's deposits represented a new value of the rate of change indicates that, for Emily, rate of change did not refer to a unique item. Instead, *Katie's earnings* and *Katie's deposits* could both be considered sub-categories of a broader category of *Rate of change* (Figure 3). Ninety–five dollars identified the specific value of Katie's weekly earnings, which provided one example of a rate of change. Additionally, 10% represented the part of that $95 that Katie deposited in her bank account, another example of rate of change.

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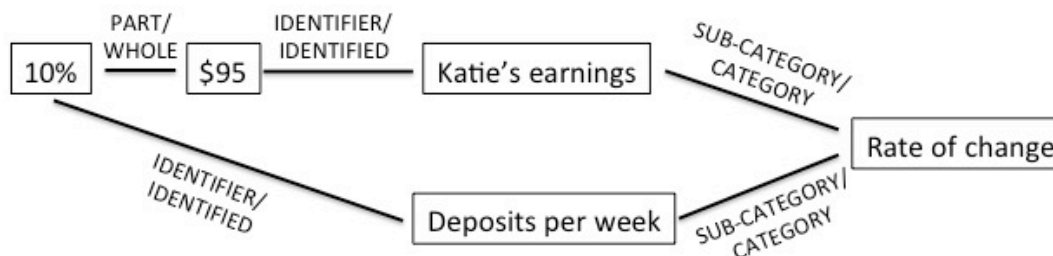
Figure 3. Semantic relations according to Emily.

In summary, there was one key difference in the semantic relations used by Tanisha and Emily, which seemed to be a source of ambiguity in their conversation. After establishing Katie's earnings as the rate of change of the function in turns 19-20, Tanisha used *Katie's earnings* synonymously with *Rate of change*. This became apparent, and also problematic, when Tanisha needed to account for the 10% deposited each week. Having already accounted for all of the parameters in the equation *y=mx+b*, Tanisha's only remaining option was to replace *Katie's earnings* with *Katie's deposits* as the rate of change. To make sense of Emily's comments, on the other hand, one can recognize Emily employed a sub-category/category relationship between *Katie's earnings*, *Katie's deposits*, and *Rate of change*, rather than a synonymous relationship. This difference helps explain Tanisha's apparent confusion about how to account for the value of 10% in the linear function.

Discussion and Conclusion

A primary point that we emphasize in our observations of Emily and Tanisha's conversation is that there is no single correct way to use specific terms in algebra. In a discussion about semantic differences in a science classroom, Lemke (1990) made an even stronger argument, noting, "words do not necessarily 'have' meanings in themselves. A word in isolation has only a 'meaning potential,' a range of possible uses to mean various things" (pp. 34–35). In this study, we saw that phrases such as "Katie's earnings" or "amount deposited" had potential for different meanings related to the rate of change of a linear function, depending on how the interactants used those phrases. Because Tanisha and Emily did not recognize the difference between their semantic constructions, when Emily proposed a new value for the rate of change, Tanisha attempted to accommodate this information into her existing semantic construction.

A natural question to pose is whether Tanisha or Emily was more or less correct in her semantic constructions. Emily, in fact, agreed with Tanisha when Tanisha suggested, "that [Katie's weekly earnings] is the rate of change." Only later did Emily propose that the $9.50 that Katie deposited per week would represent the rate of change. However, we keep in mind that the use of the term *is* can be ambiguous in how it relates terms or ideas (Schleppegrell, 2007). For Tanisha, to state that $95 *is* the rate of change seemed equivalent to stating that a dog *is* a canine; in each case the two phrases are synonyms for the same thing. For Emily, to agree that $95 *is* the rate of change seemed equivalent to suggesting that a dog *is* a mammal. In this case, the two terms are not synonymous. Instead, a dog is one subcategory of the category of mammals, which also includes many other things. Examples of this subcategory/category relationship can be identified in other areas of mathematics as well. For example, a square *is* a rectangle in the sense that squares constitute one subcategory of rectangles. From this study, we see that a construct such as "rate of change" can be viewed as a category of things, with multiple different subcategories. Because small differences like this in semantic relations are often left implicit in interaction (Lemke, 1990), research making these distinctions explicit is necessary to support teachers and students to recognize the nuances in their use of spoken language.

Beyond individuals' use of spoken language, this study highlights the way in which any given scenario can be represented by multiple different functions. The task as it was given required

students to write a function representing how much money Katie had in her bank account each week. Given the same information, one could represent how much money Katie had earned as a function of the number of weeks, using Tanisha's initial equation $y=95x+60$. Alternatively, one could represent the amount of money Katie had deposited as a function of how much she had earned using the equation $y=.10x+60$, similarly to Tanisha's third attempt at constructing the linear equation. Students in a variety of settings struggle to translate problem contexts into linear equations (Capraro & Joffrion, 2006). Moreover, students' errors in setting up linear equations stem partly from their attempts to make sense of the meaning of a problem (MacGregor & Stacey, 1993). We see it as a strength that students like Tanisha make connections between algebraic symbols and the scenarios they represent. To be able to correctly express a problem context symbolically, students must be able to interpret the relationship between the various quantities and components of that context. Students need opportunities to develop understanding of a context using informal language in order to be successful representing that context with symbols (Kieran & Chalouh, 1993). The specific function requested in the task could have been somewhat arbitrary, and an alternative question could have been posed for which one of Tanisha's linear equations would have been appropriate. The more that students can develop the skills to understand the semantic relationships between components of a problem context, the more equipped students will be to select an appropriate symbolic representation of that context.

Finally, because low-achieving students often opt out of whole-class discussions in mathematics classrooms (Baxter, Woodward, & Olsen, 2001), individualized or small group settings can be ideal for struggling students and students with learning disabilities to engage in conversations about mathematics (Woodward, 2006). In order to make the most of these opportunities, it is essential to understand how students and instructors work together to construct mathematical meaning through their talk. The analytic tools of SFL, and specifically analysis of the semantic relations among ideas in a text, provide a resource for ensuring that students and instructors are able to understand one another's meanings.

Endnotes

[1] We use pseudonyms for all names and institutions.

[2] Lemke (1990) and Herbel-Eisenmann and Otten (2011) use the more formal linguistic terms meronym/holonym to describe the part/whole relationship. We use the more colloquial terms here for brevity and clarity.

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AN EXAMINATION OF RACIAL COMPOSITION IN CULTURALLY RELEVANT MATH STUDY GROUPS ON MATH LEARNING OUTCOMES

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This paper proposes to use communalism, an asset-driven strategy effective with Black learners, with racially mixed study groups. This study found that students who studied in racially mixed groups performed better when told that individuals would receive a reward for scoring the highest in their group. Also, the White students' perceptions of how much they mattered differed based on the racial composition of the study groups where those studying with only White students felt they mattered more than those who studied with White and Black participants. Implications of the importance to have culturally relevant and responsive resources for racially diverse classrooms are discussed.

Keywords: Elementary School Education; Equity and Diversity; Instructional Activities and Practices; Affect, Emotion, Beliefs and Attitudes

Purpose

Currently, there have been no longstanding viable solutions to the problem of the achievement gap between Black students and their White counterparts. As revealed by the Nation's Report card (NAEP, 2015), though both group's scores have increased in the past 30 years, the White-Black learning gap has persisted with no significant gains made by Black students to close it. Black students' learning gap is a more severe problem when compared at the global level, where students in the United States are falling behind the scores of students from other developed nations (Boykin & Noguera, 2011). Students in countries such as Singapore, South Korea, Japan, Belgium, The Netherlands, Hungary, and Russia are all out-performing students in the United States in both fourth and eighth grade mathematics. To close this international achievement gap, culturally relevant and responsive resources targeting the performance of Black (and Brown) students both nationally and internationally must be implemented (Blanco-Álvarez & Luisa Oliveras 2016), as the number of K-12 students of color are projected to increase consistently in the coming years (NCES, 2011). Mathematics achievement, in particular, is imperative--as it has been noted as the "new" civil rights movement (Paige & Witty, 2009)--where students' achievement is linked to opportunities in the global political economy (Blanco-Álvarez & Luisa Oliveras 2016).

This study hypothesizes to increase scores in students of colors' mathematics performance by utilizing cultural assets that students possess stemming from norms and values in their home and surrounding communities (Hurley, Allen, & Boykin, 2009). Based on work that flows from "communalism" (Boykin & Noguera, 2011), the authors sought to explore the effects of an afro-cultural learning context in a racially mixed classroom setting. Specifically, the goals in this study were to examine A) the effects of racial composition of study groups and B) learning context/communalism on A)math performance, B)group processes/behaviors, and C)students' perceptions of how much they matter in their study groups. Implications from findings are to add more depth to the literature on the importance of group learning contexts, such as communalism, as a culturally responsive paradigm geared to empower nondominant youth's learning ecologies.

Theoretical Framework and Relevant Literature

In order to increase the performance of Black students, Boykin and Noguera (2011) posited that it was important to utilize students' assets in educational reform strategies. Asset-focused factors deal with the contextual conditions in which teaching, learning, and engagement are manifested (Boykin & Noguera, 2011). They are called assets because they involve learning exchanges that

build on what students bring to the classroom. Within the asset-focused factors are strategies that strive to align the values, interests, and learning priorities of both teachers and students (Boykin& Noguera, 2011). The authors describe meaningful learning and cultural resources as two major strategies that can create this alignment. However, for the purposes of this study, the researchers paid specific attention to cultural resources. The use of cultural resources, like culturally relevant pedagogy, understands that schooling does not take place in a vacuum, and students' cultural orientations can be valuable resources in the classroom. Out of these resources, one of the most successfully implemented cultural resources has been communalism (Boykin, 1986).

Boykin (1986) defines communalism as "a commitment to social connectedness which includes an awareness that social bonds and responsibilities transcend individual privileges" (p. 61). Similarly to collectivism (Triandis, 1995), communalism is an ethnographic paradigm that stands in contrast to that of individualism or—for this study—interpersonal competition. Where interpersonal competition focuses on individuals placing an emphasis on their own performance and accountability, communalism places individual performance as a responsibility of the group where all members are held accountable for the group's success (Boykin, 1986; Johnson et al., 1981). A number of empirical studies have found that Black students' performance has increased when they were placed into communal group learning contexts across a number of learning tasks. This was shown in with performance measures such as: text recall, multiplication performance, math-estimation, vocabulary, geography, and learning transfer (Dill & Boykin, 2000; Hurley, Allen, & Boykin, 2009; Hurley, Boykin, & Allen, 2005).

When students perform better academically after studying in a group, whether communally or cooperatively, there must be some underlying group processes at work that cause students to work well together where they would not have done as well working alone. To understand these processes, Hurley, Allen, and Boykin (2009) examined group processes as a means to see the differences between communalism and other group-based learning contexts. Indeed, the researchers found that there were significant differences in students' ratings for learning context. Students in the communal condition were rated higher in group behaviors (e.g. accountability for actions, {a lack of} task hindering behaviors) than did those in a competitive group condition. Moreover, a marginally significant difference was found for the interaction between condition and ethnicity. In this case, Black and White students achieved their highest ratings for positive group behaviors when they were in different learning contexts, Black students in the communal condition and White students in a more individual oriented learning context condition (Hurley, Allen, and Boykin, 2009). Furthermore, both groups of students performed best in the conditions that yielded their highest group ratings. Though not significant, the findings cannot go unnoticed.

In addition to understanding the link between communalism and group processing, the present authors sought to examine students' perceptions of whether they matter during their group work. The mattering literature had its start in the mental health field first examined by Rosenberg and McCullough (1981), who defined mattering as the "direct reciprocal of significance" (1981). The scholars further described mattering as a "motive: the feeling that others depend on us, are interested in us, are concerned with our fate, or experience us as an ego-extension exercises a powerful influence on our actions." Wicker (2004) further describes mattering as how one perceives himself as being important to another person, or group of people. Simply put, mattering asks the question: How important do I feel I am to you (Wicker, 2004)? Mattering has been divided into three components: attention, importance/appreciation, and dependence (Elliot, Kao, & Grant, 2004).

In its 30 years of research, perceived sense of mattering has been tied to important psychological outcomes such as self-esteem, relatedness to others, sense of purpose, meaningfulness in life, and job satisfaction in higher education (Gibson and Myers, 2006; Rosenberg and McCullough, 1981). For example, in a study on students attending The Citadel, a military college in the southeastern United States, researchers found that perceived mattering was positively related to a measure of Total

Wellness, particularly as it relates to the Social Self subfactor where the underlying constructs were 'love' and 'friendship' (Gibson and Myers, 2006). In a particularly disturbing study, Elliot, Colangelo and Gelles (2005) found that mattering was a significant predictor of adolescent suicide ideation; where the further participant responses for mattering went below the mean, the further the instances of suicide ideation increased. As the researchers looked more deeply into the data, they found mediating properties of other variables such as depression, self-esteem, and religiosity. Researchers found a model where depression and self-esteem mediated the effects of mattering on suicide ideation being that the addition of depression and self-esteem completely erased the established relationship between mattering and suicide ideation (Elliot, Colangelo, and Gelles, 2005). To confirm the literature that says mattering is an issue across one's life span, Dixon (2007) explored the relationship of mattering and important wellness variables for older adults; here the author found that mattering to others was a significant predictor in participants' overall wellness (positively related to purpose in life and overall wellness and negatively related to depression).

To understand why students perform differently in different learning contexts, the purpose of this study was to examine how afro-centric learning contexts, specifically utilizing communalistic cooperative learning settings, relates to group processes/behaviors and student mattering. Moreover, although numerous studies have empirically proven communalism's utility in improving African American students' achievement in performance tasks, further research is needed to examine its utility with racially mixed student demographics where African American learners are present. Thus, this study sought to answer: what are the effects of racial composition of study groups (All Black students or all White students in the same group) and mixed racial groups (Black and White students in the same group), and communalism as a learning context on the math performance, group processes, and perceptions of mattering for 5th grade Black and White students? The study also sought to understand what, if any, were the relationships between academic performance, group processes, and perceptions of mattering.

Methodology

The research reported here stems from a math intervention project that sought to investigate the math performance, perceived group processes, and perceived mattering for students as a function of learning context and the racial composition of their study groups. Data from this study was collected from a math performance task and survey measures in the southeastern United States. The sample consisted of 110 Black and White 5th grade students. These students were sampled from three racially integrated elementary schools. Each school was a part of a large school district in a large metropolitan area.

For this design, all procedures were conducted in a simulated classroom setting excluding any students who were not Black or White. Prior to the manipulation of the independent variables, participants took a pretest to assess their skill level in mathematics-estimation. Their scores were used as a covariate to control for individual/condition variation and prior knowledge before the learning phase of this study. To orient them to mathematics estimation, the researcher took the students through an example problem on the first two pages of the math estimation study packet. The students were required to follow along and answer any questions that the example problem asked. The orientation portion of the study took approximately 5 minutes. After the orientation, the researcher read the instructions for the math estimation study packet to the participants and then read the prompt for the group's respective learning context.

Prior to the administration of this study, students were randomly assigned to group learning contexts that were either the communal condition or the interpersonal competition condition. For both the communal and interpersonal competition learning contexts, students were placed in groups of four, sat together in a cluster of desks, and given one math estimation study packet. Students in the communal condition were encouraged to work together, share, and do well for the good of the

group. Students in the interpersonal condition were told that the student who performs the best on the quiz would win a prize, and that they should each worry only about themselves as they studied. The students then began the 20-minute mathematics estimation study session. Students were also randomly assigned to either participate in a racially homogeneous group or a racially mixed/heterogeneous group.   The homogeneous racial groups had students of all one race working together in the two learning context conditions. Students' study groups were either all Black or all White. In the heterogeneous racial groups, there were two Black students and two White students participating in the study session together. Students then completed the opposite form of the math estimation performance measure individually to ensure individual performance was being measured.

Students completed two surveys, the Process Loss Questionnaire (PLQ; 20 items on a 1-4 4-point likert scale; alpha reliability coefficient was .75, 13 items were used for analysis) to assess group processes, and the Perceived Mattering Questionnaire (PMQ; modified version included 15 items on a 1-5 likert scale; the three components of the mattering index reached the proper level of internal consistency with reliabilities of .78, .82, and .72 for attention, importance, and dependence, respectively). The PLQ (Hurley, Allen, & Boykin, 2009) is a measure that is built to understand children's specific group-work related behaviors. The scale pays attention to behaviors that are "presumed to be manifestations of the participants' perception of their accountability, and of the level of task hindering group dynamics that occurred in their study groups." The PMQ is a modified version of Elliot, Kao, and Grant's (2004) Mattering Index. It contains three components of the mattering construct (attention, importance, and dependence), and where the original mattering index only asks questions about students mattering as a general state, the items in the PMQ were modified to reflect the participants' sense of mattering in the group context during their study session. To ensure the students understood the questions in the surveys, the first author read each question, and explained each questionnaire's answer choices.

Results

The study sought to examine the effects of three variables (learning context as a cultural resource, racial composition of the study group, and race) on the math performance, group processes, and mattering of 110 fifth grade children. The data were analyzed by conducting a 2 (Learning Context) X 2 (Racial Composition of the Study Group) X 2 (Race) Analysis of Covariance on the dependent variable, math performance, using pretest performance as a covariate. Secondly, the group processes and mattering variables were entered into a 2 (Learning Context) X 2 (Racial Composition of the Study Group) X 2 (Race) Multivariate Analysis of Variance. Lastly, the dependent variables were entered into a correlation analysis to test the relationships between them.

For the first set of analyses, there was a two-way interaction between learning context and racial composition on performance ($F (1, 101) = 5.285$, $p < .05$). Based on pair-wise comparisons as a post hoc analysis, students who studied in the interpersonal competition performed significantly better than their communal studying counterparts (11.69 and 8.75, respectively). No other combinations of the variables' effects on math performance were statistically significant. Figure 1 below displays the means for performance in highlighting the aforementioned two-way interaction.

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.
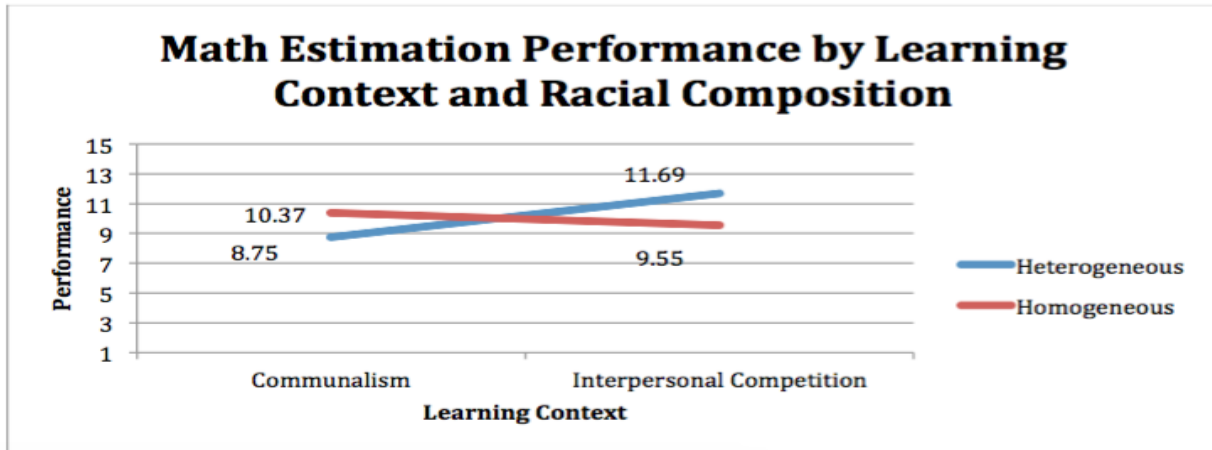
Figure 1: Math performance by learning context and racial composition.

For the next set of analyses, there was a significant two-way interaction between race and racial composition on participants' perceived mattering ($F (1, 101) = 6.529$, $p < .05$). In this interaction, post hoc analyses indicated that participants who studied in the homogeneous groups felt they mattered more (3.55) than those who studied in the heterogeneous groups (2.94). However, no other differences between the combinations of race and racial composition were statistically significant. Figure 2 displays the means for the two-way interaction.
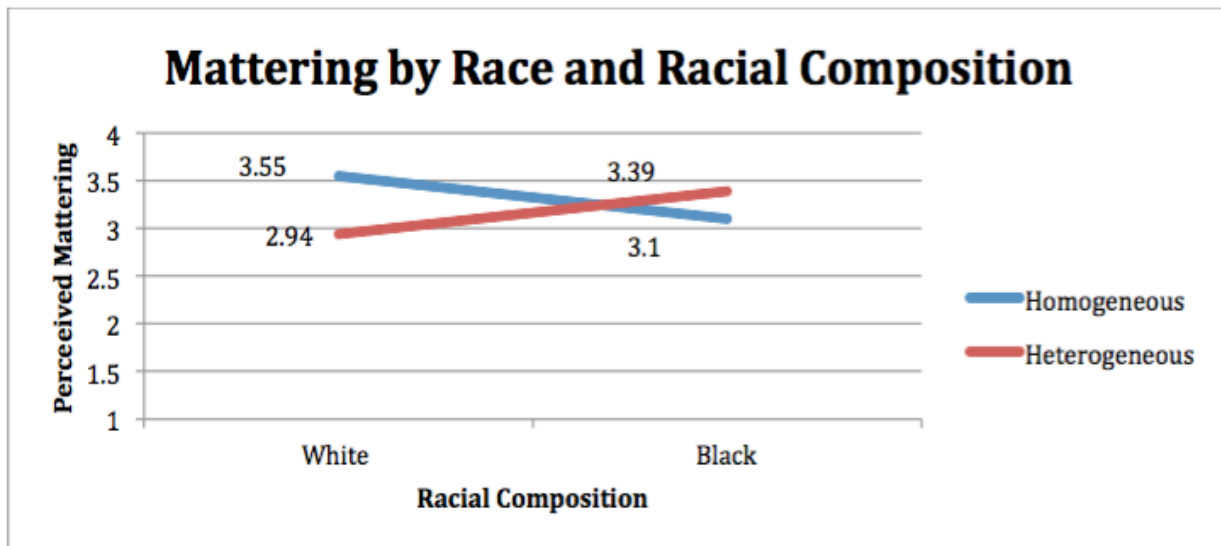
Figure 2: Mattering by race and racial composition.

Lastly, the researcher ran a correlation analysis for the dependent variables including the three subscales for the mattering survey. This analysis found that there was a negative relationship between the attention component of mattering and group process ($r = -.248$, $p < .01$) suggesting that as students perceived themselves to hold other's attention more and be noticed more by their group members, they exhibited less process loss behaviors. The table below highlights the correlation coefficients for all of the variables of interest.

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

	1	2	3	4	5	6	7
1. PRE	1						
2. POST	.365**	1					
3. GRP	.037	.168*	1				
4. MATT	.222*	.143	-.103	1			
5. ATTN	.168	.124	-.248**	.890**	1		
6. IMP	.196**	.140	.076	.841**	.620**	1	
7. INT	.242*	.110	-.078	.841**	.620**	.713**	1

Note: PRE = Pretest; POST = Post test; GRP = Group Processes; MATT = Mattering; ATTN = Attention; IMP = Importance; and INT = Interdependence. ** = Correlation is significant at the .01 level. * = Correlation is significant at the .05 level.

Figure 3: Correlation analyses of dependent variables.

Discussion and Implications

To date there has been no work exclusively examining the studying of Black and White students *together* in a communal learning context. The added layer of racially heterogeneous study groups to the communalism research led to this experiment having an exploratory nature. Finding that students studying in the heterogeneous groups with an incentive for their individual performance performed better than those who studied communally was a very interesting outcome to add to the communalism research. The pattern of findings is not an indictment of the previous communalism literature (Boykin, 1986). The findings could be attributed to the possibility that working communally in a heterogeneous group presented African American students with difficulty in code-switching, or a sense of culture shock that disrupted their abilities to work well together (Baugh and Graen, 1997; Stahl et al., 2010). On the other hand, this was only apparent for the communal groups and not the interpersonal competition groups. The students-theoretically—had to work together in this condition. However, they did not have the same impetus for cooperating with one another. The students only were required to focus on their own wellbeing when it came to the performance on the math estimation task. This competitive reward structure could also be more typical of their traditional group work, and could have led the interpersonal competition condition to be more comfortable for them.

This study also found that students' perceived sense of mattering was lower when they studied in racially heterogeneous study groups than those who studied in the homogeneous study groups. From this set of analyses, it seems that White students experienced some pro White bias regarding when they felt they mattered most based on whether they studied with students of their same race or with Black students. Contrarily, Black students experienced no differences based on racial group composition. One explanation from previous literature could potentially shed light on this pattern of findings. First, some of the mattering research examines people's feelings of marginalization and how they relate to perceptions of mattering. In this sense, experiences of marginalization have led students to feel that they matter less. Interestingly enough, in the mattering research, the marginalization-mattering connection has been made for participants of color including Black, Latino, and Middle-Eastern students attending school in the United States (Huerta & Fishman, 2014). However, in this study, perhaps the students felt some sense of marginalization as they worked in the heterogeneous groups that led to a lower perception of their own mattering. The possibility exists that students have been more comfortable working exclusively with their own race whereas Black

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students are not affected by a change in racial composition as feelings of marginalization occur both in and outside of the classroom setting in racially mixed contexts. Also, even though the groups had the same number of Black and White students, White students possibly felt some feelings of exclusion, what some students of color feel on a daily basis.

Lastly, there was a relationship between group processes and the attention component of the mattering construct. This relationship indicated that as one felt that they were noticed more by their group members they exhibited less process loss behaviors characterized by a lack of accountability of one's own actions but a proliferation of task hindering dynamics. This result may be due to a link in how one perceives his or herself and how he or she then interacts with the group as they study. One possibility is that as students perceive themselves to be "invisible" in a setting, they may seek out behaviors that led to lack of positive group processes. Attempts to be noticed may work, but they may ultimately sacrifice the wellbeing of the group's interactions as they work together on the given task. Accounting for this outcome requires further research.

Conclusion

This research has opened the door to the complexities of communal research when one attempts to use a sample perhaps more closely aligned to the average make up of the classroom across the nation. Placing both Black students and White students together in a communal learning context when studying mathematics estimation did not benefit both races significantly. However, with the current trends of the student population for this country, it is important to change the focus from what has historically worked for and in favor of White students and utilize what has been effective for African American and other students of color. Doing so can solidify achievement solutions for generations to come. There is promise in how the students behaved while they were in their study groups regardless of the condition. Researchers expected some arguing and scrambling for materials displayed by the students, especially in the interpersonal competition condition. However, this occurred minimally, and most participants either shared the materials completely of used some form of taking turns during the study phase of this experiment. Therefore, with more time, performance could follow the example of how the participants acted towards one another, especially in a communal setting. Lastly, the examination of mattering as a variable in the classroom setting shed light on how students in diverse classrooms' perceptions can change based on different learning contexts. Future research should plan to understand how students' sense of mattering operates to potentially help or hinder their wellbeing and/or academic performance in the school environment.

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A QUEER TURN IN MATHEMATICS EDUCATION RESEARCH: CENTERING THE EXPERIENCE OF MARGINALIZED QUEER STUDENTS

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Evidenced by the lack of research considering queer students, mathematics education researchers are continuing to marginalize the experiences of queer youth and the only resolution is to center the queer student experience in the mathematics context. To accomplish this, I choose to dwell in the borderlands between queer theory and mathematics not only to challenge the border between them, but also to push on the borders of mathematics education research. In order to do so, I offer ways in which mathematics education researchers can take a queer turn in mathematics education research by generating queer curriculum, engaging in queer pedagogy, and queering mathematics content.

Keywords: Equity and Diversity

> The knower and the known are intertwined, for realizing that a science without humanity - without values, purposes, beliefs - is a false science, the false science of the spectator who always stays outside the arena of action, removed from the existential happenings of life - the 'pure' scientist as the phrase goes. This spectator view of science, of knowledge and of teaching is a thing of the past. (Doll, 1989, p. 248)

Introduction

William Doll foreshadowed the sociopolitical turn in education, that would not take root in mathematics education until more than a decade later (Gutiérrez, 2013). Another turn, a queer turn, began in education more than a decade ago (e.g., geography (Elder, 1999); biology (Snyder & Broadway, 2004); English (Greene, 1996)) and is invigorated with attention today (e.g., Gowlett & Rasmussen, 2014). This call to include queer theory in mathematics education research is not independent of the call for the inclusion of gender and sexuality topics in pre-service and in-service teacher education (Hansen, 2015; Martin, 2014; Robinson & Ferfolja, 2008; Vavrus, 2009). Nevertheless, the mathematics education research community has yet to engage with queer theory in earnest. In this theoretical exploration, I argue that mathematics education researchers continue to marginalize the experiences of queer youth by not focusing on queer students in their research, and the only resolution is to center the queer student experience in the mathematics context. At the expense of oversimplifying, queer as an identity will be used as a non-exclusive umbrella term for lesbian, gay, bisexual, and transgender (*à la* LGBT). *Queer* is discussed at length in the following section. Research centering queer students is necessary if we hope for equitable opportunities to learn mathematics for all students (Esmonde, 2007).

Mathematics education has a *significant* role in students' development of learner identities (Gates & Jorgensen, 2009) and student positioning has a lasting effect on children's perceptions of their available social roles (Cannela, 2008). In light of this research, hegemonic discourses within mathematics, such as *mathematics as masculine* (Mendick, 2006) and the *white male math myth* (Stinson, 2013), are particularly damaging. Despite this concern, the mathematics education research community cannot begin to identify equivalent discourses which oppress queer students if it continues to marginalize the queer student experience. This tension is evidence of the borderlands created between sexuality and mathematics, queer theory and mathematics education. The mathematics classroom is held to be culture-free or unbiased, despite literature indicating the oppressive nature of content and pedagogy (e.g., Kumashiro, 2004; Ladson-Billings, 1997). The ability to consider sexuality irrelevant in the mathematics context is a heteronormatively privileged

position; for queer students, their queerness, indeed *other*ness, is intersectional across social contexts. As Anzaldúa (1987/2012) does, I choose to dwell in these borderlands, for me the borderlands between queer and mathematics, to leverage my "outsider within" status as a queer individual in the mathematics education context not only to challenge the border between mathematics and queer, but also to push on the borders of mathematics education research. In order to do so, I offer ways in which mathematics education researchers can take a queer turn in mathematics education research by generating queer curriculum, engaging in queer pedagogy, and queering mathematics content.

Queer(ing) Identity

Queer does not have a fixed definition; it is "relational, in reference to the normative" (Letts, 2002, p. 123). Historically, queer was linked with insult and shame. Today, queer has become the "rallying point" not only for young gays and lesbians concerned with the homonormative images of gay men and lesbian women, but also for those whom wish to identify themselves with the anti-homophobic movement (Butler, 1993). By combining the works of Butler and Anzaldúa (1991), queer is understood as a false unifying umbrella, useful for solidarity with the necessary error of homogenizing and erasing differences to yield a temporarily totalized identity, which then necessarily fails to represent the person. Queer as an identity, henceforth, will be used as an umbrella term for lesbian, gay, bisexual, transgender, and other gender and sexual minorities.

Queer students can and do have academic and emotional success; Robinson and Espelage (2011), however, highlighted the higher incidence of suicidal thoughts, suicide attempts, victimization by peers, and unexcused absences for queer students compared with their non-queer peers. Additionally, Toynton (2007) documented the alienation of queer science students while Yoder and Mattheis (2015) followed this phenomenon of marginalization into STEM workplaces. I contend that mathematics education researchers are in a key position to work against the alienation of queer students and their underrepresentation in STEM careers by challenging discourses and structures which position queer students as *other*.

Beyond the classical components of identity (e.g., gender, sexuality, racialized group, etc.), Bishop (2012) offered mathematics identity as the collection of ideas one has about who they are, and the way they should act, in the mathematics context. Bishop offers identities as multiple, flexible, and fluid, and "ways of acting" (p. 39) suggests compatibility with identity as performative (Butler, 1993), but Bishop does not offer an operationalization of performative identity. To that end, Darragh (2015) operationalized the notion of a performative mathematics identity as the repeated performances that shape student recognition of themselves as certain types of learners of mathematics and, in particular, focused her research on whether students could see themselves in the descriptions of performances of good at mathematics that they identified. The notion of intersectional identities, introduced next, yields that studying mathematics students' identities, in particular their mathematics identity, without consideration of queerness, will be necessarily incomplete.

Intersectional Identities

Envez de dejar cada parte en su región y mantener entre ellos la distancia de un silencio, mejor mantener la tensión entre nuestras quatro [sic] o seis partes/personas [Instead of keeping each part [of our identities] separate and maintaining a "distance of silence" between them, it would be better to hold in constant tension our four or six identities/*personas*]… There is no way that I can put myselves through this sieve, and say okay, I'm only going to let the "lesbian" part out, and everything else will stay in the sieve. All the multiple aspects of identities (as well as the sieve) are part of the "lesbian." (Anzaldúa, 1991, p. 252-3)

In the first half of this quotation, Anzaldúa not only highlights the multiplicity of our identities by mentioning a rhetorical four or six parts of identity but also advocates against maintaining a

"distance of silence" between them. Furthermore, Anzaldúa's assertion that each part of her identity is also a part of her lesbian identity, highlights the intersectionality of identity and, in turn, the necessity to consider student queerness in mathematics education research. I connect the second half of Anzaldúa's quotation, the analogy of the sieve, to the notion of researcher positionality (e.g., Foote & Bartell, 2013). Just as Anzaldúa highlights the impossibility of realizing some idealized form of a lesbian writer by leaving her Chicana identity unacknowledged, so too it is impossible for the mathematics education research community to leave all aspects of our identity when we perform mathematics education research. In particular, Milner (2007) provided seen, unseen, and unforeseen dangers that might emerge if researchers *detach* themselves from their research. In a sense then, through this article I "come-out" as a queer theorist with queer intentions as a mathematics education researcher. I offer my abbreviated positionality as a white, queer, assigned-male-at-birth individual that is able to *leer y escribir en español* and instead of maintaining a distance between these facets of identity, I keep them in tension, bringing my queer identity to bear on my mathematics education research.

A Queer Turn in Mathematics Education Research

Thus far, I have motivated the necessity to consider the needs of queer students in the mathematics education context. In this section, I introduce queer theory and investigate the intersection of queer theory and mathematics education research. Like Letts (2002), who held queer theory "still long enough to get a good look at it," I do not claim to present queer theory as a defined package. Instead, I first provide a theoretical overview of queer theory followed by *instantiations* of queer theory. These instantiations are like a photograph that freezes-in-time one instance of queer theory in practice and necessarily reports it back in an incomplete way. I introduce queer theory here for three key reasons. First, existing research fails to address the experience of the queer student in the mathematics context, which queer theory enables. Second, the notion of intersectional identities yields that each facet of identity is part of each other facet and, as a result, studying mathematics students' identities (e.g., racialized or gendered) without consideration of queerness will be necessarily incomplete. Finally, I introduce queer theory to offer an illustration of the types of work that queer theory can enable in mathematics education.

Queer Theory

Queer theory: is an epistemological stance (Letts, 2002); appropriates the forms of curriculum and pedagogy to investigate how sexuality is organized, how sexuality is identified, how knowledge is unfixed, unstable, and how knowledge "unfolds…subject to individual insights and cultural contingencies" (Davis & Sumara, 2000, p. 832; Sumara & Davis, 1999); is deviant and critiques normativity (Jourian, 2015); signifies action and is unstable and multiplicitous (Britzman, 1995); and is about making normal queer, revealing the socially constructed nature of truths and selves, and ultimately asks "what can be, rather than what is" (Gunckel, 2009, p. 63; Snyder & Broadway, 2004). The eight authors cited in this list present eight different descriptions of what queer theory does; queer theory *appropriates, investigates, critiques, identifies, deviates, signifies, reveals*, and ultimately, *queers*.

In *A Critical Introduction to Queer Theory* (2003), Sullivan provided a detailed history of socially constructed sexuality and gay and lesbian rights activism. Two key movements, the assimilation and liberation movements, contextualize the strands within the queer theory movement in education. The assimilationist groups did (and continue to) fight for social acceptance; often their methods involve minimizing differences and emphasizing sameness through essentialization. Dissatisfied with hiding, gay and lesbian liberationists such as Wittman offered a different perspective: "Liberation for gay people is to define for ourselves how and with whom we live, instead of measuring our relationships by straight values… we must govern ourselves, set up our own

institutions, defend ourselves, and use our own energies to improve our lives" (Wittman, 1970, as cited in Adam, 1995). These two camps, the assimilationists and the liberationists, continue to exist in tension today.

Although queer theory has its history in gay rights activism, there are "homophobic and marginalizing implications of leaving much queer studies work to queers themselves" (Marshall, 2014). That being said, if one cannot understand the implications of being queer as a result of one's own non-queerness, "the only way to act with integrity is to follow the leadership of those who are oppressed in that way, [to] support their projects and goals" (Indigenous Action Media, 2015). Having presented several understandings of queer theory, I now present queer curriculum and queer pedagogy. Queer curriculum and queer pedagogy represent only two instantiations of queer theory– examples of how queer theoretical concepts have been applied in the past, but not in an exhaustive nor definitive way. These examples do not wholly constitute queer theory but instead serve as starting points for mathematics education researchers to take a queer turn.

Queer Curriculum

The *queer curriculum movement* is my term for the collective efforts of researchers across the disciplines interested in developing queer-inclusive curriculum and queer curriculum theory. I contend that activity within the queer curriculum movement is the modern, educationally-relevant, reincarnation of the assimilationist's work in that queer curriculum is often of the "add-queers-and-stir" (Rands, 2009) variety. In other words, queer curriculum often features homonormatively-inclusive examples. As researchers work towards a queer curriculum, that curriculum must obtain additional demands; curriculum must now meet both the demands of content standards while including socially aware queer content. Scholars have argued that curriculum is obligated to interrupt the heteronormative, that sexuality is an analytic category appropriate to curriculum studies, and that curriculum might be better suited to unpack the "heterosexual closet" than to elaborate on queer identities (Davis & Sumara, 2000). By considering queer students and children with queer families or family members, culturally relevant mathematics that seeks to legitimize students' experiences in the "official" curriculum converges interest with queer curriculum (Ladson-Billings, 1994; Rands, 2009).

Such moves for queer-inclusive curriculum is not without theoretical basis. Sumara and Davis (1999, 2000) have contributed substantially to the development of a queer curriculum theory. They argued for curriculum theory to seek to understand desire, pleasure, and sexuality (1999), in part through creating *heterotopic* events, events which comprise the juxtaposition of not-often-associated objects (e.g., positive queer role models would be heterotopic to a dominant discourse of queers as perverse; the positive-queer association being heterotopic). In particular, these researchers investigated examples from queer literature curriculum and highlighted ways in which heterotopic events from the readings were juxtaposed with students' lived experiences. In a similar way, mathematics education should provide heterotopic events for students, not only for queer students by juxtaposing *visible inclusion* with their *lived hiding* (e.g., being "closeted") but particularly for non-queer students by juxtaposing queer voices with the dominant discourse of queer silence. Luecke (2011, p. 117) eloquently summarized Style's "Curriculum as Window and Mirror" (1996) as a call that "All children need curricular mirrors to see themselves reflected and thus feel safe in being themselves, and they also need curricular windows to feel safe with the differences of others."

There is a paucity of research in mathematics education using queer theory; in fact, searching ERIC for "Queer Theory" and "Mathematics" returned zero relevant articles (and five irrelevant ones; Google Scholar corroborated such findings). Broadening the search to "Queer" and "Mathematics" increased the total results to six and yielded Rands' seminal piece in queer theory and mathematics education. This article was published in *Sex Education* yet focuses exclusively on the mathematics classroom and exclusively offers mathematical examples. As Rands was, and remains to

be, one of the few authors addressing the intersection of mathematics and queer theory, I present in detail two of her pieces here to center this previously marginalized work.

Add-queers-and-stir mathematics. In "Mathematical Inqu[ee]ry: Beyond 'Add-Queers-and-Stir' Elementary Mathematics Education," Rands (2009) began by recounting a story from personal experience as an elementary teacher and the choice to queer literacy, through heterotopic readings, prior to queering mathematics. Rands' literature review referenced several anthologies of queer theory in education, but none of these included mathematics. Addressing this gap, Rands provided six examples across 1st and 3rd-5th grades of how one might queer mathematics curriculum. These examples fall into one of two categories: "Add-Queers-and-Stir" or "Mathematical Inqu[ee]ry" (Rands, 2009). The former aligns well with the assimilationists while the latter begins to advance the liberationist's goals. To elucidate, Add-Queers-and-Stir examples would generally have a goal of inclusion in homonormative ways. One such example by Rands involves finding the area of queer symbols (such as the rainbow flag, pink triangle, etc.). This approach fails to consider sexuality as a dimension of "teacher practice, student learning, and the production of knowledge" (Letts, 2002, p. 119). We can contrast the superficial inclusion of this area example with the latter category of Mathematical Inqueery.

Mathematical inqueery. In Mathematical Inqueery, students appropriate the role of liberationists and there are often tones of teaching mathematics for social justice (viz., Rands, 2013). For example, consider Rands' (2009) fifth-grade mathematical investigation on income of married couples. In this investigation, students used mathematics to challenge the hegemonic structure of marriage by considering the intersectionality of sexuality, gender, and class. Men and women in same-sex married couples, due to the income inequality between men and women, would be structurally positioned financially ahead of/behind each other. This example clearly moved beyond the superficial inclusion of the area example and began to address systemic inequalities which oppress queer lives. By following this task about income inequality with an action component, one could see this progress towards action research (Cammarota & Fine, 2006) and, in particular, I see a clear connection to teaching mathematics for social justice.

In fact, in 2013, Rands published "Supporting Transgender and Gender-Nonconforming Youth Through Teaching Mathematics for Social Justice" in the *Journal of LGBT Youth*. In this piece, Rands "synthesize[d] perspectives on gender-complex education, teaching mathematics for social justice, and research on students' development of proportional reasoning and statistical concepts, and then propose[d] a mathematics project for middle schoolers to facilitate their agency in challenging transphobia and gender oppression in their schools" (p. 106). The benefit of such activities is two-fold: the benefit of inclusion and the benefit of the social justice/action component. There is a clear need for research on additional teaching mathematics for social justice projects which consider the queer student experience. We turn our attention to a novel concept in queering the content of mathematics, and specifically queering geometry with fractal geometry.

Queering mathematical content. Both Add-Queers-and-Stir and Mathematical Inqueery work within the prescribed mathematical context whereas queering mathematical content rejects existing borders and offers an alternative reality. The following example goes beyond the understanding of queer as an identity, and queer theory's minor goal of queer-inclusive curriculum, to queer as opposition-to-the-normative and queer theory's major goal of challenging the normative. Instead of providing queer-inclusive curriculum and working within the given mathematical structure, Davis and Sumara (2000) challenge the dominant position that Euclidean geometry holds in school mathematics. By reframing geometry as a "systematic reduction of all phenomena to fundamental particles, root causes, and original principles" (p. 823), the authors challenge the necessity of Euclidean rigidity particularly in light of fractal geometry which evades it. Furthermore, by making a connection between knowledge and fractals, the authors continue the analogy between fractals and grain size: regardless of individual, social, or cultural foci, each are nested within or wholly contain

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others. It is important to note that Davis and Sumara align themselves with post-modern curriculum theory in their piece, not explicitly queer theory. This, however, does not change the interpretation presented. Characterizing additional work *ex post facto* as having converging interest with queer theory would only further emphasize the lack of explicit attempts to queer mathematical content.

Queer Pedagogy

Moving beyond the queer curriculum movement, into what most closely aligns with the liberationist's work, is the queer pedagogy movement. In fact, at its center, queer pedagogy disrupts the normative and its reproduction of oppressive structures (Luhmann, 1998), by adding goals of social justice to the educational system. Such an approach of working within structures, however, echoes assimilationist motives. Nonetheless, as mentioned above, I am particularly optimistic toward projects which consider a queer turn in teaching mathematics for social justice. Furthermore, little work has considered a uniquely queer pedagogy and none has considered a queer mathematics pedagogy. One such queer pedagogy which might be suitable for the mathematics context is presented next.

Beyond aligning existing movements such as teaching mathematics for social justice with queer pedagogical goals, scholars such as Britzman have described features of a uniquely queer pedagogy. For Britzman (1995, p. 165), a queer pedagogy is a pedagogy that "refuses normal practices and practices of normalcy,… begins with an ethical concern for one's own reading practices,… is interested in exploring what one cannot bear to know,… [and is] interested in the imagining of a sociality unhinged from the dominant conceptual order." Britzman provided us with four key features of a queer reading pedagogy, which I offer as transferrable to the mathematics education context: a queer mathematics pedagogy is one that: rejects the normative, has an ethical concern at its center, explores "what one cannot bear to know" (p. 165), and decenters normative structures and discourses. In other words, I propose that a queer mathematics pedagogy is one that, from a center of ethical concern for queer students, rejects the heteronormative systems, structures, and discourses by bringing to light queer experiences excluded by the heterosexual understanding. There is a tangible need for research on what might further constitute a queer mathematics pedagogy and what that pedagogy might offer in practice.

Concluding Remarks

Looking ahead, I see several key directions in which mathematics education researchers must move. First, mathematics education researchers should strive to operationalize performative and intersectional notions of identity (e.g., Darragh, 2015); such a distinction challenges the normative (separable) notion of identity. Incremental changes to existing research trajectories, such as simply drawing on updated notions of gender identity within the current context of gendered research in mathematics education, will still fail to include queer students. Second, I challenge the mathematics education research community to push against the borders of mathematics education research by centering the experience of queer students in their current research while simultaneously advocating that more mathematics education researchers must adopt a queer theoretical stance to accomplish this centering, not only in the mathematics context, but within the overall education system. Finally, I urge mathematics education researchers to develop and engage with queer curriculum and queer pedagogy, despite a theoretical basis outside of mathematics education and the relative lack of adoption in mathematics education research thus far. In addition to simply including queer examples, however, mathematics education researchers should continue to question the nature and boundaries of mathematics itself to challenge the notion of mathematics as fixed, neutral, and culture free. By embracing the tension between queer and the self-imposed, artificial borders of mathematics education research, the mathematics education research community can only continue to grow as more mathematics education researchers draw on the queering notions of queer theory. Taking a

queer turn in mathematics education research is the most direct path toward the safe and equitable education of queer students.

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SPATIAL ACTIVITIES AT HOME AND THE CRITICAL TRANSITION TO KINDERGARTEN

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In this research we explored the relationship between spatial activities in the home and spatial ability at the start of formal schooling at age four. In total, 30 children participated in this research. Data sources included a demographic questionnaire, a survey of at home spatial activities, and standardized testing in verbal and non-verbal spatial ability. Our results showed that the spatial activities in the home did not predict spatial ability at age four. Gender differences were observed with boys scoring higher in verbal- and non-verbal visual-spatial ability. Given the highly malleable nature of spatial ability and the importance of spatial ability to future STEM-based learning, intentionality in spatial learning by teachers is important.

Key words: Cognition, Early Childhood Education

Introduction

Studies have found that experiences in early childhood have important academic implications for children at the start of schooling and beyond (Alexander & Ignjatovic, 2012). Yet, for both children in early learning settings and those in a home care setting, there may be great variability in the kinds of opportunities for learning (Marope & Kaga, 2015). Expectations for social, emotional, and intellectual engagement may differ significantly. For some children, particularly those in early learning settings, the expectations and routines may not be that vastly different. The differences may be substantial and may even be partially cultural (Brooker, 2010). Consequently, the start of formal schooling is a critical transition for many young children.

Readiness for mathematical learning may be a particularly type of critical transition. Studies report that often relatively little emphasis is placed on mathematics in the home (Blevins-Knabe & Musun-Miller, 1996) or in early learning centers (Early et al., 2005). However, early mathematical ability is related to future success in mathematical learning (Geary, 2011; Sasanguie, De Smedt, Defever, & Reynvoet, 2012). The kinds of activities young children engage prior to formal schooling and in their homes in are linked to future academic outcomes in mathematics (Anders et al., 2012). For example, studies have shown that higher levels of mathematical and spatial talk (Gunderson & Levine, 2011), block play (Hanline, Milton, & Phelps, 2010), and puzzle play (Levine, Ratliff, Huttenlocher, & Cannon, 2012) are related to mathematical ability or spatial ability.

A recent study by Aunio and colleagues (2015) showed that schooling does *not* mediate for lack of prior experience or knowledge, particularly for those children who begin schooling with less basic mathematic understanding than what might be anticipated. Children's (age in months, M = 74.55; SD = 3.50) number knowledge was tested at three time points during kindergarten. While lower achieving children showed the most gains, children who started school with high mathematical abilities remained high – and higher than those children who tested at medium or low mathematical ability. In short, schooling did not help the lower achieving children catch up to the higher achieving children. This study illustrates the importance of a strong mathematical start and highlights that starting school *is* a critical transition in terms of mathematical learning – albeit perhaps unrecognized by teachers or parents. This research suggests that schools may need to rethink assumptions about a

young child's mathematical-readiness for schooling and how different levels of readiness need to be supported by teachers.

The early mathematics advantage is not exclusively about basic number knowledge. Studies show that spatial ability also has lasting implications for future success in mathematics and even in participation in STEM-based disciplines (Lubinski, 2010; Wai, Lubinski, & Benbow, 2009). Gunderson and colleagues (2012) found that spatial ability at age five contributed to level of number knowledge at age eight. Moreover, spatial skills in kindergarten have been found to be a more powerful predictor of mathematics ability in grades one and two than even mathematics skills in kindergarten (Frick, Möhring, & Newcombe, 2015). Childhood spatial activities are linked to spatial skills and mathematics grades in adulthood (Doyle, Voyer, & Cherney, 2012).

Spatial ability is defined broadly as the understanding of relations about (i.e., properties and attributes) and between objects and spaces including transformations in either two or multiple dimensions (i.e., mental rotations) (Clements, 2004; Linn & Peterson, 1985). Often spatial ability is associated synonymously with "visual-spatial ability." That is the case with our use of the term and that is clear in our choice of standardized tests described shortly.

In this research we look specifically at the spatial activities of four-year-old children and how the types of activities/toys children engage with intersects with spatial ability at the start of kindergarten. Our research questions were as follows: (1) *What are the sorts of spatial activities four year olds children engage in, as reported by their parents?* (2) *Do these activities contribute to children's spatial ability at the start of formal schooling in kindergarten?*

Given that approximately 60% of young children are cared for at home (Barnett, Carolan, Fitzgerald, & Squires, 2012), we analyze our results further by considering whether early childhood education and care is connected to the experiences reported by parents and spatial ability at the start of formal schooling. We also analyze our results by gender because of the numerous studies that report a gender advantage to boys in spatial ability (Frick et al., 2015; Linn & Peterson, 1985).

In our jurisdiction age four represents the start of formal schooling with a full-day kindergarten program and thus this age is the critical transition point between home and school or early learning/childcare setting and school for these young children. Given the highly malleable nature of spatial ability (Uttal et al., 2013), an understanding of how home environments contribute to the development of spatial ability is important to subsequent spatial ability but also to future mathematics learning.

Recent research exploring spatial ability in girls (approximate age 6) by Dearing and colleagues (2012) found that home spatial activities were related to the girls' spatial skills but not proximal predictors of spatial skills. Other recent research using longitudinal data found that spatial activities in the home of children age four to seven, were predictive of spatial ability when tested on the Block Design subtest from the WPPSI-IV (Frick et al., 2015). This research also found a gender difference with boys engaging more in spatial activities than boys. In our research, we consider whether the same trends observed in these other studies are evident at age four and whether these trends hold true for boys and girls. Moreover, we consider both non-verbal and verbal measures of spatial ability (cf. Block Design used by Frick et al., 2015) and a wider range of proposed spatial activities.

The theoretical framework for this research is *critical praxis*. Drawn from Tilleczek (2012), critical praxis "denotes critique and interrogation of the theory and practice surrounding childhood transitions" (p. 13). The objective of the critical praxis is to enhance the experiences of young children by inspiring "schools and people to function as communities which build bridges between students, parents, teachers, and communities" (Tilleczek, 2012, p. 17). In the present research, theory intersects practice through an analysis of the juxtaposition of home practices against young children's spatial ability as a possible precursor for future spatial learning in a school setting.

Methods

Participants

Thirty junior kindergarten children (*M* ₐgₑ = 49.9 months; *SD* = 3.2; *Range* = 45 to 56 months; 19 males) were recruited from six classrooms within one elementary school. While mandatory schooling begins in grade one, or age six, full day schooling is available for all children starting at age four. Out of the 30 children, 27 parents (23 mothers, 4 fathers) participated in the parental portion of the study. The other three parents chose to not complete any of the questionnaires and thus these children were removed from the study. The available demographic data for the 27 participants indicated that English was the most frequent language spoken at home for all children. The socioeconomic status (SES) of the participants was determined by using mother's highest education level (Catts, Fey, Zhang, & Tomblin, 2001). The highest education level attained by mothers was as follows: 4% completed some high school, 19% high school, 30% college/trade, 41% university, and 7% completed graduate/professional education.

Procedure

Children's visual-spatial ability was assessed by the Stanford Binet Intelligence Scales for Early Childhood, Fifth Edition (SB5; Roid, 2003). The children were administrated the SB5 in the school library. Each child was assessed one-on-one by either the second author or a trained research assistant. The study also consisted of a parental portion, in which parents were asked to complete a demographics questionnaire, and an at-home spatial activities questionnaire (Dearing et al., 2012).

Stanford Binet Intelligence Scales for Early Childhood, Fifth Edition (SB5; Roid, 2003). The children were administrated both the Nonverbal and Verbal Visual-Spatial Processing subtests from the SB5. Age normed scaled scores (*M* = 10, *SD* = 3) were calculated for each subtest. The Nonverbal subtest includes 16 items and requires the child to fit shapes and combine shapes into a plastic board that has forms of shapes cut out in it. These items measure the child's visualization ability of visual-spatial processing. For the last eight questions on the Nonverbal subtest, the children are presented with a two-dimensional image and are required to recreate that image with three-dimensional geometric figures. The Verbal Visual-Spatial Processing (Position and Direction) subtest is comprised of 12 items. The Verbal subtest evaluates children's understanding of position and direction by following verbal instructions and moving objects into the certain positions. The child is presented with an image and is asked to place a block on a certain direction or position with relation to the image.

Demographic Questionnaire. The participating parent completed a demographic questionnaire indicating their child's gender, date of birth, the language most frequently spoken at home, their relationship to the child, mother's highest education level, and how many hours the child spent in child care from ages one to three. Children were grouped into low (college and under) or high (university and above) SES groups.

Child Spatial Activities Questionnaire. The spatial activities questionnaire was adopted from Dearing and colleagues (2012) and it measures the frequency of children's engagement in spatial activities and toys at home. The spatial activities questionnaire (Cronbach's Alpha = .85) has good reliability (Dearing et al., 2012). Parents were asked how often their child engages in 20 spatial activities (e.g., builds with construction toys such as building blocks, LEGO®, and magnet sets). The parents indicated the frequency of each activity on a five-point scale (1 = *never*, 2 = *seldom*, 3 = *occasionally /a couple times per month,* 4 = *often/weekly,* 5 = *many times per week*). An average was computed in order to obtain an average frequency score for the spatial activities. Children were also grouped into either a high and low spatial activity group based on the average median cut-off.

Results and Discussion

We examined the types of spatial activities parents engage in with their child. The most frequently occurring spatial activities were as follows: playing in parks, playing with action figures or cars, colouring, painting, or drawing free hand, and building with construction toys. Parents also indicated that play with blocks and puzzles also occurred often or weekly (See Table 1 for mean frequencies of each activity).

Table 1: Spatial Activities Engaged in at Home

Spatial Activity	Mean (*SD*)	Min – Max
1. Play in parks or green spaces when the weather permits	4.7 (.52)	3 – 5
2. Play with toy soldiers, action figures, cars/trucks, planes or trains	4.4 (.89)	2 – 5
3. Colour, paint, or draw free hand (not filling-in outlines)	4.3 (.86)	2 – 5
4. Build with construction toys (such as building blocks, Legos, magnet sets)	4.2 (.65)	3 – 5
5. Set up play environments with toy furniture, toy buildings, train tracks or building blocks	4.1 (.81)	2 – 5
6. Play with puzzles (such as picture puzzles, tangrams, slide puzzles, 3-D puzzles)	4.1 (.84)	2 – 5
7. Race toy animals or cars on the ground or around obstacles	4.1 (.84)	3 – 5
8. Do arts and crafts projects	4.0 (.78)	2 – 5
9. Explore woods, streams, ponds, or beaches or search for plants, bugs, or animals outdoors when the weather permits	3.7 (.76)	2 – 5
10. Use a computer/video games to do drawing, painting or matching and playing with shapes	3.7 (.90)	2 – 5
11. Play paper and pencil games (such as mazes, connect-the-dots)	3.3 (1.1)	1 – 5
12. Set up obstacle courses, tunnels, or runways for kids or pets	2.8 (1.0)	1 – 5
13. Build dams, forts, tree houses, snow tunnels, or other structures outdoors when the weather permits	2.8 (1.1)	1 – 5
14. Use tools (such as hammers or screwdrivers) to make things or take things apart to see how they work	2.7 (1.3)	1 – 5
15. Play with flying toys (such as kites, paper airplanes)	2.5 (.8)	1 – 4
16. Fold or cut paper to make 3-d objects (such as origami, paper airplanes)	2.5 (1.1)	1 – 4
17. Climb trees when weather permits	2.1 (1.2)	1 – 5
18. Draw maps (such as treasure hunt maps)	2.0 (1.1)	1 – 5
19. Draw plans for houses, forts, castles or other buildings or layouts	1.8 (1.2)	1 – 5
20. Use kits to build models (such as airplanes, animals, dinosaurs, doll houses)	1.7 (.65)	1 – 3

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Note. Mean frequency and minimum and maximum for each spatial activity. Parents indicated the frequency of each activity on a five-point scale (1 = never, 2 = seldom, 3 = occasionally /a couple times per month, 4 = often/weekly, 5 = many times per week).

Children were grouped into high- and low-frequency of spatial activity groups. There were 14 children in the low spatial activity group and 13 were in the high spatial activity group (*Range* = 2.05 to 4.15, *Median* = 3.40). Children in the high spatial activity group had higher scores on their nonverbal visual-spatial ability; however, children in the low and high spatial activity group had similar scores for their verbal visual-spatial ability (See Table 2 for means). This research did not explore actual levels of spatial talk in the home which have been found to be related to spatial thinking (Gunderson & Levine, 2011). This may have explained the lack of differences between the groups.

Table 2: High- and Low-frequency Spatial Activity Groups

Spatial Task	Spatial Activity Group		Years in Child Care			
	Low	High	0	1	2	3
Nonverbal Mean *(SD)*	11 *(3.1)*	12.2(3.8)	9.5(6.3)	10.7(2.8)	11.5(4.3)	12.5(3.1)
Verbal Mean *(SD)*	12.3 *(2.6)*	12.2 *(2.6)*	12.5(0.7)	11.7(2.4)	11.6(3.4)	12.9(2.5)

Note. Mean scaled scores and standard deviations for nonverbal and verbal visual-spatial ability examining high and low spatial activity groups and years in child care.

We also examined whether attending child care prior to entering formal schooling influenced children's spatial abilities. There were two children that never attended child care, seven children attended for one year, six children for two years, and 12 children attended child care for three years. During the first three years of life every additional year spent in child care showed an increase in their average nonverbal visual-spatial ability (See Table 2 for means). In terms of verbal spatial ability there was not an evident pattern of increase based on years spent in child care, however children who attended child care for all three years had the highest scores (See Table 2). This would be expected given the many studies that report that early childhood education, particularly that of a higher quality, can have important implications for future academic outcomes (Alexander & Ignjatovic, 2012).

We found that boys (*M* = 11.79, *SD* = 3.6) had higher nonverbal visual-spatial scores compared to girls (*M* = 10.45, *SD* = 3.5). Girls (*M* = 12.36, *SD* = 3.1) had slightly higher verbal visual-spatial scores compared to boys (*M* = 12.16, *SD* = 2.6). Boys and girls frequency of spatial activities were also compared. Boys (*M* = 3.4, *SD* = 0.4) and girls (*M* = 3.16, *SD* = 0.4) had similar mean frequencies, with boys showing a slightly higher frequency. A Mann-Whitney test showed this to be a non-significant difference (*U* = 46.7, *p* = .08). Parental reports of higher frequency of spatial activity in boys is consistent with reported results (Frick et al., 2015).

Correlations were conducted for nonverbal and verbal visual-spatial ability between the following factors: gender, SES, child care attendance, and at home spatial activity average. There were no significant correlations (*p* > .05). There was also no correlation between nonverbal and verbal visual-spatial ability, therefore indicating that doing better one subtest did not mean children were more likely to do better on the other subtest.

To examine if there were any significant effects or interactions between spatial activities, child care attendance, SES and gender a 4 (child care: 0 years vs. 1. vs. 2 vs. 3) x 2 (spatial group: low vs. high) x 2 (SES: low vs. high) x 2 (gender: boys vs. girls) Analysis of Variance (ANOVA) with nonverbal visual-spatial ability as the dependent variable was conducted and revealed no significant main effects or interactions ($p > .05$).

The same ANOVA was conducted with verbal visual-spatial ability as the dependent variable. There were also no significant effects or interactions for verbal visual-spatial ability ($p > .05$). Similar to Dearing and colleague's research (2012), spatial activities were not predictors of spatial ability girls but also boys. Dearing et al.'s research considered six year olds. Our research provides further evidence. Different than Dearing et al. is that we did not find any correlations between the spatial activities and spatial ability.

The lack of significant findings could be explained by the low sample of children across groups. This is recognizably a limitation of this research. However, there are other possible explanations. Frick and colleagues (2015) recently reported that spatial activities in the home at age five were related to spatial ability at age eight. According to Uttal and colleagues (2013), spatial ability is highly malleable and thus children in Frick et al.'s study likely benefited from formal school instruction or a malleability effect. Additionally, the higher levels of spatial play that were reported by parents may not have been sufficiently complex for the child to move the child along in their continuum of learning (Gregory, Kim, & Whiren, 2003; Hanline et al., 2010; Lee, Kotsopoulos, & Zambrzycka, 2013). Finally, parents' perception of a "high" level may vary – despite the Likert scale qualifiers. Some naturalistic observational data would be helpful to offset self-reports on such frequencies, as well as larger sample sizes.

Conclusions and Educational Implications

Research has shown that spatial skills are important for future success in both mathematics and in STEM-based disciplines (Lubinski, 2010; Newcombe, 2010; Tolar, Lederberg, & Fletcher, 2009; Wai et al., 2009). Understanding the degree to children's spatial ability is developed at the start of formal schooling and the kinds of activities in the home environment that contribute to the development are important. The results from the present research suggest that home activities are not necessarily mapping directly onto spatial ability at the start of formal schooling. These results have important implications for teachers – and perhaps lasting implications for children. Schooling itself does not contribute to mathematical resilience in the absence of intentionality (Aunio et al., 2015). A lack of intentionality may be even more problematic for spatial ability. Recent research reports that relatively little time is spent overall on spatial ability (or geometry) in schools compared to other curricular strands such as numbers, data management, and measurement (Mullis, Martin, Foy, & Arora, 2012; Organisation for Economic Co-operation and Development, 2012). Consequently, intentionality in nurturing spatial ability is important.

Such intentionality would involve having a clear understanding of the beginning points of the child's spatial understanding and then building pedagogical moments to advance the child's understanding based on their unique starting point. Teachers may be challenged in determining the starting point because early mathematical development, including the development of spatial ability, may be underemphasized in their professional training. Future research should explore the ways in which professional programing prepares future teachers and early childhood educators to understand early spatial and mathematical development.

From a critical praxis perspective, the importance of occasioning spatial learning and in fostering a deeper understanding of spatial learning amongst teachers should be evident. By linking the home-school connections, the potential for enhancing opportunities for learning are increased. In the case of spatial ability and even mathematical ability, the need to occasion such bridges is paramount given the clear research that shows an academic advantage to those young children that come to school

knowing more mathematical concepts (Aunio et al., 2015; Geary, Hoard, Nugent, & Bailey, 2013; Romano, Babchishin, Pagani, & Kohen, 2010).

Acknowledgments

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DIS/ABILITY AND MATHEMATICS: THEORIZING THE RESEARCH DIVIDE BETWEEN SPECIAL EDUCATION AND MATHEMATICS

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Based on an analysis of 408 mathematics research articles published in 2013, this presentation theorizes the current divide between research in mathematics education and special education using Disability Studies in Mathematics Education. For those without disabilities, mathematical learning was understood primarily through constructivist, sociocultural, and sociopolitical perspectives, the research was both quantitative and qualitative, and almost 50% of the research was focused on the role of the teacher in learning. For those with disabilities, mathematical learning was understood primarily from medical and behavioral perspectives. This research was predominantly quantitative, and rarely focused on the teacher. We contend that this divide constructs and reifies the notion that there are two categories of mathematics learners who need different kinds of mathematics.

Keywords: Equity and Diversity, Learning Theory, Research Methods

Introduction

Students with disabilities are offered fewer opportunities to engage in meaningful mathematics, as special education classrooms and curriculum are focused on procedural rather than conceptual instruction (Jackson & Neel, 2006). Based on a content analysis of current research in both mathematics education and special education mathematics, we contend that disparities in access are influenced by a research divide between these two fields, which differ in the epistemologies, methodologies, and pedagogies used to understand learners. The research divide constructs and reifies what many consider to be a "common sense" assumption: children with and without disabilities are different, and should be educated differently in mathematics.

While this paper/presentation will describe that research divide using data from a research content analysis, our focus is to theorize that divide. Theory matters, because for some, including perhaps many at this conference, this divide may seem unremarkable. After all, disability is a medical condition, and mathematics education researchers are not experts in disability. Even in the rhetoric of equity in mathematics education research for marginalized student groups, students with disabilities are often not part of such conversations (Tan, 2014). We believe that Disability Studies can illuminate the historical reasons for this divide and challenge such borders moving forward. The conference theme of Questioning Borders is a generative location for our analysis, as we seek to deepen our understanding not only of the borders between these two academic fields, but the areas of intersection as well.

Conceptual Framework

Disability Studies in Education

Activists with disabilities pioneered the academic field of Disability Studies, advocating replacement of the medical model of disability with the social model (Union of Physically Impaired Against Segregation [UPIAS], 1975). While individuals may have cognitive or physical differences, disability is created through society's response to these differences. As Siebers (2008) writes, "the medical model defines disability as an individual defect lodged in the person, a defect that must be cured or eliminated if the person is to achieve full capacity as a human being" (p. 3). In contrast, Disability Studies "defines disability not as an individual defect but as the product of social injustice, one that requires not the cure or elimination of the defective person but significant changes in the

social and built environment" (p. 4). The response to difference is not to seek to cure it in the affected individual, but to understand how unjust social systems create or exacerbate differences. Applied to schools, Disability Studies in Education (DSE) examines disability in schools as a social construction that results in social exclusion and oppression (Gabel, 2002).

Disability Studies in Mathematics Education (DSME) (Tan, 2014) reimagines the structures and processes of teaching and learning mathematics. Students with disabilities (and all students) are representative of the diversity of human experiences for which all educational environments should be designed. DSME also draws from critical mathematics education (CME) perspectives. CME is concerned with the social and political aspects of the learning of mathematics and how students and teachers operate in a social system rife with hegemonic power (Gutiérrez, 2002). CME aims to broadly (1) develop within individuals a political awareness of individual's position in a system (e.g., classroom, school, or community), and (2) motivate individuals to enact change toward advancing social justice (Powell & Brantlinger, 2008). DSME troubles conventional mathematics research by involving students with disabilities and by surfacing and questioning power differentials. This process includes challenging "hegemonic narratives about who can do mathematics and to reconstruct the role of mathematics in the struggle to empower learners whose mathematical powers have been underdeveloped" (Powell & Brantlinger, 2008, p. 425). In utilizing DSME, action towards more just practices is led first and foremost by students with disabilities where their lived experiences and voices are privileged in the conversation. Additionally, we situate our analysis within Dis/Crit, which highlights how race and disability intersect (Annamma, Connor, & Ferri, 2013).

Historical Roots of Mathematics Education

Mathematics education has been influenced by (and has influenced) larger epistemological stances about learning and learners such as behaviorism, information processing, constructivism, and sociocultural theory (Woodward, 2004). Gutiérrez (2013) has called for mathematics education to address learning from a sociopolitical perspective as well, to better understand how larger social forces, power and positioning affect not only learning, but access to learning. We found that constructivist and sociocultural approaches to learning currently dominated research in mathematics education. In general, constructivist theories of learning understand individual learners as active participants in constructing knowledge through experience and reflection (Draper, 2002); sociocultural theory expands analysis of learning to the product of interactions between two or more people (Lerman, 2000). Learning is situated in contexts, and mediated by tools that include mathematical discourse.

Historical Roots of Special Education

Special education philosophy and research have historical roots in psychology and medicine guided by behaviorism and positivism orientations to develop and test interventions for students with disabilities (Paul, French, & Cranston-Gingras, 2002). Paul and colleagues noted that despite advancements and evolution in social science perspectives, special education researchers have maintained a strong commitment to a positivist epistemology. Mathematics research in special education continues to be heavily influenced by behaviorist theories of learning such as direct instruction (Woodward, 2004).

In the field of special education, evidence-based practices must include experimental control (Cook, Tankersley, Cook, & Landrum, 2015), significantly reducing the kinds of possible research that could influence teaching practice. Subscribing to such standards of what constitutes research evidence-based practices means only quantitative methods are valid. Research designs involving students with disabilities tend to focus on evaluating instructional practices on children's learning with limited focus on teachers' pedagogical understanding and curriculum design (e.g. Griffin, League, Griffin, & Bae, 2013) and limited focus on how students construct mathematical learning.

Direct instruction in mathematics is widely endorsed and qualify as being evidence-based (Gersten et. al, 2009). Hence, researchers grounded in conventional special education epistemologies who desire to advance knowledge in their field are strongly pulled in the direction of the "evidence." Teacher directed and explicit instruction represents mathematical knowledge that must be transmitted from "knowers" and reifies acquiescence to societal power structures (Charlton, 1998). Thus, the prevalent and long-standing stance of who benefits and who does not benefit from certain mathematics pedagogies establishes and is established by omnipresent borders across societal structures that influence research and practice restricting what is possible for students with disabilities.

Methods

In order to understand the present state of this research divide, we conducted a content analysis of research articles published in 2013 that focused on mathematics and PK-12 education (Lambert 2015a; Lambert & Tan, 2016). Our research question asked how research on mathematical learning of children with and without disabilities differed in terms of academic fields, methodologies, mathematics content, and participants. Additional information on methodologies and findings can be found in previous publications.

We limited the sample to mathematics education research articles published in peer-reviewed journals in English in 2013. We excluded research that focused exclusively on mathematics at the undergraduate level, unless the participants were pre-service teachers. We found articles through searches of educational databases (ERIC, JSTOR, & PsychINFO) looking for descriptors and keywords of mathematics, math, and numeracy. We did a hand search through all journals mentioned in an analysis of equity in research published by Lubienski and Bowen (2000). The resulting data set was 408 articles. In the first stage of research, we coded based on the title, abstract and keywords for each article. We coded for academic field of the journal, methodology of article, participant focus, equity groups mentioned (such as race or disability), mathematical content focus, and pedagogy. Some articles did not present enough information in the title, abstract or keywords for us to code, particularly in methodology and pedagogy. In some categories, it was possible to combine more than one content area or pedagogy. Inter-rater reliability of coding was 97.3%. In the second stage of research, we looked more closely at two subsets of the data: (a) the articles that included disability (n=42) and (b) the articles that focused on problem solving (n=45). We read these articles in their entirety, coding again if necessary. Our study has several limitations. We only included research published in English. While this iteration only included one year, we are beginning a second round of coding that will include 2013-2015. We were not always able to determine coding from the title, abstract or keywords.

Findings

Of the entire set of articles (408), 42 included disability, or 10.3% of the sample. Much of the following data compares the set of articles that included disability (42) to the set of articles that did not include disability (366). We do not report on all aspects of our findings, focusing here on academic field, methodology, participant focus and pedagogy.

Academic Field. We found that mathematics research on students with disabilities was overwhelmingly published in special education or psychology journals, with very little included in mathematics education journals. Articles that did not include disability were primarily published in mathematics education journals (68.3%). Only 8.5% of those articles were published in special education or psychology journals. 90.5% of articles that included disability were published in special education or psychology journals. Mathematics education journals only published 2 articles that included disability in 2013.

Methodology and Participants. Research on the mathematical learning of students with disabilities was predominantly quantitative (83.3%), with only 9.5% qualitative. In contrast, research on the mathematical learning of students without disabilities was more evenly distributed: 32.8% was quantitative and 40.2% was qualitative.

For each article, we determined whether the unit of analysis was learners in preschool, elementary school, middle school, high school or teachers. Research on the learning of students with disabilities was focused on younger learners, with over half of the research on elementary-aged students, while research on learning of students without disabilities was more evenly spaced across ages. For articles that did not focus on disability, the most frequently researched participant category was teachers (48.6%). In contrast, only 11.9% of articles that focused on learners with disabilities focused on teachers.

Disability is a highly diverse category, not only racially and culturally, but in wide variety of differences that fall under the umbrella of disability. Research on mathematics learning did not reflect that diversity, as most articles about disability and mathematics focused on one category: students with mathematical learning disabilities (71.3%). Of those articles, 40% focused not on learning, but on diagnosis. A very limited sample of people with disabilities was included in mathematical research at all.

Pedagogy. Coding the articles for pedagogy was a critical part of our investigation since we were interested in whether or not students with disabilities were understood differently in the research literature. However, coding for pedagogy was complex. Following Woodward (2004), we first identified Behaviorist, Information Processing, Constructivist and Sociocultural as theories of pedagogy that have influenced mathematics and special education, using his description of the differences between these categories as coding indicators. We added two additional categories: (a) Sociopolitical/Critical (Gutiérrez, 2013) to capture an emerging focus in mathematics education on analysis of wider contexts and processes that affect classrooms and learning, and (b) Medical. We added the category of medical because we found a significant number of research articles that understood learning as mediated or controlled by psychometrics alone. Articles could be coded for more than one pedagogical perspective. Additional information on how we coded for these categories will be available at the session.
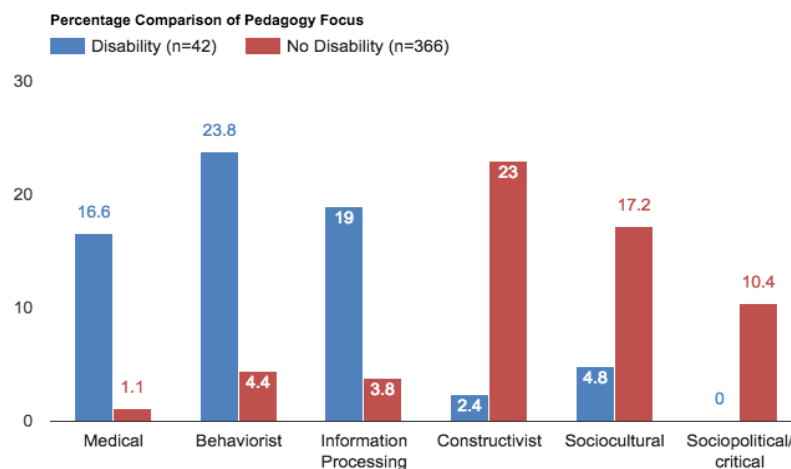
Figure 1. Percentage of Articles by Pedagogical Focus.

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

Figure 1 suggests that learners with disabilities and those without are conceptualized differently in mathematical research. We recognize the limitations of this particular analysis, particularly that we determined pedagogy through the title, abstract, and keywords alone, which meant that we did not assign a code to all articles. However, we found similar percentages in this analysis of all articles, and a follow up analysis of the full-texts of all articles that focused on problem-solving for both learners with and without disabilities.

Discussion

Based on this analysis of research published in 2013, there was significant differences between mathematics educational research focused on learners with disabilities, and that which was focused on those without disabilities. For those with disabilities, mathematical learning was understood primarily from medical, behavioral and information processing perspectives, the research was predominantly quantitative, and rarely focused on the teacher. For those without disabilities, mathematical learning was understood primarily through constructivist, sociocultural, and sociopolitical perspectives, the research was both quantitative and qualitative, and almost 50% of the research was focused on the role of the teacher in learning.

How might this matter? One way to understand the impact was to look closer at one content focus. In the articles focused on problem solving, most research articles including students with disabilities focused on "word problems," while for those without disabilities, "problem solving" was more common. Much of the research in word problem solving for students with disabilities used a schema-based approach, which builds at least partially on constructivist research on how children approach different problem types (Carpenter et al., 1999). Articles in our sample, however, reformulated that research to better fit a behaviorist model of both learning and research. Children in these studies were given "explicit instruction" on the problem types in a scripted intervention. Jitendra et al. (2013) made a clear distinction between the two pedagogies, "Standards-based instruction is characterized by an inquiry-based, student-directed approach, whereas SBI [Schema Based Instruction] incorporates an explicit, teacher-mediated approach" (p. 257). We do not intend to devalue particular research methodologies or pedagogies, rather we seek to question why certain methodologies are used for certain groups almost exclusively.

Analyzing the research published on problem solving for learners without disabilities, we found eight studies of problem posing, which was defined as a process in which students used their experiences to "construct personal interpretations of concrete situations and from these situations formulate meaningful (i.e., non-trivial) mathematical problems" (Harpen, & Presmeg, 2013, p. 119). This article, along with two others in a special issue on problem posing in our sample, made explicit claims about the connection between problem posing, creativity, and mathematical giftedness. Do articles that seek to understand the relationship between problem posing and mathematical ability further deny access to those who are not currently seen as mathematically able to make sense of their worlds using mathematics.

We argue that this research divide in methodologies and pedagogy continually reinscribes an assumption that students with disabilities are a completely different kind of learner. Learners with disabilities are understood through a medical model that seeks to identify psychometric deficits that can inform remediation. These remediations are typically designed through a behaviorist lens, focusing on simplifying mathematics by breaking mathematics into tasks, teaching students procedures to solve word problems. Policy initiatives such as Response to Intervention (RTI) ask that interventions be evidence-based, yet the definition of evidence privileges particular quantitative methodologies.

In these 2013 articles, however, we have two examples of research that challenge these borders. One article from a constructivist perspective focused on using artifacts such as grocery store flyers to develop students' problem posing (Bonnoto, 2013). This article did not create a distinction between

ability and mathematical creativity, and found that children with histories as underachievers were able to engage deeply in problem posing when artifacts were relevant to their lives. All children were expected to make mathematical meaning as long as they could connect their own lived experience. Bonnoto (2013) rejects understanding mathematical creativity as incommensurate with disability. In another article from the sample, Heyd-Metzuyanim (2013) analyzed the co-construction of learning disabilities in mathematics through interaction between a mathematics teacher and a student. As in Lambert (2015b), disability was contextual, produced through interaction. These studies suggest connections between sociocultural analysis and disability studies that should be further explored.

Implications

As mathematics education researchers, we must honor our long-standing commitment to equity for marginalized groups of students. Both activists and academics who identify with the disability rights movement increasingly demand that diversity include disability (e.g. Siebers, 2008). DSME and Dis/Crit provide mathematics education researchers theoretical frameworks that shift perception of disability as a deficit toward viewing disability as a difference. We seek a deeper analysis of disability in mathematics through these lenses, including analysis of how disability intersects with race and genders (Annamma, Connor, & Ferri, 2013).

We call for including disability, not only to improve the lives of those with disability, but to improve mathematics education. As de Freitas and Sinclair (2014) suggest, mathematics education could benefit from more deeply considering the perspective of learners with disabilities, as exploring the mathematical world through these diverse learners can help us better understanding the relationship between embodiment and knowing in mathematics.

We believe that shifting mathematics research towards learners with disabilities will allow our field to rethink assumptions that privilege the mythical "normal" mathematical learner. The borders between these academic fields police a distinction between students without disabilities and those with disabilities, who are not recognized as competent and able mathematics learners. These learners are separated from inquiry and problem-solving pedagogy and curriculum, which can affect not only learning, but identity development, or who students with disabilities are learning to become in mathematics (Lambert, 2015b). Non-disabled peers also stand to academically and socially benefit as classrooms shift to recognize and develop a wider range of mathematical competencies.

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COMPENSATION: REWRITING OUR UNDERSTANDING OF MATH LEARNING DISABILITIES

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Research has yet to make measurable progress toward understanding how to help students with math learning disabilities (MLDs) overcome their persistent difficulties. Prior research has traditionally framed MLDs as cognitive deficits and studied these deficits by analyzing failing students' errors. In this paper, we provide an alternative. We explore a student with an MLD who has compensated so effectively that she was able to major in statistics. Eight videotaped interview session were conducted. We identify how symbolic notation was inaccessible for her and how she developed ways of compensating. This research pushes boundaries not only by breaking away from the traditional deficit model, but also by removing the delineation between researcher and participant. The case study participant (second author) was an active member of the research team collaborating in the design, analysis, and dissemination of this work.

Keywords: Equity and Diversity

How do you solve the problem: 8x3=? Most adults simply retrieve the answer from memory, requiring only a fraction of a second. This problem took "Dylan," a statistics major with a mathematical learning disability (MLD), over 10 seconds to solve. She later explained her calculation process, "two eights is sixteen, and then I'm adding [another] eight. Sixteen plus what equals 20? Sixteen plus four. Now the eight, minus the four, and there is four left over. Twenty plus four, twenty-four." Rather than retrieving this solution from memory[1], she was solving four independent calculation problems each with an intermediate sum. Although it is well documented that students with MLDs have difficulties solving basic number fact problems (Geary, 2004; Swanson & Jerman, 2006), research has rarely, if ever, examined the ways in which students, like Dylan, might be solving problems differently.

In this study we take an in depth look at Dylan's mathematical understanding and problem solving approaches. She is particularly worthy of study because despite having an MLD, she was incredibly successful in navigating upper division mathematics classes as a statistics major. By examining the ways in which she compensates, we can begin to understand the unique difficulties students with MLDs may experience and explore avenues to consider when designing instruction for students with MLDs. This paper provides a novel vantage point on MLDs by drawing upon a Vygotskian notion of disability and emancipatory research approaches used in disability studies. Through videotaped interviews, we explored the nature of Dylan's difficulties and the ways in which she compensated. In this paper we focus on one predominant category of compensatory strategies that emerged from the data that we termed "rewriting." Although there were various kinds of "rewriting," each involved representing numbers or symbols in an altered form, which enabled Dylan to remember, understand, or solve problems more effectively.

Prior Research on MLDs

Math learning disabilities are neurologically based differences in how an individual processes numerical information, which lead to significant difficulties learning and doing mathematics (Butterworth, 2010). Although it is estimated that 5-8% of students have MLDs (Shalev, 2007), the field lacks methodological approaches to accurately identify students with MLDs (Mazzocco, 2007). Currently, researchers classify students as having MLDs if they fall below a researcher-established achievement threshold and conduct statistical analyses to establish the ways that the students

classified as having MLDs are deficient as compared to their typically achieving peers (Lewis & Fisher, in press). Because of the focus on documenting deficits and the reliance upon low achievement as a proxy for disability, the field has made little progress towards understanding the characteristics of MLDs or identifying potential strategies to help students overcome their difficulties.

Context of This Study

This research endeavor pushes on traditional borders between "researcher" and "participant". This work is aligned with the principles of "emancipatory research" – in which the individual with disabilities is an active member determining the goals, design, analysis, and dissemination of the research (e.g., Walmsley, 2004). Emancipatory research is an important step forward to address issues with the traditional researcher-participant dichotomy, which is oppressive to individuals with disabilities (Walmsley, 2004). Collaborating with individuals with disabilities is a major step forward because it shifts research "on" individuals with disabilities to research "with" individuals with disabilities (Charlton, 2000; Ginsburg & Rapp, 2001). This collaborative research project was initially undertaken 5 years ago by Dylan – an undergraduate statistics major with an MLD – and Katie – a math education graduate student. Although not the focus of the present analysis, it should be noted that Katie has a diagnosed language-based learning disability (i.e., dyslexia) and therefore, in some ways, also has an "insiders" perspective on disability.

Dylan (second author) initially contacted Katie (first author) in an attempt to learn more about research on adults with math learning disabilities so she could understand what strategies were available to help overcome her difficulties. Katie informed her that unfortunately there was almost no research on MLDs beyond elementary aged students engaged in basic arithmetic (Lewis & Fisher, in press). Furthermore, the field had no documented cases of an individual with a math learning disability who had majored in the field of mathematics. Dylan, therefore, had unique insight into the nature of difficulties she experienced across a range of mathematical topics (including upper division math classes) and had a wealth of compensatory strategies she employed to adjust for the atypical ways her brain processed numbers. Together, we decided to embark upon a research project to document both the nature of Dylan's difficulties and the ways in which she learned to compensate. Our shared goal for this research was to identify the particular compensatory strategies in order to inform the design of instruction for students with MLDs. Because the typical terms of "researcher" and "participant" are insufficient for this kind of collaborative and co-constructed work, Dylan is referred to as the "expert" and Katie is referred to as the "inquirer" (see Knox, Mok, & Parmenter, 2000 for similar terminology).

Theoretical Framework

In this study, we draw upon a Vygotksian theoretical framing of disability, which stands in stark contrast to the deficit model predominantly used in research on MLDs (e.g., Geary, 2010). Vygotsky (1929/1993) argued that a child with a disability is not *less* developed than his/her peers, but has developed *differently*. Vygotsky's understanding of disability was aligned with his general theory of human development. He argued that human development progressed along two lines: biological and the sociocultural. In children without disabilities these two lines of development intertwine and are mutually constituted. In children with disabilities, mediational tools (e.g., language, symbols), which have developed over the course of human history, often do not serve the same function. For example, printed text may be inaccessible to a blind child, and therefore this standard mediational form does not serve the same function in the blind child's development of literacy as it would for a child who could see.

Central to Vygotksy's theory is that the disability creates the impetus for the development of *compensatory* processes. For example, a blind individual who cannot rely upon visual stimuli to

navigate may naturally begin to echolocate (i.e., use clicking sounds to navigate; e.g., Thaler, Arnott, & Goodale, 2011). The biological difference (e.g., blindness), therefore has led to the recruitment of alternative resources and the same task is accomplished with compensatory processes. To understand a disability researcher must not only document the student's difficulties, but also the student's strengths and the ways in which a student compensates. In this study we use this Vygotkisan framing to explore MLDs. We specifically focus on identifying how standard mediational forms (i.e., numerals, symbols, representations) may be inaccessible to Dylan and the ways in which she compensates.

Methods

Classification of MLD

Given the difficulty involved in accurately identifying students with MLDs recent research that attempts to differentiate low achievement from MLDs have suggested tests of numerical processing (e.g., Dyscalculia Screener, Butterworth, 2003) or timed calculation tests (Mazzocco, 2009) be used to help identify students with cognitively-based numerical processing problems. Both of these measures were used to establish that the student, Dylan, met the qualifications for having a mathematical learning disability. On the Dyscalculia Screener (Butterworth, 2003), she received a classification of "dyscalculiac tendencies with compensatory aspects", and measures of her timed arithmetic performance indicated that she processed single-digit addition and multiplication problems slowly, (averaging 2.355 and 5.235 seconds/problem respectively). Given her performance on the Dyscalculia Screener and on the timed calculation test, Dylan is considered to have a MLD.

Data Collection

Fourteen hours of videotaped interview data was collected during eight separate sessions. During these sessions we explored various mathematical domains, including: basic arithmetic, fraction operations, algebra, and statistics. We began each session by clarifying any outstanding questions from the previous session and making any needed modifications to the agenda we had planned in the previous session. As we worked through our agenda for the day, Dylan's role in the sessions was as an expert informant, someone who was able to reflect upon and demonstrate the kinds of difficulties she experienced and explain the ways in which she was able to compensate. Katie's role in the sessions was to listen attentively and ask questions to better understand the scope of the difficulties or compensatory strategies Dylan reported. We concluded each session by collaboratively deciding our agenda for the following session. After each session Katie wrote up notes from the session, which included scanned copies of all written artifacts and a general description of what was discussed. In many instances the production of these notes resulted in Katie posing several clarifying questions, which were discussed at the start of the next session. Dylan reviewed these notes to ensure the accuracy of Katie's descriptions.

Analytic Approach

All videos were transcribed and all artifacts were scanned. The research team, comprised of the authors and one graduate research assistant, conducted an open coding on the transcripts then met to discuss preliminary coding categories. This analysis identified several predominant themes in the data. Several of these themes can be loosely classified as instances in which Dylan used the compensatory strategy of "rewriting." We explore the various ways in which Dylan employed this strategy and how it enabled her to compensate for the particular difficulties she experienced when using numeric or symbolic notation.

Results

The results focus on several different ways in which Dylan used the compensatory strategy of "rewriting". Through different kinds of rewriting Dylan accomplished several different goals, which directly addressed her difficulty processing, manipulating, and remembering symbolic representations of numbers. The first kind of rewriting enabled Dylan to kinesthetically encode numerical information that she found difficult to remember. The second kind of rewriting enabled Dylan to connect the symbols to their underlying meaning by translating the symbols into words. The third kind of rewriting involved addressing notational ambiguity by rewriting the problem in a consistent form. In each case she reflected on why the particular compensation was needed. These episodes, therefore, illuminate both the ways in which standard mediational forms were inaccessible and also how Dylan learned to compensate through various kinds of rewriting.

Rewriting For Memory

The first form of rewriting allowed Dylan to compensate for difficulties remembering numerical information. Dylan had significant difficulties using and remembering numbers throughout her life. She reported that she had difficulty remembering her pin number, the number and zip code of her street address, and historical dates. She recalled that when she was young, "I could remember what street name I lived on, but for the life of me I could not remember the house number." Dylan developed ways of compensating for her difficulties memorizing numbers. For example, to remember her address she described, "I would write it out a ton of times. And even if I couldn't actually remember it, I would remember the sensation of that movement, so that I could replicate that movement on a page," "so even if I couldn't recall it, if I had a piece of paper, and I wrote it out and then I could read it to you." She clarified that she was remembering the kinesthetic experience of writing the numerals, rather than the numerals themselves, "So you're not actually inherently remembering the thing, you are just remembering the feeling of creating it. And then once I see it again, then I remember, but it isn't until it's written."

This kind of rewriting involved kinesthetically encoding information represented with numerical digits. It is worth noting that this kind of compensatory strategy appeared to be used primarily in cases where the digits themselves did not have any quantitative property. For example, the house number for "1610 Main Street" does not represent one-thousand six hundred ten of anything. Similarly, dates and pin numbers use numerical digits but similarly do not represent quantities.

Rewriting to Connect to Meaning

The second form of rewriting involved Dylan rewriting mathematical symbols in words in order to help her connect the mathematical symbols to their underlying meaning. Dylan reported that she had difficulty being able to "read" mathematical notation particularly when she was learning new mathematical content. When she was very young and learning arithmetic, she explained, "I'd write it out as a sentence, I guess, is like the best way to think about it. And in sentences you don't use symbols, you use words." For example, when learning to solve a problem like 3+2= she would translate the symbols into words and write underneath it "three plus two equals." She explained, "Because the word, like: T - H - R - E - E , has much more meaning to me than these two little backwards Cs laying on top of each other." She clarified that it was not the auditory word "three," but actually the written English word "T H R E E," that gave the symbol meaning. She described it as a sequence of steps she needs to go through to decode the numerical symbol, "If I can go from the symbol to the actual written word, then I can go to relating it to something in real life." "*Three* - I have that word and now I can think of three objects."

Although she no longer needed to translate arithmetic problems into words, she explained that, "in my higher division math courses I will actually still write these things out." She gave an example from her probability class, "this (writes "P(A|B)") is so incredibly short but it actually translates to

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the probability of event hap – event A happening, given event B has happened. I actually have to physically write that down every time I do one of these." These examples suggest that this kind of rewriting was necessary not only for the numerical symbols zero through nine, but for other symbols that had a mathematical or quantitative meaning.

Rewriting to Remove Ambiguity

The third form or rewriting involved rewriting problems in a standardized form to remove notational ambiguity. Dylan identified symbolic notation as being particularly problematic for her. Although many students experience difficulties as symbols take on new meaning, these notational issues caused persistent issues for Dylan and her ability to engage with the problem. For example, when solving a problem presented as: $\frac{1}{2} \times \frac{1}{5} =$ (see Figure 1), she immediately rewrote the problem using parentheses and said, "Because I have taken algebra, and I know that x can in fact be a variable and not necessarily multiplication… I always use parentheses now for multiplication." Because of the ambiguity around the meaning of the "x" she found it necessary to rewrite the problem before solving it.

Figure 1. Scanned artifact of the presented problem "1/2x1/5=" and Dylan's rewritten problem form.

Dylan also found that she frequently needed to rewrite parts of the problem while in the process of solving it. For example, as she was solving the problem 123-47=, she rewrote the problem in multiple parts to be able to clearly "see" the borrowed values (see Figure 2 and Figure 3). She noted that the standard superscript notation was ineffective and problematic for her, which resulted in her rewriting parts of this problem three times in the process of solving it.

Figure 2. Step-by-step illustration of Dylan's solution process for the problem "123-47=".

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Figure 3. Scanned artifact showing Dylan's work for the problem "123-47=".

Commonalities of Rewriting

Dylan used the compensatory strategy of rewriting in several different contexts with different goals (memory, meaning, and resolving ambiguity). In each case her compensatory strategy of rewriting required more time and more effort. It is precisely because rewriting is *not* efficient that it indicates that this is truly a way in which Dylan is compensating for particular characteristics of her disability. Attending to instances in which Dylan uses rewriting, highlights both how she is compensating and suggests potential areas of difficulty.

Endnotes

[1]We are not arguing that deriving arithmetic facts is not desirable or productive for many students, however when students use derived facts to solve a problem, the answer is often achieved in a couple seconds. Our point here is that the time and process that Dylan used to solve the problem "8x3=" is unusual for a statistics major.

Conclusion

Dylan reported difficulties in remembering numbers, connecting symbols to their underlying meaning, and dealing with symbolic ambiguity. Mathematical symbols can be thought of as being at least somewhat inaccessible to Dylan. She reported several different ways in which she used the compensatory strategy of "rewriting" to accomplish the same goals. Although the three forms of rewriting presented here involved different features to accomplish different goals, in each case she rewrote something in a more accessible form. Similar to her strategy for solving "8x3=", Dylan's strategy produces the correct answer, but time consuming and more cognitively demanding.

In this paper we have attempted to push the boundaries of how MLDs have traditionally been conceptualized in two specific ways. First, we rejected the deficit-model of the learner and adopted a Vygotskian notion of disability focusing explicitly on a student who has developed sophisticated ways of compensating. Second, our collaborative effort represents break down the traditional researcher-participant hierarchy. Dylan was positioned as the expert and was a meaningful collaborator in the conception, design, analysis, and dissemination of this research. We believe that pushing on these boundaries enables possibilities for innovative research where new questions are posed and new avenues pursued because the participant has a voice and power over determining the direction of the research.

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"A BANK ON EVERY CORNER":
STUDENTS' SENSE OF PLACE IN ANALYZING SPATIAL DATA

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This paper describes the role of sense of place in students' analysis of spatial data toward understanding the distribution of financial services in their city. High school students participated in a 10-session module about their city's two-tiered financial system of banks and alternative financial institutions. The paper analyzes two class sessions organized around the use of ratios, or intensive variables, to understand the distribution: an embodied distribution activity atop a large floor map and individual exploration of scalable, data-rich digital maps. Analysis investigates the role of students' sense of place as they grappled with these ratios. Students drew on their senses of place to interpret data and generate their own sets to associate in ratios. Abstract measures and data visualizations contained in the digital maps were less accessible to students in this iteration of the module.

Keywords: Equity and Diversity

The concept of ratio cross cuts school mathematics at all levels and is an important tool for making sense of the world, in relating pairs of quantities and as building blocks of rates. Reasoning with ratios "plays such a critical role in a student's mathematical development that it has been called a watershed concept, a cornerstone of higher mathematics, and the capstone of elementary concepts" (Lesh, Post, & Behr, 1988 as cited by Lamon, 1993, p. 41). Ratios are integral to investigations of issues of social (and spatial) justice, as they are useful for analyzing equitable or fair-share distributions of resources.

This paper examines student work with associated sets ratios in the context of a spatial justice investigation around the theme of local access to financial institutions. Banks and alternative financial institutions (AFIs, such as pawnshops, wire transfer, and check-cashing stores) form a two-tiered financial system. Banks offer credit-building opportunities as well as lower rates for services than their AFI counterparts, but they tend to cater their services to people with better financial credentials and more flexible income (Servon, 2013). As part of a larger project, a curricular module was designed for high school students to use mathematics to contrast the interest rates of banks and AFIs and to build on that contrast in analyzing the local spatial distribution of these institutions. This paper focuses on the latter component and pursues the following research question: How did students' sense of place support and complicate their use of ratios in evaluating the spatial distribution of financial institutions in their city?

Guiding Frameworks

Place, Space, and Social Justice

Teaching mathematics for social justice entails reading and writing the world with mathematics; "reading" with a critical perspective on the social and political circumstances that structure the world, and "writing" with agency to respond to or change inequities or injustices (Gutstein, 2006). This paper presents findings from a mathematics module oriented around an extension of teaching mathematics for social justice: teaching mathematics for spatial justice (Rubel, Lim, Hall-Wieckert, & Sullivan, 2016).

Traditional notions of "space" in mathematics refer to geometric abstractions in which the concept of "justice" would seem to lack pertinence. This paper draws on social rather than geometric

definitions of "space" (Soja, 1996; Tuan, 1977). With such an understanding, space encompasses social, historical, and geographical dimensions, each contributing to the world's ongoing state of affairs. Abstract "space" comes to be bounded by human experience and imbued with historical meaning, taking on character as a particular, locatable "place" (Tuan, 1977). The associations that people have with a particular place, or "sense of place," can color their perceptions of the world around them (Lim & Barton, 2006; Tuan, 1979). Maps and GIS-enabled digital presentations of data can present authoritative, bird's-eye-view perspectives about place (Wood, Fels, & Krygier 2010) but can misrepresent the reality of being on foot in a particular location, and can conflict with people's sense of place (Monmonier, 1991; Pickles, 2004).

The extent to which the multi-dimensional relations that structure space and place are fair or unfair constitutes spatial justice (Soja, 2010). Spatial justice provides a context for mathematical questions and tools focused on uncovering and examining hidden biases and agendas which structure the spatial relations people inhabit. An example of this can be seen in a critical mathematical examination of a state's lottery, which can analyze lottery ticket sales by location to investigate relationships between lottery spending and income level (Rubel et al., 2016).

Intensive and Extensive Variables

Variable quantities of space can be categorized as extensive (i.e., 'quantity of space,' such as area) or intensive (i.e., ratios of extensive variables, such as population density) (Lawvere, 1992, p.18). For example, banks per square mile is an intensive variable that relates two extensive variables (number of banks and area in square miles) and coordinates a single measurement to formulate a property description of a place. Extensive and intensive variables can be used to frame data for mapping (Goodchild & Lam, 1980), as in choropleth maps, which correspond color shades to ranges in numerical data (Buckley, 2013). Since geographic areas vary in shape and size (and other properties), comparing spatial data typically necessitates normalization of data using intensive variables. In the case of this project's digital maps, extensive variables that quantify numbers of banks and AFIs by neighborhood demand normalization, in relation to other properties like area (e.g., pawnshops per square mile) or number of households (e.g., households per bank), or by comparing them against each other (e.g., banks per AFI).

As associated sets, intensive variables like these are known to be accessible to students in the way that the relationship between the two extensive measures can be imagined more concretely than in other categories of ratios (Lamon, 1993). Students' sense of place could serve as a resource for concretization of the intensive variables necessary to analyze the spatial justice of a distribution.

Methods

This paper presents findings from the first iteration of a 10-session mathematics curricular module. The module was piloted in a 10th-grade advisory class taught by a mathematics teacher at a high school located in Harwood, a low-income, predominantly Latin@ neighborhood.

Researchers audio recorded and collected detailed field notes for each class session, and recordings were used to clarify and enrich fieldnotes. The focus here is on sessions five and seven because of their emphasis on using intensive variables to normalize data. The fifth session ("Big Map Day") featured an embodied distribution activity to highlight the conceptual underpinnings about normalizing data, atop a big floor map of the city (Edelson, 2011). The seventh session ("Ratio Map Day") engaged students with custom-made digital maps that were produced as part of a suite of digital tools for the larger project.

Whole group discussions from these two sessions were transcribed from audio. Focal students' work (3 groups) with digital maps was captured using Camtasia (Techsmith, 2010), which video-records students' screens, including the actions of their cursor, in sync with audio/video of students at work though the computer's built-in camera. The Camtasia recordings were used to produce

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narrative descriptions of students' actions with the digital maps with corresponding transcription of spoken utterances. Two main sets of codes were used to analyze the data: 1) the various ways to read and compare ratios (i.e. comparing numerators additively), and 2) the variables students identified to explain distribution as well as how students used the idea of proportionality to make an argument about fairness. A collaborative process was used to generate and apply the codes to the data.

Results

Big Map Day (Session 5)

Students began with a free-form exploration of a walking-scale, 140-square-foot map of the city temporarily installed on the classroom floor. Next, standing in and representing respective counties, students received various scaled props to demonstrate relative numbers of pawnshops and banks in these spaces and were prompted to discuss their reactions to these various distributions (Figure 1). As we have noted elsewhere (Hall-Wieckert, Rubel, & Lim, 2016), the activity engaged students' sense of place (Lim & Barton, 2006) about the city. This place-based knowledge served to support students in understanding the need to normalize data and to begin to coordinate such measures with demographic characteristics.
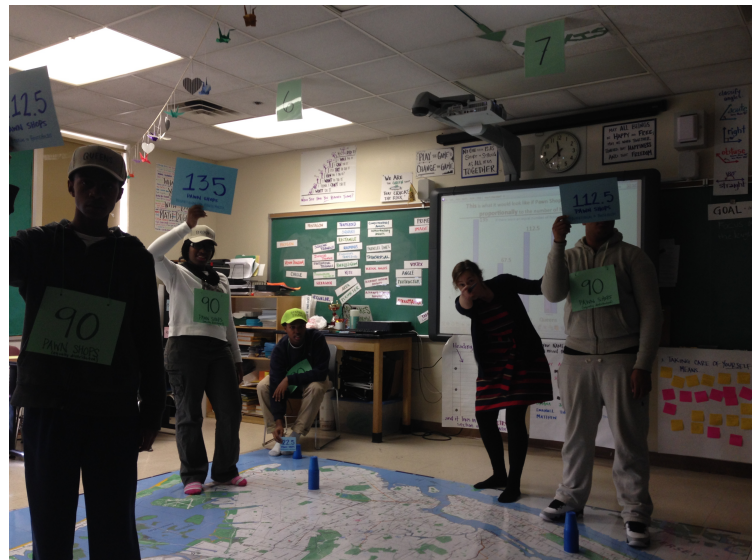
Figure 1. Students participate in embodied distribution on large map.

The activity started with a hypothetical equal distribution of pawnshops by county. A student, Sheeda, almost immediately refuted this as not making sense, saying that "some places are bigger than others, and some places are poorer than others," which suggested that the normalization should compensate for these distinctions. The teacher highlighted the former part of Sheeda's observation and led an activity distributing the institutions by area, then progressed to a distribution normalized according to households by county. Finally, the teacher revealed the actual distribution, which corresponded to none of these hypothetical and seemingly fair arrangements. When the actual distribution was shown, students confirmed the distribution using their sense of place. Sheeda explained the large number of banks in one particular county by pointing out, "...he got a bank on every corner though." This idea of normalizing by "corner" seems to suggest that she recruited her sense of the density of banks there to confirm the skewed distribution and add further nuance to the distribution of banks at a finer level of spatial scale.

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Students also drew on their sense of place to generate their own variables associated with the distribution of banks and pawnshops. Some students conjectured that some areas have more banks because they have a higher concentration of stores or more expensive stores. Several students associated variables related to their sense of place in terms of income inequalities across their city. For example, Rebecca stated, "I think that the ones that have the most pawnshops is where people have less money." Rebecca later reacted to the idea that pawnshops and banks might not be distributed fairly, and pointedly asked the teacher, "Why they never fix that, Miss?" In this case, "fixing" or redistributing institutions for equitable access and services would involve a transformative "rewriting" of the world, a line of thought that is encouraged by teaching mathematics for social (or spatial) justice, but beyond Rebecca's query is not further picked up by the teacher or other students in this discussion.

Ratio Map Day (Session 7): Whole-class Discussion

A suite of digital maps enabled students to explore the distribution of financial services in the context of various socioeconomic variables. The suite included three types of map layers: 1) points representing the locations of banks, AFIs (pawnshops, wire transfer, and check-cashing stores), and McDonald's, which was included as a service likely more familiar to youth than financial institutions; 2) demographic variables (e.g., median household income); and 3) ratio maps, which used intensive variables to normalize distribution of institutions by relating the quantity of institutions to area, number of households, or to another category of financial institution (e.g., banks per AFI). The demographic and ratio maps were choropleth layers that could be displayed concurrently with the location points. In other words, the intensive variables were displayed using a color scale by intensity and could be layered underneath the view of the extensive variables, showing the count and location of each category of institution (see Figure 2). Session 7 was the second day that students engaged with the digital maps and the first day that the ratio maps were introduced.

At the start of this session, the teacher conducted a whole-group discussion orienting students to the McDonald's maps as a way of priming them to conduct similar explorations about financial institutions later in the session. Of course, the notion of a fair distribution of McDonald's is not equivalent in significance to a fair distribution of financial services, and some might consider the presence of McDonald's as a disservice, but the example was intended to serve as a tangible, familiar variable for students to situate their reasoning about distribution in relation to area and households.

The teacher-led exploration began with a choropleth visualization of the number of McDonald's per square mile. Where on the huge map, students' sense of place about locations in the city served to prompt their association of variables toward normalizing the distribution, in this session, they recruited their sense of place as a way of interrogating the accuracy of the digital maps being presented as authoritative and official (Monmonier, 1991; Pickles, 2004). Immediately, Lina questioned this normalization and wanted to know how many blocks were equivalent to a square mile. Lina's understanding of units of measurement in city distance understandably did not conform to the traditional measure of square miles used in the maps. In this instance, Lina recruited her sense of place about the city's spatial arrangement to interrogate the decision to normalize density with square miles. Lina's skepticism was a shift from her participation on the Big Map Day when she generated associated variables with other students, like the density of stores or malls, to interrogating the variables presented on the digital maps.
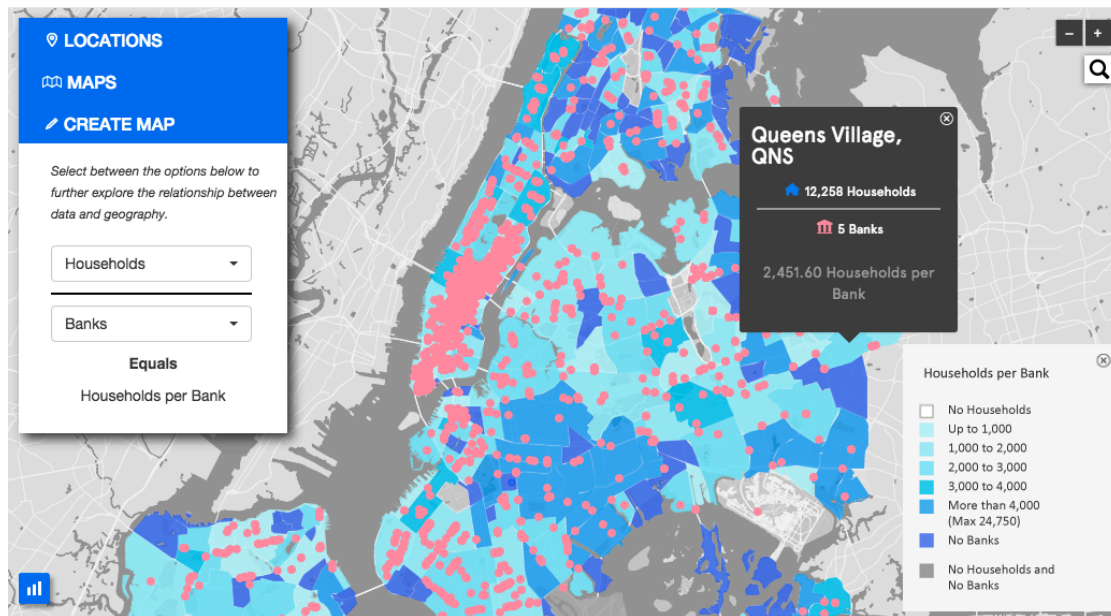
Figure 2. Location and distribution of banks in relation to number of households.

Lina's skepticism of the digital maps carried over to a questioning of the basic counts data embedded in the intensive variables like households per McDonald's. When presented with data showing that Harwood had 35,000 households for only one McDonald's, Lina exclaimed, "And one McDonald's? They lyin'!" She then proceeded to enumerate additional McDonald's located in what she considered to be Harwood; however, her sense of place in terms of the neighborhood's boundaries did not correspond with the boundaries encoded in the map.

Students' sense of place drew them to focus primarily on the map's extensive data, comparing the absolute number of McDonald's by neighborhood rather than intensive relationships. Which neighborhoods had "more" or "less," in an absolute sense, was more tangible and important. An exception to this was when Rebecca noted the crux of the relationship between the school neighborhood's McDonald's and another's: "So we only have one we share with more, and then they have way more and they share with less." This observation enabled Lina to reduce the fraction 23,000/23 to its unit ratio and follow up with, "So a thousand people go to one McDonald's?" These combined observations fed into Rebecca's expression that the distribution was unfair, and she pointedly asked the teacher again, "So why doesn't anyone fix that?" Rebecca's realization suggests that by understanding the intensive variables at work in the maps, she was able to question fairness using spatial data.

Ratio Map Day (Session 7): Map Exploration

After an introduction to the ratio maps included in the mapping tool, students explored digital ratio maps of their choice. Students, in pairs, were prompted by a worksheet to choose one map layer (i.e., one intensive variable), interpret the map's data for Harwood, notice and examine patterns across the map, and compare the data for any two neighborhoods. The different ways in which students engaged their sense of place contributed to differences in their sense making about the predetermined intensive variables to evaluate the distribution of financial institutions across the city.

Green pair. Miguel spent most of the session silently clicking through and across all of the different ratio maps as Rafaela watched, and engagement with their senses of place was not apparent. His clicks focused on each map's dark-shaded neighborhoods; that is, the neighborhoods with the highest ratios for each intensive variable. This does not necessarily mean he was looking at the most-serviced neighborhoods. For example, darker shading in the households per bank map layer would

mean a lower rate of services. In his explanation for choosing the banks per square mile variable, he stated, "Because we wanted to see how many banks were in each square mile"; and he reported that in Harwood, "for each 1.81 square miles, there's only 3 banks." The use of the word "each" in both of these statements indicates an understanding of the multiplicative relationship between the extensive quantities. Miguel's interpretation of the map shading did not coordinate between a neighborhood's number of banks and number of square miles and referred only to the extensive variable of the counts of banks. He wrote, "The darker the color becomes, the more banks you will find in that location. And the more lighter the color becomes, the less banks you will find."

Orange pair. Sheeda and Miriam engaged their sense of place by focusing on comparing Harwood with Montgomery, another familiar and adjacent neighborhood. These students ignored the worksheet, and instead spent most of their time exploring the absolute numbers of financial institutions by neighborhood. As a typical instance, upon examining the pawnshops per square miles map, Sheeda clicked on Montgomery and noted, "There's 3 pawnshops," by reading only the numerator from the fraction shown in the pop-up box. She ignored the square miles in the denominator and the resulting ratio of pawnshops per square mile. Again, examining the ratio of AFIs per bank for Harwood, Miriam compared the quantities in the numerator and the denominator additively: "We got more pawnshops than banks." At no point did Sheeda or Miriam demonstrate thinking about the intensive quantities, and they persisted in engaging only with the extensive components, even with adult intervention. A researcher guided the students to observe that Harwood was smaller in area but had more AFIs than Montgomery. When the researcher asked the students to explain what this means, Sheeda focused only on the extensive variables and responded, "That means Montgomery is bigger than Harwood."

Purple pair. Rebecca worked alongside an assistant teacher and focused on the households per bank map. Rebecca wrote an analysis of Harwood's data that went beyond Miguel's by not only interpreting the ratio terms but also how it related to the unit ratio: "The data says that there are 35,521 households for each 3 banks. So 11,840 people share each bank. It says that Harwood shares each bank with a lot of people." After reading the number of households and institutions in her chosen neighborhoods, she stated, "So this one [Portmore] has less households and it has more banks. And this one [Easington] has more people and just one bank." She concluded, "They [Easington] should have put … had more [banks]."

Rebecca did not generalize her interpretation beyond making sense of specific data points. When prompted to explain what a higher ratio meant for her chosen variable, Rebecca said, "When it says the ratios are higher, I think it means where the banks are more at." She did not recognize that in the case of households per bank, a higher unit ratio would indicate a lower proportion of banks per household. Rebecca's sense of place confounded her analysis in that her hypothesis that the number of banks was related to a neighborhood's income level was a distraction from interpreting the given legend. Rebecca stated that she "just wanted to know where was the more banks for houses with less money," and she expected to be able to answer this question through a single map layer. This expectation led her to try to inject income into her analysis of the households per bank map and read the categories in the legend as referring to ranges in household income rather than ranges in number of households.

Discussion

Sense of place contributed to students' understanding of the rationale for launching a mathematical exploration of the distribution of financial institutions. The idea of dividing up the number of institutions equally by county evoked a voicing of ideas about variables, related to their sense of place, that they considered important in analyzing this distribution. Students' consideration of these variables provided a launching point for the teacher to introduce data normalization.

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In contrast, students' senses of place challenged the representations in the digital maps. Some student-generated associated variables, like relative size of different parts of the city, connected well to ratio measures of distribution like density per square mile, which were included on the digital maps. However, many of the suggested variables by students, like the density of stores in different parts of the city, were unanticipated and not available as intensive ratio maps. The map tool did not allow for students to follow through with their own ideas about what variables might be salient and thereby engage in more authentic data explorations.

In addition to a fixed set of variables to examine, the way that the maps were designed did not allow for flexibility in approach. The mathematization of the selected variables in these maps were imposed; for instance, the choice of square miles made sense to the designers as a unit of areal measure, but blocks or "corners" could have been a more accessible unit. Similarly, neighborhood delineations on the map did not always correspond with students' senses of boundaries. For Rebecca, her desire to examine a direct relationship between income and number of banks on the map was constrained by the designers' decision to include demographic maps as separate static maps to be compared, but not combined, with the chosen ratio maps.

Sense of place served as an anchor for students to grasp the tangible aspects of the given variables; at the same time, it distracted them from engaging with more abstract, intensive variables. During the whole-class discussion and in their investigations with digital maps, students tended to compare extensive variables–in this case, the discrete number of institutions. Scaffolding concrete representations of abstract intensive measures, such as households per institution as the number of people sharing one institution, aided in students' interpretation of these variables and supported their mathematical arguments about fairness. Miguel and Rebecca, for example, were successful at interpreting intensive variables with respect to specific neighborhoods but struggled to interpret the more abstract choropleth layers.

Conclusion

Students' sense of place engaged them in considering and analyzing spatial data. The open-endedness of the huge floor map enabled students to understand the conceptual basis of and suggest strategies for normalization. Students drew on their sense of place to imagine concrete representations of intensive variables, which allowed them to interpret data points, make comparisons, and even formulate arguments about fairness. These concrete representations could serve as building blocks toward making sense of the data represented more abstractly in the map's choropleth layers, which were more elusive for students to interpret.

The closed nature of the larger project's digital maps did not enable students to use the maps to fully explore variables that connected to or built on their sense of place. Rather, students' sense of place became a tool to critique the point of view presented by the seemingly authoritative digital maps. More support towards activities that combine the open-endedness of the big map activity with the abstraction and mathematization of the digital maps is needed. A future goal is the design and implementation of digital mapping tools that enable students to generate their own intensive variables or ratios, based on their sense of place, which can then be visualized in authoritative map layers. The authors are currently investigating new designs for activities to engage students in map-making of this kind.

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HOW BORDERS PRODUCE PERSONS: THE CASE OF CALCULUS IN ENGINEERING SCHOOL

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Mathematics has long been understood to be a "gatekeeper," which functions to restrict access to educational opportunities due to (a) its central role in formal educational sequences, and (b) the high failure rates often observed in mathematics courses. Failing is an encounter with a type of border: a border that divides a continuous scale into two categories, passing and failing. Typically, this border is understood as restrictive. We present a more nuanced view of the role of borders in mathematics education. In particular, we argue for a view of borders as fundamentally productive objects, and suggest that analytical attention should be focused on who or what gets produced, rather than on who or what gets restricted.

Keywords: Affect, Emotion, Beliefs, and Attitudes, Learning Theory, Post-Secondary Education

Mathematics occupies an interesting position in engineering education. On the one hand, the calculus sequence occupies a central role in the curriculum. On the other hand, the calculus sequence is widely recognized by students, faculty, and researchers as a "gatekeeper," which often functions to push students out of engineering. The gatekeeping role has been ascribed to the calculus sequence due to (a) its centrality in engineering curricula, and (b) the astonishingly high rates at which students fail calculus courses (Seymour & Hewitt, 2000). In this paper, we examine the centrality of mathematics in the engineering curriculum, and we analyze what happens when students fail. This is, by now, well-worn territory. However, our analysis offers new insight into this process.

Failing is an encounter with a type of border: a border that divides a continuous scale into two categories, passing and failing. Typically, this border is understood as restrictive, in that it is used to restrict access to future course work. From this perspective, the work summarized above can be understood as attempts to document the restrictive nature of the border, and to help more students land on the passing side of it. We do not disagree with these efforts. Our goal with this paper, however, is to present a more nuanced view of the role of borders in mathematics education. In particular, we argue for a view of borders as fundamentally *productive* objects, and suggest that analytical attention should be focused on *who or what gets produced*, rather than on who or what gets restricted.

Theoretical framework

We take a critical, situated perspective on learning, in which learning is understood as an ontogenetic process of *becoming* a member of a community of practice. Traditionally, such analyses focus on how both people and communities of practice are produced and re-produced through collective, often asymmetric, participation in cultural practices. In this perspective, *trajectories of membership* are often invoked to describe a person's "changing participation in changing practices" (Lave, 1996, p. 150), and such trajectories are analyzed to document the ways in which participants are produced as new kinds of people (e.g., Nasir & Cooks, 2009). We adopt a similar focus, but augment it by considering also the role of *objects*— that is, "stuff and things, tools, artifacts and techniques, and ideas, stories, and memories […] that are treated as consequential by community

members" (Bowker & Star, 1999, p. 298)—in the production of persons and communities of practice.

We draw on two key theoretical constructs to examine the relationship between objects and participation in the production of people and communities of practice. *Trajectories of membership* (Bowker & Star, 1999; Lave & Wenger, 1991) describe the adoption of and conferral of practice-based identities upon newcomers. *Trajectories of naturalization* (Bowker & Star, 1999) describe the ways in which objects—including categories and categorization systems and their attendant borders—enter into and become naturalized within a community of practice.

Such a dual focus is crucial in understanding membership because the naturalization of objects is the process through which norms and values become embodied in practices of everyday life. In this perspective, interactions among people are always mediated by objects, because it is through objects of various kinds that people become seen as a certain type of people:

> The relationship of the newcomer to the community largely revolves around the nature of the relationship with the objects and not, counterintuitively, directly with the people. This sort of directness only exists hypothetically—there is always mediation by some sort of object. Acceptance or legitimacy derives from the familiarity of action mediated by member objects (Bowker & Star, 1999, p. 299).

Research method and context

Our research focuses on the experiences of students in a diversity program in the engineering school at State U, a flagship state university in the Western United States. To capture student experiences and the ways in which those experiences were organized within this community of practice, we conducted field-based ethnographic work centered on students, faculty, and staff in the program. We used a variety of fieldwork methods including ethnographic observations of routine activities, ethnographic interviews, and focus groups. Our data include fieldnotes, meeting minutes, and video and audio recordings. Our analysis involved concurrent engagement in data collection and data analysis, using Constant Comparative Analysis (Glaser & Strauss, 1967). We analyzed data from initial fieldwork early in the research process, leading to a preliminary "grounded theory," which led in turn to further fieldwork to refine the theory, and so on through multiple iterative cycles.

The engineering school at State U is predominantly composed of white, male, middle- and upper-class students. *Access* is a program that seeks to broaden access to the college by admitting a cohort of approximately 30 "next-tier" students to the college each year. Students in the Access cohorts were initially denied admission to the engineering school, but were accepted via the Access program after a second round of admission screening. The Access program has explicit diversity goals, and is composed almost entirely of women, students of color, and first-generation college students. Although these students are admitted directly to the engineering school, they are enrolled in a "performance-enhancing year," in which they take remedial courses to prepare for courses in the engineering school.

Our initial field work pointed to the centrality of Calculus in the organization of the experience of Access students. Calculus is a naturalized, taken-for-granted object within the community of practice. How did this come to be, and how does calculus organize the trajectories of membership for students? In the sections below, we document the trajectory through which calculus came to be a naturalized part of engineering education, and we describe how one Access student's trajectory of membership into engineering school became torqued by this trajectory of naturalization.

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Trajectory of naturalization: Mathematics as a naturalized object in the engineering curriculum

In the late 19th century, an American engineer beginning his career (in reality, it was almost always "his" career) likely would have started on the shop floor and learned his craft through a practice-based apprenticeship. In those days, engineering was a craft, practiced by masters who relied on "design experience and rules of thumb" (Seely, 1999, p. 289). By 1965, aspiring engineers would experience something very different. Instead of a shop floor, they were likely to find themselves sitting at a desk in a classroom at a four-year university. Gone was the practice-based apprenticeship, replaced by a curriculum based in theory (viz. math and science). Gone too was the image of an engineer as a craftsman, replaced by an image of the engineer as a rational technicist. As Seely (1999) traces, over the course of 80 years, mathematics became naturalized in engineering education.

Three broad trends propelled mathematics along its trajectory of naturalization in engineering education. First was the emergence in the U.S. of new technologies such as electricity that appeared to defy the commonsense rules of thumb that had previously guided engineering design. Second, the expansion of land grant colleges, along with an emerging belief in the power of math and science to make the world better, began a general trend to push professional preparation out of apprenticeships and into universities. Even so, university engineering programs at the start of the 20th century were dominated by practical concerns. While these programs involved some math and science, the curriculum remained grounded in practice, and students spent a good deal of time in machine shops and at drafting tables (Reynolds, 1992; Seely, 2005). Finally, an influx of European engineers—who were trained in physics and mathematics, and who approached engineering problems as applied exercises in these disciplines—into the U.S. occurred during the early part of the 20th century.

Concerned by what they perceived as American engineers' poor technical training, European engineers turned their sights to education. Here the trajectories of European engineers became coordinated with those of U.S. students. Students were entering universities to learn engineering, and European engineers were entering universities to teach it. Guided by a belief that engineering was a technical discipline resting on a foundation of basic mathematics and science, European approaches to engineering began to transform engineering education at the same time as the rise of the rational thinker in the American imagination (Seely, 1999).

The transformation of engineering education was not a smooth one, however, and debates over the nature of engineering and engineering education were "loud and prolonged" (Seely, 2005, p. 116). While this debate took place largely on university campuses, it can be read as a struggle over the nature of engineering itself, as a practice-based trade or as a theory-based profession. Geopolitics entered this debate in the form of the Second World War. Pre-war, engineering faculty were primarily occupied with teaching. Research was rare, and what research was conducted was small-scale, practical research driven by the needs of local industry. During and post-war, the Department of Defense, which was determined to maintain an arsenal of cutting-edge vehicles and weaponry, demanded basic research into explosives, propulsion systems, and materials. Engineering faculty were determined that they, rather than researchers in the basic sciences, should get this funding and its associated prestige. Soon basic research was woven into the fabric of engineering schools, and curricula were twisted to match the demands of the research. Machine shops were replaced with classrooms, drafting tables with desks, and practical experience with math and science. By 1965, mathematics and science were fully naturalized objects of the engineering curriculum (Seely, 1999, 2005).

Today, mathematics is a naturalized, taken for granted part of the infrastructure of the engineering school at State U, encoded into such objects as curriculum flowcharts, shown in Figure 1. One way to see this naturalization is to examine the role of math courses on the one hand, and projects courses on the other. First, notice the centrality of math courses. This can be seen by examining the dependency trees for courses with pre-requisites. All of these dependency trees

include math. Now, compare this with the two "projects courses" that are part of the first and fourth year. As shown, these courses are widely separated and generally disconnected from the rest of the curriculum. Within math and science courses, the course called "Calculus 1 for Engineers" stands out. This is arguably the most important course in the flowchart, as it is part of every dependency tree, and it is the only course for which this is true. Furthermore, it is the first course in each of these dependency trees. In other words, the first step to accessing any course with a pre-requisite is passing Calculus 1.
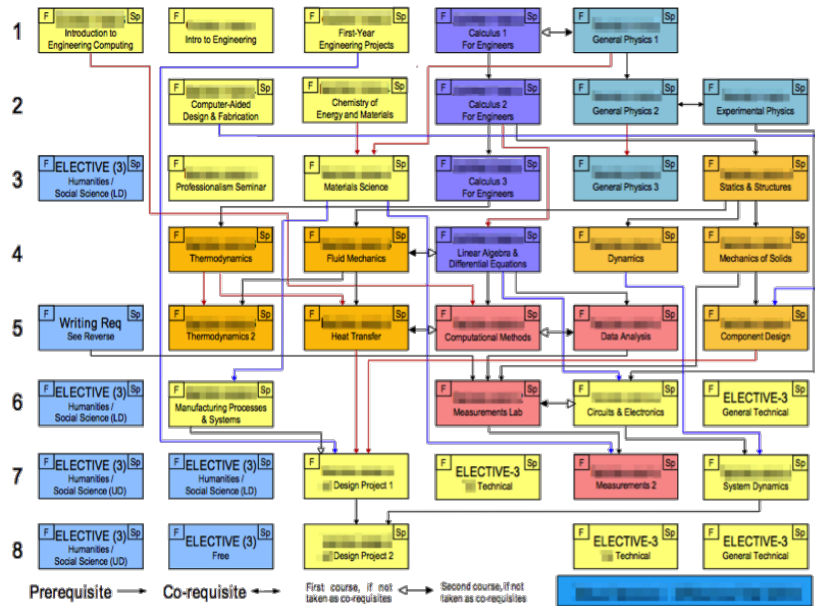
Figure 1. Engineering curriculum flowchart at State U. Course numbers are blurred to preserve anonymity. Numbered rows indicate academic semesters.

Notice the logic inscribed in this object, and the role of borders in producing and maintaining this logic. The rows of the flowchart segment time into semesters. The rectangles segment content into courses, with borders that bound both time and content. The borders between courses can only be breached in a particular sequence, signified on the flowchart with arrows. Taken together the flowchart inscribes a logic of learning as a linear process that occurs within bordered units of content and time.

Moreover, the flowchart necessitates a set of institutional practices for producing and policing borders. Chief among these "*border-policing practices*" are *testing* and *grading*, as these are the practices through which students are classified into courses, that move students from course to course. In overview: tests are composed of problems; students provide solutions to these problems by making inscriptions—a process through which students translate some piece of themselves into marks on paper; and these inscriptions are mobilized into a separate room where instructors translate them into scores—a different set of marks on paper. These scores are then aggregated and translated into a grade, yet another mark on paper. Final grades, then, are produced via a cascade of inscriptions. Students produce inscriptions in response to test problems, these inscriptions are assembled and translated into a numerical score, and these numerical scores are assembled and translated into a final grade.

These grades serve as the mechanism by which the institution polices the borders inscribed in the flowchart. Depending on the grade a student receives, she is either granted or denied passage through the flowchart. Passage from one course to the next is binary—the border either opens or it doesn't—

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and this gives certain categories heightened importance. For Access students, the category that determines passage into the next math class is a B-.

In addition to doing institutional work, the flow chart also serves as a resources for students to locate and construct themselves in relation to the institution (Nespor, 2007). The flowchart translates time (measured in semesters) to space, and produces a standardized "path" onto which students reckon themselves in spatial terms. For example, students talk about being "ahead" or "behind." These reckonings are value-laden; it is much better to be "ahead" than "behind," and being "behind" brings with it costs in terms of money, time, and status. Thus the borders of the flowchart don't just serve to segment content and time, they also serve to produce students' institutional identities. As such, the borders do both institutional and identity work, as we highlight in the case below.

Trajectory of membership: Mary

Mary is a white female student whose experience in the Access program has been shaped by mathematics and its associated boundary-policing practices. She came into the program having taken calculus in high school, and expected that she would take calculus during her first semester. She never had the option. In order to understand why, we have to examine the history of the Access program, and its relationship to calculus.

The Access program has enrolled six cohorts of students. From the beginning, the program has faced opposition from some faculty in the school of engineering, who see the program as a sign that the school is "lowering standards." This opposition became especially salient for Professor Turner, the director of the Access program, in the program's third year when it became clear that students in the Access program were not passing Calculus at a rate commiserate with the rest of the school of engineering. In a focus group she explained, "when we saw our kids failing [Calculus], were like, 'oh crap, we're about to lose this program.'" Professor Turner recognized that at State U, calculus doesn't just legitimize students, it legitimizes *programs*.

Calculus I for Engineers at State U has a 30% failure rate, and, because it's so central in the flowchart, this leads to attrition in the engineering school. Still, it's largely untouchable. From Professor Turner's perspective, "changing calculus is like moving an elephant—a mountain." So rather than change calculus, the Access program changed their remedial sequence. They worked with the math department to create a new Pre-calculus course with the explicit goal of making Access students "calculus ready." Because passing calculus is so important for engineering students (and the programs they populate), Professor Turner decided to enroll all of the students in Mary's Access cohort in Pre-calculus, regardless of each student's history with mathematics.

While Professor Turner made the decision to enroll Mary and the rest of her cohort in Pre-calculus in order to enhance the legitimacy of Access students and the Access program, the decision initially had the opposite effect for Mary. This is because the nuance of such a decision cannot be captured on the flowchart, which enforces a standardized pathway through the curriculum. Professor Turner's decision pushed students like Mary outside of the borders of this pathway, into into the margins of the flowchart. Mary explained:

> Almost all of the classes here for engineering are based off of calc, so if you're not caught up in calc you're gonna be behind in all of your other classes as well. And there's like no getting around it. So that means I'm definitely way behind other people in the engineering school.

One the one hand, such a position is expected for Access students. It's a natural outcome of the "performance enhancing year" that is at the heart of the program, and of Professor Turner's decision to enroll all Access students in Pre-calculus. From this perspective, Mary, who took Calculus I in the fall semester of her second year at State U, isn't behind at all. She's exactly where the Access Program expects her to be. However, such a perspective fails to account for the lived experience of

students like Mary, who took what she calls "freshman classes" during her sophomore year. During an interview, her voice cracked as she described what this is like:

> [It's] kind of embarrassing almost, cause you feel like you can be smarter than this, and like moving on, but you're not. Yeah. I don't know. I, it's just, I don't know... I don't know (softly). I can't- I don't know how else to describe it.

The margin of the flowchart is not a neutral space. In the margin, Mary questioned her legitimacy as an engineering student.

This is not the only way that mathematics and its associated border-policing practices affected Mary's sense of self. She describes herself as someone who tries to "think things through," and "show all the steps." For her, this is how to make learning "stick." However, at State U, Mary experienced mathematics as an exercise in answer-getting under time pressure. In the segment below, she talks about doing slow, careful work in mathematics.

> Well, it's not good if you want to actually get stuff done. [Last semester] I would just run out of time to do stuff, because I spent so much time like, going through… And it's not a good thing for exams because that's kind of like, you've got to know it, you've got to write it out really fast. And you've got to know all the stuff, you can't just like stop to think for too long, otherwise you run out of time.

As Mary describes, borders are important in testing. Tests are temporally and spatially bounded. Inscriptions that are produced outside of these borders are not allowed in. Hence, "you've got to write it out really fast," because inscriptions made outside the temporal border aren't mobilized into the grading room. If you "run out of time," you bump up against the impermeable temporal border of the test.

Mary "ran out of time" on her tests in Pre-calculus, and this turned out to be consequential. At the end of her first semester, her grade was below the binary cut-off and she therefore had to repeat the course the following semester. For Mary, mathematics was refusing to be naturalized, and this pushed her further into the margins of the flowchart. In large part, this was due to the way that mathematics was construed at in the engineering school at State U. Because the border-policing practices privileged answer-getting and speed over reasoning and thoughtfulness, her pensive style went unrecognized by the institution.

Mary "passed" Pre-calculus after her second semester. Looking back, she describes Pre-calculus as "very important" for calculus. In large part, this is because exam problems *made* it important:

> [Pre-calculus] was very important. The first test was like over all things pre-calc basically. With a couple of calc things. […] They had a question on absolute value. And you learned like a lot about absolute value and translations and stuff [in pre-calc]. And you needed that information to solve a certain problem on the calc exam.

Under these circumstances, Pre-calculus helped Mary to perform on the consequential knowledge displays (Stevens, O'Connor, Garrison, Jocuns, & Amos, 2008) that police the borders of the flowchart. As such, it helped propel Mary out of the margins of the flowchart and onto the standard pathway.

But it is in the margins, after Mary failed her first semester of Pre-calculus, that something even more extraordinary happened. Because of her non-standard position as a student repeating Pre-calculus, Mary was in a small class of students. In this environment, her instructor came to know Mary and her thoughtful style. This led the instructor to ask Mary to be a Learning Assistant (LA) for the Pre-calculus course in the following year. LAs are undergraduates who help to facilitate small-group interaction in large-enrollment courses. As an LA, Mary was positioned as an expert in mathematics. Students looked to her for help, this helped to affirm Mary's legitimacy:

You feel like, kinda, proud of yourself and like, um, really happy that you got to help someone and like, make their life a little easier. Um, it's just, yeah, a really good feeling.

In addition to helping students, she worked with and socialized with Pre-calculus instructors, she helped to grade exams, and she conducted an independent education research project that she presented at a poster session on campus.

In some ways, mathematics—and the boundary-policing practices that allow it to be institutionalized—pushed Mary into the margins of the engineering curriculum, causing Mary to question her identity as a member of the engineering community. This began with the Access program's decision to enroll Mary in Pre-calculus—a decision that ensured that Mary would always be behind on the flowchart, but which, in retrospect, helped Mary gain legitimacy in calculus. It continues with the ongoing violence that timed tests inflict on Mary's sense of self. At the same time, the margin became a space of possibility for Mary (hooks, 1989). In the margin of the flowchart, mathematics pulled Mary into a position of some power and authority within the engineering school, allowing her to build relationships and participate in practices that affirm her legitimacy in the community.

Conclusion: The role of borders in the production of persons

Foucault (1978) presented a radical new vision of power. While the received view was (and continues to be) that power was coercive, Foucault argued that, on the contrary, power was fundamentally *productive*. This was not an argument that power was benign, just that power produced people and society in particular ways.

In this paper, we advanced a similar argument for the role of borders in mathematics education. We first documented how, in its trajectory of naturalization within engineering education, mathematics became intertwined with institutional borders. Next we illustrated how these borders (and their associated border-policing practices), are fundamentally productive objects that constantly work to produce students in particular and complex ways. To be sure, borders do violence to Mary. They create margins and produce suffering, as Mary questioned her legitimacy as an engineering student. But at the same time, the very same margin became a place of possibility for Mary, and helped to produce her as a privileged member of the community. There is little doubt that Mary has been produced in complex ways by the borders associated with mathematics.

Mary's story is just one of the many we could have told about the complex ways that borders become productive in mathematics education. For many of the students that we have worked with, borders have produced both injury and possibility. For example, another student in the Access program, driven by fears that his marginality (due to mathematics) will lead to him getting "kicked out of the engineering school," designed and patented a "high-end upgrade for 3-D printers," which he is currently selling online. Another collection of Access students formed a student-led group to argue for more individuation in the Access program, largely in response to Professor Turner's decision to enroll entire cohorts in Pre-calculus.

We stress that when we argue that borders are productive, we are not arguing that borders are benign. Borders produce students in complex ways that cannot be encapsulated *a priori* using simplistic categorizes like "good" or "bad." Instead, our argument is, to paraphrase Jean Lave (1993, p. 8): *that borders are productive is not problematic. What gets produced is always complexly problematic*. We argue, then, that more analytical attention should be focused on what gets produced by the borders in mathematics education.

Acknowledgments

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CROSSING THE LITERACY/QUANTITATIVE LITERACY BORDER

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Researchers have described the relationship between quantitative literacy (QL) and literacy in various ways: from distinctly separate, with quantitative literacy as the mirror image of literacy (e.g. Cockcroft, 1982) to quantitative literacy as one subset within a broad literacy conceptualization (e.g. Suda, 2000). However QL scholars conceptualize their work with reference to literacy, they invoke historical narratives about literacy and its scholarship. We wanted to embrace literacy scholarship as having the potential to productively inform QL scholarship. In this paper, we share some of what we have learned so far from acknowledging and investigating that potential by crossing the literacy/QL border. We conclude by crafting a working definition of QL that we think helps illuminate challenges in effecting it pedagogically.

Keywords: Equity and Diversity, Curriculum, Technology

Steen (1997) claimed that quantitative literacy (QL) had emerged as a partner to literacy and that supporting people's development of QL has similarly emerged as a crucial educational endeavor. For instance, Steen (1997) claimed that "an innumerate citizen is as vulnerable as the illiterate peasant of Gutenberg's time" (p. xv). Bass (2003) argued that most scholars engaged in QL research are interested in effecting QL rather than etymology, so it is neither productive nor urgent to develop a precise definition. We believe, in contrast, that attending to literacy scholarship has helped us begin to construct a more precise definition that helps illuminate complexities of and challenges to effecting QL.

How Literacy Informs Quantitative Literacy

We want to share three findings from our exploration of literacy research that have helped us inform our conceptualization of quantitative literacy. First, literacy is *deictic*, or continuously changing, due to technological changes and emergent social practices (e.g. Gee, 1989; Leu, Kinzer, Coiro, Castek & Henry, 2013). Second, *critical* literacy is an ideology that empowers people to both read the word and read the world; that is, people become critically literate by reading and writing language as a means to interpret, reflect on, and change their world (Freire & Macedo, 1987). Finally, being *willing* to read and write the world requires a focus on people's dispositions that needs to be attended to pedagogically (e.g. Giroux, 1988).

Changing Quantitative Information

Leu and colleagues (2013) described the transformative power that technology generally, and the internet specifically, have had on literacy. The authors claimed the internet drives the emergence of new social practices of literacy both by creating new literacies and systematically transforming existing literacy practices. Steen and colleagues (2001) made their *Case for Quantitative Literacy* on the premise that the 21st-century is, and will continue to become, a significantly more quantitative environment than any previous time in history. Steen (1997) claimed "as the printing press gave the power of letters to the masses, so the computer gives the power of number to ordinary citizens" (p. xv). Computers and the internet have also transformed QL by changing what quantitative information people see in several ways. Many researchers in statistics education have acknowledged the prevalence and pervasiveness of statistics, data, and representations in the digital information age (e.g. Gal, 2002). The nature of those statistics, data, and representations has not been static over time, however; for example, *infographics* that combine graphics, text, mathematics and statistics. Bertini,

Perer, Plaisant, & Santucci (2008) have worked to unpack how people understand infographics and data visualizations and have found that different evaluation strategies are used for these new representations.

Bezemer and Kress (2008) used social semiotics to investigate the evolution of creating multimodal representations of information with potential for learning. By arranging the information as an infographic, words with particular mathematical or statistical meanings (a percent, relative sizes, etc.) are often translated into and shared in graphical form, further connecting QL and literacy. The move of multimodal representations from static to dynamic recontextualizes the social meaning of the representation. Specifically, the social structures in which knowledge is generated, presented, shared and validated, affect and are affected by the mode of representation (Bernstein, 1996). Infographics online, therefore, are qualitatively different than their previous static, paper counterparts due to the social structure of the internet; infographics online can be shared with comment, and traditional consumers of information have become creators who reconstitute and validate the information as knowledge.

Infographics are one example of how technology can change the ways people encounter quantitative information. Integrating quantitative information into politics, businesses, and other social decisions also changes those decisions. This integration raises dilemmas: ethically, politically, socially and legally (e.g., Kitchin, 2014). For example, an overreliance and overvaluation on big data sets can give less space for and attention to dissent and counter-narratives that come from thick data (Dove & Özdemir, 2015). How do we acknowledge the hope that people not only understand the quantitative information, but also these dilemmas?

A Critical Quantitative Lens

Critical literacy differs from literacy because critical literacy refers "only [to] where concerted efforts are being made to understand and practice reading and writing in ways that enhance the quest for democratic emancipation" (Lankshear & McLaren, 1993, p. xix). Mathematics and related disciplinary lenses can powerfully function to "understand, identify, and tak[e] action on social justice issues" (Esmonde, 2013, p. 349). Scholars have done an impressive amount of work unpacking and examining critical theory in domains related to QL. These works include: teaching for social justice (e.g., Gutstein, 2003), criticalmathematical literacy (e.g., Frankenstein, 2001), mathematical literacy (e.g., Jablonka, 2003), critical mathematics education (e.g., Skovsmose & Niss, 2008), and work in equity which is informed by a Freirean critical stance toward mathematics education (e.g., Gutierrez, 2012).

Just knowing how to do the mathematics and statistics that you face during your life does not capture what we consider a critical quantitative lens. A critical lens more clearly includes what Gutstein (2003) calls writing the world with mathematics. Understanding the mathematics and statistics in how the unemployment rate is calculated, budgeting for their property tax, or the logic backing a legal decision is important. But, a critical lens toward these quantitative arguments involves acknowledging dilemmas; why is a person who has stopped looking for work excluded from the labor force during the calculation? Should education funding be tied to property tax? Are the premises of a legal decision true? These questions represent some of what Frankenstein (2001) meant when she delineated understanding mathematical knowledge from understanding the mathematics of political knowledge and understanding the politics of mathematical knowledge, and we think they clarify what we mean by a critical lens.

Jablonka (2003) stated "mathematical literacy focusing on citizenship should refer to the aim of critically evaluating aspects of the surrounding culture – a culture that is more or less colonised by practices that involve mathematics" (p. 76). Expressions of QL should similarly extend beyond understanding the ways mathematics and statistics have embedded into contemporary life. Critical

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

expressions of quantitative literacy embrace the power of mathematics and statistics as lenses to read and write the world.

Willingness to Engage

What effects do changing quantitative information and attending to a critical stance have pedagogically? Leu et al. (2013) argued that new literacies further complicate professional teaching by requiring a shift in role and by placing new demands on teacher knowledge. Specifically, Leu et al. (2013) claimed that a pedagogy reactive to the speed at which literacy changes creates an untenable demand on teachers to stay technologically current. The authors proposed that teachers move from dispensers of literacy knowledge to facilitators of productive literacy strategies and practices. Supporting critical literacy development also requires particular forms of pedagogy. Giroux (1988) pointed out that participating in direct forms of instruction is contradictory to a goal of helping students become critically literate. Likewise, careful attention to pedagogical strategies which support a person's disposition to engage in reading and writing the world with mathematics is necessary.

Wagner and Herbel-Eisenmann (2009) described the socio-psychological implications of choosing dispositions as a learning goal. Dispositions are more stable aspects of a person's identity than ways students might position themselves while in the classroom. In particular, the authors argued that dispositions are those socially constructed positions that people reify and transfer between contexts. Steen et al. (2001) claimed that students would develop a quantitative literacy habit of mind "only by encountering the elements and expressions of numeracy in real contexts that are meaningful to them" (p. 18).

Changes to pedagogy in a classroom devoted to critical QL involve more than developing contextual mathematical problems; students need opportunities to decide the problems they want to pursue in order to develop an active sociopolitical consciousness and a willingness to read and write the world with mathematics (Gutstein, 2003). Gutstein's (2003) students developed a sense of agency as actors who could change their world with mathematics by developing mathematical power. The challenge of creating a high quality curriculum is not met by developing artificial problems meant to meet mathematical goals (Romberg, 1992). To develop mathematical power and agency, a curriculum should reflect students' sociopolitical realities wherein mathematical and statistical understanding can be leveraged. Gutstein (2003) relinquished sole authority of problem posing; instead, he shared the decisions about which problems to pursue with his students.

A Working Definition and Implications

As a working definition, we take a person's *quantitative literacy* to be their willingness to engage a critical quantitative lens towards their changing world. We think this definition helps to illustrate particular challenges to effecting QL: (1) the world about which we want people to be quantitatively literate is transformed by and itself transforms quantitative information, (2) our goals for effecting QL extend beyond understanding the quantitative information they encounter to critically reading and writing their world, including its quantitative information, and (3) critically reading and writing the world with QL is not just an ability, but also an action: a person expresses QL not just by being able to read and write their world, but by actually doing it.

In this paper we crossed the QL/literacy border with the goal of using literacy scholarship to inform QL scholarship. We think that important differences between QL and literacy exist, however, and suggest that scholars continue illustrating those differences. We also think that QL scholarship can help inform some literacy scholarship, for instance bringing a QL lens to infographics can contribute to investigating multimodality in literacy scholarship.

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BLACK GIRLS IN HIGH SCHOOL MATHEMATICS: CROSSING THE BORDERS OF DEFICIT DISCOURSES

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This paper explores the discourses of three Black girls in one high school Algebra 1 mathematics classroom. We investigate how these students navigate a border crossing from deficit discourses about student success and toward success in mathematics. We further argue that the teacher's classroom practices, which facilitated this border crossing, were striving toward equity. The classroom practices we found included: (a) assigning competence, (b) valuing multiple solution paths, (c) helping group members, and (d) explaining mathematical thinking.

Keywords: Classroom Discourse, Equity and Diversity, Gender, Instructional Activities and Practices

Significance and 2016 PMNEA Theme Connections

One of the most pressing issues in American education is the underperformance of Black girls and women in mathematics across the P-20+ pipeline (Herzig, 2004; National Science Foundation [NSF], 2013). Deficit-based discourse about this population's performance has exacerbated poor performance (Solorzano & Yosso, 2001). Deficit-based discourse is a "border" that has significantly limited marginalized students' access to rigorous mathematical content and pedagogy. This discourse segregates Black girls to mathematics classrooms that under or devalue their ways of knowing, thinking, and engaging with mathematics, decreasing their opportunities to learn. Deficit discourses can also include how standardized tests are talked about and are made the center of knowledge production when it comes to identifying who succeeds and who fails. Deficit discourses also include conscious and unconscious low teacher expectations and negative beliefs about Black girls, which can be shaped by broader narratives in media and the literature (Russell, Viesca, & Bianco, in press). All of these deficit discourses can contribute to a socially-constructed border that teachers and students must confront, cross, and/or tear down to achieve success for all in mathematics. We argue that certain mathematics teaching practices can be enacted to help Black girls navigate crossing such borders. We explore the discourses of Black girls in an Algebra class who began our study achieving at lower levels, when compared with most ninth graders at their school and district, but who, over time, shifted their self-perceptions of competence because of their teacher's attention to equitable teaching practices. We seek to begin a conversation of this "border crossing" for Black girls in U.S. mathematics classrooms (Anzaldúa, 1987; Newman, 2003). We investigate how three Black girls navigated border crossing into self-identified success in mathematics through equitable mathematics teaching practices that sought to disrupt normalized expectations for competence. We adopt Esmonde's (2009) definition of equitable teaching and learning as "a fair distribution of opportunities to learn" for all students (p. 1008). We raise tensions, challenges, and we put forth generative possibilities for successful border crossings.

Research Questions

This study asked and answered the following questions:

1. What teaching practices do Black girls report as influencing their mathematics learning?
2. How do the reported practices reflect equitable mathematics learning?

Theoretical Framing

In order to navigate the perspective of our Black girls, we draw on a framework that combines status and positioning theories. This framework attends to how and whether students are positioned as competent (Campbell & Dunleavy, 2016) and cross the border from deficit-based discourses toward successful self-identification in mathematics (Newman, 2003). Status theory offers an explanation for how generalized expectations for competence develop from the academic power held in a given community (Webster & Foschi, 1988). Positioning theory offers the opportunity to understand the moment-to-moment discursive acts that create storylines over time (Harré & van Langenhove, 1999). We describe a successful border crossing as students and teachers moving away from deficit-discourses about who can be successful and toward equitable learning outcomes. We use this framing to understand who has academic power in a given classroom, how the power is distributed, and which discourses and practices affect change.

Participant Selection, Data Collection, & Analysis

We situated this study in a racially, ethnically, linguistically, and socioeconomically diverse school. The teacher, whom we call Ms. Martin, was invited to participate because she described making pedagogical choices that fostered student-to-student discourse (Cazden, 2001), interdependence (Lotan, 2003), and countered status issues (Cohen, 1997). This study represents the voices of 3 of the 14 Black girls from the class, each of whom completed all interviews. The girls featured are Jaelyn, a 10th grader, and Neesha and Irene, 9th graders.

Each participant completed 2 individual interviews between November 2011 and January 2012 and 1 group interview in February 2012. Other data sources included (a) field notes taken during classroom observations, (b) daily qualitative records, (c) video recordings of focal classroom sessions, (d) classroom work, and (e) video recordings of each interview.

We sought to understand which classroom practices these Black girls found influential in their mathematical learning in their Algebra 1 class. Initial analyses of classroom and interview data took place by coding field notes and interview transcripts, looking for how the participants identified competence for themselves and for their classmates. In a second round of analysis, the teacher's classroom practices were identified, including (a) assigning competence, (b) valuing multiple solution methods, (c) helping other group members, and (d) explaining mathematics thinking. Once initial findings emerged, we analyzed the students' border crossing away from deficit discourses by examining students' self-perceptions over time.

Findings

We investigated the classroom practices that emerged to look for evidence of equitable teaching and learning. Overall, our findings indicate that each Black girl described mathematics classrooms practices that positively shaped her self-perception of mathematics learning. We argue that these practices translated to positive experiences for these girls, which afforded each girl to disrupt previous negative conceptualizations of her mathematics learning. We further found these disruptions facilitated a space for grand narratives about their border crossing toward self-described success in mathematics.

We found that Jaelyn, Neesha, and Irene's interpretations of their experiences with mathematics drastically improved during first semester in Algebra 1. We coded interviews for how students described their mathematics experience. In the January and February interviews, each student shared a positive outlook of mathematics and of her interpretation of her abilities. Each student's change in self-perceptions of competence is shared in Table 1.

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Table 1: Students' Changed Self-Perceptions of Mathematics Competence

Student	Initial Self-Perceptions	End-of-Study Self-Perceptions
Jaelyn	"Last year I didn't like to go to class."	"I was curious to get to class and see what everybody else's answer was."
Neesha	"I used to shut down." "I'm not very good at math."	"Math is one of my favorite classes." "I am. I am good at Algebra."
Irene	"Since 7th grade math has not been my strongest subject. It is either confusing or I just don't understand it."	"Now that I'm in this class, it's actually kind of easier. I don't get as much stuff wrong with math now. This year, I get it."

After coding interviews for students' interpretations of their mathematics abilities, we looked to the classroom practices that Ms. Martin implemented. We were curious about whether there would be a link between classroom practices and what students' reported influenced their learning. Four classroom practices emerged from our analyses, including: assigning competence, valuing multiple solution methods, helping group members, and explaining mathematical thinking. See Table 2 for an illustration of these classroom practices.

Table 2: Classroom Practices that Influenced Learning

Structure	Student	Evidence
Assigning competence	Jaelyn	Ms. Martin told students to "bug Jaelyn" about how to use Lab Gear to explain simplifying an algebraic expression.
	Neesha	"Ms. Martin won't give you the answer, she'll listen and eventually say, 'I knew you'd [get it]!' "
	Irene	"Ms. Martin might have come over and asked us to start sharing about that equations. She came over and she saw [mine]. I was supposed to put mine in the middle to share with the table. And after she left, I continued with that."
Valuing multiple solution paths	Jaelyn	Jaelyn described students regularly putting their papers in the middle to compare solution methods.
	Neesha	"[You have to] learn your own way and your group members' way[s] to solve a problem."
	Irene	"After Ms. Martin leaves, it seems like more people start to talk. And then, you get more like, you get different kinds of ways people do it, and you get different answers and you figure out how to get the same answer."
Helping group members	Jaelyn	"As a learner, if you don't understand it, this is where a group comes in…Ms. Martin will make everybody in your group help you out."
	Neesha	"[We] wanna make sure everyone gets it."
	Irene	"It helps more when we're on the same problem, because they'll just stop and like, 'wait, what'd you get for this problem?' 'how'd you get that?' "
Explaining mathematical thinking	Jaelyn	"When you get everybody going 'Oh!' that just makes you move through the process easier. Like, when I was in that group, we all had it down. We'd stop, one of us would be confused for a minute and then [by explaining], you flowed through it easier."
	Neesha	"When somebody else doesn't understand it, you have to find a different way to explain it to them."
	Irene	"We had to go back and help her, and explain how we did it. And then sometimes, when I would explain and people would catch that I said the wrong thing or I got the wrong thing, they would explain how they did it."

Linking Equitable Teaching, Learning, and Border Crossing

Many classroom episodes highlighted connections between Ms. Martin's classroom practices and equitable learning opportunities for Jaelyn, Neesha, and Irene. Each classroom practice from our analysis supported our Black girls' border crossing from deficit-discourses toward self-identifying as competent mathematicians. In one instance, Ms. Martin assigned competence to Jaelyn by

positioning her as the group-member who was competent to explain Lab Gear to her group members. In another, Irene described Ms. Martin's encouragement that she physically moved her work to the center, and she admitted that once Ms. Martin left, she continued explaining the mathematics to her group. Neesha reported that Ms. Martin regularly listened to groups come to a mathematical understanding themselves, after which time she would reaffirm their competence, telling them, "I knew you would!" Each of these instances is representative of the many times these girls described Ms. Martin assigning them and their classmates competence. By regularly assigning competence to these students, Ms. Martin influenced these students' perceptions of competence (Cohen & Lotan, 1997). Changing self-perceptions of competence gave our Black girls the tools to counter deficit notions of self, aiding them in dismantling false borders of incompetence. Moving toward improved self-perceptions of competence also highlights evidence that these students, by the end of the study, saw themselves as mathematics authorities in their classroom, which moves her classroom toward equity (Dunleavy, 2015).

Crossing the Borders of Deficit Discourses

Our findings provide existence proof of how these Black girls spoke to countering deficit narratives and crossed the border toward seeing themselves as competent mathematicians. Further, we offer evidence for which classroom practices facilitated their border crossing toward self-described success in mathematicians. Future research may continue to explore border crossing, including how classroom teachers assist students in countering deficit discourses and how students' ways of knowing are related to what counts as meaningful mathematics.

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LOOKING AT VERSUS LOOKING THROUGH: WHEN DESIGNS UNDERMINE STUDENT REASONING

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This paper reports on two rounds of a design-based research study whose goal was to support mathematical problem solving in the context of an immersive game using the concepts of rate, ratio, and proportion. Findings from these implementations suggest that while all students learned in both years, students in year 1 scored significantly higher than students in year 2. Based on this analysis, we concluded that the changes from year 1 to year 2 had a negative effect on student reasoning about ratio, particularly due to the ways that a ratio tool focused students' and the teacher's attention on separate versus coordinated reasoning about ratio.

Keywords: Design Experiments, Middle School Education, Instructional Activities and Practices

Introduction

A persistent challenge of complex problem solving involves balancing between problem solving activity and underlying conceptual understanding. Complex problem solving involves what is often described as a *dance of agency* (Pickering, 1995) between the conceptual agency of decision-making and the disciplinary agency of leveraging appropriate methods and strategies to solve the given problem. Successful problem solving therefore fundamentally requires conceptual understanding of strategies; that is, understanding not just *how* to execute a procedure, but also what a procedure might tell you and why it might be useful in a particular situation (Gresalfi, 2015; Gresalfi & Barnes, 2015; Santos-Trigo, 2007; Schoenfeld, 2007). This kind of conceptual understanding is notoriously difficult to support, for reasons that range from the complexity of underlying ideas, to more mundane challenges such as instructional time.

This paper reports on two rounds of a design-based research study whose goal was to support mathematical problem solving in the context of an immersive game using the concepts of rate, ratio, and proportion. There is significant agreement that the concepts of ratio and proportion are critical topics of mathematical study and also notably challenging for students to understand (Beckmann & Izsák, 2015; Behr, Harel, Post, & Lesh, 1992). Great attention continues to be focused on the nuances of understanding ratio and proportion (Beckmann & Izsák, 2015) and how to best support student learning of those topics (Lobato, Ellis, & Zbiek, 2010). Although there are myriad elements that compose the concepts of ratio and proportion, many learning goals in schools involve articulating definitional distinctions and setting up an algorithm that can be resolved using cross multiplication. Like any procedural approach to mathematics, students can remember steps but struggle with the underlying conceptual ideas (Cramer & Post, 1993; Lesh, Post, & Behr, 1988; Lobato et al., 2010).

Project Overview

We report overall findings of the first two round of design and implementation, and present a detailed analysis of the second round of implementation, focusing in particular on the design of a *ratio tool* intended to support students' conceptual understanding of proportion.

Problem Solving Game

We studied mathematical reasoning and problem solving in the context of an educational video game called *Boone's Meadow,* based on a project-based mathematics activity from the *Adventures of Jasper Woodbury* (Bransford, Zech, Schwarz, Barron, & Vye, 2000; Pellegrino et al., 1992). The game begins when students are told that an endangered eagle has been shot in Boone's Meadow—a

place that cannot be reached by car and takes 6 hours to hike by foot. Multiple decisions need to be made to rescue the eagle, including which route to take, which plane to fly, the length and time of the journey, how much gasoline will be required (and where to stop to get it), and whether any additional cargo is necessary (or feasible) given the weight limit of the small aircraft.

Ratio tool. The first iteration of the game did not include an explicit scaffold for proportional reasoning based on earlier pilots with 9-10 year olds, who successfully engaged with problem solving tasks in the game using multiplicative comparisons. However, in year 1 we discovered support was needed, as described below. We thus design a *ratio tool* to support proportional reasoning as a double number line (see Figure 1) for the following reasons. First, a double number line makes clear the *coordination* between two quantities. We included an additional design feature to attune students to this coordination—the paired numbers are enlarged when the slider is moved to their position (see 30 and 20 in Figure 1). Second, compared to cross multiplication, a double number line has a spatial component making the *co-variation* between the two quantities more explicit (Confrey & Smith, 1995; Thompson & Carlson, in press).
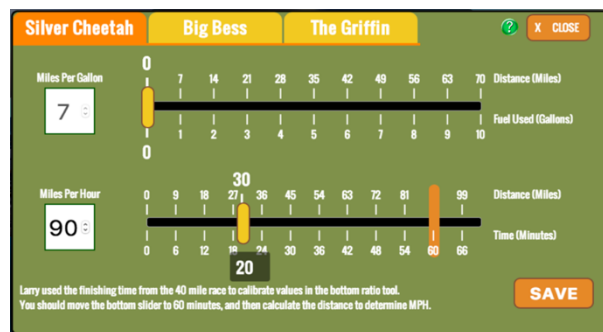
Figure 1. The Ratio Tool Designed for Round 2.

Methods

Participants and Data Collection

Data come from two years of implementation in the same 7th grade teacher's classroom (Ms. Lynn) at a diverse middle-school (92% free and reduced lunch, 30% English language learners) in a southeastern U.S. city. The students generally performed at least two grades below level; in both years students struggled with fundamental understanding of multiplication and relied on a multiplication chart when conducting calculations.

Although the game changed slightly from year 1-2, the flow of the implementation was similar. Four days of class time were devoted to game play both years. Each day, class began with a warm-up and a discussion before computers were given to students. These discussions varied in length, lasting as long as 50 minutes of a 1.5-hour class period. Follow-up discussions after game play occurred some days, but not all. During both years, data was collected in the form of pre/post tests, video of pairs of a subset of students working with the game, and video of the teacher in whole class discussion and working with individual students during gameplay.

Analysis

For years 1 and 2, pre and post tests were scored by a minimum of two coders, and compared for gain within year and difference across years. To examine how students were interacting with the ratio tool (year 2), we examined videos from two thirds of focus groups (six pairs), selected randomly. Transcripts were made of interactions with the ratio tool while conducting six unit rate calculations, and coded to determine whether students were reasoning with ratio in a separate or coordinated way. This distinction came from literature, which has identified seeing ratios as a single coordinated unit

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rather than two separate units moving together as a core concept in order to build proportional reasoning (Lobato et al, 2010). In addition, transcripts of the teacher's talk about ratio were coded using the same separate vs. coordinated coding scheme.

Results

A simple t-test revealed that all students learned in both years ($t=6.00$, $p<.001$). There was a significant effect for year of study on posttest scores, although not in the hoped for direction; a multiple regression revealed that students in year 1 scored significantly higher than students in year 2 ($t= -2.31$, $p <.05$). This was surprising, as we observed higher levels of engagement in year 2, and were asked help far less frequently. To better understand why the ratio tool might have contributed to lower post test scores in year 2, we examined videos of student and teacher interactions, focusing specifically on times when the ratio tool was the object of activity.

Student Interactions

Of the 36 questions (6 questions per group) that students solved while playing the game, six questions were lost due to video malfunction loss, and 12 were coded as copy answer, meaning that students had recorded answers from prior discussions or other groups. The remaining 19 questions were coded as separated reasoning (11) or coordinated reasoning (7), see Table 1. However, the difference between coordinated versus separate was more disparate than it might seem at first glance; a single pair of students (second row of Table 1) was responsible for 5 out of the 7 questions coded as coordinated (2 of which were prompted by Ms. Lynn or a researcher). The other 2 questions coded as coordinated reasoning were in separate groups and both heavily depended on help from researchers. Thus, students did not seem to naturally use the tool in a coordinated manner. Instead, many students cued into separate patterns within the tool, that the ratio tool was undermining a basic property of ratio, attending to and coordinating two quantities (Lobato et al, 2010). Rather than building the conceptual understanding of proportion, the tool became the object to look at and solve, not learn with (Greeno & Hall, 1997).

Table 1: Counts Of Student Solved Problems

	Total questions	Separate	Coordinated	Copy Answer	Lost
All Groups	36 (6 per 6 groups)	11	7	12	6
One Group	6	1	5	0	0

Teacher Interactions

As seen in Table 2, the ratio tool also seemed to reorient Ms. Lynn's discourse from dominantly coordinated with twice as many coordinated utterances as separate, to dominantly separate with half as many coordinated utterances as separate.

Table 2: Counts Of Utterances Coded As Separate Or Coordinated

	Separate	Coordinated
Without Ratio Tool	23	49
With Ratio Tool	70	35

Discussion

The overall goal of this research project is to support mathematical problem solving that involves making decisions about how to solve problems, and considering the impact of those mathematical decisions. The data shared here describe two rounds of a design-based research study that attempted to support student conceptual understanding by building in a tool that, we hoped, would support students' reasoning about ratio. Based on measures of pre-post change, we found that although students learned, they did not learn as much with the addition of the ratio tool. Although it is not

clear that the introduction of the tool caused this lower level of learning, the results of our in-depth analyses around the tool, documenting the dominance of separated talk about the proportion, suggest that the tool did not support understanding of proportion.

It is possible that this problem is particular to this population of students due to their lack of facility with multiplication. Perhaps the tool attunes students to the relationship between numbers only when the relationships that are being identified are easily uncovered; that is, students might notice that the relationship between 2:10 and 5:25 only if they are easily able to identify both 1:5. Thus, it is possible that the ratio tool undermines *some students'* reasoning about ratio, but not all. This is a topic of continued investigation as we work with more teachers and students. Regardless, our next round of design needs to consider whether and how we can support students' whole-number multiplicative reasoning while simultaneously supporting them to connect that reasoning with a focus on invariant (i.e. proportional) relationships.

Acknowledgments

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UNDERSTANDING UNDERGRADUATES' MATH-RELATED PERCEPTIONS

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In this study, we surveyed 323 undergraduate students at a large public university in the Northeastern region of the United States. We explored the relationship between these students' career interest in STEM with their beliefs and self-perceptions related to mathematics, including: mindset, anxiety, identity, and efficacy. Our findings suggest statically significant correlations between the four mathematics constructs. In addition, mathematics identity and self-efficacy were positive predictors for students' STEM career interest. Furthermore, family interest and gender are important measures for predicting student career interest.

Keywords: Affect, Emotion, Beliefs, and Attitudes, Gender, Post-Secondary Education

Introduction

The goal of this study is to understand the relationship between mathematics related affective measures, including mindsets in math, math anxiety, math identity, and math self-efficacy, and to investigate the predictive nature of these constructs toward students' career interest in STEM. Previous research has identified these affective factors important to understanding people's choice, behavior or learning in mathematics (e.g, Bandura, 1997). Mindset orientations in math have been defined to include fixed mindset and growth mindset: people with a fixed mindset tend to view math intelligence as unchangeable while people with a growth mindset view math intelligence as malleable that can grow with effort (Dweck, 2000). Math self-efficacy focusses on one's perceived confidence in performing math related tasks. Math anxiety, a concept closely related to math self-efficacy, normally refers to one's unpleasant feelings (e.g., fear) associated with math task performance (Ashcraft, 2002). Finally, math identity refers to how students see themselves in relation to mathematics based upon their perceptions and navigation of everyday experiences with mathematics (Enyedy, Goldberg, & Welsh, 2006).

Prior work related to mindset, anxiety, identity, and self-efficacy have shown the constructs to play an important role toward students' engagement in math. For example, it has shown that math self-efficacy affects students' math course choices and math-related career choices (e.g., Anjum, 2006; Hackett, 1985). Research has also found that mindset is not only predictive of math achievement in adolescents (Blackwell, Trzensiewski, & Dweck, 2007) but also predicts the level of math courses that students take as they transitioned through middle school (Romero, Master, Paunesku, Dweck, & Gross, 2014). Likewise, math anxiety has been shown to influence students' self-perceptions related to their math ability, expectancies for success in math, and perceived value of math (Meece, Eccles, & Wigfield, 1990). Other work indicates that math anxiety is correlated with which math courses students take in high school (Ashcraft, Krause, & Hopko, 2007).

Mathematics identity is also an important construct that explains students' persistence in STEM. Research in the area of math identity has shown two factors as having a direct, positive effect on students' math identity development (Cribbs, Hazari, Sadler, & Sonnert, 2015). Recognition is one of the factors and is defined as how individuals view themselves in relation to mathematics. The second factor, interest, is defined as an individual's desire to think about and learn mathematics. While research has shown connections between math identity and students' career goals (Cribbs, Cass, Hazari, Sadler, & Sonnert, 2016), there is limited work exploring how math identity correlates with other affective measures that influence students' engagement with mathematics. As math anxiety and math identity both have the potential to influence students' choices for future engagement in math or

math-related activities, it is possible that these constructs are correlated in some way. Likewise, students' math self-efficacy could also be correlated with math identity, particularly when considering the strong correlation that students' perceptions about their ability to perform in and understand math has to the sub-constructs of math identity, recognition, and interest (Cribbs et al., 2015). How the above constructs correlate to mindset orientations in math is also worth exploration as little research has focused on this question. In addition, we are interested in examining which of these math related constructions are better predictive of student interest in STEM-related careers.

The following research questions were used to guide this study: 1) Are students' self-perceptions and beliefs about mathematics correlated, and 2) What is the relationship between students' career interest in STEM and math mindset, anxiety, identity and self-efficacy?

Methods

Data were collected through surveying undergraduate freshman in the fall of 2015 at one university in the northeast region of the United States. A total of 323 completed responses were received. The survey was piloted and revised in the spring of 2014 and includes questions on student demographic information, career goals, and perceptions related to mathematics, including math mindset (Dweck, 2000), anxiety (Betz, 1978), identity (Cribbs et al., 2015) and self-efficacy (Fennema & Sherman, 1976). Previous research has established the validity and reliability of each affect construct measured. In addition, the internal consistency for each construct was calculated and found to be appropriate in the present study: mindset (0.90), anxiety (0.87), identity (0.95) and self-efficacy (0.95). Before the construct variables were calculated, all negatively worded items were reverse coded so that a higher score indicates a more malleable view of intelligence (growth mindset) and a higher level of anxiety, identify, or self-efficacy.

Pearson correlation tests between each of the four mathematics beliefs and self-perception constructs were run to address the first research question. The second research question was addressed using logistic regression, where a model was developed for each math construct. Students' STEM career interest, which is a dichotomous variable, was the dependent variable. Table 1 summarizes the number of responses to the career interest variable.

Table 1: STEM Career Interest

Are you interested in pursuing a STEM-related career?	Female	Male	Total
Yes	69	42	111
No	175	37	212

For the correlation analysis and regression models, a proxy for each mathematics construct was developed by averaging participants' responses. Furthermore, the following demographic information was entered as control variables: gender, mother and father's highest level of education, family support and interest, and race. List-wise deletion was used for missing data.

Results

Table 2 details the results for research question 1: Are students' self-perceptions and beliefs about mathematics correlated? Each of the mathematics constructs had statistically significant correlations with the others: growth mindset and anxiety were moderately negatively correlated, $r(313) = -.35$, $p < .001$; growth mindset and identity were weakly positively correlated, $r(313) = .29$, $p < .001$; growth mindset and self-efficacy were moderately positively correlated, $r(311) = .46$, $p < .001$; anxiety and identity were strongly negatively correlated, $r(317) = -.72$, $p < .001$; anxiety and self-efficacy were strongly negatively correlated, $r(315) = -.84$, $p < .001$; and identity and self-efficacy were strongly positively correlated, $r(316) = .81$, $p < .001$.

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Table 2: Correlation Test Results

Construct	Mean	SD	Mindset	Anxiety	Identity	Self-efficacy
Mindset	3.48	.83	-			
Anxiety	2.92	.94	-.35***	-		
Identity	2.99	.19	.29***	-.72***	-	
Self-efficacy	3.44	.97	.46***	-.84***	.81***	-

***p<0.001

In order to address research question 2, what is the relationship between students' career interest in STEM and math mindset, anxiety, identity and self-efficacy, a series of regression models were created. Mindset and anxiety were not significantly predictive of students' career interest. However, identity and self-efficacy were both predictive of students' career interest in STEM as shown in Table 3 and 4. Additionally, gender, mother's highest level of education, and family interest in math as a career (my family saw math as a way for me to have a better career) were all positive predictors for students' career interest in STEM.

Table 3: Relationship between students' career interest in STEM and identity (N=314)

	Estimate	SE	Odds Ratio	Sig.
Intercept	-4.212	0.653		***
Controls				
Gender (Male=1, Female=0)	1.061	0.316	2.890	***
Mother's Highest Education	0.281	0.124	1.325	*
Family Interest in Math as Career	0.349	0.084	1.418	***
Mathematics Identity	0.273	0.120	1.313	*

*p<0.05 **p<0.01   ***p<0.001

In addition to indicating that self-efficacy is predictive of students' career interest in STEM, several control variables were also positive predictors, gender and family interest in math as a career (my family saw math as a way for me to have a better career).

Table 4: Relationship between students' career interest in STEM and self-efficacy (N=315)

	Estimate	SE	Odds Ratio	Sig.
Intercept	-3.862	0.616		***
Controls				
Gender (Male=1, Female=0)	1.024	0.294	2.784	***
Family Interest in Math as Career	0.433	0.083	1.542	***
Mathematics Self-efficacy	0.313	0.147	1.368	*

*p<0.05 **p<0.01   ***p<0.001

Discussion

Results from this study indicate that math-specific mindset, anxiety, identity and self-efficacy are all correlated with one another. However, some of the variables are more highly correlated than others. Specifically, the strong correlation between math self-efficacy and anxiety supports previous theoretical and empirical research on their relationship (e.g., Bandura, 1997; Griggs, Patton, Rimm-Kaufman, & Merritt, 2013). In addition, the strong correlation between math identity and self-efficacy was not surprising either given prior work related to the constructs of self-efficacy and math identity (Cribbs et al., 2015). Math identity and math anxiety also had a strong yet negative correlation. The strongest correlation that mindset had was with self-efficacy, followed by anxiety

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and identity. When investigating how predictive each of the affective measures were with students' STEM career interest, mindset and anxiety were not found to be predictors. However, mathematics identity and self-efficacy did significantly predict students' STEM career interest.

No prior research has integrated these math related affective factors into one model and explored their relationships with each other and with STEM career interest, as explored in the present study. Additional research is needed to better understand the relationship between math-specific mindset, anxiety, identity and self-efficacy by exploring possible mediating effects among the variables to STEM interest. Particularly, structural equation modeling could provide important information as to how the constructs are related.

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LEARNING FROM TRANSGENDER INDIVIDUALS IN STEM EDUCATION

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This paper outlines a gender-complex framework and narrative research methods to investigate how transgender and gender non-conforming students' experiences in STEM education have varied as their gender presentation has evolved. The theoretical framework draws from post-structural feminism, queer theory, and gender performance theory. This is summarized in the gender oppression plane, in which one axis measures gender category privilege or oppression, and the other axis measures gender conformity privilege or gender transgression oppression. Feminism addresses the former axis, and queer theory addresses the latter axis. Gender performativity theory helps to explain an individual's placement on the plane, and how that placement can change with time or context. Proposed research methods include interviews and reflective journaling, which will then be analyzed narratively.

Keywords: Gender, Equity and Diversity, Post-Secondary Education

Gender is a category that pervades almost every aspect of our lives. To most people, the fact that the world is divided into two groups – male and female – is seen as unproblematic, even among those opposed to discrimination against one group. This practice of taking gender and sex (which are often equated) for granted is one that pervades many research studies seeking to discover or explain gender differences in mathematics education (Damarin & Erchick, 2010). The borders between sex and gender, between male and female, and between man and woman need to be questioned. Motivated by a desire to interrogate these borders and an interest in telling the stories of those who have crossed them, I outline a framework and methods to explore the following research question: "How have transgender and gender non-conforming students' experiences in STEM education varied as their gender presentation/performance has evolved?" My framework draws from post-structural feminism, queer theory, and gender performativity theory.

Rationale

It is an oft-cited fact that women are underrepresented in STEM fields (National Science Foundation, 2015). There have been many theories to try to explain this fact, such as stereotype threat (e.g., Brown & Josephs, 1999) and differences in learning styles (e.g., Geist & King, 2008). Nearly all of these studies take gender as a well-defined, unproblematic, binary category (Damarin & Erchick, 2010). However, studies that include gender identity have found it to be a better predictor than sex (e.g., Hackett & Betz, 1989; Severiens & ten Dam, 1997). This study will examine the experiences that transgender students have had in STEM education, and focus on their interactions with others in that setting.

Definitions

A framework that includes many subtleties surrounding the concept of gender necessitates the careful definition of such terms as gender identity, gender presentation, and gender attribution. *Gender identity* refers to an individual's "sense of self as a boy or girl, woman or man (or, as we are increasingly realizing, as a nongendered, bigendered, transgendered, intersexed, or otherwise alternatively gendered person)" (Tranzmission, n.d., p. 10). *Gender presentation* refers to the intentional and unintentional ways that an individual communicates their gender to others. *Gender attribution* is the way in which an individual interprets the myriad cues available to designate an

individual as male, female, indeterminate, etc. It is gender attribution rather than gender presentation that determines a person's gendered treatment by others.

I define a *transgender* person as anyone whose gender identity is not the same as their assigned sex at birth. Defining transgender in this way includes those who are intersex and were forcibly assigned a gender with which they do not identify, those with a non-binary or fluctuating gender, and those do not present as their preferred gender identity.

Theoretical Framework

I borrow the concept of a gender oppression plane from Kathleen Rands (2009), although the visual depiction of it is my own (see Figure 1). In this approach, there are two dimensions along which one might experience privilege or oppression based upon one's gender. The first, *gender category oppression*, describes the ways in men experience systematic privilege and women experience systematic oppression. This is based on whether one is perceived as a man, a woman, or another gender category. The second dimension is *gender transgression oppression*. This refers to the systematic privilege experienced by those who do not challenge heteronormativity, and the systematic oppression of those who do. While my focus is on these two types of gender oppression/privilege, it is important to also acknowledge that they interact with other types of oppression and privilege, such as that based on race or class. Race and class (and other bases on which our society discriminates) would be on axes perpendicular to the gender oppression plane.
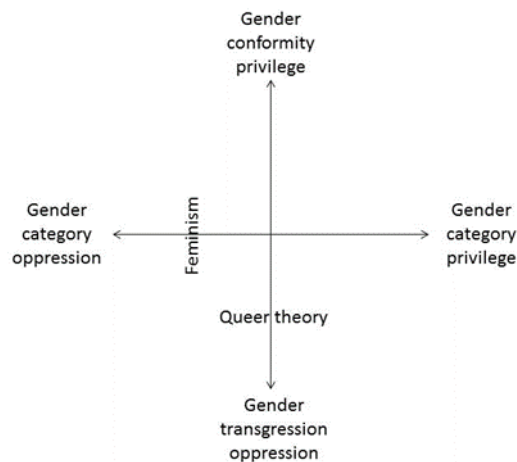
Figure 1. Gender oppression plane. Privilege and oppression based on gender category interact with that based on gender conformity/transgression.

My theoretical framework is a critical one, since I am interested in issues of privilege and oppression, and especially how those are operationalized by those who have power (such as instructors and administrators). The four pillars of my theoretical framework are the gender-oppression matrix (Rands, 2009), feminism, queer theory, and gender performativity theory. Feminism is concerned with the gender category axis of the gender oppression matrix, and queer theory is concerned with the gender transgression axis of the gender oppression matrix. Gender performativity theory helps to explain where an individual might be situated on this plane, and how that position may change with context, time, or circumstances.

Judith Butler (1993) conceives of gender as performativity, an idea which I incorporate into my own framework. In this theory, one's gender presentation is an act of communication, both conscious and unconscious, but always present and always political. It is likely to be dynamic rather than static. But like any act of communication, it is dependent upon the reception and subsequent interpretation

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of social signals as much as upon the sending of those signals. Thus, the gender attribution is dependent upon the observer as much as the actor.

Methods

Narrative inquiry is a type of qualitative research that focuses on telling the stories of people's lived experiences, or their biography. In narrative inquiry, the focus is on empowering the research subject – not as someone who is subjected to research, but as a subjective human being whose experiences are considered to be valuable (Chase, 2011).

Participants

My participants will be individuals whose gender presentation has significantly changed during their experiences in STEM education. This may include gender nonconformists as well as transgender students. I will primarily recruit graduate and undergraduate students, but I also welcome the participation of transgender faculty members in STEM fields and recent alumni of STEM programs. I will use snowball sampling, relying heavily on gatekeepers and insiders in the LGBTQ community, similar to the process described by Winkle-Wagner (2009).

Data Sources and Collection

I will draw from a series of three open interviews with each participant and reflective journals in order to help construct my participants' narratives. I will conduct three phenomenological interviews with each participant, as in Seidman (2013). Each interview will focus on a different time period: the first focused on the past, the second focused on the present, and the third focused on the future and connections across time. I expect each interview to be approximately 90 minutes long. For the reflective journals, I will provide prompts, such as, "What does it mean to be treated as a man in STEM education? What does it mean to be treated as a woman?" Participants will also be free to write about topics other than the prompts. These will be completed between interviews. All interviews will be audio-recorded in order to capture the exact words of the participants as exactly as possible.

Data Analysis

All interviews will be transcribed and reviewed by the participants for the faithfulness of the interpretation. I will use the restorying process described by Cresswell (2008). This process involves three steps. First, the researcher transcribes the interviews. This is the raw data. Second, the data is re-transcribed with codes added for each element of the story: setting, characters, action, problem, and resolution. Finally, the researcher presents these key elements in a sequence, such as setting, characters, and then events in chronological order. I will debrief with peers and conduct member checks for the themes, in order to ensure that they are both analytically rigorous and true to the intended meanings of the participants.

My research will be framed by Clandinin and Connelly's (2000) three-dimensional space of narrative inquiry as consisting of interaction, continuity, and situation. These are derived from Dewey's theory of experience. Interaction entails both the personal and the social. This involves movement inward and outward: inward toward internal conditions, and outward toward the environment. Continuity entails the past, present, and future. This involves movement forward and backward, chronologically. Situation entails place. This involves the physical and cultural locations of participants. Researchers can look at any of these dimensions individually, or at the intersections of them. This could include intersections within a dimension as well, such as the intersection of the personal with the social. Since it is a critical study, I will pay particular attention to any actionable findings, such as institutional bias that could be remedied by a change of policy. Participants will also have the chance to reflect on the final version of the narratives and discussion, particularly

because confidentiality may be an important concern for them, but also because I want to include their opinions on the final analysis.

Conclusion

This study will shine a light on a neglected area of study, the experiences of transgender students in STEM education. I will use a critical lens informed by feminism, queer theory, and gender performance theory to investigate issues of privilege and oppression associated with gender category and gender transgression. By telling these stories, I hope my audience will question the border between male and female and gain more respect and understanding for those who cross that border.

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HOW DOES PARENTAL ATTITUDE TOWARD MATHEMATICS PROMPT STUDENT ACHIEVEMENT?

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Parents, K-8 teachers, and 4th-8th grade children partnered as learners in math-focused parental involvement through the Math and Parent Partners (MAPPS) program. We hypothesize that participation in MAPPS improved parent attitudes towards mathematics motivating children to learn at school.

Keywords: Informal Education, Affect, Emotion, Beliefs, and Attitudes

Background and Research Questions

Low parental involvement in disadvantaged schools has been related to a gap in mathematics achievement (Jackson & Remillard, 2005) while strong parent-child communications and parents' aspirations for their children's mathematics education have been found beneficial for student achievement (Aldous, 2006; Robinson & Harris, 2014). A school district and university in the Southeast partnered to boost student achievement in Title I schools through the Math and Parent Partners (MAPPS) Program. Knapp, Jefferson, and Landers (2013) found standardized test scores to rise significantly over a three year period for those participants attending regularly for at least one Minicourse. To build on this result, we ask, *How might improvement in student understanding and achievement occur within a standards-based mathematics PI program such as MAPPS? In particular, do parents' attitudes related to mathematics improve?*

Theoretical Framework: Funds of Knowledge

From its inception, MAPPS has been grounded in the socio cultural theory of Funds of Knowledge/Communities of learners (Rogoff, 1994; Allexsaht-Snider & Bernier, 2003). The assumption that families possess bodies of knowledge used for daily living, "is critical in terms of reconceptualizing households, not as the source of barriers to educational attainment, but as repositories of resources that can be strategically tapped" (González, 1996, p. 3). Civil (2007) refers to these resources as "Funds of Knowledge". According to Civil, Funds of Knowledge build on students' and parents' knowledge and experiences as a resource for schooling and values community-based teaching as well. Such Funds involve pedagogically mathematizing household knowledge such as carpentry, repair, or folk medicine for classroom use.

Participants and Context

In this vein, the Math and Parent Partners (MAPPS) program engages parents in mathematical tasks to equip them as mathematical resources for children and schools. The five-part MAPPS curriculum was developed with National Science Foundation project ESI-9901275 funding K-8 parents to collaboratively explore mathematics learning at school as well as connecting to parents' existing Funds of Knowledge. A particular MAPPS program located in the Southeast invited all parents, instructional staff, administrators, and children from certain schools to participate in the study (Knapp, Jefferson, & Landers, 2013). Children in 4th-8th grade accompanied their parents for Mini-courses comprised of eight weeks in two hour sessions while young children participated in childcare with mathematical activities and games. Eight separate Mini-courses on number, geometry, algebra, and data were offered over the course of three years. Instructors were practicing teachers studying mathematics education. MAPPS Mini-course structure followed the National Council of Teachers of Mathematics (2000) process standards by engaging participants in content

and pedagogy through learning communities of parents, children, and teachers to solve tasks, use manipulatives, and present solutions to the entire group (Knapp et al., 2013). In total, 59 parents, 33 teachers, and 115 children from four main Title I elementary schools attended at least one Mini-course on a regular basis. About double attended less regularly. Most attendees were single mothers and held low-income jobs. They were approximately 40% Caucasian, 40% African-American, and 20% Hispanic.

Methods, Data Analysis, and Coding Tallies

Impacts of the MAPPS Mini-courses were ascertained through pre/post attitude inventories taken by parents and teachers. A focus group of parents, teachers, and children also participated in 95 pre/post interviews lasting approximately 15 minutes probing for improvement in mathematical knowledge and on parents' ability to assist their children in mathematics. Interviews were coded for factors that might affect their mathematics achievement, including aspects of the home environment related to Funds of Knowledge. Code identification examples are given in bold in the results section. After coding the interviews and pre/post surveys, we tallied the 59 codes to identify areas of participant growth as well as factors prompting that growth. For quantitative attitude analysis, the researchers administered a modified version of the Attitudes Toward Mathematics Inventory (ATMI) (Tapia & Marsh, 2004) to parents and teachers before and after each Mini-course. The inventory consisted of 25 items to reflect five affective mathematics dimensions (confidence, anxiety, value, enjoyment, and motivation). The ATMI was found to be reliable (alpha = .948) for parents and teachers. Parents were asked to rate statements such as "Mathematics is a very interesting subject" on a Likert scale (Tapia, 1996). Attitude surveys were analyzed using paired samples *t*-tests. This mixed methods study was of quasi-experimental design as parents self-selected to the program.

Results and Discussion

Question #1: How might improvement in student understanding and achievement occur?

The MAPPS **learning community** environment yielded many benefits. First parents, teachers, and children assisted one another in learning mathematics. At times**, children helped parents** figure out problems, which became a source of pride and motivation for the children, especially when they could present their solutions to the group. **Enjoyment** of mathematics was the second product of the learning community. "Before you leave, you're laughing because you've learned. The average 8 and 78-year-old learning together," said one parent. She explained that people come to MAPPS for the enjoyment of learning. Children also expressed enjoyment such as in seeing their teacher and parent interact. The third product of the learning community included **motivation**. The learning community motivated parents to increase parent-child interaction around mathematics as they observed other parents engage with their children. As one parent put it, she was motivated to explicitly budget time at home for helping her child with mathematics. Motivational factors for children included 1) parental presence at MAPPS, 2) parental interest in what the children were doing, 3) a non-traditional, ungraded learning environment at the university, and 4) a location that they found exciting. Seeing their parents **value mathematics** also motivated children to value it. The MAPPS-fostered learning community among parents, teachers, and children motivated children not just to learn mathematics at MAPPS and at home, but in school as well. One teacher reported:

> I think as they [children] saw things that we did outside of class [at MAPPS] in our class, it motivated them because they could share their experience with their peers. They could say, 'I understand this because I've seen it before.' It built their confidence because when they knew how to do something, people [MAPPS participants] looked to them for help.

A child expressed her gained mathematical confidence and motivation by saying, "You would get smarter and feel more confident. You would just have fun with that". Thus, parental involvement in mathematics along with the MAPPS learning community appeared to spawn increased motivation and confidence for children to learn in the school setting. Analysis of the qualitative data appears to show that the MAPPS environment improved classroom learning for children, an improvement which may have impacted student achievement over time.

Question #2: Do Parents' Attitudes Related to Mathematics Improve?
The quantitative attitude survey supports the qualitative evidence gathered about improved **confidence** and **motivation** of parents with respect to their own mathematics learning and teaching of their children. Mean **attitude** scores improved during most sessions (See Table 1).

Table 1: Parent and Teacher Attitude Surveys- Mean Scores out of 125

	n	Pre	Post	Change
2008-2009 Session1	18	100.5	101.6	1.1
2008-2009 Session 2	32	97.2	98.6	1.4
2009-2010 Session 1	24	90.7	85.5	-5.2
2009-2010 Session 2	20	92.6	98.3	5.7
2010-2011 Session 1	4	90.8	95.0	4.2
All years 1st Mini-course taken	65	93.0	94.2	1.2
All years 1st-Last Mini-course	65	93.6	96.2	2.6

Parents showed improved attitude toward mathematics when comparing the first time the inventory was taken to the last (some participants took several Mini-courses and thus took the survey multiple times) ($p = 0.101$, $d = 0.160$). Adults (parents and teachers) analyzed together improved significantly when comparing the first time they took the inventory to the last ($p = 0.084$, $d = 0.125$). We chose *alpha* = 0.1 as the limit level due to the small sample size. Thus for the p-value of 0.101, we have moderate evidence to say it is significant and the p-value of 0.084 is considered significant (Borenstein, 2012). An increase in parent attitude toward mathematics may have contributed to the improved motivation of children to learn mathematics (See Table 2).

Table 2: Parent and Teacher Attitude Scores- 125 points possible

	n	Pre	Post	Change	Result
1st-Last Mini-course Parents	46	93.6	96.2	2.6	*p*=.101
		sd	sd		*d*=.160
		14.1	11.3		
1st-Last Mini-course Parents & Teachers	65	93.1	95.2	2.1	*p* = 0.084
		sd	sd		*d* = 0.125
		17.7	16.0		

Conclusions
In conclusion, parents' improved attitudes toward mathematics and their confidence in explaining it appeared to impact **parent-child interactions** and thereby impacting student achievement. Parent-teacher relationships forged through the MAPPS learning community also impacted student motivation and consequently sustained mathematics learning. *Children's interactions with parents fueled by the MAPPS learning community prompted children's motivation and confidence to learn mathematics at school, possibly leading to student achievement gains.* The *familial bond* to and with the child may channel motivation to learn mathematics. Indeed, top codes from data analysis

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revealed improved parent-child interaction around mathematics and parent enjoyment and valuation of MAPPS; certainly children would have picked up on these cues. Through mixed methods, this mid-sized quasi-experiment points to parental attitude toward mathematics as a possible predictor of student achievement. Other possible factors in the study may have been changes in parent and teacher knowledge and parent-child interaction among other factors (See Knapp et al., 2013; Knapp & Landers, 2012). Further quantitative study is needed on larger scale and in multiple contexts relating parent attitude and student achievement. Additionally, a correlation study is needed to compare parental attitude towards mathematics and student attitude toward mathematics.

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INTERVENTION IN PARTICIPATION SUPPORTING STUDENTS WITH LEARNING DISABILITIES IN STANDARDS- BASED MATHEMATICS CLASSROOMS

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Understanding learning disabilities (LD) through neurodiversity, this literature review found that students with LD were supported towards equal participation in standards-based mathematics through teachers trained in Mathematical Knowledge for Teaching and multi-modal curriculum that was responsive to individual student thinking. Rather than focusing solely on intervention in content, research in the mathematical learning of students with LD must address participation gaps with intervention in participation, striving to deepen the participation of students with LD in meaningful mathematics.

Keywords: Equity and Diversity, Classroom Discourse, Instructional Activities and Practices, Problem Solving

Introduction

Considering the creativity and entrepreneurship of individuals with LD (Von Karolyi, Winner, Gray, & Sherman, 2003), increasing participation of students with LD in these fields will not only benefit those individuals, but society as a whole. Understanding LD as a natural and beneficial aspect of *neurodiversity* (Boundy, 2008), this presentation presents the results of an integrative research review (Torraco, 2005) describing studies that investigated the participation of students with LD in standards-based mathematics classrooms, presenting findings across the studies and future directions for research.

This review takes a sociocultural view of participation as engagement in practices such as discourse and problem solving (Nasir & Hand, 2008). Building proficiency in mathematics for all learners means sustained and deep engagement in practices such as problem solving, reasoning and critique. However, students with disabilities have less access to these mathematical practices (Jackson & Neel, 2006; Kurz, Elliott, Wehby, & Smithson, 2010). Even when students with LD are given access to standards-based mathematics, without additional support, there may exist a *participation gap* between students with and without LD. Differences in participation could account for achievement differences, as participation in mathematical practices predicts mathematical learning (Ing et al., 2015, Webb et al., 2009).

LD can be understood through multiple perspectives. From a medical model, LD is a processing deficit. Through the social model, LD may be better understood as a mismatch between students and inflexible classrooms (Reid & Valle, 2004). LD can be understood as interactional in mathematics, produced through a series of iterated interactions that position learners in relationship to mathematics (Lambert, 2015; Heyd-Metzuyanim, 2013).

To be included in this integrative research review, studies need to be 1) standards-based mathematics curriculum, 2) a naturalistic classroom setting rather than a separate intervention, 3) include students with documented LD and 4) qualitative data on student participation. Studies were found through searches of "special education" and/or "disability" in combination with any one of the following terms: "participation," "mathematical practices," and/or "engagement."

Findings

In a study of the participation of students in a standards-based mathematics curriculum, students with LD did not participate in whole group mathematics discussion, and tended to focus on non-mathematical tasks such as materials management during small group work (Baxter, Woodward, &

Olsen, 2001). In another study, students with LD were called on fewer times than other students, and were less involved in small group work (Bottge, Heinrichs, Mehta, & Hung, 2002). After documenting this disparity in participation, both the research teams led by Baxter and Bottge designed future studies to explore increasing engagement and participation of students with LD in standards-based mathematics (Baxter, Woodward, & Olson, 2005; Bottge, Rueda, Serlin, Hung, & Kwon, 2007). Instead of designing intervention only in specific mathematical topics, they designed *intervention in participation*.

While some research studies offered evidence that students with LD were not participating equally in standards-based mathematics (Baxter et al., 2001; Bottge et al., 2002), other studies offered evidence of promising practices to deepen the participation of students with LD (Baxter et al., 2002; Bottge et al., 2007; Foote & Lambert, 2011; Moscardini, 2010). Some of these studies offered glimpses of classrooms in which participation in mathematical discussion and problem solving could not be predicted by whether or not a student had a disability.

The design of curriculum may be critical. Students were supported in standards-based mathematics classrooms in which they were presented with tasks that offered multiple solution paths (Bottge et al., 2007; Foote & Lambert, 2011; Moscardini, 2010). Students were supported by curriculum that was responsive to student thinking, so that teachers designed the tasks based not on a predesigned sequence but on the current understandings of their students (Carpenter et al., 1999/2014). Students were supported when mathematics curriculum provided multi-modal representations of content, and multiple ways of engaging in mathematical activity (Baxter et al., 2005; Bottge et al., 2007; Foote & Lambert, 2011).

These studies suggest that teacher moves are critical in increasing student participation. While some studies documented significant differences in participation between students with LD and those who were not (Baxter et al., 2001; Bottge et al., 2002), there were classrooms in which participation was equalized (Foote & Lambert, 2011). In these classrooms, teachers supported student problem solving through strategies such as supporting student participation in discussion. Other strategies included rewording problems during problem-solving (Moscardini, 2010) and supporting equity in small group work (Bottge et al., 2007). Teachers supported the equal participation of students with LD by giving equal status to presentations that included the notebook and manipulatives as supports (Foote & Lambert, 2011). Researchers noted that consistency in classroom routines may have increased the participation of students with LD.

Effective use of teacher moves are predicated on mathematical knowledge for teaching (MKT) or developing understanding of how learners come to understand complex mathematical ideas (Hill, Rowan, & Ball, 2005). All studies that reported successful participation of students with LD either provided professional development in MKT for teachers, or used teachers with already developed MKT. The issue of MKT is particularly relevant to special education. Special education training for teachers has long focused on understanding learner characteristics rather than content-specific pedagogy (Woodward & Montague, 2002). Based on their training, special education teachers often see pedagogy as generalized strategies that work across content areas, such as using flashcards to support memorization both for math facts and letter recognition. These generalized strategies are quite different from discipline specific strategies, such as in mathematics, supporting students' ability to build on known facts using the properties of operations. This situation continues even as research in mathematics education has demonstrated the importance of MKT for effective mathematics teaching (Hill et al., 2005). The studies reviewed here suggest that shifts in the training of special education teachers could allow students with LD greater access to successful participation in standards-based mathematics. While special education teachers are not trained sufficiently in MKT, general education teachers report being underprepared for including students with disabilities in their classrooms. Moscardini (2013) found that the general education teachers in his study reported that basing instructional decisions on the thinking of children was more effective with students with

learning disabilities, and these general education teachers reported being more receptive to including learners with disabilities in their mathematics classrooms.

Research in the mathematical learning of students with learning disabilities must focus attention on *intervention in participation*. This review included only a small number of studies that suggest a future direction for research in this area. Such research could provide a much needed focus on what learners who are labeled LD *can* do within standards-based mathematics, rather than what they *cannot* do. With increasing evidence that such students can construct effective strategies on their own (e.g. Peltenburg, Heuvel-Panhuizen, & Robitzsch, 2012), educators can no longer assume that standards-based mathematics will not work for these students based on perceived deficits. Nor can we assume that simply including students with learning disabilities in standards-based mathematics classrooms will lead to higher achievement. Instead, we must learn more about how to support all learners for full participation in standards- based mathematics classrooms. These supports are likely to be helpful for a wide range of learners, assisting teachers in making standards-based mathematics more equitable for all.

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PARENTS' INVOLVEMENT IN EARLY YEARS MATHEMATICS LEARNING: THE CASE OF JAPANESE IMMIGRANT PARENTS

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A meaningful collaboration between schools and homes can enhance students' opportunities to learn mathematics. The goal of this study is to understand how parents experience their involvement in children's mathematics learning and how they describe their relationships with schools and teachers. This study utilizes the data collected from semi-structured interviews with Japanese immigrant families in Canada. Findings identified active parental involvement in children's mathematics learning among this population. At the same time, findings also suggested the invisibility of school mathematics learning for those parents. This study proposes creating boundary objects that can meaningfully bridge homes and schools.

Keywords: Equity and Diversity, Informal Education

Background and Literature Review

Parents can influence mathematics education, especially in a child's early years and in elementary schools. For example, in Canada, which is the context of this study, a group of parents recently organized a petition to the Alberta government to push mathematics curriculum "back to basics" (Tran-Davies, n.d.). Understanding parent's discourse surrounding mathematics learning is important for mathematics teaching.

This study investigates how parents experience their involvement in children's mathematics learning and how parents, particularly immigrant parents in this case, describe their relationships with schools and teachers. Previous research tells us how certain forms of parental involvement affect students' academic achievement in school (Galindo & Sonnenschein, 2015; Organisation for Economic Co-Operation and Development, 2012). According to those studies, influential forms of parental involvement include engaging in discussions that facilitates critical thinking and setting a good example for academic engagement. Because parental involvement can affect students' academic performance, it is important to identify how parental involvement at home is promoted or hindered.

The current study focuses on Japanese immigrant families living in an urban city of Canada. Previous studies on the topic of immigrant parents' engagement in mathematics learning shed light on immigrant parents' mathematics knowledge and resources embedded in their cultural practices (Civil, 2007; Willey, 2008). This line of research has also highlighted the conflicts and struggles that immigrant parents experience in their relationships with schools (Abreu & Cline, 2005; Civil & Bernier, 2006; Crafter, 2012). This study adds a new cultural and historical context to this body of literature. Based on the experience of Japanese immigrant families, I will explore possible ways to facilitate a meaningful collaboration between homes and schools.

Theoretical Framework

In order to investigate the relationship between schools and homes/communities for students' mathematics learning, I will examine the presence or the lack of boundary objects. The concept of boundary objects was proposed by Star and Griesemer (1989) and defined as:

> Boundary objects are objects which are both plastic enough to adapt to local needs and the constraints of the several parties employing them, yet robust enough to maintain a common identity across sites. (…) They have different meanings in different social worlds but their

structure is common enough to more than one world to make them recognizable, a means of translation. The creation and management of boundary objects is a key process in developing and maintaining coherence across intersecting social worlds (p.393).

Boundary objects can connect communities together for a common goal (Wenger, 1998). Boundary objects allow us to examine how collaboration among different parties become possible without a prior consensus (Star, 2010).

Method and Methodology

The data are derived from the ethnographic study I conducted with Japanese immigrant families in an urban city of Canada. Here I focus on findings gained from the interviews with 14 Japanese parents, first generation immigrants to Canada. All the parents were raising school-aged children. Each interview lasted approximately 60 minutes and elicited backgrounds of the parents (education, language, and immigration), involvement in children's school education, education at home, mathematical cultural practices in which parents were involved in Japan and in Canada, and sense of belonging and social networks in Canada.

For the current analysis, I focused on the following two aspects described in the interview: 1) how parents were involved in their children's mathematics learning at home and 2) how parents described their relationships with their children's schools and school teachers. The following findings section is organized around these two aspects and I will first describe how the participants in this study were involved in their children's mathematics education. Subsequently, based on the concept of boundary objects, I will discuss whether and how boundary objects existed between schools and homes for immigrant families.

Drawing from the framework of Discourse (with a capital D) proposed by Gee (1990), my analysis focuses on identifying common threats that characterizes the collective discourse of Japanese immigrant parents living in Canada. Discourse recognizes that people's ways of talking and acting signifies their involvement in certain social groups. By using Discourse as an analytic tool, I tried to shift my focus from individuals who speak, to socially and historically defined discourses identified in Japanese parents' accounts on mathematics teaching and learning.

Findings

Overall, Japanese parents in this study were actively involved in their children's mathematics education at home and in their community. There were forms of parental involvement described both at the community level and at the individual level. At the community level, Japanese immigrant parents were organizing space for children to learn arithmetic. Many elementary school children were learning to use the Japanese abacus in informal settings, from Japanese immigrant parents. One of the teachers (who is also a parent) of the community abacus class explained that the class started spontaneously based on the needs from parents. The teacher/parent also explained that it was a way of making a contribution to the community: "My strength I can use to contribute to the Japanese Canadian community is abacus, so…(laughter)." In addition to teaching abacus and arithmetic, parents also hoped to teach Japanese language through the abacus classes. The arithmetic children learn at the abacus class is advanced, compared to the regular school curriculum. For example, when I visited one of the abacus classes, Grade 3 students were working on 6-digit multiplication.

Similarly, at the individual level, many parents were teaching mathematics by using Japanese mathematics textbooks or workbooks that they brought back from Japan. Parents were concerned that their children might fall behind academically due to their limited exposure to English at home. They decided to provide additional support to enrich their children's mathematics learning opportunities. For example, a parent explained, "I was concerned that my children might academically suffer or might come to dislike school because their English is not strong. So, I wanted to teach mathematics

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even for a bit, for them to gain confidence academically, even when they don't understand English well."

Overall, parents in this study were actively involved in mathematics teaching at home. At the same time, they expressed concerns regarding the disconnection between home and school. One of the concerns was the difference in mathematics pedagogy between what parents experienced in Japan and what their children were learning in Canada. For this issue, a parent said, "In Canadian schools, I don't think they place so much value on quick computing, because computation is just a tool. It's more important to know how to use the tool and why the particular tool is used in a particular situation."

At the same time, parents felt there were limited opportunities to learn about what children are learning at school. For all the parents in this study, school mathematics was invisible in Canada. Recalling the Japanese education system, parents named several culturally-specific practices that made Japanese school curriculum visible to parents. For instance, there were opportunities for parents to join a regular class to observe (*jyugyou sankan*). During these occasions, teachers often explained the curriculum and pedagogical goals to parents. Also, in Japan, each child receives a textbook and studies at home. Parents and children are generally able to communicate about what is learned at school, via textbooks (if parents and children have the time and capacity to do so). Most of the parents in this study were interested in supporting their children's academic success, especially in mathematics in early years. However, they felt limitations in supporting their children because they did not have a clear idea about the curriculum and pedagogy at school. All the parents I interviewed felt these culturally-specific practices were missing in Canadian contexts and the lack of these practices were contributing to the invisibility of school mathematics for parents.

Discussion and Implications

Parents in this study were actively involved at home and in their communities to support children's learning mathematics, especially during the early years. Parents' academic involvement tended to focus on number sense and computation, as represented in the community abacus classes. Some parents were concerned about the pedagogical gap between what children are learning at school and what they are teaching at home and in the community. All the participants expressed the invisibility of school mathematics due to the lack of organized practices, which could help to make school mathematics visible to parents. This finding is compelling especially that parents can have considerable impact on mathematics curriculum and pedagogy. This finding suggests the need to build a bridge between schools and homes/communities and to facilitate a dialogue between teachers and parents.

Designing boundary objects is one way of filling this gap. Parents' interviews in this study revealed some effective boundary objects they observed in Japan. For example, *jyugyo sankan* (lesson observation) can allow teachers to explain and enact their teaching philosophy and curriculum goals by going through the details of a lesson. Communicating via textbook is another example. Parents hoped to learn more about what children were learning at school so that they could provide appropriate academic support to them. As Stigler and Hiebert (1998) maintain, teaching is a cultural activity that is situated and deeply rooted in history, policy and everyday practices. It will be important to understand the history, policy and practices that surround these boundary objects. Designing boundary objects that can bridge homes, communities and schools in a meaningful way can enrich the dialogue and collaboration for students' mathematics learning.

This study examined a new context, which has not yet fully been investigated in the body of immigrant parents' engagement in mathematics (e.g., Abreu & Cline, 2005; Civil & Bernier, 2006; Civil, 2007; Crafter, 2012; Willey, 2008). Several unique issues were revealed through this study on Japanese immigrant parents living in Canada. One of the issues was discussed above and all the parents in this study acknowledged the boundary objects that could bridge homes and schools, by

referring to their experiences in Japan. Another interesting initiative observed in this community was the abacus classes that parents organized. Further examination will be beneficial to understand how these children experience mathematics learning across the mainstream mathematics classes in Canada, the community abacus classes and learning at home.

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WHAT CONTRIBUTES TO POSITIVE FEELINGS TOWARDS MATHEMATICS?: EXAMINING MATHEMATICS AUTOBIOGRAPHIES

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In this paper, we present how the participants in our study (post-secondary students) described what contributed to fostering their positive feelings towards mathematics. Drawing from mathematics autobiographies completed by the participants, we present some of the contexts wherein participants described positive feelings toward mathematics. We discuss a) encounters with teacher dispositions and pedagogical practices, b) experiencing the joy of engaging in mathematics, and c) external validation from teachers and parents, and consider whether each of these contexts sustained participants' positive feelings towards mathematics.

Keywords: Affect, Emotion, Beliefs, and Attitudes

Literature Review and Theoretical Framework

The interplay between the affective domain (beliefs, attitudes, and emotions) and teaching and learning mathematics has been explored over the past 30 years. Many of the studies investigating affect and mathematics in the field of cognitive psychology tend to focus on negative aspects, such as "math anxiety," associated with mathematics (e.g., Ahmed, Minnaert, Kuyper, & van der Werf, 2012; Young, Wu, & Menon, 2012). Also, when conducting large survey-based studies on the affective domain of mathematics learning (e.g., Vandecandelaere, Speybroeck, Vanlaar, De Fraine, & Van Damme, 2012), treating emotion as fluid, dynamic and changing becomes a methodological challenge so the construct tends to be portrayed in static ways and as an attribute of an individual. As Di Martino and Zan (2011) maintain, the mathematics education community tends to treat emotion in a more holistic way by considering a wider range of emotions and the importance of social aspects. For example, researchers in the mathematics education community pay close attention to the conditions and instructional contexts in which students' emotions towards mathematics can be changed (e.g., Evans, Morgan, & Tsatsaroni, 2006; Hannula, 2002). However, such investigation is still often limited to single case studies.

Our study further extends these earlier investigations by analyzing a large number of autobiographical narratives from Kindergarten to post-secondary level students. In this paper, we focus in particular on the data from participants enrolled in post-secondary level education, in order to give us a longer-term view of the contexts that influenced students' feelings about learning mathematics in schools and the factors that helped them to sustain positive feelings into their post-secondary years. Because the majority of research on emotions and mathematics, especially when drawing from a psychological paradigm, focuses on negative emotions, we hoped to examine other aspects of emotions, such as positive feelings and a change in feelings.

This research is framed by enactivism, a theory of embodied cognition that emphasizes the interrelationship of cognition and emotion in learning (Maturana & Varela, 1992). Enactivist thought reorients us to the significance of this mathematical milieu in shaping not only what students learn in school but also their emotional connections with the discipline.

Method

This study is part of our larger project, which investigates students' experiences learning mathematics in Canadian K-12 schools and post-secondary institutions. The data we present here were gathered by using an online submission form. We asked participants (ages 18 and older) to submit mathematics autobiographies, wherein they described their histories of learning mathematics

and their relationships with the discipline of mathematics. We also asked participants to contribute demographic information including age, gender identity, education history, and current profession. To date, we have analyzed 70 submissions, 48 of which were submitted by pre-service teachers, and 26 are multimodal autobiographies using visuals, sounds, and videos. Our data collection and analysis are still ongoing. The analysis of this paper focuses on positive feelings towards mathematics described by participants. Positive feelings were coded when linguistic markers to describe positive emotions (e.g., "like" "love" "enjoy") were used.

Findings

In this section, we present how the participants described the factors that contributed to fostering their positive feelings towards mathematics. Mainly, participants described a) an encounter with particular kinds of teacher dispositions and pedagogical practices, b) experiencing the joy of engaging in mathematics, and c) external validation from a teacher and parents.

Encounters with Teacher Dispositions and Pedagogical Practices

One of the recurring themes was how teachers made an impact on participants' feelings towards mathematics. Many participants acknowledged teachers who appreciated various ways of solving mathematical problems rather than strictly following a set of predetermined procedures. For example, a participant said: "The best teachers I had for math taught to my learning style (and this happened rarely). For me, math is highly intuitive, and I loved the idea that math answers could be arrived at using different methods." Other aspects of teaching that changed participants' disposition towards mathematics were teachers' commitment to students' success and providing sufficient help when needed. A participant recalled high school mathematics classes and said, "At lunch one could frequently find students from both classes crammed into his (the mathematics teacher's) classroom, clamouring for help."

Teachers' dispositions toward mathematics can also influence students' feelings. The following description, by a participant who considered herself to be "math anxious," is informative in showing how teaching approaches can make a difference in students' feelings.

> Specifically, I LOVED algebra!….I remember when Ms. W. spoke about algebra. She reminded us throughout the unit that doing algebra is like doing a puzzle; have fun with it!…Ms. W. had always given us extra time to complete tests, which I needed being the ball of nerves I was and still am. I, the former math anxious student, received the highest grade in the class (92%). More importantly, I approached the test as a puzzle, something I needed to take time with and figure it out. I was given that time. Time to play around with letters and numbers. Time to experiment with math!

This quote shows the significance of a teacher's dispositions towards mathematics—the importance of communicating the joy of mathematics and validating the time in which students "play around with" and "experiment with" mathematics. While many participants acknowledged great teachers, they also described encounters with a good mathematics teacher as a rather rare experience.

Experiencing the Joy of Engaging in Mathematics

In contrast with findings from our Kindergarten to Grade 9 student dataset where very few participants reported liking mathematics because of an intrinsic appeal (Hall, Towers, Takeuchi, & Martin, 2015), a third of post-secondary participants reported an intrinsic love of mathematics itself. Interestingly, the descriptions of the internal joy of engaging in mathematics described by these participants painted two very different portraits of "mathematics." Some participants conceived of mathematics as a discipline consisting of a set of procedures and correct answers to be discovered and it was this apparently "one right answer" vision of mathematics that they found compelling. The

following quote represents this stance: "I liked that I was given a formula to follow and that there was always only one right answer to each question. Everything was straight forward and to the point in my eye." These participants appreciated how the mathematics with which they engaged at school was "simple" "reasonable" and "concrete." Notably, those who perceived mathematics as a set of procedures did not further pursue a post-secondary degree that heavily involves mathematics.

Others perceived mathematics to be like a "puzzle," "an elegant language," "something beautiful," and "patterns." Many of these participants further pursued a post-secondary mathematics education through the degrees in the areas of engineering, astrophysics, biology, chemistry, and computer sciences. For example, the following quote is one representation of this group of participants: "I have associated mathematics with puzzles.…To me it is a game, with each level of increasing difficulty add[ing] another layer of beauty and technique." For these participants, mathematics was much broader than a set of procedures to follow. Another participant described the joy of mathematics as: "I actually really like doing math; it is an elegant language and I wish I could speak more of it. When I hear the word 'mathematics' I see the Universe and all of the matter within it." It is noteworthy that we observed quite opposite ways of describing the joy of engaging in mathematics among the participants.

External Validation from Teachers and Parents

External validation, appraisal, and positive evaluations received from teachers and parents contributed to participants' positive feelings to some extent. For example, a participant said, "When we had "mad minutes" (a minute to complete a sheet of multiplication questions) in grade three, I always tried to be the first one done." For some participants, the speed with which they could operate computation was rewarded in school and that contributed to their positive feelings towards mathematics. But the feelings of excitement were inconsistent for these participants. For example, the same participant said, "in Grade two, I couldn't understand at first how to subtract numbers and 'borrow' ones...I was just about in tears and felt really stupid." This participant associated her confusion with her identity. Similarly, when participants were relying on external validation from others, positive feelings towards mathematics could be easily swung to become negative ones.

> As a mathematics student in elementary school I remember liking mathematics because I liked solving problems and I was quite good at it. In junior high, I liked math even more because my family was amazed that I was able to get quite good marks in math.…I kept persisting in achieving high marks because I liked the acknowledgement that I received from teachers and my family.

This participant's grade dropped in Grade 12 and she received 65%. She said she "was very ashamed of this grade" and "I discovered that math no longer had a purpose in my life, so the hard work needed to get through the class didn't feel worth my time anymore." As clearly described in this case, positive feelings about mathematics backed up merely by external validation could be drastically changed to negative feelings towards mathematics as soon as the external validation of good marks and acknowledgement from others disappeared.

Discussion and Implications

Our analysis adds to the existing literature by detailing some of the factors contributing to students' positive relationships with mathematics. This analysis adds to earlier investigations on the complexity and dynamics of emotion (e.g., Evans, Morgan, & Tsatsaroni, 2006; Hannula, 2002) by demonstrating that an encounter with a mathematics teacher, who embodies the joy of mathematics and encourages students to explore and experiment with mathematics, can affect students' feelings towards mathematics in a positive way. However, we also came to learn that such an encounter was described as rather rare. In addition, our study suggested that positive feelings backed up merely by

external validation from teachers and parents could easily shift to negative feelings in the absence of such validation. This finding suggests the significance of teacher education where teachers can foster a wider range of mathematical dispositions and teaching methods that can positively influence students' relationships with mathematics.

Our study also suggests the importance of attending to the details and contexts when participants describe positive feelings towards mathematics. For such investigations, participants' autobiographical narratives provided contextualized accounts of how participants' feelings have changed over time and whether participants pursued mathematics-rich further education and careers. Our analysis revealed that those who enjoyed mathematics because it was perceived as straightforward, simple, and a set of procedures, did not further pursue post-secondary mathematics education. In contrast, those who were attracted by its beauty, challenge, and elegance further pursued mathematics. This is an important finding that should be further explored by mathematics education researchers.

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TEACHERS REPORT FREQUENT INTERACTIONS WITH STUDENTS TO EVALUATE PROGRESS

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Formative assessment during instruction makes a difference in students' learning outcomes (Black & Wiliam, 1998) and provides information about students' learning needs, learning styles, and development that can be used by teachers to adjust their instruction (Greenstein, 2010). Additionally, Black and Wiliam (1998) concluded that effective learning is based on active student classroom involvement. "Learning requires active and frequent interaction between teachers and students" (Black et al. 2004, p.47). In particular, teachers should determine effective teaching strategies for their students' learning needs and discuss these strategies with the students so as to maximize the quality and quantity of student achievement.

This poster provides preliminary findings of the relationships between teachers' reports on their frequent interactions with students to evaluate students' progress and algebra achievement. The data was withdrawn from the National Assessment of Educational Progress (NAEP) data across selected ten states between the years from 2009 to 2015. The dependent variable was the eighth-grade students' average algebra scale scores. The independent variables are the teachers reports on the frequency of interacting with individual students to evaluate their progress in mathematics, in the following activities: (a) Determining how to adjust teaching strategies to meet student's current learning needs and to reflect student's future goals (b) Discussing student current performance level (CPL)". The responses were given on a 5-point Likert scale: Never or hardly ever (1), a few times a year (2), one or twice a month (3), once or twice a week (4), every day or almost every day (5). Also, "teachers were asked to specify the frequency with which they interact with students in various ways to evaluate their progress" (Coley, 2010, p.9).

A one-way analysis of variance (ANOVA) was conducted to examine the differences in the students' algebra scores based on the frequency of teachers' interactions. There were no significant differences in the frequency of interacting with individual students to evaluate their progress in algebra, in the following activity: "determined how to adjust teaching strategies to meet student's current learning needs and to reflect student's future goals". However, there was a significant difference in eighth grade students' algebra scores based on the teachers' frequency of discussion of students' CPL. Notably, the eighth grade students whose teachers never discussed students' current performance level had significantly lower algebra scores when compared to students whose teachers discussed students' current performance levels "a few times year, one or twice a month, one or twice a day", or "every day or almost every day".

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EXPLORING COLLECTIVE CREATIVITY IN ELEMENTARY MATHEMATICS CLASSROOM SETTINGS

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In the field of education, and following the argument that students will need to be better equipped to successfully navigate the increasingly complex and ill-defined nature of life in the twenty-first century (Wells & Claxton, 2002), there is full agreement on the importance of creativity, and the need to reconceptualize pedagogy for purposes like teaching for creativity, learning creatively, and promoting creativity in classroom settings. But for such reconceptualization to be possible and applicable, it is important for us as researchers and educators to understand the nature of creativity in classroom settings, and to develop a description of it as it may emerge in such settings.

Although scholars in pedagogy, mathematics education, and teacher education have generated a good literature based in promoting learning for individual creativity, the fostering of individual creativity, and characterizing mathematical creativity, only a few of the current definitions for creativity are suited to the distributed and collective enterprise of the classroom. This claim doesn't mean that earlier accounts are wrong nor are they unfruitful, on the contrary they do provide food for thought on creativity. They may be just incomplete, given that they mostly restrict themselves to one path, or one vision, or one description, or one experience of creativity.

For the purpose of overcoming such an incompleteness, and based on the claim that doing and understanding mathematics are creative processes that should be considered at both the individual and the collective levels (Martin, Towers, & Pirie, 2006), my study explores possibilities and potentials of integrating a combination of theories related to creativity and theories related to collectivity with other teaching and learning practices in mathematics classroom settings in order to: investigate the nature of collective creativity in mathematics learning, offer needed empirical findings concerning collective creativity in elementary schools, explore ways in which collective creativity might be fostered in such settings, and generate understandings about the role of teachers in this endeavor. Collective creativity in elementary mathematics classroom settings is investigated in the classroom within a problem based curriculum, using a design-based research methodology.

My study is driven by the following questions: 1) what can be learned from the process of developing and refining an emergent definition of collective creativity for the elementary mathematics classroom? 2) Does collective creativity emerge in elementary mathematics classroom settings? 3) How can we foster collective creativity in elementary mathematics classroom settings? 4) And what is the teacher's role in fostering collective creativity?

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DEVELOPMENTAL MATHEMATICS STUDENTS' AGENCY IN AN IN-CLASS COMPUTER-CENTERED CLASSROOM

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Keywords: Affect, Emotion, Beliefs, and Attitudes, Equity and Diversity, Post-Secondary Education, Technology

Developmental mathematics students in community college, particularly those of color, are disproportionately unsuccessful and fail to reap the benefits of higher education (Bahr, 2008, 2010). Despite the intentions of course instructors, course designers, and students, developmental courses frequently fail to adequately support students. In-class computer-centered (ICCC) classes, where at least 80% of the content is supplied via stand-alone software, are a possible solution to this social justice issue. Such courses provide students with individual pacing and a flexible learning model. However, these courses also limit students' mathematics agency by limiting their ability to demonstrate knowledge. This paradox of limited mathematical agency and unlimited student agency provides a tension likely to yield findings on student actions.

This study explored student actions and behavior to answer the research question *how do disadvantaged students draw upon their experiences to navigate the ICCC classroom to achieve their goals* using agency–defined by Bandura (2008) as the capacity to understand, predict and alter the course of one's life's events–as an analytic framework. Students' past experiences and future goals were considered in the analysis (Emirbayer & Mische, 1998). Using agency in such a manner is novel in mathematics education research, which traditionally explored the type and extent of student agency in mathematics classes. In this study, three students were interviewed and observed working in a developmental mathematics class using MyMathLabs software.

Findings suggest that students established reasonable plans to successfully navigate their courses. However, these plans were based on assumptions, resulting from students' past experiences and ultimately undermined their intentions. These assumptions could have been formed from students' cultural capital not being aligned with the culture of community colleges.

Student success was also adversely affected by the hyper-linear environment where students could not advance in the course if they did not pass their immediate exam. Hyper-linearity prevented students from using later concepts to reinforce and further understand previous ideas, limiting opportunities for students to make conceptual connections.

This research explored how the reasonable design in ICCC mathematics classroom levied a burden on as-yet-unsuccessful students. Eliminating substantial feedback and requiring independent learning for a difficult subject, affected student's overall success, thus preventing them from gaining necessary college-level mathematical skills.

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TEACHING ASSISTANTS' AFFECT INFLUENCES STUDENT PERCEPTIONS

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Keywords: Affect, Emotion, Beliefs, and Attitudes, Instructional Activities and Practices, Post-Secondary Education

While graduate teaching assistants (TAs) are usually evaluated by students as part of a course evaluation, these evaluations are often vague or entirely quantitative and may not correlate with student outcomes (Patitsas & Belleville, 2012). While students are known to prefer male lecturers (MacNell, Driscoll, & Hunt, 2014) this may not be the case for TAs (Patitsas & Belleville, 2012). There is a known bias against TAs who are non-native English speakers (Lenio et al., 2014). The present study does not seek to address such biases and does not control for them. The focus of this poster is on student responses to open-ended questions asking students to describe their best and worst TAs. The results should be interpreted in the light of possible actions taken by existing programs to inform a TA selection, training, or evaluation process.

Method

Participants were 78 respondents to an anonymous internet survey sent in the early summer of 2015 to all students enrolled in first-quarter calculus in the fall of 2014 at a large, public, southwestern university, the intervening year assured that most had experienced multiple TA-led mathematics discussion sections at the school. This institution has professor lecturers and utilized TAs for breakout sections. Participants filled out a short survey about their experiences in TA-led mathematics discussion sections including describing what made a TA the "best" or "worst" of the TAs they had experienced. Open coding was utilized for data analysis.

Results

The greatest number of descriptors of best TAs were of personality, TA interest in students and/or the subject, or of the TA's ways of interacting – that is, of affective traits. Similarly, students describing their worst TAs frequently described affective traits such as condescending attitudes. Based on this analysis, it seems that affective and not just didactic or mechanical aspects of teaching and interacting with students would be appropriate to include in both TA training and assessments. The first and second-most mentioned TA factors mentioned – affect and preparedness/clarity– for both positive and negative TA experiences are likely related. If a TA cares about and therefore prioritizes their classes it will be obvious to their students, as will the reverse. It is of note that few students expressed gender, ethnic, or language preferences in this study, instead expressing similar views of action and attitude across TA demographics.

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WHAT ABOUT ME? EXPLORING THE IMPACT OF REMEDIAL COURSEWORK ON POST-TRADITIONAL LEARNERS' SELF BELIEFS

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Setting the Stage

More than 50 percent of students entering two-year colleges are placed into non-credit bearing remedial (NCBR) courses, a large leap over the almost 20 percent placed at four-year institutions (Complete College America, 2012). As community colleges offer flexibility to students who wish to pursue a college education but do not have the financial resources or available time to attend conventional, four-year institutions, they appeal to adults seeking to further their education. The American Council on Education (ACE) defines adult, or post-traditional, learner as the population of working age, adult students (ages 25 to 64).

Post-traditional students seek out community colleges for access and flexibility, and many are placed into non-credit bearing remedial courses, in part due to gaps in their educational careers. This study seeks to understand how NCBR courses affect post-traditional students mathematics self-efficacy and identity, and what relationship, if any, mathematics self-efficacy and identity have with their successes and struggles as they attempt to transition to college-level coursework. Bandura (1977) defined self-efficacy as the beliefs students hold about their academic abilities. Mathematics self-efficacy is the belief a person holds about her mathematics ability. This study draws on the work of Usher and Pajares (2009) to explore mathematics self-efficacy across these domains. Including identity into this analysis allows a broader focus beyond students' beliefs about their academic ability to do math, following Martin's (2000) framework for mathematics identity formation and socialization.

Findings

This poster presents preliminary findings on the effects of remediation on post-traditional students' mathematics self-efficacy and identity. Quantitative results will include changes in self-efficacy using an adapted version of Usher and Pajares (2009) instrument for mathematics self-efficacy ($\alpha = .798$). Results from a repeated measures ANOVA will show that post-traditional students exhibited a significant improvement in mathematics self-efficacy over a 16-week, NCBR course ($F[1,34] = 7.683, p < .01$), with significant differences in the domains of vicarious experience and social persuasion. Relationships to course completion and success will be presented, as well as the impact of the course on enrollment of post-traditional students in a mathematics course for the subsequent semester. Interview results are presented in the form of vignettes, illuminating struggles and successes post-traditional students experience in NCBR courses with the intent that future reform and innovation in remediation include the ever-increasing population of post-traditional students frequently left underserved.

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ANALYZING MATHEMATICS IDENTITIES IN AN UPPER-LEVEL UNDERGRADUATE MATHEMATICS COURSE

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Real Analysis is a course that often presents a shift in content toward constructing definition-based proofs, and is recognized to be a conceptually difficult course (Weber, 2008). At the large U.S. public research university where this study was conducted, it acts as a gateway, as it is the only upper-level course that is required for all mathematics majors regardless of concentration. Given its high stakes, this course offers an important opportunity for examining the development of students' mathematics identities, which may help contribute ideas for both improving student learning and fostering identities of belonging (Solomon, 2007).

This study consisted of classroom observations as well as interviews with 11 students in one section of Real Analysis (initial enrollment of 27 students). All 11 students confirmed the make-or-break nature of the course; as one student described it, "I heard that it is terrible. I heard that it's very hard and very challenging. But the general consensus is, after you've taken it, it's like you're like a veteran math student." This poster specifically examines students' responses to an interview question asked at both the beginning and end of the semester about how they see themselves in relation to mathematics.

First, these data were coded using Anderson's (2007) four faces of math identity (engagement, imagination, alignment, and nature), in order to see generally how students responded to the open-ended question, and what aspects of identity were most relevant to them. All 11 students commented on aspects of *engagement*, largely in positive ways such as "I actually think it would be fun to continue proving stuff in other classes." Interesting differences arose in that juniors in the course tended to focus on *alignment* (e.g., "I'm a real math major now") whereas seniors tended to focus on *imagination* (e.g., "My job isn't going to use stuff from the math major specifically"), suggesting that the course served different roles depending on students' stage in their degree. The *nature* component of students' responses was the most complicated and conflicted, requiring further analysis, and suggesting that students were wrestling with their own narratives about their ability and confidence in mathematics.

Second, students' responses were analyzed for change over time. Students in the pure math concentration and those who were double majors (5 of 11) tended to have the most positive shifts (e.g., "I feel like I can do anything in math now"), whereas students in statistics, education, or who were math minors (6 of 11) expressed decreases in confidence (e.g., "when it comes to this class I don't think I'm good at math"). These findings suggest the need to re-evaluate the structure and goals of critical gateway courses like this one in order to better support students beyond traditional mathematics majors in developing positive and productive math identities.

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CLARITY, COMPASSION, INTEGRITY: A MODEL FOR SOCIAL JUSTICE MATHEMATICS PEDAGOGY

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Teaching mathematics for social justice is understood in multiple ways - teaching about social justice issues using mathematics, teaching mathematics with a social justice lens, or teaching students to use mathematics to challenge social injustices (Gutierrez, 2002; Gutstein, 2006). In order to do this, besides having fairly in-depth content and pedagogical knowledge, teachers need to be familiar with real-world contexts (Ball, Lubienski, & Mewborn, 2001) and be able to facilitate 'bringing the world into the classroom' (Gutstein, 2006). Teachers often reject the idea because of the perception that paying attention to social justice issues means moving away from rigorous content and academic achievement for which they are held accountable (Erchick & Tyson, 2011). This poster provides findings from a study undertaken to push the boundaries of such perceptions and provide vivid descriptions of effective social justice pedagogical practices.

The Study: Methodology, Context, Data Analysis

A two-year longitudinal ethnographic case study was conducted in a 5th grade classroom in a culturally, ethnically, and linguistically diverse school with 90% low SES students. Researcher provided support to Ms. Lara (pseudonym) in teaching mathematics to ELL students. Ms. Lara is experienced, with Special Education, Educational Philosophy and Montessori Education background. Video-recorded lessons, student work samples, field notes, audio-recorded and transcribed interviews with Ms. Lara provided data. Data were coded using a codebook grounded in research and developed over three studies, covering a range of equitable instructional practices in teaching mathematics for equity and social justice.

Findings

Key features of Ms. Lara's pedagogy surfaced: (a) critical thinking as a personal process; (b) informative assessment; (c) intentionality in knowing students' context, aspirations, and using knowledge to make instructional decisions; (d) building mathematical and social identities; (e) students as locus of authority; (f) attention to "connected knowledge". Findings led to developing the model. (Codebook and framework displayed on poster.) The "codebook" and the framework can inform understanding of and teaching about, with, and for social justice.

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VOICES OF MARGINALIZED YOUTH: AN EXPLORATION OF LIMITED CHOICE, HIGH MOBILITY, AND MATHEMATICAL LEARNING

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Keywords: Equity and Diversity, Policy Matters, High School Education

School choice is touted as a solution to alleviate inequities in students' schooling experiences, but just like in mathematics, inequities persist. School choice can be seen as a way out of under-performing neighborhood schools. However, due to students' and parents' limited knowledge of charter schools and credit recovery schools catering to marginalized students, it can be very challenging for marginalized students to leave an under-performing neighborhood school and transfer into a school of choice (Rangel, 2013). Therefore, as marginalized students try to understand the school choice landscape, they may engage in strings of transfers increasing the chance that they will drop out (Rumberger & Larson, 1998). What is not well understood is the impact transferring has on students' mathematical learning. This poster examines how highly mobile, marginalized youth describe their experiences learning mathematics.

Using an analytic framework that foregrounds students' mathematical empowerment (Alsop, Bertelsen, & Holland, 2006), I analyzed one student's experiences engaging in high mobility and mathematical learning. The data included eighteen study sessions, a pre- and post-interview, and the student's school records. Using interpretive methods, I analyzed talk in each interview transcript, coding for components of my analytic framework, and wrote analytic memos across the data, striving to understand the individual student's empowerment cycles.

In this poster, I will focus on my findings of Felix, a seventeen-year-old junior at the time of the study. He was a Hispanic American who emigrated from Cuba with his parents when he was two. Felix did not receive equitable mathematics instruction, as he was often passed along without learning mathematics. In turn, Felix often experienced mathematical passing empowerment without considering his limited mathematical learning. The opportunity structures of his elementary and middle schools allowed him to "never think about" his mathematical knowledge (post-interview, line 387). When Felix became highly mobile, transferring five times, during his first two years of high school, his limited mathematical empowerment only intensified. It was not until Felix transferred to a charter school of good fit that he was finally mathematically empowered both in terms of learning mathematics and earning mathematics credit. For the first time since third grade, Felix experienced both passing and learning empowerment. In summary, this study sought to reveal some of the challenges highly mobile, marginalized youth face learning mathematics. These findings are important for mathematics educators, policy makers, and school officials as students' mathematical learning experiences are evolving with increased school choice.

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BLURRING THE BOUNDARIES BETWEEN HOME AND SCHOOL: SUPPORTING PARENT AND STUDENT LEARNING WITH FAMILY MATH

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Keywords: High School Education, Informal Education

Parent involvement has been identified as an important factor in the academic success of students (Sheldon & Epstein, 2005). However, DePlanty et al. (2007) found that parent involvement declines as students' progress through secondary school, citing intimidation and a lack of content knowledge as potential barriers. In addition, Becker and Epstein (1982) suggest that fewer parent involvement strategies are being used by secondary educators, and Abreu and Cline (2005) found that parent's confidence may be shaken when their mathematical approaches are not equally valued by the school. Finding a strategy for high schools to overcome these barriers for parents is often overlooked in the literature, and provides the purpose of this study.

In reviewing parent involvement strategies, Family Math Night (FMN) emerged as a potential fit for our purpose. FMNs have been described as opportunities for families to enter a non-threatening environment where they can demystify a subject and share in new challenges (Schussheim, 2004), and as a means to address curriculum needs, make school resources available, or celebrate mathematics (Szemcsak & West, 1996). With FMN as our selected strategy, we asked the following questions: (1) Is FMN appropriate and welcomed by parents and students in an urban high school? (2) What are parent and student takeaways from FMN?

To address these questions FMNs were implemented at an urban high school over the course of four years. Critical to the development of the FMNs were the following design features: inform parents about changes to students' curriculum and assessments, and provide activities and materials that emulate *open tasks* and the standards for mathematical practice. After each FMN parent and student survey data were collected. The data were analyzed using descriptive statistics and open-coding to better understand participants' perspective and voice.

From preliminary analyses, we have found a shift in parents' and students' views about mathematics to a subject that is useful and approachable. In addition, parents' expressed how they did not realize that mathematics included much more than problems found in textbooks. Parents and students also suggest that FMN was welcomed, beneficial, interesting, challenging, and informative. We see FMN as a parent involvement strategy that challenges many of the potential barriers to parent involvement in high school and helps blur the boarders between the home and school.

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EMOCIONES EN LA RESOLUCIÓN DE PROBLEMAS MATEMÁTICOS EN ESTUDIANTES DE BACHILLERATO

HIGH SCHOOL STUDENTS' EMOTIONS ON MATHEMATICAL PROBLEM SOLVING

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Importancia de las emociones en el ámbito escolar

Entre otros, Collell y Escudé (2003) plantean que gran parte del fracaso escolar de los alumnos no es atribuible a una falta de capacidad intelectual, sino a dificultades asociadas a experiencias emocionalmente negativas que tienen múltiples manifestaciones, por ejemplo, comportamientos problemáticos y conflictos interpersonales.

Dada la complejidad de los aspectos emocionales, diversos autores han propuesto su definición para la emoción, de las cuales se adoptó para este trabajo la conceptualización que ofrecen Ortony, Clore & Collins (1989): las emociones surgen como resultado de la interpretación de situaciones por parte de quienes las experimentan, pueden ser consideradas como "experiencias valoradas de eventos, agentes u objetos, es decir, están determinadas por la forma en que se interpreta la situación construida".

Las características de la estructura cognitiva de las emociones aluden a términos como "cualidad emocional" e "intensidad emocional" y son, por lo tanto, dos parámetros generales con los que se intenta definir una reacción emocional (Cano, 1989).

McLeod (1989) menciona que en resolución de problemas los estados emocionales se caracterizan por su brevedad, los alumnos experimentan un bloqueo en el plan de acción puesto en marcha para llegar a la solución de un problema, experimentan de forma casi inmediata respuestas emocionales intensas pero muy breves.

Considerando el rechazo a la resolución de problemas matemáticos que se observa en los estudiantes y el correlato emocional que el sujeto experimenta como resolutor de problemas, este trabajo tuvo como objetivo: Analizar las emociones de estudiantes de nivel medio superior cuando resuelven problemas de matemáticas.

Método

Está investigación es de tipo cuantitativo. Participaron 87 estudiantes de nivel medio superior de un instituto privado de la ciudad del Estado de Puebla, México. Sus edades eran 16-17 años. Para evaluar las emociones de los estudiantes al resolver problemas de matemáticas se empleó la Prueba de Positividad de Fredrickson (Cortina & Berenzon, 2013) la cual está compuesta por 20 reactivos que evalúan las emociones con base en la proporción entre el afecto positivo y el negativo, cabe mencionar que esta prueba se modificó al contexto de resolución de problemas matemáticos.

Resultados

Se encontró que las experiencias emocionales más frecuentes, y por lo tanto más constantes en situaciones de resolución de problemas son el orgullo, asombro, estrés y culpa por no estudiar. Los resultados del estudio muestran que el estudiante no tiene un manejo adecuado de emociones positivas, los cual es un aspecto importante ya que predominan las emociones negativas como estrés y culpa al resolver problemas matemáticos, lo cual es una implicación en el proceso de enseñanza-aprendizaje de la materia.

Keywords: Cognition, Affect, Emotion, Beliefs, and Attitudes, Problem Solving

Importance of Emotions in Schools

Among others, Collell and Escudé (2003) suggest that much of the students' school failure is not attributable to a lack of brainpower, but difficulties associated to emotionally negative experiences that have multiple manifestations, for example, problem behaviors and interpersonal conflicts.

According to the complexity of the emotional aspects, several authors have proposed their own definition for emotion, which was adopted from Ortony, Clore, & Collins (1989): Emotions arise as a result of the interpretation of situations by those who experience them, they can be considered as " valued experiences of events, agents or objects, that is, they are determined by the way the built situation is interpreted."

The characteristics of the cognitive structure of emotions refer to terms such as "emotional quality" and "emotional intensity" and are, therefore, two general parameters with which one attempts to define an emotional reaction (Cano, 1989).

McLeod (1989) mentions that in problem solving, emotional states are characterized by their brevity. Students experience a blockage in the action plan to reach the solution of a problem. They experienced intense but very brief emotional responses immediately.

Considering the rejection of solving mathematical problems observed in students and the correlative emotion that the subject experiences as a problem solver, this study aimed to analyze the emotions of high students when solving math problems.

Method

This study is quantitative. Eighty-seven students of higher average level from a private institute located in Puebla, Mexico participated in the study. Their ages were from 16 to 17 years old. We evaluated the students emotions to solve math problems with *Fredrickson's Positivity Test* (Cortina & Berenzon, 2013) which has 20 items that assess emotions based on the ratio of positive and negative affect, it is important to mention that the test was modified to fit the of mathematical problem solving context.

Results

We found that the most frequent and therefore more consistent emotional experiences in problem solving situations are pride, amazement, stress, and guilt for not studying. Results show that students have no proper management of positive emotions, which is an important aspect as it is dominated by negative emotions such as stress and guilt while solving mathematical problems. This has implications for the teaching and learning process of the subject matter.

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LAS IDEAS DEL "NOLANO" GIORDANO BRUNO, CONDICIÓN NECESARIA (QUIZÁ SUFICIENTE) PARA MATEMATIZAR EL MOVIMIENTO

THE IDEAS OF "NOLAN" GIORDANO BRUNO, NECESSARY (AND MAYBE SUFFICIENT) CONDITIONS TO MATHEMATIZE MOVEMENT

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Palabras clave: Educación Primaria, Educación Secundaria, Resolución de Problemas

Deseamos evidenciar la importancia de ideas descritas por el Nolano contemporáneo de Nicolás Copérnico y Galileo Galilei, sugiriendo que aparezcan en los programas y libros de texto quizá desde nivel básico. Encontramos similitud en lo que hoy conocemos como "Relatividad de Galileo" basada en el movimiento de un barco (Hacyan, 1999, p. 13). Este hecho fue descrito por Bruno en "La Cena de las Cenizas", con años de anticipación (Schettino, 1972, p. 161). Compartimos la idea de lo expuesto por De Angelis y Espirito (2015) respecto a la contribución de Bruno al principio de la relatividad espacial (p. 246). El reconocimiento del movimiento de los cuerpos, debería lograrse antes de la resolución de problemas vía fórmulas, considerando el desarrollo histórico de las ideas que dieron paso al establecimiento de las mismas. Partimos de la pregunta, ¿Qué ideas describen los estudiantes de nivel básico, que permitirían descubrir las relaciones involucradas en el movimiento de los cuerpos?

Usaremos la teoría de los Modelos Mentales y Modelos Conceptuales en la enseñanza de las ciencias, aporte de Greca, Moreira y Rodríguez (2002), al explicar que: "Las personas operan cognitivamente con los modelos mentales. Entender un sistema físico o fenómeno natural, por ejemplo, implica tener un modelo mental del sistema que le permite a la persona que lo construye explicarlo y hacer previsiones con respecto a él" (p. 37). Un ejemplo que hace referencia a la cinemática encontrada en Giordano es el siguiente: "Con la tierra se mueven, por tanto, todas las cosas que se encuentran en ella. Por consiguiente, si desde un lugar fuera de la tierra se arrojara algún objeto hacia ella, perdería la rectitud debido al movimiento de ésta". (Schettino, 1972, p. 161) Elegimos un estudio cualitativo, aplicado en tercer grado de Primaria donde estudian las Ciencias Naturales y en segundo grado de Secundaria, en la materia de Ciencias II con énfasis en Física. Usaremos situaciones encontradas en la historia que permitieron despertar el interés de los antecesores del Nolano sobre el movimiento de los cuerpos. Nos interesa establecer que el uso de ecuaciones sobre el movimiento en la resolución de problemas de cinemática, debe hacerse una vez comprendida la situación física, poniendo énfasis en la comprensión del movimiento. Es necesario resaltar la importancia del Sistema de Referencia al resolver problemas de cinemática, causa de la existencia de valores negativos en velocidad, aceleración, tiempo y signo del valor de la gravedad. Categorizaremos los Modelos Mentales surgidos en estudiantes de nivel básico al dar respuesta a las situaciones históricas planteadas.

Keywords: Elementary School Education, Middle School Education, Problem Solving

We aim to highlight the importance of ideas described by "Nolan," contemporary of Nicolaus Copernicus and Galileo Galilei, suggesting that these appear in programs and textbooks, perhaps, since the very basic level. We find similarity in what is now known as "Galileo's Relativity" based on the movement of a boat (Hacyan, 1999, p. 13). This fact was described by Bruno in "The Supper of Ashes" years in advance (Schettino, 1972, p. 161). We share the idea of the comments made by De

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Angelis & Espirito (2015) regarding Bruno's contribution to the principle of spatial relativity (p. 246). The recognition of the movement of bodies should be achieved before solving problems via formulas, considering the historical development of the ideas that gave way to the establishment of these. We begin with the question: what ideas do basic-level students describe which would allow them to discover the relationships involved in the movement of bodies?

We will use the theory of Mental Models and Conceptual Models in science education, contribution made by Greca, Moreira and Rodriguez (2002), explaining that: "People operate cognitively with mental models. Understanding a physical system or natural phenomenon, for example, involves having a mental model of the system allowing the person who builds it to explain it and make predictions in relation to it" (p. 37). An example that refers to the kinematics found in Giordano is as follows: "Together with the Earth, all things that are in it move. Therefore, if from a place off the Earth an object was thrown to it, it would lose the straightness due to its movement" (Schettino, 1972, p. 161). We chose a qualitative study, applied to third graders where they study natural science and eighth grade in secondary schools, in Science II with emphasis on Physics. We use situations encountered historically that allowed to spark the interest of Nolan's ancestors on the movement of bodies. It is important to establish that the use of equations of movement in solving kinematics problems should be carried out once the physical situation is understood, with an emphasis on understanding such movement. It is necessary to emphasize the importance of the Reference System to solve kinematics problems because of the existence of negative values in velocity, acceleration, time and sign of a gravity value. We categorize the Mental Models emerged in basic-level students to respond to the referred historical situations.

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FACILITATING CONVERSATIONS ABOUT EQUITY THROUGH IMAGERY

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Increasingly there have been calls for mathematics teacher educators (MTEs) to address issues of equity and diversity (e.g., Strutchens, et al, 2012). Addressing issues of equity in the context of mathematics teacher preparation, however, can be a challenging task (Vomvoridi-Ivanovíc & McLeman, 2015). One of the most discussed challenges is that of resistance (e.g., Aguirre, 2009). Aguirre noted that pre-service teachers (PSTs) can resist discussing issues related to race or power. Even when PSTs are willing to examine biases present (e.g., in curriculum), they might still be resistant to examine how race and other physical characteristics might provide people advantages in the world.

To counter some of these challenges and in an attempt to cross borders between the literature world and the mathematics world, we explored the use of comics to facilitate conversations about equity in our courses. Mallia (2007) argued that comics have the intrinsic potential of being a valuable affective and cognitive tool and can be used in instruction for, among other things, motivational and retention purposes. As such, we engaged in an exploratory study to address the following question: How does the inclusion of visual imagery, comics in particular, help MTEs facilitate classroom discussions about issues of equity?

As MTEs of courses in various contexts, we selected comics that depicted conceptions of equity and equality and included them in our syllabi. Further, we used the comics to frame discussions about key course objectives. Utilizing components of grounded theory (Glaser & Strauss, 1967), we analyzed course artifacts, student work, and our self-reflections.

In this poster, we present initial findings regarding the use of comics in our differing contexts. Preliminary findings show that the images enabled us to engage our students in difficult dialogues about equity in mathematics. Furthermore, our findings suggest that the use of comics allowed students from multiple contexts (e.g., education level, major, course focus) to meaningfully enter into conversations about equity in a safe manner.

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LA COMPRENSIÓN TEXTUAL DE PROBLEMAS MATEMÁTICOS VERBALES EN NIVEL BÁSICO

TEXTUAL COMPREHENSION OF BASIC LEVEL VERBAL-MATHEMATICAL PROBLEMS

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Palabras clave: Cognición, Educación Primaria, Educación Secundaria, Modelación

Representaciones Mentales

En 1986, W. Kintsch presentó un estudio, en el que niños de primaria resuelven problemas aritméticos y se ocupa de cómo los niños de este nivel entienden problemas aritméticos verbales. Con ese trabajo, el autor concluyó que las modificaciones lingüísticas menores afectan en gran medida la capacidad de los estudiantes para resolver un problema, comentó que no sólo la formulación lingüística del problema es importante, también es necesario comprender la situación descrita en el problema. Mencionó un problema de aritmética con dos formulaciones: "(1) Hay 5 pájaros y 3 gusanos, ¿Cuántos pájaros hay más que gusanos?", "(2) Hay 5 pájaros y 3 gusanos, ¿Cuántos pájaros no obtendrán un gusano?". El problema 1 fue resuelto correctamente por el 39% de los sujetos, mientras que el problema 2 por el 79%. En el problema 2, la expresión lingüística permitió a los sujetos formar un modelo situacional definitivo (pájaros comiendo gusanos) que les proporcionaba la estructura aritmética correcta para llegar a la solución. En el problema 1, sin embargo, el término relacional abstracto "más que" no proporcionó tal ayuda. Dado lo anterior, nuestro objetivo es analizar cómo ciertas modificaciones en el texto base influyen en la construcción del modelo situacional de un problema verbal de matemáticas en el nivel básico y la pregunta de investigación que nos planteamos es: ¿Cómo influye la modificación en el texto base en la construcción del modelo situacional y en la solución del problema?

Método

Para alcanzar el objetivo planteado se aplicaron dos problemas con modificaciones en su estructura lingüística y matemática y se compararon los modelos situacionales representados en los dibujos que se les solicitaron a los alumnos inmediatamente después de haber leído los problemas. En esta investigación se analizaron las construcciones mentales de los estudiantes a través de los dibujos. El análisis es de tipo cualitativo. Se aplicaron los dos problemas que mencionamos de Walter Kintsch (1986) a 117 estudiantes de los tres grados de secundaria y a 87 estudiantes de los primeros tres grados de primaria. El estudio se realizó en la ciudad de Puebla, México.

Análisis y Conclusiones

Las respuestas obtenidas en secundaria no coinciden con las de Kintsch, ya que la mayoría de los estudiantes contesta correctamente ambos problemas, esto se debió seguramente al grado de estudio, ya que el autor trabaja con niños de primaria y esto se nota en el estudio que se hizo en primaria, donde nuestros resultados coinciden con los del autor. Al analizar los modelos situacionales en los dos niveles, se logra ver un modelo más coherente en estudiantes que trabajan con el segundo problema, en donde la claridad de la expresión lingüística es mucho mejor. Podemos concluir lo mismo que Kintsch, que la complejidad del modelo del problema requerido para la solución del problema 1 confunde a los sujetos.

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Keywords: Cognition, Elementary School Education, Middle School Education, Modeling

Mental Representations

In 1986, W. Kintsch presented a study in which elementary school children solved arithmetic problems and deals with how the children of this level understand verbal arithmetic problems.

With this work, the author concluded that minor linguistic modifications greatly affect the ability of students to solve a math problem, he states that not only the linguistic formulation of the problem is important, but that it is also necessary to understand the situation described in the text of the problem. He mentioned an arithmetic problem with two formulations: "(1) There are five birds and three worms, how many more birds are there than the number of worms?" and "(2) There are five birds and three worms, how many birds will not get a worm?" Problem 1 was solved correctly by 39% of the students, while problem 2 by 79%. In problem 2, the linguistic expression helped students to form a definitive situational model (birds eating worms) that provided them the correct arithmetic structure to solve the problem. In problem 1, however, the relational abstract term "more than" did not provide the same help.

In view of the above, the purpose of this work is to analyze how certain modifications in the base text influence the construction of the situational model of a verbal math problem at the basic level. Our research question is: How does modifying the base text influence the construction of the situational model and the problem solving outcome?

Method

To achieve the established objective two problems with modifications in their linguistic and mathematical structure were applied. We then compared the situational models represented in the drawings that were requested to students immediately after reading the problem. In this work the students' mental constructs were analyzed through the drawings. The analysis is qualitative. The two problems mentioned by Walter Kintsch (1986) were applied to 117 students grades 7, 8 and 9, and 87 students from the first three grades of elementary school. The study was conducted in the city of Puebla, Mexico.

Analysis and Conclusions

The answers obtained from students in grades 7 to 9 did not match Kintsch's results since most students solved both problems correctly; this was possibly due to the fact that his study was at the elementary school level, not middle school. However, we obtained similar results to the author's in the study with elementary school students. Our conclusions are the same as Kintsch's, namely that the complexity of the model of the problem required for the solution of problem 1 is confusing for the students.

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PEER ROLE MODELS IMPROVE SELF-PERCEPTION OF MATH ABILITY

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Keywords: Gender, Equity and Diversity, Affect, Emotion, Beliefs, and Attitudes, Post-Secondary Education

Student persistence in the STEM disciplines continues to be a national problem, especially for women and underrepresented minorities. Female and male high school students are equally prepared in their mathematics coursework, yet only half as many incoming college female students plan to major in a STEM field (NSF, 2010). One key factor is that women typically do not encounter enough female peer role models in their mathematics classes for them to feel they belong (Lockwood, 2006). For underrepresented students, belonging is crucial for participation and performance (Walton & Cohen, 2007). We present results of an ongoing study involving the use of peer role models to increase women's, (esp. Latina), persistence in the calculus sequence. Marx and Roman (2002) define peer role models as inspiring in-group members who defy negative stereotypes associated with their group (e.g., mathematically-capable females).

In a large, ethnically diverse university, half of the Calc I break-out sections were visited twice by female peer role models (treatment condition) whereas the other half were not (control). At the end of the semester, we administered a Likert survey on feelings of belongingness, personal mathematical ability, attitude toward mathematics, as well as, indicating their intention to continue in the calculus sequence. The results were statistically analyzed as a function of section type (peer role model vs. control) and student gender (male/female). As a measure of persistence, we also looked at looked at how many students enrolled in Calc II/repeated Calc I.

In the control condition, female students' beliefs about their math ability were significantly lower than male students ($p < 0.051$). In the treatment condition, female students' beliefs about their math ability were just as high as that of male students. Regardless of student gender, students who saw a role model were more likely to be enrolled in Calc I/Calc II than those not exposed to a role model. This finding shows promising evidence that exposure to a female peer role model may improve women's mathematics experiences.

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EXPLORING STUDENT VOICE IN A HIGH SCHOOL MATHEMATICS CLASSROOM – EMPOWERING STRATEGIES FOR MARGINALIZED STUDENTS

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Keywords: Classroom Discourse, Equity and Diversity, High School Education

In traditional math classes, people of color, people from low socioeconomic communities, women, English language learners, students with resource needs, first generation graduates, and other marginalized groups of people are negatively impacted by the lack of opportunities to exchange ideas and engage in rich discussions (Gutierrez, 2002). Social inequalities that are perpetuated by public school systems need to be disrupted by providing access to higher-level mathematics education to all learners (Apple, 1995). For the purpose of teaching and empowering traditionally marginalized students, I explored the following research question: *how do students react to a teacher's attempts to implement discussion-based mathematical practices?* This poster addresses preliminary findings through the example of one classroom observation of several students publically grappling with the context of a math problem. Assuming the role as researcher-teacher, I used sociocultural classroom practices such as individual journal writing, partner talk, small group collaboration, and whole class discussions. As the teacher on record, I used daily field notes, daily audio-recordings, monthly student-journal entries, tri-annual survey questions, and daily lesson plans to provide evidence of how students responded to discussion-based lessons. These artifacts will be systematically coded to determine patterns of student voice in our math class. Similar to others who have researched their own teaching practice (e.g., Lampert, 1985), excerpts from field notes are used to portray specific examples of how students engaged in a whole class discussion.

My findings highlight the extent to which marginalized students engage in mathematical discussions and relate it to their previous experiences. Some students engaged in the practice of arguing, critiquing, and making sense of each other's ideas. Although the participants focused more on the context of the problem than on the mathematical concepts, they were publically sharing their ideas and practicing sociomathematical norms. Additionally, students used the classroom environment to collaboratively discuss mathematical situations while incorporating a variety of different races, socioeconomic backgrounds, genders, and skill levels. The classroom norms created a space for students to simultaneously perform social skills alongside engaging in the CCSS math practices. As the teacher, I was able to use this discussion as a formative assessment of my students' previous interpretations of using graphs to solve problems. The successes and challenges of creating a discussion-based classroom environment can be further analyzed to help math educators determine how to effectively break down traditional math classroom norms.

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REPRESENTACIÓN DEL ASPECTO AFECTIVO EN PROFESORES UNIVERSITARIOS DE MATEMÁTICAS DEL ÁREA TÉCNICA

REPRESENTATION OF THE AFFECTIVE ASPECT IN COLLEGE MATH TEACHERS IN THE TECHNICAL AREA

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Palabras clave: Afecto, Emoción, Creencias, y Actitudes

Por lo general, en la enseñanza matemática se deja de lado el considerar los aspectos afectivos de manera explícita, ignorando las repercusiones que esto acarrea en el aprendizaje matemático. Este trabajo parte de este supuesto, analizándose esta situación en el área técnica del nivel universitario (Rivera, 2014; Rivera & Lezama, 2014a). Se parte también del hecho de que el ser humano actúa ante la realidad, no conforme a lo que ésta es en verdad, sino conforme a la representación que de ella se forma en su mente (Moscovici, 1979). Esto se traslada al quehacer docente en matemáticas, resaltándose entonces la importancia de determinar qué representación tienen ellos respecto a esto. Para corroborar la hipótesis propuesta, se plantean dos preguntas de investigación: ¿Qué es lo afectivo en el aprendizaje de las matemáticas en el área técnica? y ¿qué representación sobre el aspecto afectivo tienen los docentes de matemáticas del área técnica?

Con este fin, se aplican entrevistas en profundidad en forma de grupos focales, a cinco colectivos de profesores de matemáticas de diversas carreras del área técnica de una universidad estatal. Ubicada ésta al lado este de México, en el estado de Veracruz. Se emplea la Teoría de Representación Social (Moscovici, 1979), como soporte teórico en la parte experimental; así como la metodología de grupos de discusión con entrevistas en profundidad y la Teoría Fundamentada (Strauss & Corbin, 2012), para las etapas de captura, codificación y análisis de datos respectivamente. Los hallazgos permiten construir una representación de lo que es lo afectivo para el colectivo de profesores en cuestión. Igualmente, pueden determinarse aspectos dejados de lado o minusvalorados por ellos, que afectan el aprendizaje matemático, como mostró ser el caso específico de la categoría de Emociones; misma que mostró –en este caso-, ser parte protagónica también presente en el núcleo central de la representación construida.

Keywords: Affect, Emotion, Beliefs, and Attitudes

Generally, in the teaching of mathematics, considering affective issues in an explicit form is left aside; it is common to ignore the repercussions this has on the mathematical learning. This work assumes this postulation, it analyses this situation in the technical area of college level (Rivera, 2014; Rivera & Lezama, 2014a). It is also accepted as true, that human beings act in the face of reality, not according to what this really is, but according to "its representation" –the they have in their minds-, (Moscovici, 1979). This transfers to the work of teaching mathematics, highlighting the importance of determining what representations teachers have about it. To test the proposed hypothesis, there are two research questions: What is affective in math learning in technical area? and, What is the representation of the affective aspect that teachers in technical area have?

For this purpose, in-depth interviews are applied in focal groups. There were five groups of teachers of mathematics from different careers of technical area participating in this study. They all were from a state university in Eastern Mexico, in the State of Veracruz. Social Representation Theory is used (Moscovici, 1979) as theoretical framework for the experimental section. We used

focus groups with in-depth interviews to collect data, and Grounded Theory (Strauss & Corbin, 2012) for coding and data analysis. Findings allow to build a representation of the affective aspect for the group of instructors involved, and to determine issues left aside or undervalued that affect the mathematical learning. An example is the category of "Emotions", which also turned out to play a main role in the core of the built representation.

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CULTURE IN BETWEEN: MUSLIM AMERICAN STUDENTS' CULTURE AND IDENTITIES IN MATHEMATICS CLASSROOMS

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Keywords: Equity and Diversity, Teacher Beliefs, Classroom Discourse

In the wake of domestic terror events and the prevalent anti-Muslim rhetoric, Muslim American students have found themselves living in cultures that are in conflict (Sirin & Fine, 2007) where their cultural and mathematics identities are being positioned in the dominant regimes of representation and constructed as different and as 'others' (Hall, 1990). Informed by the postulations of culturally relevant pedagogy CRP (Ladson-Billing, 1995), I argue that anti-Muslim rhetoric and Islamophobia, if not countered, problematize equity, obstruct the development of students' mathematical identity, and hinder engagement in mathematics classrooms for Muslim students living in the *borderlands* (Cobb & Hodge, 2010).

Theory and Methods

This narrative inquiry study (Clandinin & Rosiek, 2007) uses semi-structured interviews with ten Muslim high school students to explore how they position their mathematics identity when navigating multiple cultural contexts particularly when school system precludes Islamic culture and/or when curricula do not counter Islamophobia. Using narrative inquiry analytic lenses (Chase, 2005), interviews are coded and deconstructed into themes and sub-themes.

Preliminary Results

Preliminary analysis shows that students perceive their interactions in schools as racialized and marginalized as Muslim students. They perceive mathematics as decontextualized space in which enculturation is absent. Narratives of fear, distrust, and the "need to do better to prove self" dominated the conversations as well as references of "them/they" versus "us/we".

Implications

These experiences hold implications for school curricula and instructional practices to reconceptualize educational equity and social justice in mathematics classrooms for Muslim students. Fostering a feel of belonging and inclusion through teaching mathematics with *cultural eyes* is crucial for the development of students' mathematics identity and their success in school.

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AGENGY AND PERSISTENCE OF FRESHMAN AND SOPHOMORE UNDERGRADUATE STUDENTS

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The purpose of this research was to examine freshman and sophomore undergraduate mathematics socialization and identities and how these related to agency and persistence in undergraduate mathematics. A cross-case, qualitative analysis was conducted using classroom observations, interviews, and survey data, to answer the research question: What social forces impact the mathematics identities, agency and persistence of undergraduate freshman and sophomores? One case study was of an African-American female (Joy) who entered the university through a Student Support Services (SSS) program and was enrolled in a required College Algebra corequisite course. The second case study was of an Asian male and first generation immigrant (Sam) who was enrolled in Calculus III, a non-required advanced mathematics course.

The research was guided by Danny Martin's (2000) Multilevel Framework for Analyzing Mathematics Socialization and Identity Among African-Americans. This framework was developed to examine socio-historical, community, school and intrapersonal forces that contribute to mathematics socialization and identities, agency and persistence (Martin, 2000). Survey and interview questions explored students' mathematics socializations and their beliefs about social forces that impacted their mathematics identities, agency and persistence. Personal identities and goals, personal motivations, and beliefs about mathematics; beliefs about school and classroom forces pertaining to mathematics learning such as classroom experiences, available school supports, and relationships with professors and peers; and the influence of family members, mentors and peers were all examined.

Fitting with Martin's findings (2000), both participants exhibited positive agency in the face of negative social forces. Joy did not have a positive mathematics identity which she attributed to negative school forces in secondary mathematics, but she did have a strong academic identity which she attributed to positive intrapersonal and community forces. She actively sought family and SSS support when faced with negative forces in her undergraduate mathematics class and credited her persistence and success in mathematics to them. Sam had a strong, positive mathematics identity which he attributed to positive school forces in secondary mathematics. When faced with negative forces in his undergraduate mathematics class, Sam persisted independently and credited his success to intrapersonal forces. While Sam's mathematics and academic identities were closely connected, Joy's were not. This finding seems contrary to Martin's and points to the need for further research on successful mathematics students who have weak or negative mathematics identities, as well as the resources students with differing mathematics identities draw on in order to persist and succeed.

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MATHEMATICAL PROBLEM-SOLVING IN UNIVERSITY-EDUCATED ADULTS ON THE AUTISM SPECTRUM

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Prior research on mathematics and autism has primarily focused on young children or professors of mathematics. In my reading of the literature, I have found little focused on college students or graduates on the autism spectrum. Exploring this subject and finding particular characteristics or traits of mathematical thinking among such adults is the goal of my present research.

Research Setting

After an initial period of background information and anything else in particular my interviewees wished to share about their perspectives on mathematics, I gave various mathematical tasks to elicit more specific responses. Some of these were used from prior research on a general student population, such as the Magic Carpet Ride sequence used by Wawro et al. (2012) and paradox tasks used by Mamolo and Zazkis (2008).

There were several reported characteristics of people on the autism spectrum which I thought could be promising for mathematics education research. Use of special interests and geometric approaches were drivers of initial questions, but more factors appeared during interviews, particularly with paradox-related tasks. I will give examples of these with a comparison to the particular findings relevant to them in the cases of Joshua and Cyrus.

Method and Results

Particularly due to the work of Temple Grandin, one of the most famous people on the autism spectrum, there is often considered to be an association between the spectrum and visualization or spatial reasoning (Grandin, Peterson, and Shaw, 1998). I found a strong preference for visual, spatial, or geometric reasoning in the interviews I conducted with Joshua, but found the opposite pattern with Cyrus. My suspicion is currently that there may be stronger variance or preference in types of reasoning, but that it is not all necessarily toward the geometric type.

I have also presented several paradox tasks during my interviews; one reason for this was the interplay between visual and algebraic explanations seen in some student responses to these paradoxes. Like many of the students in previous studies (of a general student population without consideration of autism), the people I interviewed found these to be strange and paradoxical. However, the response was notably more positive than those from most students. I also found it notable that I did not see any tendency toward rejecting the mathematical facts after they had been presented from Joshua or Cyrus, unlike in many of the students in the prior studies.

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EXPANDING THE BOUNDARIES TO COMMUNITY COLLEGE MATH EDUCATION THROUGH A SITUATED PRACTICE APPROACH

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Keywords: Equity and Diversity, Post-Secondary Education, Affect, Emotion, Beliefs, and Attitudes

Community Colleges represent an unchartered territory with multiple possibilities for the field of math education research. They are a popular choice in higher education institution. Approximately 34% of the post-secondary enrollment in the 2013 fall semester was attributed to 2-year institutions (National Center for Education Statistics, 2013). Mesa, Wladis and Watkins (2014) noted that math education research in community colleges has been limited, disorganized and in lack of attention to questions of teaching and learning. Based on their thorough analysis of the extant work at these institutions, Mesa et al. (2014) described community colleges as a "new research frontier" (p. 180) and proposed an agenda that would include more practitioner-led research.

This study responds to the observed need for more research on teaching and learning at community colleges, particularly in the area of equity in math education. Community Colleges' classrooms are rich in diversity. Their first-year college students are more likely to correspond to a typically underserved subpopulation (e.g. non-white, low income, no exposure to higher level high school math courses) than their counterparts in four year universities (Mesa et al., 2014). Moreover, the majority of college remediation is done at community colleges, where students not only have to relearn math, but they also have to unlearn misconceptions about what it means to learn math (Mesa et al., 2014). Research from practice in these classrooms offers great insights on equitable math teaching practices with direct implications on college completion and increased educational access for students from underserved populations.

The work presented here is considered to be on-going because data continue to be collected directly out of real-teaching experiences for the past four years at a community college in an urban setting. Though the researcher's practice spans sixteen years at both the high school and the community college level, the research work was formally conducted through journaling, memos and triangulation (Merriam, 2009) to students' feedback. Results confirm the need to create lessons that are flexible and open to re-construction based on students' individualities and their experiences inside and outside the classroom. They also confirm that mathematical misconceptions can be capitalized on to build a stronger more integrative mathematical foundation. The proposed poster would showcase classroom experiences that evidence successful practices. These practices are relevant to equitable teaching at any classroom level (Urbina-Lilback, in press).

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BREAKING LANGUAGE BORDERS IN AN ONLINE PRE-CALCULUS COURSE

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Language Use in Mathematics Courses

Technology is commonly used in the educational system (Griffiths et al., 1994; New London Group, 1996). Griffiths et al. (1994) found that the impact in 1994 of information technology on the oppressed cultures was debilitating. These researchers found these preliminary results to show that learning mathematics can be positively impacted by technology when converted to a student's native language (Casas et al., 2011).

Participants in this study enrolled in a fully online pre-calculus course offered during the 2014 fall semester. Participants were asked if they changed the software to Spanish. Participants translated the software mostly to understand the instructions or to understand the meaning of mathematical vocabulary. For the most part, they were adamant about keeping the software in English.

Learning Mathematical Topics in English

Participants felt that if they changed a particular problem to Spanish to understand the concept then they would not be able to connect this concept to future topics. They also believed that if they learned a pre-calculus topic in Spanish they might not be able to connect that topic to future math classes. "Because learning in English will avoid math language problems in future classes" stated one participant. Pre-calculus is a gateway course to STEM degrees and careers. The students were aware of this and understood that at a university in the United States, these courses will be taught in English. "It is important to work in English because, after all, we are receiving a professional education in English; we need to understand terms in the proper language."

Discussion

The skills these students learned in Spanish were continually brought to the forefront when confronted with these computations at the college level. The level of proficiency in one language does not appear to be the catalyst for success in an online math course. Participant's use of both languages is independent of the level of proficiency. What is important is how participants use the internet to make meaning of instruction and math vocabulary to comprehend math topics. The "implication therefore is that if bilinguals are strategic language users, they may also be strategic problem solvers, especially if learning to problem solve is viewed as a communication-based experience" (Dominquez, 2011, p. 309).

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LINGUISTIC CHALLENGES IN THE KENYA CERTIFICATE OF SECONDARY EDUCATION (KCSE) MATHEMATICS WORD PROBLEMS

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The language of mathematics is a multisemiotic system that draws on not only linguistic, but also symbolic and visual resources (O'Halloran, 2005). Successful readers of mathematical text, therefore, do recognize these resources in a text and understand how they interact to generate certain meanings. The first stage in problem solving is understanding the problem (Polya, 1957), meaning that successful problem solving demands a proper understanding of the problem at hand. Word problems are useful tools for investigating linguistic challenges students may have while trying to understand mathematics texts (Newman, 1977). Fang (2012) noted four hindrances to students' mathematical text comprehension, namely technical vocabulary, nominalization, long noun phrases, and symbolism and visual displays.

The Kenya Certificate of Secondary Education examinations are high-stakes exams done by all Kenyan students completing high school (ages 17 or 18). Considering all of the subjects on the exams, students have performed the worst in mathematics for the last several decades. Although numerous research studies have attributed the poor performance to a number of factors, research investigating the role of language factor in Kenyan high-stakes examination is notably missing. Yet these examinations are written in English, which is a second or perhaps the third language for most students. In doing problems prepared in a language other than their first, students face an additional challenge to comprehend the "third language of mathematics" in a second language (Kenney & de Oliveira, in press). Research has also shown that unnecessary linguistic complexity in mathematics word problems is a major cause of poor performance among English language learners (ELLs) (Barbu & Beal, 2010).

In this study, we used task-based interviews to investigate linguistic challenges faced by Kenyan students solving KCSE word problems in an attempt to empirically test Fang's (2012) framework. We engaged eight KCSE candidates – four males and four females – in solving three KCSE word problems that were deemed linguistically challenging, while responding to prompts that would elicit their understanding. Data collected included students' written work and interviews, which were transcribed. The proposed poster presents the main findings of this investigation and discusses the implications of this research.

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PROBLEM SOLVING AS CO-CONSTRUCTION OF NARRATIVE: COLLABORATION AND SHARED COGNITION IN A CALCULUS WORKSHOP

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Collaborative learning has seen a recent increase in popularity as an instructional approach in college mathematics courses (e.g., Duncan & Dick, 2000). Collaborative learning in mathematics gives students opportunities to practice the kinds of discourse that lead to effective disciplinary thinking and to ask questions, propose and test ideas, and critique the reasoning of others (Bruffee, 1984; Davidson, 1990). Students benefit from opportunities to listen to explanations from peers and subsequently engage in *constructive activity* such as solving problems and explaining solutions (Webb et al., 1995; Slavin, 1992). What is not well understood are the mechanisms by which collaboration leads to greater mathematical understanding.

We draw from theories about mathematics as narrative (e.g., Solomon & O'Neill, 1998) and about co-construction of narratives (Ochs & Capps, 2001) to conceptualize the collaborative problem-solving process as the co-construction of a "narrative." As students build the "story," they add narrative elements based on their understanding of the mathematical objects in the story, and take detours as necessary when a "subconflict" arises or when they do not have the means to resolve the primary conflict.

In this study, we examine the collaboration among three students in a problem-solving workshop for vector calculus students. In some cases, students borrow narrative elements from one another in order to add to a shared narrative. In others, a student may borrow elements to aid in the construction of a separate narrative that is not shared by the entire group. One student's difficulty in incorporating his own narrative into the group's co-construction may be explained by a difference in "genre," in that the student's narrative follows a different epistemology and style of storytelling than the group narrative.

Awareness of differences among students' narratives, and in the "genres" of these narratives, may help mathematics teachers mediate disagreements among group members so that the members themselves take the lead in comparing different stories, finding points of agreement, and resolving conflicts.

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IMÁGENES DE ESTUDIANTES ACERCA DE LOS MATEMÁTICOS Y LA MATEMÁTICA

IMAGES OF STUDENTS ABOUT MATHEMATICIANS AND MATHEMATICS

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Palabras clave: Educación Secundaria

En este trabajo se presentan los resultados de una investigación cuyo propósito fue determinar qué imagen de los matemáticos y de la matemática tenían estudiantes de nivel medio superior del Instituto Politécnico Nacional. En este estudio una imagen es una representación que un grupo de personas tiene de algo y cuya construcción está basada en elementos sociales y culturales y que son transmitidos a través de interacciones personales y con el medio, dentro y fuera de la escuela. Consistente con esta posición, Rensaa (2006) reconoce que la influencia que forma las imágenes de los matemáticos y la matemática vienen de la escuela, de los pares, parientes o medios masivos. Ella agrupa y representa cada una de estas influencias en dos grupos: sociedad pública y sociedad escolar. En este estudio se identificaron elementos de esas sociedades que formaron las imágenes de algunos estudiantes del Instituto Politécnico Nacional sobre personas cuya profesión es la matemática. El total de estudiantes que participaron en el estudio fue de 170, de ellos hubo 36 mujeres y 134 hombres. El método utilizado en esta investigación fue una variación del DAST (Draw-A-Scientist-Test) propuesto por Chambers (1983), el cual fue retomado y modificado por Aguilar, Rosas, Molina y Romo-Vazquez (2016). El interés de indagar en las creencias de estos estudiantes sobre los matemáticos y la matemática y aportar conocimiento de ello para el caso de estudiantes mexicanos. Además interesó averiguar si existen conexiones entre la imagen que estos alumnos tienen de lo que es un matemático y la posible elección de una carrera profesional como matemáticos u otra carrera afín a esta. El test se divide en dos partes, en la primera se les solicita a los estudiantes que hagan un dibujo de un matemático, en la segunda parte se les hace una serie de preguntas, algunas relacionadas con su dibujo y otras relacionadas con lo que piensan acerca de los matemáticos, como la personalidad de estos y su actividad laboral; también se les pregunta por lo que ellos consideran que es la matemática y su gusto o desagrado hacia ella. Los resultados encontrados muestran que la gran mayoría de los estudiantes tienen dos imagines que coexisten, un imagen positiva de lo que consideran un matemático (que es la que se impone y surge naturalmente) y una negativa que proviene de los medios masivos. En los 170 estudiantes la imagen representativa y positiva de un matemático fue: un hombre de aspecto joven y alegre, el cual no usa barba, bigote ni lentes, viste casual y utiliza un corte de cabello formal. En cuanto a la imagen negativa de los matemáticos que promueven medios masivos, algunos adjetivos que los alumnos mencionan son que el matemático es sarcástico, extraño, loco, sin muchos amigos. En cuanto a sus ideas sobre la matemática predominó una idea positiva sobre ella, les resultan interesantes, divertidas, fáciles y la mayoría declara que son la base de todo.

Keywords: Middle School Education

In this paper we present the outcomes of a research, whose purpose was to know what images middle school students from the National Polytechnic Institute have about mathematicians and mathematics. In our study we adopt the concept of image as: an image is a representation that a group of people has of something, and whose construction is based on social and cultural elements that are

transmitted through personal interactions and interactions with the environment, within and outside the school. Consistent with this position, Rensaa (2006) proposes that the influences that shape students' images of mathematics and mathematicians come from the school, parents, relatives, or mass media. Rensaa groups and represents each of these into two groups: public society and school societies. In this study we identified elements from these societies that shaped the images that a group of IPN students held about people whose profession is mathematics. The students number who participated in the research was 170: 36 women and 134 men. The method used was a variation of Draw-A-Scientist-Test (DAST) suggested by Chambers (1983), this method was modified and adapted by Aguilar, Rosas, Molina, and Romo-Vazquez (2016). The interest of exploring the beliefs of these students about mathematicians and mathematics is to contribute to the knowledge in the Mexican students' case. In addition, we wanted to know if there are links between the image that these students have and the university program they would choose, like mathematics or similar career path. The DAST is divided into two parts. In the first part students are asked to draw a mathematician. In the second part, some questions are asked, some about their drawing and others related to what they think about mathematicians, such as what they think about their personalities and work. Students are also asked what they think about mathematics and whether they like it or not. Our findings show that a majority of the students have two coexisting images: A positive image of what they think a mathematician is (the most common and which appears naturally) and a negative one that comes from the mass media. Among the 170 students, the most representative image from a mathematician was positive: a young looking and happy man, without a beard, mustache or eyeglasses; he wears casual clothing and sporting a formal hair style. In relation to the negative image promoted by the mass media, some adjectives that students mentioned are mathematicians are sarcastic, weird, crazy, without many friends. Regarding their ideas about mathematics, it prevailed a positive idea about this subject; students find mathematics interesting, fun, easy, and most of students express that the mathematics are the basis of everything.

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Chapter 11

Teaching and Classroom Practice

Research Reports

Brief Research Reports

Posters

USING PRECISE MATHEMATICS LANGUAGE TO ENGAGE STUDENTS IN MATHEMATICS PRACTICES

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This study examined discussions centered on precise mathematical language use in two fifth grade classrooms. Drawing on episodes from lessons in which teachers focused on encouraging mathematics reasoning, our analysis examines the relationship between precise language use and mathematical justifying. We present three classroom episodes that illustrate facets of the relationship between precise language use and mathematical argument, and explore how blurring the borders of two types of pedagogical tools (attention to precise mathematical language and engagement of students in justifying) promotes enculturation of students into mathematics.

Keywords: Classroom Discourse, Instructional Activities and Practices, Reasoning and Proof, Elementary School Education

Using mathematical language precisely and with understanding is a complex endeavor and one with which many students struggle. In order for students to use language precisely, they must have a clear understanding of the underlying mathematical meanings and relationships associated with particular terms, as well as how terms may be used differently in common language.

Theoretical Framework

Learning Mathematics Language

Learning mathematical language is complicated by many factors. Mathematical terms are frequently used for specific and narrow ideas, whereas common language often uses terms broadly. Such terms may be confusing because the mathematical meaning may differ from the non-mathematical meaning, because multiple terms may be used to describe the same mathematical concept, or because two words sound the same (Kenney, Hancewicz, Heuer, Metsisto, & Tuttle, 2005). Thus, it is possible that a common meaning of a term may be more familiar to students than the mathematical meaning.

Furthermore, using mathematical language precisely can be a challenge for students because meanings of terms may have been obscured in prior lessons and discussions. Language used in mathematics classes has often focused more on the appearance of notation than on the mathematical ideas and relationships represented. These challenges may also lead teachers themselves to struggle with consistently using terms precisely and correctly.

Word knowledge is incremental, multi-dimensional, and interrelated, and words have multiple meanings. Students need multiple encounters with words in a variety of contexts and opportunities to build background knowledge that relates to the words in use (Spencer & Guillaume, 2006). However, vocabulary is not often taught in this manner. Common instructional practices provide students with definitions or ask students to look up definitions. As many definitions are written using words that are not meaningful to students, such approaches rarely lead students to the desired understandings. For terms associated with complex or abstract concepts, which include many mathematical terms,

simply asking students to passively learn definitions has not been shown effective for robust vocabulary development.

One strategy for developing meaning for mathematical terms is to engage students in discussion of their understanding of particular concepts and of the precise meanings of various mathematical terms. In developing understanding of terms, Bransford, Brown, and Corking (1999) highlight the importance of involving learners actively in the generation of word meanings. It is important that learners connect words with concepts and with their prior knowledge. Integrating vocabulary development with other aspects of instruction, so that terms are encountered in context, and discussing connections students are making between terms and concepts are effective ways to support students in making needed connections. Through classroom mathematical discourse, students can develop understanding of mathematics terms and of the underlying mathematics relationships. Such discussions can be effectively embedded in other learning activities such as problem solving or mathematical investigations. "The richer and more varied students' experiences related to particular concepts, the more finely detailed and nuanced their understanding of related terms can be expected to be" (Spencer & Guillaume, 2006, p. 208). Barnett-Clark and Ramirez (2004) remind us,

> As teachers, we must learn to carefully choose the language pathways that support mathematical understanding, and simultaneously, we must be alert for language pitfalls that contribute to misunderstandings of mathematical ideas. More specifically, we must learn how to invite, support, and model thoughtful explanation, evaluation, and revision of mathematical ideas using correct mathematical terms and symbols. (p. 56)

Once mathematical terms are understood, their succinct meanings need to be reinforced throughout students' experiences. For many teachers, engaging students in the *Common Core State Standards for Mathematics* (CCSSM) (National Governors Association Center for Best Practices, Council of Chief State School Officers [NGA & CCSSO], 2010) eight standards for mathematical practice represents a departure from their established teaching practices. CCSSM Practice 6, *Attend to precision*, involves students in attending to precision in a variety of ways, including precise use of mathematical language. "Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning…. In the elementary grades, students give carefully formulated explanations to each other" (NGA & CCSSO, 2010, p. 8).

Teachers can use development of precise language as a way to engage students in mathematical practices. While attending to precision and engaging students in mathematical practices such as justifying are both considered desirable teaching pedagogies (NGA & CCSSO, 2010; NCTM, 2014), they need not be considered as separate, unrelated pedagogies. The borders of each can be dissolved as the two pedagogies are used in connection with one another. One aspect of this connection is the use of precise mathematical language to act as a catalyst for engaging students in mathematical justifying.

Mathematical Justification

The practice of teaching and learning via mathematical argument/justification affords a practical context with ongoing opportunities for students to deepen their understanding of mathematics and use mathematical language precisely. In mathematical argumentation, the arguer must support or refute a claim with an argument that links the claim to the underlying mathematical principles and relationships. These ideas must be specified precisely and clearly in order for the argument to be understood and accepted by teacher and peers. The goal of peer acceptance of one's argument presents a powerful motivator for students to learn precise meanings of mathematical terms and to use them appropriately.

Mathematical argumentation is itself an important practice for learning and doing mathematics. It

is a key practice of professional mathematicians and is captured in CCSSM Mathematics Practice 3, *Construct viable arguments and critique the reasoning of others*. When engaged in this practice, students

> make conjectures and build a logical progression of statements to explore the truth of their conjectures….They justify their conclusions, communicate them to others, and respond to the arguments of others…. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments. (NGA & CCSSO, 2010, p. 7)

We view a viable argument as one with a clearly stated claim which is supported by reasoning showing how the claim follows logically from mathematical definitions and ideas that are known and accepted (Adams, Ely, & Yopp, in press). Yopp and Ely (2016) remind us that the argument's viability can only be ascertained when the definitions of the operations and objects in the claim are clear and agreed-upon.

Engaging students in mathematical argumentation benefits their learning in multiple ways. Yopp (2011) suggested that proving, one form of mathematical argument, can be used to understand a mathematical definition. Researchers have found that in mathematics classrooms where students are encouraged to explain and justify their thinking, learners demonstrated both greater achievement and more complex thinking (Boaler & Staples, 2008; Kazemi & Stipek, 2001). Sharing arguments promotes the mathematical understanding of the arguer and of the listening peers. Middle school teachers observed that when making an argument, students must interact with mathematical ideas, determine why a mathematical process works, connect ideas in new ways, refine their own thinking, and develop their mathematical communication skills (Staples, Bartlo, & Thanheiser, 2012). Additionally, student justification allows teachers to discern student thinking (Knuth, 2002) and provides information they can use in making instructional decisions. The teachers in Staples et al.'s (2012) study noticed that students who engaged in justification also developed general learning skills such as perseverance, independence, critical thinking, communication, and the expectation that ideas be supported.

Research Question

Through work with teachers to support students in mathematical reasoning, we have observed many instances of teachers engaging students in discussions of the precise meanings of mathematical terms. In this study we focus on mathematical discourse that centers on the precise meaning of mathematical terms or phrases. We examine the relationship between precise use of mathematical language and student justification. Specifically, we asked, what is the role of discussion of precise mathematical language in supporting student engagement in justification?

Methodology

The findings presented in this proposal are from an NSF-funded research and professional development project that focused on engaging students in mathematical generalization and justification. As part of the larger qualitative study, we examined two fifth grade mathematics lessons that contained class discussions of mathematical language. Both lessons were videotaped and occurred during the first month of the school year in rural school districts in the northwestern United States. While both teachers had an inclination towards focusing on precise mathematical language and justification, one teacher, Ms. K, had gone through two years of professional development as a result of the NSF-funded project. The other teacher, Ms. L, had been involved in a number of unrelated professional development experiences, but had not started work with the NSF-funded project. We selected three episodes from the two lessons in which discussion of mathematical language arose in conjunction with whole-class mathematical justifying. Each episode was analyzed

to determine if and how the discussion of precise language was related to the mathematical justification. The three primary authors discussed the findings regarding the relationship between precise mathematical language and justifying until a consensus was reached.

Findings

We present here three episodes of classroom discussion that illustrate how conversations about precise mathematical language can be used to further student mathematical understandings and lead to student justification. All of the episodes depict a teacher asking students to clarify the mathematical meaning behind terms being used in class and all result in some type of mathematical justifying. The first two episodes are from Ms. K's lesson, and the third is from Ms. L's lesson. Episode one depicts a 15-minute discussion that capitalizes on students' lack of clarity to engage students in justifying why specific fractions are equivalent. Episode two depicts a 15-minute discussion that arose from a student's imprecise use of mathematical language, resulting in a justification of a commonly used generalization (attaching a zero to the end of a whole number when multiplying by 10). Episode three depicts an hour-long discussion that arose from multiple students' use of imprecise mathematical language, resulting in a justification of place value relationships. All names are pseudonyms.

Episode One: Equivalent fractions

In this episode from Ms. K's class, students were presented with a visual pie divided into 8 pieces, 4 of which were shaded, and asked, "What fraction of the pie is shaded?" Multiple students presented answers of 4/8, 2/4, and 1/2. In providing these multiple answers, students implicitly claimed that the three fractions represented the same quantity. Ms. K explicated this claim, writing on the board, "4/8 = 2/4 = 1/2". She then created disequilibrium by stating that the 3 fractions were all different. In making this challenge, Ms. K appealed to the appearance of the fractions as they were written, saying, "4/8 is not the same as 2/4. It's not the same as 1/2. What do I mean by *same*? What does he [student] mean by *same*? Four is not the same as two. … So how is this the same?" Through this challenge, Ms. K prompted students for a justification of why two fractions are equivalent. The opportunity for this justification arose from use of the imprecise and ambiguous term, *same*. While students did not actually use this term, Ms. K introduced it when explicating the students' implicit claim of fraction equivalence. It is apparent that Ms. K's intention during this episode was to highlight the distinction between the appearance of a fraction and the quantity it represents.

At the start of the justifying episode, a student used the phrase *equivalent fractions* as a justification of why the three fractions could be considered the same. While *equivalent* is a more precise term than *same* because it clarifies how exactly the fractions are the same (i.e. in the quantity that they represent, rather than appearance), Ms. K continued to encourage students to unpack the meaning of the word: "So *equivalent* has the root meaning "equal", right? But I've already said to you that 8 is not the same as 4." She continued to press for a more detailed justification by asking how the fractions could be the same. One student offered the explanation that one could divide 4/8 by 2/2 to get 2/4. Another said that all three fractions show the same ratio, saying, "The numerator is half of the denominator." Each additional student justification provided a more complete description of equivalent fractions for the class. Ms. K. continued the lesson as she elaborated on the difference between the appearance of a fraction and the value that it represents: "Four is just a digit. It's a digit. We don't know what its *value* is. … Is four always worth the same amount?" She emphasized her point that digits can represent different quantities depending on their place value by writing 40 and 400 on the board. At the end of the episode, Ms. K connected the mathematical ideas of place value and fractions by highlighting that each fraction could be considered as a division problem, which each yielded the same result.

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Episode Two: Add a Zero for Multiples of Ten

In this episode Ms. K serendipitously capitalized on a student's imprecise use of language to launch a discussion of what it means to "add a zero." She also used the discussion as an opportunity to teach conventions for precise use of mathematical language. In the process of unpacking the meaning of "adding a zero," a mathematical claim about relationships involved in finding the product of 6 and 20 was stated and evidence that could be used to justify the claim was documented. The result was a clear statement that they were not adding a zero; they were multiplying by ten. This discussion also helped students to understand that when discussing mathematics, they were expected to explicitly describe mathematical relationships and meanings of operations.

The episode began when Carl described a method for finding the product of 6 and 20: "I multiplied 6 times 2, which is 12, and then I added zero." Ms. K took the mathematical meaning of his words literally, while ignoring the implied meaning regarding how to write the answer. She wrote $6 \times 2 = 12 + 0 = 12$ and then said, "He is saying the answer to 6×20 is 12. Do you agree? Why not?" Amber said, "Because you don't add a zero. You put a zero behind the 12." Ms. K interpreted this statement broadly, writing 12   0, and said, "That's a zero behind it." She then asked, "Why is Ms. K being so silly here?" Kylee responded that sometimes students put things in the wrong places. The teacher agreed and said, "I don't like it when people say: 6×2 and add a zero. You are not adding a zero. What are you actually adding?"

The students clearly had a habit of describing changes in the way the number visually appears when written rather than the mathematical meaning or relationships involved, and therefore did not understand that Ms. K's intention was to show that they were multiplying by ten, and not adding a zero. Another student said, very precisely, "You are adding a zero to the one's place." Ms. K again took the words literally and wrote: $12 + 0$. She said, "I'm adding a zero to the one's place," and then prompted, "You know what you are doing. I want you to be explicit in your thinking" and asked them to talk with a partner.

Ms. K acknowledged the strategy of first multiplying by two as "brilliant" because "it uses multiplication facts we already know." She asked, "What are we really doing here? What's the difference between 120 and 12?" Anna replied, "Zero." Ms. K asked, "What is the value of that zero?" Zoe explained, "Zeros, when added onto a number, it kind of transforms the number. If you add on two zeros – in the tens and hundreds place, it transforms it to one hundred. Zeros are pretty much kind of magic. Cause without zeros, we would be stuck with pure numbers." This response gave Ms. K the opportunity to make explicit a mathematical convention for her class: that there are reasons for the way things happen, and it is important to uncover and talk about those reasons, and to explain why. She said, "Magic makes it seem like there is some trick to it, and not maybe a reason, and that's not true at all. … I always want you to know why you are using a trick. … So I wanted you to understand that when you say *add a zero* what you are really saying is what?" Liam responded, "I noticed something. It takes ten 12's to make 120." This idea is what Ms. K had been waiting for. She emphasized its importance by writing ten 12's in a column, counting them as she wrote. She also wrote 12×10, and repeated, "We are not adding a zero. Chloe said, "We are multiplying by ten," which ended this episode.

Episode Three: Add a Zero for Place Value

This episode began when multiple groups of Ms. L's students worked on a task and then described a pattern they noticed among a list of numbers (specifically, 3, 30, 300, 3000, 30,000, 300,000). During the class discussion, Bob said, "I was just thinking that there was ones with added zeros." Ms. L repeated his phrase "ones with added zeros," which was used by multiple students prior to the class discussion, and then asked, "Do you think that's really true though? Ones with *added* zeros? Are we adding zeros?" Bob responded, "No, you put zeros on them." Ms. L again repeated Bob's wording and implicitly asked students to explain the meaning behind the words. "You

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put zeros at the end. Okay, but I wonder what that means that we put zeros at the end." She stated, "If we start with three, and we put a zero at the end, we aren't really adding zero, are we?" and then asked the class "What are we doing if we start with three, and we write 30?" As students did not quickly answer, Ms. L encouraged them to describe the meaning of three, to which a few students responded "three ones." She followed by asking what does thirty mean, to which Alex answered "three tens." Ms. L continued the pattern and asked, what does three hundreds mean, and Frank answered "three hundred."

Continuing the justifying activity, Ms. L asked students to discuss how to "get from 3 to 30," clarifying her desire for them to use "multiplication thinking, not addition thinking." Ms. L summarized multiple student responses as she said multiply 3 by 10. Continuing the justification, Ms. L asked how students could get from 30 to 300. Following group discussions, Beth said her group multiplied 30 by 10 to get 300. Ms. L pointed to a previously recorded pattern on the board, and explained that because they said 300 is 3 times 100, they could move the parentheses and say that 300 is 3 times 10 times 10. This move is from (3 x 10) x 10 to 3 x (10 x 10), although Ms. L points rather than saying this explicitly.

As an attempt to encourage understanding of her explanation, Ms. L asked Ted about the solution to 10 times 10, and he answered "100." Another student in Fred's group added that they multiplied 3 by 10 to get 30, and 30 by 10 to get 300, and that one could keep going with that pattern. Ms. L stated that if they multiplied by another 10, it would be 3 times 10 times 10 times 10, and asked students "what is 10 times 10 times 10? " Multiple students responded, "1000," and Ms. L summarized their argument: "Here's our first 10, right? 3 times 10 is 30. And 300, you guys said just add a zero and I said I don't think we can add a zero. So then you said we're multiplying by 10. So we have 300 equals … 3 times 10 times 10. And so you guys are saying each time we multiply it by 10, we have to put in another times 10 each time, don't we?"

Ms. L rephrased her summary: "3 equals 3 times 1. 30 equals 3 times 10. And you guys told me that to get from 3 to 30, I would have to multiply 3 by 10. Then 300, you guys said add a zero, but no, we can't add a zero, so we multiplied 30 by 10 and got 300. And so we said that we could say that that's 3 times 10 times 10. So it's either 30 times 10 or 3 by 100, either way. 3000 is 3 times 10 times 10 times 10. We see a pattern developing. We have one 10 here, two 10's here. And we're not adding tens, we're doing what with our tens?" Students said "Multiplying," and Ms. L responded, "We're multiplying each time by 10. Then, you just said that 30, 000 is 3 times 100 times 100 (a student previously and silently had recorded this on the board). Can he do that? Because 3 times 10 times 10 times 10 times 10 is that the same as saying 3 times 100 times 100? Just for the sake of time, I'm gunna say it is. We will have to talk about that in a couple of days. And then another student said 300,000 is 3 times 10 times 10 times 10 times 10 times 10." A student noted, "Five times." And Ms. L capitalized on the observation: "Five times. Here we had it 4 times; here we had it 3 times; here we had it 2 times; here we had it 1 time."

As time for the class period is running out, Ms. L connected their findings to a place value chart, showing students that each time "you move to the left" you are multiplying by ten, and asked students to think about what happens if you "move to the right." Most of the students responded with "divide" and "by 10" in unison as the lesson ended.

Discussion

Eliminating the borders between the two pedagogical practices of attending to precise language and constructing viable arguments can help improve student engagement in both types of mathematical activity. For instance, the Fraction Episode illustrates how a better understanding of the mathematical meaning behind fraction notation enables students to justify why specific fractions are equivalent. The use of precise language allows for the creation of a viable argument, which subsequently allows students to better understand the appropriate use and importance of precise

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mathematical language. As these episodes illustrate, teachers can capitalize on the imprecise language often used by students as a way to engage them in unpacking mathematical meanings and justifying mathematical properties or algorithms that are often taken for granted, such as the algorithm of appending a zero to the end of number when multiplying by ten. By increasing their own awareness of students' use of imprecise language, teachers can identify opportunities to engage students more frequently in creating viable arguments or justifications. Additionally, attending to the use of precise language develops an understanding of the culture and nature of mathematical language. While many students may be able to use mathematical terms, they need the additional understandings of *how* and *when* to use mathematical terms. For instance, it is important for students to understand the distinctions between the terms *fraction* and *ratio*, and recognize appropriate circumstances for using each.

Beyond simply knowing the correct term, another insight these three episodes provide is that common language can conflict with mathematical language, and that the pedagogical tool of attention to precision allows teachers to distinguish between the two types of language. For instance, when a person in an everyday context says, "add a zero on the end of that number," it is typically understood as the action of appending a zero. However, mathematical accuracy would require one to say "multiply the number by ten." When common language contradicts mathematical language, it creates a conflict within the class and often promotes misunderstandings about the mathematics. Focusing on the correct use of mathematically precise language helps to engage students in mathematical justifying, and the resulting focus on precision will advance students' skill with future justifications. In other words, use of precise language is a requisite skill in order to develop mathematically valid justifications. Therefore, if we want students to be able to justify, we also need them to develop the ability to use precise language when they are in a mathematical context. When mathematical language conflicts with common language, teachers need to explicitly clarify the mathematical language and consistently use it in classes. This will develop students' abilities to think mathematically and use such thinking to outline mathematical arguments at the proper times. Gaining skill in using mathematically precise language will allow students to create arguments and justifications and do so using mathematical language.

The Add a Zero for Place Value episode was an hour-long lesson that developed student understanding of the precise meaning of the use of zero within the place value number system. It is clear that students assumed everyone knows what it means to "add a zero" to the end of a number, but as Ms. L asked them to unpack the meaning behind the common phrase, it became evident that the class lacked the ability to do so. This episode illustrates the difficulty of developing a mathematically accurate interpretation of common language that is used regularly and taken as understood. As conversations about mathematical meanings and the use of precise mathematical language increase in mathematics lessons, the opportunity for students to develop a deeper understanding of mathematical language and relationships increases as well.

All three episodes point to instances where students seemed to know the correct mathematical language because they used commonly understood terms at the correct time; however, when each teacher attended to the precision of the mathematical language, there was clearly more understanding to be developed. In these episodes, this was accomplished through an unpacking of the mathematical language used within the lesson. We recommend that teachers use the pedagogical tools of attention to precision and encouragement of viable arguments and justifications in conjunction with one another to help students develop deeper mathematical understandings as well as to enculturate students into the mathematical community.

Acknowledgments

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LANGUAGE REPERTOIRES FOR MATHEMATICAL AND OTHER DISCOURSES

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We start with the stance that it is important for educators to understand students' language repertoires in relation to characteristically mathematical conceptualizations and processes. The data in our study of grade 3 to 11 students' language repertoires for conjecture led our attention to competing discourses in the classroom and the consideration that the mathematical language repertoires are also repertoires for friendship, competition, etc. In this paper we use a framework for identifying authority structures in mathematics classrooms to focus on formative communication acts, and then we consider how each act might serve a purpose for positioning the students involved in terms of each discourse that we identify as in play.

Keywords: Classroom Discourse, Equity and Diversity, Problem Solving, Instructional Activities and Practices

Introduction

Because mathematics is mediated through and by language, it is important for educators to understand students' language repertoires for mathematical conceptualizations and processes. As part of a large-scale research project focused on identifying specificities of students' language repertoires, especially in contexts of mathematical investigation, we sought to identify connections between their ways of talking about conjecture and what they think their expressions mean.

In this paper, we use one episode from our data to problematize our fundamental research question. Positioning theory helps us understand the way students' communication acts connect to a variety of discourses, including mathematics. There are many discourses enacted in any classroom context. In addition to mathematics, we found mischief, romance, play, hunger, and more at work. In short, we cannot understand students' communication about mathematical processes without understanding that these acts are also part of their repertoires for the other discourses in play. We argue that the connections among the discourses indexed by students' communications may give us insight into what mathematics is for these students.

Positioning and Discourses

Position metaphors are often used to illustrate the way people relate to each other. Van Langenhove and Harré (1999) have described positioning as the ways in which people use action and speech to arrange social structures. According to this theorization, in any interaction, the participants envision known storylines to help them interpret what is happening. These storylines may be conscious or not. They are contested explicitly or implicitly. A powerful aspect of this theory is its radical focus on the immanent, its rejection of the transcendent. In other words, it considers real only that which is present in the interaction and rejects the power of exterior forces.

In an analysis of the way this theory was taken up in mathematics education research, Wagner & Herbel-Eisenmann (2009) noted that exterior forces, such as the discipline of mathematics, may be myths, but they can be taken as real in classroom or other interactions because teachers and others may be viewed as representatives of these exterior forces. The classical triad developed by the progenitors of the theory (e.g., van Langenhove & Harré, 1999, p. 18), which connected a speech act to a storyline and a positioning, was reconfigured by Herbel-Eisenmann, Wagner, Johnson, Suh, & Figueras (2015, p. 194), as shown in Figure 1.
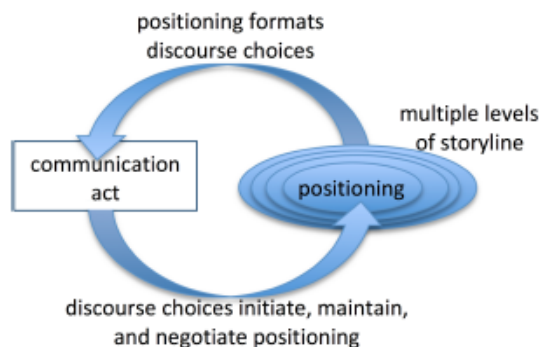
Figure 1. Illustrating positioning. (Herbel-Eisenmann et al., 2015, p. 194).

They layered storylines to emphasize positioning theory's claim that multiple storylines may co-exist in an interaction. And they used arrows to highlight the dynamic interaction between a communication act and the exterior storyline—a communication act initiates, maintains, and negotiates positioning within a storyline, and this positioning formats communication acts. For us, this recursive relationship is reminiscent of Foucault's (1982) description of discourses—"practices that systematically form the objects of which they speak" (p. 52). Thus we will use the term *discourse* instead of *storyline.*

For our analysis in this paper we elaborate the diagram into three dimensions (Figure 2).
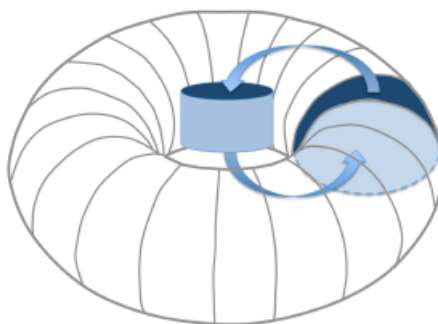

Figure 2. Extending the positioning model into three dimensions.

We are motivated by our data to avoid foregrounding any one discourse. Thus we position a range of discourses in relation to a communication act. Each discourse may be seen as a slice of the torus though the lines of demarcation between discourses would not be so clear as they are in the diagram; the various discourses are interconnected. A communication act appears as a cylinder passing through the torus. The cross section cut of this 3D figure would appear about the same as Figure 1. The communication act (a slice of the cylinder, which would appear as a rectangle in cross-section) interacts with a discourse (a slice of the torus, which would appear as an ellipse). The arrows show how the communication act constructs the discourse while the discourse constructs the communication act.

Connecting communication acts to discourses

Because positioning theory focuses on people's rights and obligations in interactions, we choose a conceptual frame developed in mathematics education that makes distinctions among such structures. Working from a large-scale quantitative analysis of the communication in mathematics classrooms, Wagner & Herbel-Eisenmann (2014) distinguished among four authority relationships. To identify personal authority, one looks for "evidence that someone is following the wishes of another for no explicitly given reason" (p. 875). Linguistic clues include the presence of *I* and *you* in

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the same sentence, exclusive imperatives, closed questions, and choral response. To identify discourse as authority, one looks for "evidence that certain actions must be done where no person/people are identified as demanding this" (p. 875). The strongest linguistic clue is the presence of modal verbs that suggest necessity—e.g., *have to, need to, must*. To identify discursive inevitability, one looks for "evidence that people speak as though they know what will happen without giving reasons why they know" (p. 875). The modal verb *going to* is a strong indicator of this structure. Finally, to identify personal latitude, one looks for "evidence that people are aware they or others are making choices" (p. 875). Linguistic indicators include open questions, inclusive imperatives, and indicators of someone changing their mind—for example, *I was going to, could have*.

We emphasize that we identify these authority relationships on the basis of particular communication acts, and try to avoid reading intention. For example, a student may say or do something because s/he thinks that is what the teacher wants, but we look for communication acts that explicitly indicate this authority structure. The intention may be significant, but our attention to language repertoires compels us to look for the communication acts that underlie or motivate this intention. We take the stance that the sense that a student is doing what a teacher wants comes from one or a series of communication acts that set up this discourse.

In this paper we work through a transcript identifying authority relationships. We focus on instances where these relationships appear to change and on aspects that are resilient to change. At the same time we identify exterior discourses that are referenced (We acknowledge that there are further discourses at play that are never explicitly addressed in the transcript, or that are addressed in a way that we do not recognize). In addition to the mathematics, we identified the following discourses in this particular group's work: school work, hobby/game playing, our research agenda, competitiveness, romance, affect in identity, clothing, friendship, body image, and physical/material resources. We considered referring to these other discourses as "distractions" but realized that this word is subjective; yes, romance may distract from mathematics, but mathematics may also distract from romance. Thus we think of the whole set of discourses, including mathematics, as competing and intertwined discourses. In our analysis, for each significant communication act, we ask how the specificities of that act might serve the person's interests within the various discourses. How is mathematics positioned? How does this act position the friendships? How does it project body images? etc.

We chose this particular group of three grade 10 students (approximately 15-years-old) because they made progress mathematically though their teacher did not have high hopes for them working well together. They were given a page with the following task and some images of cubes and cut up cubes (e.g., Figure 3): "A cube was painted red, and then cut into smaller cubes, 3 x 3 x 3. How many of the small cubes have no red faces? How many have 1 red face? 2 red faces? 3 red faces? 4 red faces? 5? 6? How about a cube cut into 4 x 4 x 4? Or 5 x 5 x 5? Or 10 x 10 x 10? Or n x n x n?" They were also given a set of twenty-seven solid white cubes snapped together in a 3 x 3 x 3 configuration. It was possible for students to pull the set apart and reorganize the cubes but they still had to visualize what sides would be painted and what would not.

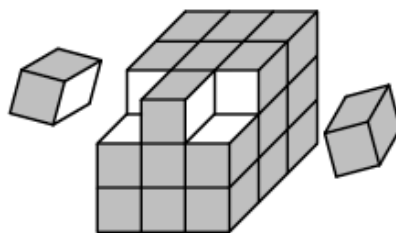Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

Figure 3. Cube task images.

Applying the Framework

The task involved working in a group and working mathematically with an object that could only be visualized. This kind of investigation work invites (perhaps 'requires') students to decide how they will organize their investigation and themselves while doing the investigation. Steven set the tone:

> 3 *Steven*: All right boys, well, let's get to work. "A cube was painted red and cut into smaller cubes, 3 by 3 by 3…"
> 4 *Peter*: I don't have one.
> 5 *Steven*: You don't have a sheet? [turning to a researcher…] Um, excuse me? He don't have a sheet. [turning back to the group] Okay.

This exchange establishes the positioning in the group. Steven's "let's get to work" (turn 3) suggests personal authority, similar to the way a teacher might demand work from students; he told the others what to do and did not tell them why it would be a good idea or why he had the authority to do so. First we consider how this positions mathematics. It appears that mathematics comprises performance at the behest of someone in authority, performance of certain kinds of procedures or processes (involving shape and number in this case, and probably more generally). This authority structure does not align with conventional views of mathematics being a bastion of reason. Steven also indexes the friendship discourse with "All right boys" (turn 3), and positions himself as a leader in this group of friends (however, we were told by the teacher that these boys were not friends). Connecting these discourses, mathematics is positioned as something one can use to establish authority in friendship, and vice-versa—friendship can be used to establish authority in mathematics. The other discourses we identified in this episode were not yet explicit. We pick up the interaction when the group begins to engage with the task:

> 12 *Steven*: twenty-seven have no red on them. Oh wait, no, that's wrong. [*he holds the large cube and points at small cubes when counting*] 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, … oh, fuck.
> 13 *Doug*: No look…
> 14 *Steven*: It's like, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, …
> 15 *Doug*: All this would, here, here, and here [*pointing*].
> 16 *Steven*: Just the outer layer. Just the outsides.
> 17 *Doug*: Yeah all the other.

Steven was in control of the cubes while Doug and Peter watched him. Though it could appear that they were not paying attention, we know they were paying attention because of what they said later. In turns 12 to 17, there are no expressions suggesting personal authority, discourse as authority, or personal latitude. By deduction, the dialogue might suggest discursive inevitability, but we ask whether it fits the description of this category. There are no instances of *going to*, which is the marker identified by Wagner & Herbel-Eisenmann (2014). However, the students are working as though there is a correct answer that they will identify once they have worked enough at the task (which aligns with the description of this category). Linguistically, this is achieved in a number of ways. Steven counts outside of sentence structure (turns 12 and 14). This suggests that there is only one way to count the objects; no one would count them differently. If they had thought there are different ways of counting, they would qualify their counting with, for example, "if we count [this way] we would get […]." Furthermore, Doug's expression, "this would" (turn 15) suggests that he knows what will happen.

We now ask how these communication acts position the students with the discourses we have identified in the entire session. First, because of the orientation we bring to the research (as

mathematics educators) we are attentive to the expression "this would" (turn 15) because it is a way of indicating a generalization. The expression will appear multiple times later in the transcript as part of a clearer generalization. In the case above, the sentence is incomplete so it is difficult to see Doug's generalization. Second, we found no linguistic references to other discourses. There were markers of some of them throughout the interaction, however; Doug's shirt indexed his hobby, the clothing of all three of them suggested that they identify as male, their proximity and body language could relate to friendship discourses, their physical context connected them to school discourses, our presence in the room and the recording devices potentially reminded them of the research discourse, etc. Nevertheless, the grammar of their interaction, specifically the ways their language suggested inevitability within mathematics, eclipsed the other discourses that potentially could draw their intention. Indeed, their attention was drawn to the recording devices, a girl in another group, and other "distractions" at times in this interaction. This idea of inevitability suggests that there is one way of counting, for example, but it also draws attention away from other discourses—it seemed to be inevitable that they would count those blocks in the way presented in the task. As the conversation continued, there is a change away from discursive inevitability:

18 *Steven*: I wonder if you take it apart? Like this? [*As he takes the cubes apart they do not separate as he intended.*] Fuck. All these, the outer, if you look on the inside… these two sides don't, these two sides don't, take that apart, these two don't…

19 *Doug*: Yeah, but also…

20 *Peter*: Hey Steven, can I see the cube for a sec?

21 *Steven*: One sec.

22 *Peter*: Okay.

23 *Doug*: Once you take it apart you're saying all these would have red, but what he's asking is that if you take an individual cube, so this was right here right, and like

24 *Steven*: Oh.

25 *Doug*: This part would have red, this part would have red, this part would have red on it, so those three sides… er… yeah, three sides would have red.

26 *Steven*: So how many of them wouldn't?

27 *Doug*: Only three cubes… one cube

28 *Steven*: If this whole thing was painted red, how many of these would have no—that's what he's asking, how many of these? Okay, so it would go like this, one, two, oh yeah, so it would be these sides right? It would be this side, this side, this side, this side, and this side. So yeah, it would be like, how many of them? It would be this… no, this side wouldn't have…

29 *Doug*: No, that side wouldn't

30 *Steven*: All right, this side wouldn't and this side wouldn't.

Starting in turn 18, Steven's "I wonder" demonstrated an awareness that he can think about the cube structure in different ways. This suggests personal latitude because he expressed awareness of his choice about ways of seeing. This change in authority structure seems to be contagious. Doug's response "Yeah, but also" (turn 19) references personal latitude too. He agreed with Steven and identified that something could be added. And Peter then asked for the cube (line 20), indicating awareness that there is choice about who holds the cube. Steven's response in line 21 then indicates awareness that he can choose whether to honour Peter's request, and then Peter complied, likewise acknowledging that he can agree or disagree with Steven's request for a little more time. This led to the beginning of a conjecture (though not clearly worded) from Steven: "If you take…" (turn 23). Turns 24 to 30 are replete with the word *would*, which we indicated earlier as a marker of discursive inevitability. However, in this case the word has different meaning, specifically a mathematical justification, because it is paired with the subordinate conjunction *if*.

This flurry of personal latitude may be an example of interlocutors picking up discourse patterns from each other. Applied linguistics literature has identified this phenomenon in terms of people picking up each other's words, which is a little different than picking up an authority structure. Positioning theory discusses the possibility of picking up structure from others, as the theory describes (using the language of first-order, second-order, and third-order positioning) how people follow or resist positioning established by others. Nevertheless, as we ask what happened to switch the dialogue from a stream of discursive inevitability to this flurry of personal latitude, we note the possibility that the authority structure was part of the students' repertoires the whole time just waiting to be triggered. It appeared to be triggered by Steven's insight, but one could say this too was triggered by the task inviting a certain kind of thinking. It could also have been triggered by certain classroom norms that honour this kind of interaction. There are lots of possibilities.

Steven's "If you take" (turn 23) is the beginning of a generalization in which he imagined stripping the cube of the outside layer of small cubes, which would leave one bare cube in the middle. The conjunction *if* is strongly associated with conjectures and generalizations throughout the interaction of this group especially in its pairing with *would*: "If …, then … would…" though the word *then* tended to be omitted. This and the other ways of expressing the students' choices to see the cube in different ways supports the development of their mathematical insights, and thus constructs mathematics as a discipline in which people develop insight by making choices about how to see things. These communication acts that index personal latitude can make similar impacts on the other discourses though it is not straightforward to map them because this part of the interaction was focused on mathematics. Nevertheless, the former hierarchy within the friendship discourse, in which Steven had the role of telling his "boys" what to do, was opened up to allow for the others to exercise agency. The finite physical resources were a factor in this negotiation of the object of attention though it is clear from their dialogue to come that both Doug and Peter were manipulating the cube in their imagination as they watched Steven manipulate the physical cube.

Discussion

Our research interest was initially on student language repertoires for mathematics. Our data has led us to connect this to some of our previous work that recognized and identified a range of discourses at work in mathematics classroom contexts (c.f. Andersson, 2011). In reflection we return to the repertoires. We remember that whatever repertoires people have for mathematical thinking and action, these repertoires serve purposes in other discourses as well. In fact, participation in other discourses that share linguistic resources with mathematics informs one's mathematical thinking. And vice-versa; participation in mathematics informs one's thinking and action in other discourses that share linguistic resources.

Thus we argue that mathematics teachers (with the support of mathematics education researchers) would be well-served to develop further understanding of how the characteristically mathematical expressions appear in students' discourses outside the classroom. Such insight would inform the ideas about mathematics the students would have when hearing and using the expressions in mathematics. For instance, two of the students in the focus interaction in this paper were "gamers." We are not very familiar with gamer culture, but we know that participants are in constant interaction with each other, and they are confronted with a continuous and fast-paced stream of choices that impact their progress in the game. Thus the students in our focus interaction would be using the "If … then …would …" construction in their gaming just as they used it in their mathematics. And we wonder how these two environments connect to each other for these students.

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PROFILES OF RESPONSIVENESS IN MIDDLE GRADES MATHEMATICS CLASSROOMS

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In this paper we consider how mathematics instruction that values, attends to, and builds on students' mathematical ideas is realized through discourse. We describe interactions that build on students' thinking and in which students help to determine the direction of mathematics lessons as responsive. Using a framework we developed to characterize the responsiveness of mathematics interactions, we report the variation in responsiveness across seven middle grades classrooms by describing (a) students' mathematical contributions, (b) the moves teachers enact in response to these contributions, and (c) how these two components interact. We found that there are multiple ways to be responsive to student thinking.

Keywords: Classroom Discourse, Instructional Activities and Practices

In this paper we consider how mathematics instruction that values, attends to and builds on students' mathematical ideas is realized through discourse. Discourse by its nature is responsive and relational (Bahktin, 1986; Halliday, 1978). Our goal is to consider a particular feature of discourse rooted in Bakhtin's notion of dialogism and Halliday's interpersonal metafunction—what we have termed responsiveness. We describe classroom interactions that build on students' thinking and in which students help to determine the direction of mathematics lessons as responsive. *Responsiveness* to students' mathematical thinking is a characteristic of interactions wherein students' mathematical ideas are present, valued, attended to, and taken up as the basis for instruction. Interactions can be more and less responsive, and in this proposal we document multiple profiles of responsiveness across middle grades mathematics classrooms. Our proposal addresses the conference theme of *Questioning Borders* by addressing issues of access and participation. In particular, the ways in which conversants are responsive to each other allows them to participate in certain ways, to take on different roles and identities, and ultimately affects how classroom participants engage with the mathematics at hand.

Theoretical Framework & Literature Review

To understand another is a responsive act. One must engage with another's idea and respond to it in order to understand it—though the manner and quality of engagement can vary widely. Bakhtin (1986) explains it as follows: "The fact is that when the listener perceives and understands the meaning (the language meaning) of speech, he simultaneously takes an active, responsive attitude toward it. He either agrees or disagrees with it (completely or partially), augments it, applies it, prepares it for execution, and so on" (p. 68). If understanding is responsive, then so too is speaking (and other forms of communication). In his discussion of speech genres Bakhtin says, "Utterances are not indifferent to one another, and are not self-sufficient … Every utterance must be regarded primarily as a *response* to preceding utterances" (p. 91). We focus on responsiveness because participating in mathematical discourse is inherently a responsive act, for both listeners and speakers. Specifically, we are interested in the degree of responsiveness to students' *mathematical ideas* within classroom settings and the extent to which utterances mutually acknowledge, take up, and reflect an awareness of student thinking. Below we share a brief overview of research related to responsiveness—primarily studies that focus on student thinking—and synthesize findings in order to situate our study.

Broadly speaking, research related to responsiveness can be categorized into two main types of

studies: (a) descriptive studies of classrooms in which students' mathematical thinking is either already prevalent or is developed within a classroom over time, and (b) studies identifying features of instruction that are positively related to mathematical proficiency. The first group of studies takes as a given the desirability and effectiveness of instruction that incorporates student thinking and considers what this type of instruction looks like. Some studies look at specific teacher moves such as a probing sequence of questions (Franke et al., 2009) or a reflective toss (vanZee & Minstrell, 1997). Other studies develop broader frameworks to describe the classroom contexts and features of instruction that influence or constrain whether, how, and in what ways teachers explore student ideas in their teaching. For example, Leatham, Peterson, Stockero, and Van Zoest (2015) developed a framework to identify instances when it might be productive to pursue students' mathematical ideas in-the-moment. Their focus is how to identify a particular type of student contribution—a *M*athematically Significant Pedagogical *O*pportunity to build on *S*tudent *T*hinking or *MOST*. Additionally, research on the construct of teacher noticing has explored requisite knowledge and skills teachers need to be responsive to student thinking. In particular, Jacobs, Lamb, and Philipp (2010) developed a framework for teacher noticing comprised of three interrelated components that has students' mathematical thinking as its foundation. Before a teacher can incorporate student thinking into instruction, she must first *attend* to student ideas and *interpret* their significance before *deciding how to respond*.

The second group of studies identifies features of instruction related to responsiveness that are positively related to improved mathematical proficiency. Many of these studies use quantitative tools to provide evidence that the presence of student thinking is an effective feature of instruction based on a positive relationship between student thinking and outcome measures such as achievement scores, problem solving, and improved dispositions toward mathematics (Carpenter et al., 1989; Ing, et al., 2015; Nystrand et al., 1997). For example, in their work investigating what practices 'press' students for conceptual mathematical thinking, Kazemi and Stipek (2001) found that press for conceptual understanding was positively correlated with students' understanding of fractions. They identified features of discourse that were present in high press classrooms including engaging students in explaining, justifying, verifying, and arguing about their own and their peers' thinking.

Across this literature we see a vision of mathematics instruction that values students' mathematical ideas and seeks to incorporate those ideas into instruction in productive and meaningful ways. Some of the research focuses on discursive moves that can be enacted in-the-moment. Other research considers either how to develop classrooms that value students' thinking or the skills needed to respond to students' thinking. Moreover, researchers have indicated that this type of instruction is not commonplace and is challenging to enact (Pimm, 1987). Our goal in this proposal is to develop a framework to characterize responsiveness in mathematics classrooms and use it to describe the variation in responsiveness across classrooms. The research question guiding our study is, In what ways are middle grades mathematics classrooms responsive to students' mathematical ideas during whole-class discussions?

Methods

Participants and Data Collection

The data in this report are part of a larger study investigating characteristics of productive mathematics discourse in grades 5–7. Participants include teachers and students in seven classrooms across three U.S. states. Participating teachers were recommended by district personnel, faculty researchers, and professional development providers based on the teacher's reputation for using problem solving and discussion regularly during instruction. Our participants were five fifth-grade teachers, one sixth-grade teacher, and one seventh-grade teacher. All were certified in either elementary or middles grades education with 7 to 30 years of teaching experience. For each teacher,

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we videorecorded and transcribed an introductory lesson on fractions. We chose fractions because it is an important topic and spans the middle grades required content. Lessons were filmed at various points throughout the school year based on when fractions were introduced and ranged in length from 40 to 95 minutes. In our analysis, we used both the video recordings and transcripts, which allowed us access to gestures, student written work, and other non-verbal communication in the classroom.

The Development of a Coding Framework for Responsiveness

We developed a coding framework for responsiveness in whole-class mathematics discussions (Przybyla-Kuchek, Hardison, Bishop, 2015) using the constant-comparative method. The unit of analysis is a segment, which we define as a series of turns of talk with a common focus (e.g., activity or strategy). Our framework comprises two components: (1) students' mathematical contributions and (2) the moves teachers enact in response to these contributions (Figure 1). In this section, we describe the ordered levels for each component of our framework.

		Teacher Moves		
		Low	Medium	High
Student Contributions	None — Minimal			
	Considerable			
	Substantive			

Figure 1. Whole-class responsiveness framework.

Student contributions. Our framework includes four levels of students' mathematical contributions: none, minimal, considerable, and substantive. We define a segment as *None* if it contains no mathematical student contributions (e.g., teacher monologue). *Minimal* segments are dominated by students performing routine calculations, recalling facts, and providing short responses to known-information questions. *Considerable* segments are those in which students share their strategies or other mathematical ideas without justification. Unlike minimal segments, considerable segments contain evidence that students have opportunities to make sense of mathematical content and to share their ideas. *Substantive* segments, like considerable segments, are characterized by students discussing their mathematical ideas; however, in substantive segments, student contributions also include providing justifications, making generalizations, or participating in mathematical argumentation. When considering these elements of students' contributions, we focus on the structure of these contributions rather than on correctness (from our own perspectives). A segment may be characterized by substantive student contributions even if it contains individual turns of talk that might be described as minimal or considerable.

Teacher moves. Our framework includes three levels of teacher moves that reflect the extent to which students' mathematical contributions are made public, taken up, and serve as the basis for instruction: low, medium, and high. In *Low* segments, teachers do not use students' mathematical contributions as the foundation for instruction; low teacher moves include, brushing-off, evaluating, and not reacting to students' contributions. In *Medium* segments, teachers focus on (a) understanding and highlighting individual students' thinking by revoicing student ideas or asking probing questions, or (b) asking classmates to engage momentarily with particular student ideas by asking other students to correct, evaluate, or indicate whether their thinking aligns with another student (e.g., "Who used the same strategy?"). However, teachers do not focus simultaneously on both (a) and (b). In *High* segments, teachers simultaneously focus on student thinking and explicitly direct students to engage significantly with the mathematical ideas of others. High moves include requesting comparisons across student contributions, taking up a student-posed problem as a whole-class activity, asking

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students to restate or apply another's strategy, and inviting students to ask questions of their peers. As with student contributions, teacher moves are coded holistically at the segment level.

Analysis

Prior to coding each introductory fraction lesson, a member of the research group watched the videorecording and partitioned the lesson transcript into segments. We did not code segments consisting of entirely nonmathematical content (e.g., discussions of norms, transitional time). At least two members of the research group coded each lesson independently using the videorecording and segmented transcript. Each segment was first assigned a student contribution code. Segments without mathematical student contributions (i.e., those coded as none) were not assigned a teacher moves code as there were no student ideas to which teachers could respond. Segments containing mathematical student contributions (i.e., those coded as minimal, considerable, or substantive) were also assigned one of the three teacher moves codes. Sets of independent codes were compared for each lesson, and coding discrepancies were discussed until the coders achieved consensus on a final set of codes for each lesson. In summary, each segment was characterized by a combination of codes corresponding to exactly one of the ten empty cells depicted in Figure 1; we refer to these ten coding combinations as compound codes.

Because lessons varied in terms of minutes of whole-class instruction and segments were defined by shifts in common focus, there was variation in both the number of segments per lesson and number of seconds per segment. Thus, to investigate trends and variation in responsiveness across lessons and classrooms, we weighted each segment's student contribution, teacher move, and corresponding compound code (e.g., considerable–low) according to the instructional time in seconds corresponding to each segment. We then calculated the percentage of whole-class mathematical discussion time accounted for by the respective code.

Illustrating the Framework with Excerpts

In this section we discuss three excerpts from the lessons we analyzed to illustrate different levels of student contributions and teacher moves described in the framework above (See Figure 2). In Excerpt 1, Teacher MA's 6th graders are exploring fractions using a number line on the whiteboard. After a student subdivides the interval from zero to one into 24 parts, Teacher MA asks her students about the representation. The student contributions provide little evidence of students sharing their ideas and consist entirely of short responses to known-information questions wherein the students are attempting to match particular responses predetermined by the teacher. Thus, this segment is characterized by minimal student contributions. For teacher moves, Teacher MA implicitly evaluates students by echoing students' responses and recording them on the whiteboard. When Tucker's response, "three," breaks from the format of the other student responses, Teacher MA corrects his response by providing additional information, "three twenty fourths." Thus, Excerpt 1 illustrates a segment characterized by low teacher moves. Excerpt 2 occurs in Teacher EC's 5th grade classroom, where Leena presents her strategy for determining how much candy each child would receive if five children shared eight candy bars. The student contributions provide evidence of students sharing their mathematical ideas and making sense of mathematical content. While Leena's description contains a hint of justification (e.g., "…*since* there are five students, I split it into five…"), the collective student contributions are best characterized as strategy sharing. By requesting that another student restate Leena's strategy, Teacher EC simultaneously focuses on Leena's thinking and asks other students to engage with her thinking in a nontrivial manner. Throughout the interaction with Jordan, Teacher EC asks additional questions that engage Jordan with Leena's strategy. Excerpt 2 illustrates a segment characterized by high teacher moves.

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Excerpt 1: Minimal-Low	Excerpt 2: Considerable-High	Excerpt 3: Substantive-Medium
Tchr MA: So, Mitchell, what's this first line represent?	*Leena:* I drew eight candy bars and since there are five students, I split it into five, so there's fifths. And then I colored in each color equals one person. So there's one fifth from every candy bar and all together that equals eight fifths and turned into a mixed number that's one and three fifths.	*Tchr LE:* Mia said she's adding an eighth here and an eighth here and she got two sixteenths. Right Mia?
Mitchell: One twenty fourth. [As students respond, Teacher MA writes labels on the number line.]		[Mia nods]
		Tchr LE: Okay. Um, Jack.
Tchr MA: One twenty fourth. Victoria, the second line.		*Jack:* I always, uh, thought that you couldn't add up the denominators.
		S48: Yeah
Victoria: Two twenty fourths.		*S49:* Yeah
Tchr MA: Um, Tucker, what's the next line? [overtalk]	*Tchr EC:* Very nice, very nice. So you broke each, oh, can someone explain what she just said? What did she just tell us? Anyone can put it in their own words. What did Leena just tell us? Can someone put it in their own words? Jordan? Want to give it a shot?	*Tchr LE:* Why not?
		Jack: Uh, because it's
Tucker: Oh yes. [overtalk] Uh, three?		*Jonathan:* The number would. The number would get lower.
Tchr MA: Three twenty fourths. Manny?		*Jack:* Oh yeah. The – it would get smaller. Because
Manny: Four twenty fourths.		*S50:* The fraction
Tchr MA: Ida, the next line.	*Jordan:* Um	*Tchr LE:* What it? What is it?
Ida: Five twenty fourths.	*Tchr EC:* I'll help you if you get stuck. Go ahead. What was the first thing that Leena did?	*Jack:* The denominator. Like the whole number.
Tchr MA: Five twenty fourths. Uh, Nancy the next line.		*Fiona:* Fraction (inaudible) smaller.
	Jordan: She drawed the eight candy bars.	*Tchr LE:* But how does that relate to my picture? Well two sixteenths is the same as one eighth. Does – does my fraction get smaller?
Nancy: Six twenty fourths.		
Tchr MA: Cam.	*Tchr EC:* She drew eight candy bars right. Okay, continue.	*Class:* No.
Cam: Seven twenty fourths.	*Jordan:* And then she	*Joseph:* When you add eighths, it wouldn't go up. It would – you would go to a whole number so you wouldn't add – you would just like add, um, the – the top. You wouldn't add the bottom cause then you would never get a whole number.
Tchr MA: Judy.	*Tchr EC:* So she drew the four, eight candy bars and then what'd she do?	
Judy: Eight twenty fourths.		
Tchr MA: Ron.	*Jordan:* And she took, and then she got, then she like all	
Ron: Nine twenty four.	*Jordan:* She divided them into five groups.	
Tchr MA: Gloria.		
Gloria: Ten twenty fourths.	*Tchr EC:* She divided them into five, groups or five pieces, yeah.	*Tchr LE:* What do you mean I'd never get the whole number?
Tchr MA: Claire.		*Joseph:* Cause it would just keep adding on.
Claire: Eleven twenty fourths.	*Jordan:* And then she gave each kid equal amounts but, um,	*S51:* Yeah the denominator would [overtalking]
Tchr MA: Lily.		
Lily: Twelve twenty fourths.	*Tchr EC:* How much did she give each kid?	*Jonathan:* Yeah it would just keep getting lower and lower and lower. [overtalking]
Tchr MA: Luke.	*Jordan:* An equal amount.	
Luke: Thirteen twenty fourths.	*Teacher EC:* An equal amount, okay. And what was the equal amount that she gave to each person?	*Joseph:* The denominator would keep getting. [ovetalking]
Tchr MA: Uh, Jay.		*Gracie:* number but actually it would be getting smaller [overtalking]
Jay: Uh, fourteen twenty fourths.		*Tchr LE:* Oh. If you kept adding the denominators then it would just – you'd never reach a whole.
Alex: Fifteen twenty fourths	*Jordan:* Eight fifths.	

Figure 2. Three excerpts illustrating the responsiveness framework.

In Excerpt 3, Teacher LE asks 5th graders to write equations related to their solutions for an equal sharing problem. When Teacher LE asks students to share their equations, Mia asserts that one-eighth plus one-eighth is two-sixteenths. The students in this excerpt participate in mathematical argumentation as they try to determine the validity of Mia's assertion. Jack generalizes the situation

to adding any denominators rather than considering only the particular equality in question. Throughout the segment, students collectively develop a fairly sophisticated informal proof by contradiction. Consequently, excerpt 3 is characterized by substantial student contributions. Teacher LE initially revoices Mia's claim. Her subsequent moves are predominantly probing questions rooted in understanding particular students' contributions (e.g., "What do you mean I'd never get the whole number?"). Although students engage with others' ideas throughout the excerpt, Teacher LE's moves do not direct students to engage with the ideas of others. As such, Excerpt 3 illustrates a segment characterized by medium teacher moves.

Findings and Implications

We now describe the findings from our analysis of the responsiveness of mathematics classroom discourse. We consider general trends across classrooms, variability in responsiveness both across and within classrooms, and the ways in which student contributions and teacher moves interacted in our data set. In some of the participating classrooms, the majority of the lesson was spent engaging in whole-class mathematics discussions, whereas in other classrooms students spent significant amounts of time in small groups or doing individual work. Across our classrooms, the percent of time spent in whole-class mathematics discussions ranged from 23 to 74% (mean of 49%). The analyses that follow are *only* for time spent in whole-class discussion. Figure 3 displays the distribution of the student contributions and teacher moves as a percentage of time spent during whole-class mathematics discussions in the different categories of our framework.
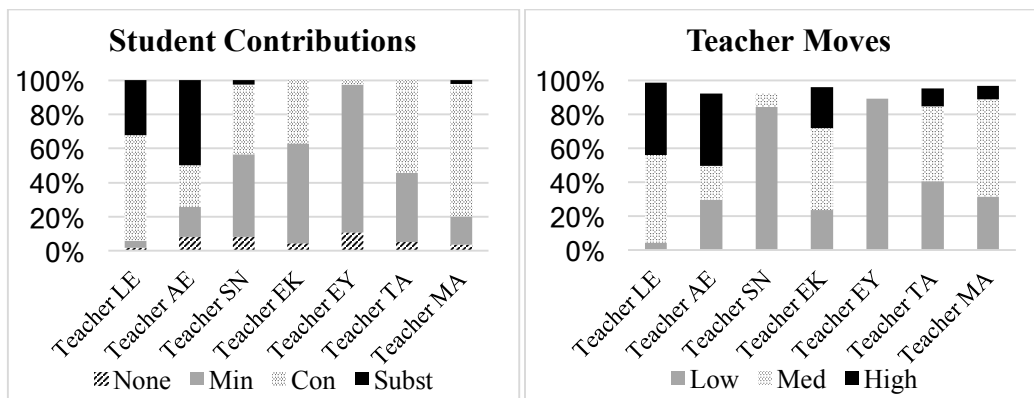

Figure 3. Percentage of time spent across the responsiveness framework categories.

We found that whole-class discussions were not dominated by teacher monologues, but that over 90% of the time students made mathematical contributions during whole-class discussions. In all but one of the classrooms, more than one-third of the time students made substantive or considerable contributions; and in four classrooms, over half the time students made substantive or considerable contributions. In most classrooms, students had opportunities to solve problems and explain, justify, or generalize their thinking during whole-class discussions. Moreover, in five of the classrooms, more than half of the teacher moves were medium or high level. These teachers not only created opportunities for students to engage with mathematics, but in many of the whole-class discussions they focused on and took up students' mathematical contributions. In addition to analyzing the student contributions and teacher moves independently, we also considered how our two framework components worked together during whole class discussions. Teacher AE's responsiveness profile is seen in Table 1.

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Table 1: Responsiveness Profile for Teacher AE based on percent of time during whole-class discussions

			Teacher Moves			
			Low	Medium	High	*Sums*
Student Contributions	None	Minimal	17.8%	0%	0%	*17.8%*
	8.1%	Considerable	0%	2.5%	21.7%	*24.2%*
		Substantive	7.5%	17.4%	20.8%	*45.8%*
		Sums	*25.3%*	*20%*	*42.5%*	

Note that the most common compound code in Teacher AE's classroom was a considerable student contribution paired with high-level teacher moves, occurring just over one-fifth of the time during whole-class discussions. The second most common combination was substantive student contributions paired with high-level teacher moves, also occurring about one-fifth of the time. Thus, in roughly 40% of the time spent in whole-class discussion, Teacher AE supported students' engagement with their classmates' mathematical ideas in order to make sense of, explain, justify, critique, exemplify, or generalize. However, we also see a large percent of minimal-low interactions and segments of no student contributions (i.e., None) in this classroom.

After creating a similar table for each teacher, we found that for four of the classrooms, over 50% of the time spent in whole-class discussions involved considerable or substantive student contributions and medium or high teacher moves (see the shaded cells in Table 1). Each of these four combinations requires students to engage in important mathematical activities and uses the resultant student ideas as the basis for instruction. We noticed that in *all* classrooms, time was spent during whole-class conversation in minimal–low interactions (ranging from 4.4% to 87% of the time). This suggests to us that interactions at the lower levels of our framework are not necessarily negative, but that they play a role in whole-class discussion. However, we believe it is problematic if the majority of time spent in whole-class discussions falls into this category.

We also found that the responsiveness of classroom interactions varied. For example, in Figure 3 we see that the proportion of time teachers responded with high-level moves ranged from 0 to 46.5%, and the proportion of time teachers responded with low-level moves ranged from 0 to 89%. There was also large variability in student contributions with ranges from 3 to 78% and 0 to 46%, respectively, for considerable and substantive student contributions. Moreover, five teachers enacted teacher moves at all levels of the framework and four classrooms had student contributions at all levels suggesting that not only is variation in responsiveness present across classrooms but also within classrooms.

In summary, our data indicate that the kinds of instruction advocated for in existing literature is possible (e.g., Franke et al., 2009; Kazemi & Stipek, 2001; NCTM, 2014). In the interactions we analyzed we found that, for large portions of whole-class discussions, student ideas can be used to drive instruction and that there are multiple ways to be responsive to student thinking. We also found that mathematical interactions were not always in the highest categories for teacher moves and student contributions. Thus, at times, it seems appropriate and necessary to have teacher moves and student contributions at the lower levels of the framework. Additionally, our coding framework was able to adequately capture variation in responsiveness in middle grades mathematics classrooms. Given the variation present in our data, we encourage a variety of student contributions and teacher moves during whole-class discussions wherein a significant proportion of whole-class discussions are characterized by considerable or substantive student contributions and medium or high teacher moves.

In this paper we considered one aspect of mathematics classroom discourse, responsiveness to students' mathematical ideas, but acknowledge that this focus provides a narrow view of mathematics classroom discourse. We believe that there are other important discursive features that can and should be analyzed, and which would showcase our participating teachers differently. In the future we hope to apply our framework to additional lessons in our data and incorporate analyses of other aspects of classroom discourse.

Acknowledgments

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QUESTIONS OF TRUTH: ETHICAL AND MORAL WANDERINGS IN MIDDLE GRADES MATHEMATICS CLASSROOMS AND RESEARCH

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This paper describes two researchers engagement with two teachers as they taught a middle grades mathematics course, Current Events Math, in a large urban school district. The researchers share bits of data and their ethical entanglements as they entered into the site to find the truth about what works in middle grades mathematics classrooms only to realize that truth cannot be found through research. They then grappled with the question of the purpose of research and their roles as researchers in the school and the academy.

Keywords: Research Methods, Middle School Education, Equity and Diversity, Curriculum

Entanglements (Context/Site)

Suppose you were prompted to answer this question in the midst of the 2014 Ebola epidemic: What was the worst Ebola outbreak in history? How might you answer the question? What might you ask yourself? What tools might you use to determine the truth about the question? Middle school students in a charter school in a large urban school district grappled with this question and asked things like *What is worst? How do we measure worst? From whose perspective?* We too—researchers and teachers—found ourselves grappling with these and other questions: What is the interplay between the ethics and the mathematics? How do we raise and attempt to answer ethical questions in the current educational and political environment? What is the relationship between mathematics, truth, and ethics?

This study explores two middle grades mathematics teachers' concepts of ethics and truth as they engaged in a 12-week course. The course, *Current Events Math,* positioned students as researchers who studied current events and used mathematics to consider those events. Observations and interviews were conducted during the 2015-2016 school year and focused on the teachers' experiences teaching the course during 2014-2015 and 2015-2016. In this paper we consider multiple and overlapping layers of ethics: ethics of the researcher in the field, ethics of the teacher towards her students, ethics of the citizen in a community, and ethics of representation in the media.

Theoretical Perspectives/Ways of Seeing/Knowing

This study was composed through interactions between a mathematics educator, a doctoral student and former mathematics teacher, and two practicing mathematics teachers. The format and content of this study are informed by poststructural and post-qualitative perspectives. Though traditional empirical studies have their place in mathematics education, and there have been significant strides in the scope of qualitative research, some of "these versions of knowing express the sense that our inherited ideas have tended to underplay multiplicity, complexity and cultural specificities, none fully captures the postmodern analytical edge that invites a less certain space for research, pedagogy and practice" (Walshaw, 2004, p. 4).

We believe that there are multiple perspectives and versions of truth and that our ethical charge is not to attempt to boil down our impressions and data into one ultimate understanding or truth. Instead, we agree with Neyland that, "the primary ethical domain is not monotonous, regular or predictable; it is shot through with uncertainty and contradiction and cannot avoid ambiguity" (2004, p. 61). Mathematics is often given privilege as both true and unbiased, so it is particularly important to us that ambiguity is recognized within the field of mathematics education. Neyland (2004) refers to mathematics education as "paradigm case subject" (p. 62) in the postmodern ethical agenda

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because "it is the curriculum subject that can be used to make the strongest case against the project of modernity in education more generally" (p.62). Neyland further asserts that "mathematics educators have the urgent task of undertaking a postmodern re-enchantment of mathematics and a postmodern restoration of the primacy of the direct relationship of responsibility between teachers and students" (p.63). The course, *Current Events Math,* is built upon the foundation of responsibility between teachers and students, which for these particular teachers also includes an outlook towards equity in the world. In considering equity, the students and teachers had to look closely at the numbers that are used to describe and represent events that intersect the lives of students and teachers.

Although numbers are largely seen for their face value by academics and society at large and mathematics is assumed to be objective and unbiased, numbers are often complex representations that are constructed and composed. Because of the complexity of their construction and their sometimes misaligned or ineffective representation, numbers can function to distance decision makers from moral and ethical decisions. In the mathematics course, the teachers and students questioned the truth(s) that different numbers told. In our research, we do not rely on numbers as evidence of a valid and reliable study. Instead, this study brings together data pieces that begin to deconstruct the neutrality of mathematics and numbers.

Entanglements with "data" (methods)

The present study includes "data" from teacher interviews, classroom observations/field notes, photo elicitation, researcher journals, and student and teacher created documents. In considering these documents, the authors viewed them as co-constructed by the authors, participants, school, and students. These documents, from transcribed interviews to student journal entries, do not have single authors or sites of production. The following quotes speak to our views on data and representation: "a fieldnote fragment or video image – starts to glimmer, gathering our attention" (MacLure, 2010, p. 282), "always a body of statements to consider in which the individual words and sentences merely slumber" (Prior, 2003, p. 113), and

> The aim is to 'make materially visible the structure of representation as a trace of temporality and exchange, the fragments as mementos, as ''presents'' re-presented in the ongoing process of assemblage, of stitching in and tearing out (Mitchell, 1994, p. 419 as quoted in Radley, Hodgetts, & Cullen, 2005, p. 278).

The researchers used writing as inquiry following Richardson and St. Pierre (2005), "but they were always already in my mind and body, and they cropped up unexpectedly and fittingly in my writing— figurative, fleeting data, that were excessive and out-of-category. My point here is that these data might have escaped entirely if I had not written; they were collected only in the writing" (p. 970). Through writing as inquiry, the researchers troubled their identities as researchers, former teachers and mathematics educators while acknowledging the complexity of their identities as women, mothers, and friends. Through these intersecting viewpoints, "data is fluid, a chameleon, able to take different "shades" of meaning based on the perspective of the researcher" (Koro-Ljungberg, 2015, p. 47).

As they navigated this research site with colleagues and friends, "ethics explodes anew in every circumstance, demands a specific reinscription, and hounds praxis unmercifully" (St. Pierre, 1997, p. 176). One researcher wrote in her journal, "So, I am here with so much uncertainty and also some confidence in that uncertainty is productive. I am asking myself, "How do I do representation knowing that I can never quite get it right?" (Pillow, 2003, p. 176).

From interview probes, to classroom interactions and hallway encounters, the researchers questioned the ethics of their choices. In reflecting on a transcription, Susan wrote in her journal,

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> I wonder if I should have probed when she said, "If they just knew me." I paused. I thought about it. I think my reasons for moving on were two fold. One I was hesitant to get too personal on the first meeting and two I was worried about staying on topic and getting through the interview. I think that the first reason might be legitimate; the second in retrospect seems counterproductive. This adherence to the interview guide, is it productive for me? I do not think it is. Here the participant was offering me something emotional and I turned back to content, to math. I shut down this person who is saying she feels unseen. I didn't see her or wasn't comfortable seeing her. Ethically, I signaled that her story wasn't important in the research. How do I come back around to this? I cannot right the wrong, but can I move forward in a way that is productive.

Throughout the research, the researchers began to "see" what they hadn't been seeing. The classroom observations and interviews left the researchers entangled with the concepts and questions of truth and representation and feeling the persistent tug of positivism at their sleeve. Aren't numbers, data, facts materialized manipulations of the cognitive processes involved in measurement? Numbers are "arrested 'moments' of measurement captured through technical decisions" (adapted from Knowles, 2006, p. 512). The research questions became questions about the research. How do we use mathematics/research to produce truths? How do we deconstruct "truths" created through mathematics/research that are dangerous or destructive?

Assemblage(s) (Results/Findings)

During 2014-2015, the students researched Ebola, Michael Brown, gender imbalance and violence towards women in the gaming industry, and gay marriage. In the 2015-2016 school year, topics included Rand Paul, air pollution in China, New Horizons, the European refugee crisis, head injuries in the NFL, and the use of sugar in food. Following is an assemblage of data pieces stitched together by the researchers, some conversations and observations related to these topics and some conversations that got at the larger goals and questions around this course. As mentioned above, one interesting thread that wove through the data was the idea of truth.

Truth from Experience

The students engaged in a study of concussions in professional football. In this unit the students collected and analyzed data and statistics about the likelihood of concussion for particular positions. The working groups in the classrooms were intentionally constructed with one football player in each group. As the students did their analysis there was tension between the *truth* presented by the numbers and the *truth(s)* brought by the group members founded in their experiences on the football field. Is truth running full speed ahead, hearing your breath in your own ears and feeling its moisture on your face, and then a bone crunching hit from the side and the sudden scent of grass and mud? Or is this truth: cornerbacks suffered 10% of the total concussions reported by the NFL in 2013 (Breslow, 2014)? Which is more valid, reliable, and believable? Which one *counts*? How might these truths influence the students' belief in a number?

Truth, Bias, and Prejudice

Truth(s) arose out of biases and numbers and new truths were created through numbers to undo prejudice. As the class researched the shooting of Michael Brown in Ferguson and the subsequent protests, mathematics helped them to understand the injustices that had been occurring there. Elisabeth stated in her first interview,

> Looking at the race issue in Ferguson became a question of math actually. So why do some people feel like it's not fair or not equal? What we could do with Ferguson was to look at the population numbers and the arrest records. We could look at records of police stopping individuals and keep track of those statistics by race over time. As we looked at these numbers

and converted them into a percent, because that was the math we were looking at. When we equalize numbers we're not just looking at the number of people, but we are looking at a number that is kind of stabilized by percent by having the same denominator. The kids were able to say, "oh that's not, that doesn't seem fair." If 70% of the people you know, if they only represent 30% of the community but 70% are stopped, they begin to see that there is inequality there. So then you can go back to the original question of why are there riots, and kids can say "oh because it really doesn't feel fair because out of ten people, 7 of your friends have been stopped by the police, but if you're white only 3 of your friends have been stropped by the police. As a black person, you're like, "hey everybody gets stopped by the police", and as a white person, you're like, "really do we get stopped by the police?" So that piece, that particular instance was getting at the core of why is there rioting aside from the emotional piece there was math behind it. There was math that could help kids understand how somebody who wasn't like them might feel.

When we spoke to Ayesha, who self identifies as Pakistani-American, about what she hoped students would take away from the course, she also spoke about truth from bias, but from a different perspective:

Susan: What are you hoping that students will take away from the course?
Ayesha: A sense of responsibility.
Susan: To whom or to what?
Ayesha: To those around them
Susan: What would that look like?
Ayesha: People who are conscious of what is happening around them and are willing to speak up, are willing to try to get more and willing to change if, not changing the world, changing their environment. Or if you are at the airport and you have someone who is very different that comes and sits next to you being comfortable in that situation.
Susan: Um huh
Ayesha: You know I hope that that is what they get out of it. Looking at me as their teacher is learning that it is ok and if they hear something in the media then they will be like there must be a different side of the story as well or what are the numbers to help you be more conscious.

As we worked together on this research project, we began to recognize that numbers and mathematics help people to see injustices and to recognize prejudices, but that they also have effects that increase bias and prejudice. Then how do we know which number to trust?

Multiple or Conflicting Truths

As mentioned at the start of this paper, the students and teachers studied the Ebola outbreak in the spring of 2014. The class researched the outbreak and asked, what was the worst outbreak in history? To answer this question, they had to wrestle with questions like, *What is worst? And worst to whom or for whom? Does worst mean the highest total number of deaths, or the highest percentages of deaths per infection, or the percentage of the total population that was infected and died during the outbreak?* Elisabeth talked about these multiple or conflicting truths—the idea that "truth" can be used by anyone to pursue any agenda:

I worry sometimes that the idea of social conscious or justice can be used by all people to pursue any agenda. I can say there's uncertainty, and I can say everyone can find their own truth, but the bottom line is the math can help us differentiate between exaggeration and what's really there, so we talk again about the rounding situation looking at politicians and what they say. If we look at the real, the numbers we can actually know, and then the media or politics can change them or

turn them and look at them from another direction, and say no they mean this, but we just looked at them do we think that and why?

Elisabeth talks later in the same interview about interpretation in mathematics:

[Understanding] how to read a graph how to interpret numbers that people use really helps you understand data presented as fact or truth in the media, and so I think you could, I mean statistics are used to support arguments. Numbers are used all the time to validate people's positions on things because somehow that quantitative data feels nonnegotiable. You know a number is a number is a number, and you can't, you know… So understanding what those numbers mean allows you to take a more critical look at whether it's true. You know how you interpret those numbers whether you believe them or not. It's really easy. I think a lot of people come out of school feeling like-- math is hard, math is not something that I understand entirely. So they are willing to take numbers at their face value and what I know to be true is that we interpret numbers in the media and in statistics numbers are interpreted.

Ayesha spoke about her recognition that there are multiple ways to present a topic to students:

When I read something before I show it to my students, I am reading it with a very keen eye. Like what do I want the students, like so there's bias there. So, yeah, it's there. I do want them to have all of the perspectives, so sometimes I will choose something that I don't agree with, like I want them to know the truth, and so I will throw that out. But yeah, I think it does impact because even though I am being fair and providing all the things I'm sure that there are people out there that don't do that because they want that one point of view….

So, if we know there are multiple truths, then there will be conflicting truths. How might middle school students handle this? Ayesha describes a conversation she had with three students about evolution. One student said, "You can't believe in science and god, you can only do one." Ayesha replied, "I believe in science and god. I believe in evolution and god." Another student remarked, "You can't do that if you go in a church, and you tell them you want to be a scientist; they are going to say get out." Ayesha went on to wonder, "they are already starting to have these thoughts so I am wondering when they started and how they have come to this age at 11 or12 years old *knowing* these things…" As a team of women, a team of teachers, researchers, teacher researchers, etc., we found ourselves grappling with what it means to know something in mathematics and in the research we are doing around the teaching of mathematics. What happens when you have multiple and conflicting truths in mathematics and in your research "spaces"?

Students Becoming Capable and Critical Mathematicians

Susan: How do you view the students as they come to you and what's your goal for them as they…?

Ayesha: So, I actually, having taught some of the students before, I do have preconceived notions of some when they come, and I have concerns or expectations based on those preconceived notions. Some of them hold true and some of them don't, but my goal for each of them is that they find some joy. I guess in doing math that was a little bit challenging, so that they could know something, so that they could learn something bigger than math out of it.

Susan: What's bigger than math?

Ayesha: That they could know something that was real or true about the world because of math. And I, or that they could just know that they could know. Like, "I can figure this out. I've got this skill that allows me to know this thing or to think more deeply about this thing." I think that's the piece that feels… I want the kids to feel empowered to be able to ask questions when they see numbers anywhere and to know like, "oh, they got those numbers from

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somewhere. I could get those numbers and check. I could know that. I could do that thing that they did. Maybe I can't do it right now, but someday I could do it." Anyone of them I would want them to think like, "I could be a NASA scientist. Really, there's so much more to it than just this rate business. But I see now that it's not as scary as it seems to take 5.88 billion and divide by 460,000 to find out the km per minute. I could do that." So, I want them to feel like-- one, I can do that, and --two, I want to do that. I would love for them to think to be thinking I want to do this.

As Elisabeth talked about the course and her hopes for the students, truth came up as well.

Reflection leads us to know things about ourselves that we maybe didn't know before and I think of that in terms of truth, I believe something about this situation and looking at the numbers about it. It may have changed and that for me is now the true thing about this story. And math informed that or helped inform that true thing about the story. That idea of value in math, there are some numbers that are just the numbers and you don't, there's not much you can do but the truth of it I guess, the truth of those numbers is for kids is I can know this I can figure out how to do this. I can know this and that knowing feels like the truth in some ways like it's not a mystery anymore even if it is a mystery, there is a door open to the mystery so I can go through it and figure it out.

What does it mean to prepare students as mathematicians? As critical citizens? As truth seekers? Is questioning all numbers productive or is it crippling? When is questioning too much? When does it just hurt?

Discomfort in Ethical and Moral Work

As they planned the final unit of their course, Susan sat in on the planning session. The teachers began by saying that they did not want to take on a topic that was too depressing. They had just finished 10 weeks of difficult discussions during which students and the teachers were asked to bring more of themselves into the work of school than is typical. Halloween was just around the corner, so the teachers decided to begin a unit on candy. As they got further into this unit, they found that looking critically at this issue was also quite troubling. They watched videos about the use of sugar to hook consumers on particular products, and the damaging effects of sugar on our bodies. How is it that they always ended up in the place where they were discussing things that were troubling? Are we finding disturbing *truth(s)* because we are looking for them? Or are they there whether we look or not, and it is our ethical duty to deconstruct them? What are the ethical and moral repercussions of bringing students and other teachers into the work of looking beyond the singular truth represented? Looking beyond the number and beyond the norm?

Denying Access to Truth

Susan witnessed a conversation between the two teachers where one teacher was relaying to the other that she would not be able to teach enrichment math, *Current Events Math,* in the next term; she would be teaching a remediation math course instead. She was disappointed to have this switch and that there would be no enrichment math classes in the next term. She also expressed understanding that there were 7th graders that really needed support in their math and therefore it was okay to teach the remediation course. Susan asked the question, why is it that we think that we have to teach a remediation mathematics course in a different way than an enrichment mathematics course? Weren't the skills taught in the *Current Events Math* course important for those students in the remediation course as well? How would teaching basic or foundational skills out of a workbook to the 7th grade students function? Would they see themselves as distant from the mathematics, consumers of it, rather than as in relation with it? Don't we want all students to be critical readers of mathematical truth(s)?

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Wonderings (Implications?)

Though we resist the term findings, there are important questions that this study brings to light that could be asked in other places and spaces. The multiple and overlapping layers of ethics: ethics of the researcher in the field, ethics of the teacher towards her students, ethics of the citizen in a community, ethics of representation in the media, are ever present, and even as we make moves, the context changes and we wonder how to move responsibly. In Todd's thinking about responsibility, ethics and relationships, she stated,

> what counts as ethical in Levinas's thought is not encapsulated within rule-governed behaviours, ethical codes, or moral precepts that can be secured through stable significations. Rather, the ethical lies within the very ambiguity of communication, within that which slips our cognitive grasp and possession…. For Levinas, communication is inherently ambiguous because it gestures beyond any stable meaning toward the very otherness of the other that marks her as radically distinct from myself. And it is this relation to the other as one of unknowability where the ethical promise – and risk –of ambiguity lies. (2003, p. 33)

If we think back to the football players, we wonder about the truth of experience understood through relationship with others. Should the other students have believed a person they had a relationship with who had experience with the topic or the number derived from many instances and published by scientists?

Is the responsible, ethical thing to do to continually create new interpretations and representations based on the particular context? We are still stumbling and stammering in the moment knowing there is no "right" to get to, no certainty that can calm us. Is the calm then in the acceptance of the ambiguity as promise and the willingness to continue to question, and to ask, and to wonder, and to disrupt?

Rather than concluding we resist conclusion following Koro-Ljungberg (2010). "Instead, unpredictable attentiveness and unexpected relationship with the Other could activate researchers' responsibility and thus enable open and humble data interpretations, as well as study conclusions that avoid definite closure" (p. 608). We question and question and question. In what ways do experience and "truth" interact as we take up teaching—and researching the teaching of—mathematics? How does our distance from a particular construction of a number influence how we read that number and its truth? How does bias impact what we doubt and what we believe? In what ways does the mathematics we "know" interact with these biases? What happens when faced with a problem with multiple or conflicting truths? Do ethics drive the solution? Or mathematics? What, or who, wins? By asking these and similar questions and remaining ambiguous in our understanding of how they might be answered, perhaps we can begin to answer Neyland's (2004) call to "re-enchant" mathematics and mathematics education.

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"COMPLICANDO ALGO TAN SENCILLO": BRIDGING MATHEMATICAL UNDERSTANDING OF LATINO IMMIGRANT PARENTS

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The purpose of this paper is to demonstrate the mathematical understanding of Latino immigrant parents in curricular and pedagogical practices in elementary school. The paper seeks to counter widely spread deficit discourses about the parental involvement of Latinos in education. Using data from the Agency and Young Children project, a video-cued ethnographic study (Tobin, Wu, & Davidson, 1989; Tobin, Hsueh, Karasawa, 2009; Tobin, Arzubiaga, & Adair, 2013), we highlight aspects from home that schools can use as resources to build a bridge that supports children's mathematical understanding.

Keywords: Curriculum, Elementary School Education, Equity and Diversity, Mathematical Knowledge for Teaching

Introduction

According to Suárez-Orozco (2001), immigrant children lie "in the margins of two cultures" (p.92). As the children, so too their parents find themselves in this borderline situation. This paper presents data collected from 14 focus groups in two different states. We showed parents a video of a typical day in a first grade classroom to stimulate conversation about their thoughts, ideas, and recollections from their own and their children's academic experiences in schools in the United States and in their country of origin. Although they share a variety of topics and experiences, we focus our analysis for this paper on their experiences related to mathematics instruction.

In this paper we argue that there is a real need of bridging communication between school and parents, particularly in mathematics instruction. We share the view that "parents are underused resources as partners in mathematics learning" (Aguirre, Mayfield-Ingram, & Martin, 2013, p.87). More common, unfortunately, is the deficit view that characterizes parents as lacking interest and involvement in their children's academic experience. Therefore, there is a need to strengthen the relationship between parents and schools to counter these debilitating characterizations. Adding to these already worrisome views, a common misconception arises regarding mathematics instruction: that mathematics is a universal language because it is a language of symbols, therefore culture-independent, and hence the ideal subject for ELL students because matters of language should not substantially affect understanding of the content conveyed (Moschkovich, 2007).

This work presents data that challenges deficit views regarding parent interest and involvement in their children's academic preparation. Latino immigrant parents interviewed for this project are attentive to and aware of their children's academic experiences during mathematics instruction. Parents are willing to support their children's mathematical learning at home, and they understand that teachers and schools do not always recognize their efforts. Parents are conscious of the dissonance between the two cultures and how it interferes in the ways in which they can support their children's learning at home. According to parents in this study, mathematics was the subject area in which parents can see the most concrete differences in methodology and symbology between the country of origin and new country.

The data presented will show not only the cultural differences in the mathematics classroom, but also the importance of communication between teacher and parents in order to improve the academic achievement of students. On this basis, we can start to build a bridge that facilitates the exchange of ideas, concerns, and support between the school and home. We propose that the construction of this

bridge start at the school with teachers prepared to recognize and incorporate the cultural variety present in their classrooms.

Our present work answers the following research questions: How could teachers better support and communicate with Latino immigrant parents when teaching mathematics to their children? How can teachers' understanding of immigrant families' mathematical knowledge potentially improve their practice? How can teachers bridge mathematical understanding and academic support at home?

Literature Review and Theoretical Framework

There is an extensive body of research that examines what a teacher needs to know for teaching mathematics in the elementary school. Many of these studies illustrate how mathematical knowledge does not necessarily translate into mathematical instruction designed for students to develop a deep conceptual understanding of mathematics as envisioned in current reform documents by the National Council of Teachers of Mathematics (2000). NCTM (2000) emphasizes contexts for engaging students in learning mathematics, such as problem solving, reasoning and proof, representations, communication, and connections. The underlying idea is that by engaging students in the practice of mathematics, they should develop conceptual understanding and the ability to reason.

Two research-based theories, Constructivism and Sociocultural Theory, illustrate how students learn. The constructivist approach asserts that students come to the classroom with previous knowledge. Students acquire this knowledge outside the classroom based on experiences with the surrounding world. Students "construct" their own learning. The sociocultural approach suggests that the student can also "construct" knowledge, but typically with the assistance of others who are more "knowledgeable" (Vygotsky, 1978, 1986). Still, both approaches see the student as an active agent in the learning process.

These theories hold important implications for immigrant students' opportunities to use in the classroom environment their previous knowledge from outside the classroom, and for what types of support they receive from more knowledgeable peers. Zevenbergen (2000) has documented how the preferences, behaviors, and attitudes of dominant groups rarely fit with those of the marginalized groups. Her work illustrates a disconnection between theory and practice particularly in the contexts of classrooms where different cultures mix. According to Zevenbergen (2000), cultural and social differences strongly influence how individual students construct their identities when learning mathematics. Civil (2005) explains that some difficulties in learning mathematics may derive from a student's particular social or cultural group and how others perceive their mathematical practices, or from the relationships marginalized groups have to dominant groups.

Extending the constructivist and sociocultural approaches for teaching mathematics, our work stresses the need to provide immigrant students with opportunities to bring their culture into the mathematics classroom. Along with Civil (2005) we highlight the importance of immigrant parents as intellectual resources in their children's education. Moreover, we reject the deficit model that frequently points to the home as the root of academic failure (Civil, 2005). In particular: *"In our view, meaningful inclusion and interaction with students necessitate knowledge of their personal, family, and community backgrounds as well as their social realities"* (Aguirre, et al., 2013, p. 9). But given that teachers and parents are aware of these differences, the task remains of creating a productive mode of communication between school and home.

In the context of the United States, immigrant parents are often stereotyped as uncaring and unengaged in their children's education when they do not participate in the traditional activities that traditionally defined appropriate parental involvement, such as volunteering for school, chaperoning for field trips, and PTA membership (Chavkin, 1993; Riojas-Cortez & Flores, 2009). Latino immigrant parents particularly face individual and institutional barriers when trying to participate in their children's educational experiences: for example, individual characteristics such as socioeconomic status, limited English proficiency, and lack of understanding of cultural norms

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(Arias & Morillo-Campbell, 2008; Lareau, 1987; Nieto, 1985; Rueda, Monzó, & Arzubiaga, 2003). Also parents face institutional barriers that include schools' deficit views of immigrant parents, the under-preparation of teachers to serve culturally and linguistically diverse families, and the different expectations that minority parents and schools hold for children (Arzubiaga, Noguerón & Sullivan, 2009; Graue, 2005; Graue & Brown, 2003; Riojas-Cortez & Flores, 2009; Baum & Swick, 2008; Whitehouse & Colvin, 2001). Individual and institutional challenges interfere in how parents communicate and participate in their children's schooling. Often deficit discourses in school describe immigrant parents not only as unengaged in their involvement at school, but also as incompetent in helping their children with schoolwork at home. However, there is empirical work that highlights the great value Latino immigrant parents place in their children's education (Valdes, 1996; Epstein, 1995). Lopez (2001) suggest the need for schools to broaden their idea of parental involvement and recognize Latino immigrant parents' culture as important resources at school.

The general need to broaden the borders of involvement for Latino immigrant parents in schools naturally includes specific application in mathematics instruction. It is commonplace to hear non-immigrant parents complaining about the "new" ways for teaching mathematics that widely differ from the "old" ways (the way they learned). One can imagine how such dissonance might be magnified for those parents who grew up in different countries and try to make sense of their children's mathematics homework in the effort to help them. Researchers such as Perkins & Flores (2002) and Moschkovich (2007) have documented the differences in symbols, algorithms and methodologies across different countries. In addition to these differences, there is also a cultural difference. As Civil (2005) documented in her study, immigrant parents traditionally have a great sense of respect for the teacher and tend not to "interfere" in what the teacher has established.

In this paper, we use a Funds of Knowledge (Gonzalez, Moll, & Amanti, 2005) framework. Latino immigrant families bring with them knowledge and experiences, or funds of knowledge, that are not always appreciated and therefore not always conducive to helping them understand the U.S. schooling system. However, the work of Gonzalez and colleagues (2005) views the linguistic and cultural practices of minority communities as capital, particularly when teachers recognize those "funds" and are able to bring them into the classroom. In this study, parents are recognized as knowledgeable of their children's education and their funds of knowledge valued as they share their opinions about, expertise in, and concerns regarding the education their children receive in the U.S.

Methodology

Participants

Participants selected were Latino immigrant parents with children enrolled in U.S. schools from Pre-K through 3rd grade. For the study, fourteen focus groups were conducted in Texas and California with 55 Latino immigrant parents from Mexico, Peru, Guatemala, Chile, Venezuela and Honduras. The sites in Texas included 10 focus groups in two urban and one rural area and 4 focus groups in two urban areas in California. We interviewed 50 mothers and 5 fathers, with levels of formal education that ranged from 5th grade to a Master's degree.

Data Collection

The focus groups were conducted in rural and urban cities for purposes of comparison. Focus groups in Texas were conducted on school grounds during and after school hours in central and southern cities and border towns. The California focus groups were conducted in the Bay and East Bay areas in the homes of participants who agreed to host a focus group with friends and acquaintances who met the above criteria.

Focus groups began with a short explanation of the project and a film. The film represents a typical day in a first grade ESL classroom with a large number of Latino immigrant children. The 20-

minute film captures a variety of scenes throughout a school day, with math, reading, writing, small and large group work, centers and recess, to provide parents with a glimpse of what a school day looks like in a first grade public school classroom in Central Texas. Each focus group watched the same film. Focus groups were audio and video recorded for transcription and translation purposes. Parents viewed the film and discussed what they saw. A member of the research team facilitated each conversation guided by a series of pre-established questions. These questions were used with all focus groups and included aspects of pedagogy, curriculum and parental involvement.

Method

The research team transcribed all focus group interactions. We then coded the transcriptions and used content analysis to identify patterns in the codes. We used debriefing (Merriam, 1998) to think deeply about our initial codes and how to categorize them into more defined themes. Throughout this entire qualitative process, we continually reflected on the data collection and analysis (Creswell, 2003; Merriam, 1998; Stake, 1995; Yin, 2003).

Findings

The data we present in this section focus on two major categories that highlight parents' concerns and ideas about mathematics instruction and curriculum, and the support they provide for their children. These findings are further explored below.

Latino Immigrant Parents' Support

During the focus group interviews, we asked the parents about the school and home responsibilities when it comes to learning. Parents expressed as part of their responsibility the need to help and support their teachers' work at home. In the Bay Area, Perla shared how important it is for her that school and home work as a team.

> *Perla:* [Communication is important] at all times, either because of behavior or academics, and to work as a team, that way we can learn about what they do in the classroom and how we can help at home. And to always keep that communication open. [About] things that are important at home and especially during the summer, because it is mostly when... like that commercial, that makes me laugh, where the kid kind of shakes his head and the letters come out of his ear. It seems terrible to me, but at the same time it is so graphic, it is so… very, very well depicted if I can say that. It is all about keeping a routine as much as we can. I mean, you are not going to be sitting with a notebook doing math every day, but we can play with numbers and find games and make them practice some things, and keep reading. To me that is crucial.

It is important to note in this mother's response that she places high value on open communication and teamwork between the school and home. For her it is also important to know not only about her children's behavior but also about their academic progress and needs. The parent also stresses finding out how she can help at home providing learning opportunities that reinforce what her children are learning at school, especially during the summer months.

In El Naranjo Elementary School in a rural border town in Texas, the focus group discussed ways parents support their children's schoolwork at home. Parents frequently mentioned helping their children with homework. In the following transcript they shared some of the strategies they used while helping their children with math work.

> *Kiyomi*: I have a question, because one of the things that teachers generally ask parents is, "How do you help your child at home with the homework assignments?" Have they asked you this question: "How do you help your child?" I don't know, so I'll ask, how you help your child at home with the homework. Do you help with the homework?
>
> *Marcela*: I help him with the homework.

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Kiyomi: With the homework too.

Jazmin: I explain to them, like when doing mathematics and let's assume they are adding, maybe adding 9 plus 3, I tell them make little dots, 9, and then make 3 and then count them all together. That way he is learning, or I show him how to use his fingers... I teach them too.

Marcela: That's how I teach them, with the fingers.

Kiyomi: We all teach that way.

Marcela: But it's that it is different because my mathematics is really different from hers. I actually taught her mathematics, in fact. When she started 2nd grade, I taught her how to divide, because... it is not that I want to brag, but she turned out to be very intelligent. I taught her to divide; then she learned how to multiply. Now, at the Valparaíso High School, forget it ma' [referring to her daughter], I have no idea, that didn't come up in my time.

This transcript shows how parents use a variety of pedagogical methods to teach their children mathematics. Jazmin uses drawings and her fingers as tools to support her children's understanding of addition and subtraction. Marcela shares her story about how her child first learned how to divide and then how to multiply. Traditionally in schools, students first learn how to multiply before they are exposed to division problems. However, research on children's mathematical thinking (Empson & Levi, 2011) has shown that children in grades as early as Kindergarten can solve division problems without having any formal instruction in multiplication facts.

Mathematics, Instruction and Curriculum

During the focus group interviews we asked parents if they noticed any aspects of teaching that were different when compared to their own learning at school. In Las Rosas Elementary School while referring to the video, parents shared some aspects of teaching mathematics that were very different compared to their own learning. In the film first graders were asked to pair and represent the number 10 in different ways by combining number cards and then write the number combinations on a piece of paper.

Ricardo: The way they teach mathematics, I mean, it seems… it seems they are making complicated something so simple.

Liliana: Yes.

Olivia: Yes, I was going to say the same thing (laughs).

Kiyomi: Let me understand, explain it to me: what? how? Give me an example.

Ricardo: That for example, for what they were doing in the video, to add 10, how many options do we have?

Parents at Las Rosas have strong ideas about how mathematics should be taught at school. In this particular case parents disagree with what the teacher is asking children to do. This is a good example of how communication between school and home is important so parents can understand how children are learning and the rationale for the learning experiences at school. The idea behind the task is to develop the student understanding of base-10 concepts. Once students see the different ways in which they can construct the number 10 they can extend this idea to other numbers and develop an understanding of arithmetic operations. In mathematics instruction children are encouraged to explore with numbers. Decomposing numbers in such a way is a fundamental activity for their own mathematical understanding (Carpenter, Fenemma, Franke, Levi, & Empson, 2015).

In Las Rosas, parents continued to discuss mathematical instruction and the following transcript shows the difference in mathematical procedures between Olivia and her first grader Javier while helping him to complete his mathematics assignment.

Olivia: You haven't seen the ones with 6 and 7 that Javier gives me, a bunch of numbers. And I'll tell him, "Kiddo, it was only a division problem," because I know, I help him... nothing but a

division problem: 72 fits in there and that's it, these… . "No, kiddo, it looks like I have to do everything. Enough! Do it yourself!"… because really, when it's like that, I don't know how to do it that way.

Kiyomi: Then, the way they are taught mathematics is different.

Olivia: It turns out more complicated, when really it should be much easier.

In this transcript Javier is completing his homework and his mother is helping him. However Olivia expressed frustration as Javier is expected to solve the problem while documenting the process. She does not understand why he needs to add that step, as it complicates the assignment but does not change the outcome. Parents are willing and able to help their children with schoolwork, but have trouble when their own academic experiences do not match school expectations. It would be helpful if they had a better understanding of the rationale for the assignments their children are required to complete.

Discussion and Conclusions

Mathematics is widely thought of as a universal language. Data from the focus group interviews reveal that, while mathematics is the same across contexts of instruction in various countries, the pedagogical methods can differ in important ways. Parents expressed the willingness and desire to help their children with schoolwork and to provide support whenever possible, but they also expressed a lack of understanding of, and frustration with, the rationale behind the schoolwork. Parents' own school experiences do not necessarily match their children's, causing them to feel helpless in supporting their children academically. This counters deficit discourses that usually portray Latino immigrant parents as uncaring and unengaged. Both teachers and school need to improve support of and communication with families by explaining the curriculum or how to access related information. We argue that this communication needs to be open, such that teachers reach out to families to share and explain the pedagogical practices and methods they implement. Teachers must understand that there are cultural and methodological differences whose acknowledgement and inclusion can potentially enrich their teaching of mathematics. These can broaden the cultural understanding of the teacher and the class. Parents, if invited to the schools, can provide that knowledge, so that teachers can learn and better understand the cultural richness and mathematical thinking that parents and their children bring with them.

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PARENT WORKSHOPS FOCUSED ON MATHEMATICS KNOWLEDGE FOR PARENTING (MKP): SHIFTING BELIEFS ABOUT LEARNING MATHEMATICS

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The purpose of this study was to investigate the extent to which parents of first, second, and third grade students who attended a two-day workshop on mathematics strategies shifted beliefs about learning mathematics as compared to parents who did not attend the workshops. Parents impact their children's mathematics learning when the students are at home working on homework. This can be an important barrier to overcome. The results suggested parents could benefit from workshops focused on solving mathematics problems in multiple ways, similar to ways their children are learning mathematics.

Keywords: Affect, Emotions, Beliefs, and Attitudes, Classroom Discourse, Elementary School Education, Number Concepts and Operations

Introduction

Social media outlets indicate parents are frustrated with the way their children are learning mathematics, such as using multiple solution strategies to solve problems that seem to include unnecessary steps (Garland, 2014; Richards, 2014). This is important because when the children are at home, their parents often take on the role of teacher. Parents want to help their children with homework, but their own personal experiences with learning mathematics were likely procedural (Garland, 2014; Richards, 2014). The purpose of this study was to determine if completing a two-session workshop on mathematics pedagogy and content related to whole number concepts and operations shifted parents' beliefs about learning mathematics. Beliefs were measured by the abbreviated Mathematics Beliefs Scales (aMBS), created by Fennema, Carpenter, and Loef (1990) and later adapted by Capraro (2005). More specifically, the goal was to answer the following question: *To what extent do parents who attend a mathematics workshop on whole number concepts and operations differ, on average and over time, in their beliefs about learning mathematics as compared to parents who do not attend?*

Background

When an elementary-aged child asks his or her parent for help on homework, the parent thinks they should be able to help their child, but then they realize mathematics is being taught differently today than when they were in school (Richards, 2014). Some parents do not understand the justification behind why these new strategies are being used, and think they are a waste of time. Teachers have opportunities to gain a deeper understanding of the mathematics they are to teach through education classes or professional development, but parents may not have the same opportunities. By creating workshops that allow parents to work through the strategies their children use, this study investigated how those workshops influenced parents' beliefs about how students should learn mathematics. After all, parents are their children's teacher when the children are at home. Previous research findings from qualitative studies suggest that parents who participated in parent workshops felt like they (a) increased their content knowledge and improved their attitude towards mathematics (Knapp, Jefferson, & Landers, 2013), and (b) improved relationships between parents and their child (Kreinberg, 1989; Mistretta, 2013) and between parents and the school (Kreinberg, 1989). Although parents may not be confident when helping their child with mathematics homework before participating in the workshops, after participating in workshops parents were more

confident with their mathematics ability when helping their child (Cotton, 2014; Kreinberg, 1989; Marshall & Swan, 2010).

The researcher proposes a connection between parents and preservice teachers (PSTs) due to the lack of quantitative research regarding shifts in beliefs and learning about mathematics by parents. The way PSTs think in terms of solving mathematics problems could be tangentially related to the way parents think about solving mathematics problems before participating in workshops. PSTs come to their education courses with experiences in learning mathematics procedurally, similar to parents. Having a productive disposition in mathematics includes being able to see mathematics as useful and worthwhile. Opportunities to help learners make sense of mathematics may influence their beliefs, especially when the beliefs are called into question. This shift is important because "changes in beliefs are assumed to reflect development" (Oliveria & Hannula, 2008, p. 14). Teachers can do this by solving problems in ways that call existing beliefs into question or by making mathematical discoveries on their own (Liljedahl, Rolka, & Rosken, 2007). Changing teachers' beliefs about mathematics can be difficult, however, because many preservice teachers think mathematics is about memorizing formulas and procedures (Szydlik, Szydlik, & Benson, 2003).

A person's beliefs evolve over time, and are influenced by his or her experiences (Op't Eynde, DeCorte, & Verschaffel, 2002). Richardson (2003) implied there were three different experiences that influenced beliefs: personal experiences, experiences in an instructional situation, and experiences with specific content knowledge. These beliefs shape how a person views different situations, one being the way children learn about mathematics (Richardson, 2003). Most parents of elementary grade students were taught mathematics in more teacher-centered classrooms, where the teacher was the one who determined whether or not the student had the correct answer (Garland, 2014; Richards, 2014). When most of these parents were elementary students they were not encouraged to work with their peers to find more than one solution strategy, but according to "best practice," classrooms today are more student-centered, where the teacher acts as a facilitator and the focus of instruction is on making sense of problems and solutions with lesser emphasis on judging correct answers (Ertmer & Newby, 2013).

Research Design

The current study used a quantitative, quasi-experimental, non-equivalent control group design. The research design was chosen because participants self-selected either the control or intervention groups. The control group included participants who were unable to attend the workshop, but still participated in testing. For both the control and intervention groups, all interactions were conducted face-to-face at the same school site. The school site allowed the researcher to set up a table during school hours to meet with participants in the control group. The control group attended two face-to-face meetings during a time convenient for them. During the first meeting participants completed pretests, and during the second meeting they completed posttests. The majority of control group participants completed pretests and posttests within a two-week period, similar to the time between pretest and posttest for the treatment group.

Parents or guardians of 1st, 2nd, and 3rd grade students at multiple neighboring public elementary schools in Central Florida were invited to participate. For the purpose of this research, "parents" will include any person who takes on that role. First, second, and third grades were chosen because this is when mathematics homework first represents strategies that are likely different from instruction most parents received when they were in elementary school (Garland, 2014; Richards, 2014). The sample was one of convenience, given the location of the participating schools was near the researcher's residence, and parents volunteered to participate.

The abbreviated Mathematics Beliefs Scales (aMBS) (Capraro, 2005), was used to measure parents' beliefs about learning mathematics. The aMBS, originally created by Fennema, Carpenter, and Loef (1990), was developed to measure the mathematical beliefs of teachers. Responses to

questions were measured using a five point Likert scale, ranging from strongly agree to strongly disagree. The original aMBS had 48 items, and because researchers commented that participants complained about the length and repetitiveness of the instrument, an exploratory factor analysis was run on all 48 items (Capraro, 2001). The 18 questions that were chosen because of the analysis explained 46% of the variance and could be split into three factors with six items in each. The three factors were student learning, stages of learning, and teacher practices. One sample question in the student learning subscale is, "Children will not understand an operation (addition, subtraction, multiplication, or division) until they have mastered some of the relevant number facts" (Capraro, 2005, p. 86). One sample question in the stages of learning subscale is, "Children should understand computational procedures before they master them" (Capraro, 2005, p. 86). One sample question in the teacher practices subscale is, "Teachers should allow children to figure out their own ways to solve simple word problems" (Capraro, 2005, p. 87).

The treatment group participated in two workshop sessions created to help parents engage in learning mathematics in ways similar to their children, according to "best practice" such as encouraging students to come up with multiple strategies that would allow them to think flexibly about mathematics. The workshop was repeated a total of three times to allow for flexibility to allow more parents to participate. Each workshop was held for two hours, including the time spent on pretests and posttests, so participants in the treatment group were engaged in learning mathematics content for approximately 2.5 hours, as the other 1.5 hours were used for administrating tests. Pretest for the treatment group was administered at the beginning of the first day of the session, and posttests were administered at the end of the second day.

Following the pretest, participants were asked to solve the following problem, "*Andrea has 14 cookies. Jamie gave her some more. Now Andrea has 32 cookies. How many cookies did Jamie give Andrea?"* Participants were encouraged to use two different methods to solve this problem. Solving the problem using a second strategy was a difficult task initially. All participants solved by subtracting 14 from 32 using the standard algorithm and did not know how to solve using a second strategy. Through questioning techniques participants engaged in using drawings, concrete tools, and other computational strategies to solve problems. Participants made sense of base ten blocks, ten frames, the hundred chart, and an open number line. Then, participants were given another problem. After participants tried to solve a problem on their own using at least two different strategies, they shared their strategies with a partner, and tried to made sense of the different strategies. Participants used the tool that made the most sense to them, and, after sharing with a partner, they had a better understanding of other strategies. The researcher chose the order in which participants shared their strategies from more concrete to more abstract, to help them make connections between the different strategies. Then, possible student errors were discussed, first with a partner then with the whole group. The first error, which involved regrouping, was difficult for participants to communicate where the student made their mistake. They knew the answer was incorrect, but were unable to explain why. Through questioning techniques, participants were able to make sense of why the answer was incorrect. The goal was to encourage participants to "think outside the box" when supporting their children at home.

During the second workshop session, participants were given multiplication and division problems and asked to solve them in two different ways. For example, participants solved the multiplication problem "*Amy has 4 boxes. Each box has 7 bags of chips. How many bags of chips does Amy have?"* First, a participant shared a strategy that involved using repeated addition. The participant drew four boxes and put the number 7 in each one. Then the participant wrote $7 + 7 = 14$, $14 + 7 = 21$, and then $21 + 7 = 28$. Another participant wrote 7, 14, 21, 28 and we had a discussion on the similarities and differences between the two strategies. Later strategies included finding doubles. For example, one participant wrote $7 + 7 = 14$, $14 + 14 = 28$. Participants made a connection between this strategy and the first two strategies, repeated addition and skip counting. Because participants

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had worked through several solution strategies in the previous workshop, they were able to construct new strategies much easier than when they began the first session. Participants were comfortable sharing with their partner, and then sharing with the group when asked by the researcher. If a tool was not brought up, the researcher would ask a question like, "how could we use ten frames to help us solve the problem?" Through discussion, participants made sense of using an open number line, ten frames, an array, and an area model, in addition to other number strategies. The researcher helped participants to make connections between multiplication and division problems through questioning techniques. In addition, participants made connections between division and addition as well as division and subtraction. By allowing participants to share strategies and make sense of other strategies, they were engaged in the discussion. Finally, participants were given example solutions to determine student errors while using the standard algorithm for multiplication and long division. After the group discussion, participants were given the posttest.

Findings

Participants in this study had students who attended one of four neighboring public elementary schools, all with similar school demographics. The majority of participants had children who attended the school site where the workshops were held, with the second largest population of from the school approximately one mile away from the school site. The breakdown of the number of participants in each of the three treatment groups and the control group is listed in Table 1.

Table 1: Participant Information (Frequencies and Percentages)

		Total
Workshop (42%)	Series 1	6 (21%)
	Series 2	4 (14%)
	Series 3	2 (7%)
Non-Workshop (58%)		17 (58%)

Demographic information was obtained from the 29 parent participants, and the researcher discovered the majority of participants: identified as being "white ($n=22$), were in the 36-45 age range ($n=22$), were either married or in a domestic partnership ($n=24$), earned at least a bachelor's degree ($n=24$), were working full time ($n=21$), and never had a teaching related position ($n=27$). All participants said their child brought home mathematics homework, and some said they helped their child everyday ($n=10$), while others said they never helped their child ($n=5$). The rest of the participants were split on all other options ranging from one to four times per week.

The purpose of this study was to determine the extent to which parents or guardians who attended a workshop on mathematics strategies differed on average and over time with parents who did not attend the workshop regarding beliefs. When statistical significance was found on the aMBS, additional analyses were run on the three different factors to determine which factors had a statistically significant difference.

A two-factor split-plot (one within-subjects factor and one between subjects factor) ANOVA was conducted. The within-subjects factor was time (pretest or posttest) and the between-subjects factor was group (treatment or control). Assumptions of baseline equivalency, homogeneity of variance, independence, normality, and sphericity were tested, and were met for everything except normality and sphericity. The violation of the assumption of normality suggested the increased likelihood of a Type II error, however the ANOVA is a robust test. The assumption of sphericity was violated for each separate test so statistics from the Greenhouse-Geisser conservative F test were reported when analyzing statistical results. Additionally, effect sizes according to Cohen's values (1988), for small (.01), moderate (.06), and large (.14) effect sizes will be reported.

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There was a large effect size (.201) and sufficient power (.708) for the interaction for the between-within factor for the entire aMBS. Additionally, there was a statistically significant within-between subjects interaction effect between group and time ($F = 6.771$, $df = 1, 27$, $p = .015$). ($M_{pre\,x\,control} = 54.41$, $SD = 10.186$; $M_{pre\,x\,treatment} = 50.33$, $SD = 6.401$; $M_{post\,x\,control} = 53.41$, $SD = 10.278$; $M_{post\,x\,treatment} = 56$, $SD = 7.604$). This statistically significant result suggested that there were differences, on average, between treatment and control group over time regarding beliefs reported from the entire aMBS.

Then the aMBS was split into the three factors identified by Capraro (2001), student learning (factor 1), stages of learning (factor 2), and teacher practices (factor 3). While there was a non-statistically significant result for factors 1 and 2, there was a statistically significant within-between subject interaction between group and time ($F = 7.48$, $df = 1, 27$, $p = .011$). ($M_{pre\,x\,control} = 21.71$, $SD = 4.18$; $M_{pre\,x\,treatment} = 20.50$, $SD = 3.12$; $M_{post\,x\,control} = 20.47$, $SD = 4.24$; $M_{post\,x\,treatment} = 23.25$, $SD = 3.33$) for factor 3. Additionally, there was a large effect size (.217) and sufficient power (.751) for the interaction for the between-within factor for aMBS-factor 3. This statistically significant result suggested that there were mean differences, on average between groups over time regarding beliefs about teacher practices. Results are displayed in Table 2 and Table 3.

Table 2: Greenhouse-Geisser Results for Statistically Significant Beliefs

Variable	Source	df_1	df_2	F	p	Partial η^2	Power
aMBS	Time * Group	1	27	6.771	.015	.201	.708
aMBS –F3	Time * Group	1	27	7.48	.011	.217	.751

Table 3: Mean and Standard Deviations for Pretest vs. Posttest Split by Group

	Pre			Post		
	Total	Treatment	Control	Total	Treatment	Control
N	29	12	17	29	12	17
Beliefs (RQ2)						
Mean	52.72	50.33	54.41	54.48	56	53.41
Standard Deviation	8.92	6.40	10.19	9.21	7.60	10.28
Belief Factors (AQ3 – F3)						
Mean	21.51	20.5	21.71	21.62	23.25	20.47
Standard Deviation	3.76	3.12	4.18	4.07	3.33	4.24

Figure 1 and Figure 2 displays the means from the pretest and posttest, between the treatment and control groups for the entire aMBS and the aMBS-Factor 3. These figures provide a visual representation of the statistically significant interaction between time and group as reported from Table 2 and Table 3.

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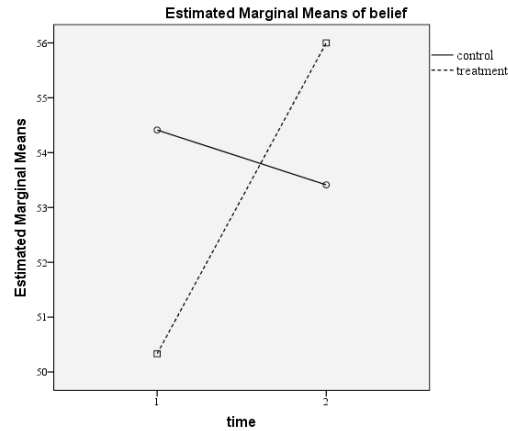
Figure 3. Profile Plot for Beliefs.
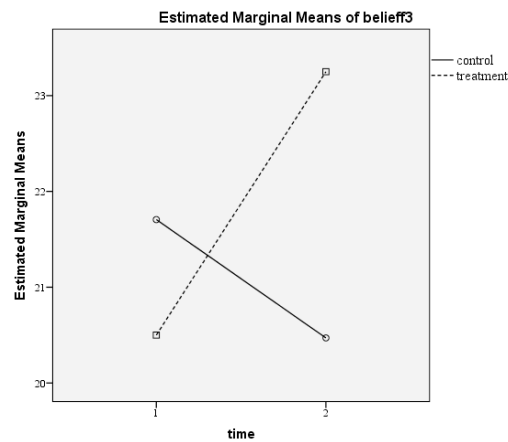
Figure 2. Profile Plot for Beliefs, Factor 3.

Conclusions

Results were statistically significant between groups and over time with a large effect size. The treatment group changed their beliefs to ones that were more focused on students constructing their own knowledge. Additionally, there was no statistically significant difference between groups over time for factor 1 (student learning) and factor 2 (stages of learning), but there was a statistically significant change between groups over time for factor 3 (teacher practices) with a large effect size. Parents in the treatment group had beliefs that leaned more towards students learning about mathematics in a learner-centered environment after completing the workshops. Parents in the control group did not shift their beliefs from the pretest to the posttest. This indicates that the workshops may have shifted parents' beliefs about student learning to beliefs that students should learn in a learner-centered environment.

These statistically significant results indicate that parents may have shifted beliefs about student learning through participation in workshops. This supports previous findings related to preservice and inservice teachers (ISTs) that suggest belief change could occur through participation in an instructional situation (Liljedahl, Rolka, & Rosken, 2007; Richardson, 2003). This significance could be explained because parents were participating as learners in a student-centered environment, which was different from the direct instruction they may have experienced as young learners (Garland, 2014; Richards, 2014). Through participation in the workshops, which were learner focused, parents may have understood the importance of allowing *their* child to learn mathematics in a student-

centered environment instead of one focused on the parent guiding their child to the answer. Previous research indicates beliefs are influenced by a person's experiences (Op't Eynde, DeCorte, & Verschaffel, 2002).

One limitation of this research was that participants completed a belief instrument where their beliefs were self-reported. Researchers have found that teachers need to be observed multiple times to determine their underlying beliefs, which can be different from their self-reported beliefs (Cross & Hong, 2012; Leatham, 2006). Due to the similarities of teachers and parents regarding helping a student with a mathematical task, this may be true for parents as well. Because the parents were asked to complete the instrument, their underlying beliefs may not have been apparent. Additionally, instrumentation validity may be a threat because while there was some evidence supporting the reliability and validity of the aMBS (Capraro, 2005), this instrument was not tested on parents and the small sample size in the current study did not support testing statistical validity and reliability evidence for the scores from the instrument. The parents in the study did not represent a random sample because they were selected based on convenience and self-selection, which may have introduced self-selection bias. The lack of random selection from the population limits the generalizability of the study findings. Additional workshops will need to be conducted and data will need to be collected and analyzed in different parts of the country to be able to generalize these findings.

Future research could be conducted in undergraduate and graduate programs. If PSTs and ISTs are involved in this type of research, they might have a better understanding of why it is important to get parents involved in this manner. This could, in turn, help PSTs and ISTs determine common errors students might bring to whole class discussions, and why those errors arise by allowing parents to share solution strategies. Parents may be the biggest supporter for a teacher when working with their child on homework, but parents need the tools that will be most helpful.

Although the duration of the workshops was short, parents who participated in the workshops indicated a shift in their beliefs about learning mathematics to a more learner-centered environment. These statistically significant findings indicated that workshops for parents were beneficial. By giving parents the opportunity to engage in mathematics in ways similar to the way their children learn in the classroom, beliefs about mathematics were challenged. Prior to the workshops, and according to the belief instrument that parents completed, parents' beliefs leaned more towards working *with* their child on homework instead of allowing their child to come up with their own strategies. The findings in this research study suggest more research on parent workshops is crucial.

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THE SECOND DERIVATIVE TEST: A CASE STUDY OF INSTRUCTOR GESTURE USE

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We present a case study of how five instructors used gesture when introducing the second derivative test in a first semester calculus class. The second derivative test and optimization naturally evoke hand motions while teaching, making this a fertile ground for studying gesture use in the classroom. Each of the five instructors used a classic optimization problem as a primary example of how one would use the second derivative test to verify that a found value would be a maximum or minimum. We observed the instructors making connections between the algebraic, numerical, and graphical representations of this concept through gesture. The quantity and types of gesture varied greatly by instructor, but there were two key links that every instructor made.

Keywords: Instructional Activities and Practices, Classroom Discourse, Post-Secondary Education

Introduction

There is a large body of work in cognitive science focused on embodied cognition; our knowledge is shaped by our experiences and interactions with the world around us (Lakoff & Nunez, 2000; Nemirovsky & Ferrara, 2009; Nemirovsky, Tierney, & Wright, 1998). Through bodily experiences, such as gesture, our understanding of complex concepts is shaped. As calculus can be considered the study of motion, it is a natural place to examine gesture. There are several problems that require students to visualize/imagine situations involving rate(s) of change – related rates and optimization are two examples. Diagrams and graphs are means of visualization that may facilitate the understanding of many concepts and problems in calculus. While a student may read an example in the textbook and model a solution based on this text, an example presented in class provides the learner an additional modality for learning: the instructor's gestures. An instructor's gestures can give life to a static diagram or a point on a graph. There is a growing body of research that suggests that an instructor's use of gesture can have a positive impact on student learning (Alibali & Nathan, 2007; Hostetter & Alibali, 2010).

Gesture use has recently become the focus of many educational studies. While the literature related to gestures made in undergraduate mathematics classroom is limited (Marrongelle, 2007; Rasmussen, Stephan, & Allen, 2004; Wittmann, Flood, & Black, 2013; Yoon, Thomas, & Dreyfus, 2011), there is a plethora of studies examining gesture in K-12 classrooms (Alibali & Kita, 2010; Arzarello, Paoloa, Robutti, & Sabena, 2009; Cook, Duffy, & Fenn, 2013; Cook, Mitchell, & Goldin-Meadow, 2008; Edwards, 2009; Goldin-Meadow, Kim, & Singer, 1999; Maschietto & Bussi, 2009; Nemirovsky et al., 1998; Roth & Thom, 2009). Goldin-Meadow (2000) stated, "[a] task for the future is to determine how gesture can best be harnessed to improve communication in classrooms" (p. 235) and Roth and Lawless (2002) noted, "little is known about the role of gesture in learning and instruction" (p. 285).

Gesture use in the calculus classroom has not been extensively studied. We seek to address this gap in the literature by examining how instructors use gesture in natural teaching environments. In this paper, we address the questions: What examples does an instructor use to communicate mathematical ideas and concepts? While working through the chosen examples, what key ideas are linked? What gestures does the instructor make while making links for students? This study examines the answers to these questions at one point in time – when the instructor introduces the second derivative test.

Theoretical Perspective

Mathematics instructors often make connections between ideas and concepts. In this research, we focus on what Alibali et al. (2014) defined as a linking episode: segments of discourse in which teachers connect ideas. Note that a linking episode may contain several distinct links. For example, we observed instructors making two distinct links in a single linking episode. We examined the linking episodes that occurred naturally during calculus instruction and the extent to which instructors used varying modalities, particularly gesture, to express links between ideas.

Gestures are a natural part of communication and may convey additional information to the listener to foster comprehension. Yoon et al. (2011, pp. 891-892) indicate "gestures are a useful, generative, but potentially undertapped resource for leveraging new insights in advanced levels of mathematics" (p. 891-892). They further advise that instructors should model gestures for students in lecture. Hence, we want to observe how instructors use gesture to highlight the ideas that they are connecting during a linking episode.

Methods

A qualitative case study methodology (Cohen, Manion, & Morrison, 2011) was used to examine how five instructors used gesture during natural classroom activities that involved the second derivative test. Each lesson in which the second derivative test was introduced was transcribed and broken down into linking episodes as described by Alibali et al. (2014). These linking episodes were then analyzed to determine how many and what types of gesture were used in conjunction with the links being made. Links were coded as multi-modal if they were made using some combination of speech, writing, and gesture, and were coded as uni-modal if they were made in speech alone. In the transcription, speech made in conjunction with gesture are in bracketed pairs: [speech uttered][gesture description].

For this study, gestures were coded according to the static and dynamic categorizations defined by Garcia and Engelke Infante (2012, 2013) and the pointing and writing gestures defined by Alibali et al (2014). Engelke Infante and Garcia examined how students use gesture when solving related rates and optimization problems in first semester calculus and identified two broad categories of gesture use: dynamic and static (Garcia & Engelke Infante, 2012, 2013). Dynamic gestures consist of moving the hands to describe action that occurs in the problem or movements made to represent mathematical concepts. Within dynamic gestures two subcategories were identified: dynamic gestures related to the problem (DRP) and gestures that are not related to the problem (DNRP). We added a third subcategory to dynamic gestures, Dynamic Algebraic. Dynamic Algebraic (DA) gestures are made to indicate how we manipulate quantities in an equation. For example, we often speak of "moving x to the other side" and make an under/over swooping motion with a hand.

Static gestures are made to illustrate a fixed value (length, radius, volume, etc.) or to illustrate properties of a geometric object. Note that while the term static is used, that does not necessarily mean that there is no motion involved. Again, the category of static gestures was further sub-categorized into static gestures related to a fixed value (SFV) and gestures related to the shape of a graph (SSG). SFV and SSG are both primarily types of iconic gestures but may also be metaphoric. Garcia and Engelke Infante (2012, 2013) observed static gestures primarily being used to facilitate diagram construction.

As defined by Alibali et al. (2014), pointing gestures are those used to index objects, locations, and inscriptions in the physical world. Writing gestures are those in which writing or drawing is integrated with speech in a way in which a writing instrument is used to indicate content of the accompanying speech (such as drawing a circle around a term in an equation while stating 'this term'). We further categorized points as either static points (SP-pointing to one static object on the board, such as a point on a graph), tracing points (TP-using a finger, hand or other body part to trace the shape of an object on the board, such as a finger tracing along a tangent line), or generic points

(GP-points directing attention to something that was done before, but not a 'look at this particular thing' type of point).

Results

Here we examine how the five instructors introduced the second derivative test. In each case, the second derivative test was introduced in conjunction with the solution to an optimization problem. All of the instructors used a similar example problem to illustrate the second derivative test – construct a fence that maximizes area for a given perimeter (Instructors A, B, and C) or construct a fence of minimum perimeter for a fixed area (Instructors D and E). Instructors A, B and C introduced the second derivative test on the first day of instruction on optimization while the other two instructors introduced the second derivative test on the second day of instruction on optimization.

We observed that there were two key links made by every instructor. First, every instructor made explicit the link between the sign of the second derivative and the concavity. Second, every instructor made a link between the concavity (shape of the graph) and the optimizing function having a maximum or minimum value. Often, these two links occurred in succession as a single linking episode (see rows two and three of the table). Table 1 summarizes how these key links were made by each instructor and indicates whether the link was accompanied by gesture. Observe that Instructors A, B, and C made all of their links multi-modally while Instructors D and E made about half of their links multi-modally.

Table 1: Links made by calculus instructors while explaining the second derivative test

Linking Episode	Representations Linked*	A	B	C	D	E
f'' – concavity	A-G	1+ (SP - SSG)				
f'' negative – CCD – max	A-G-N	2+ (SSG-SP)	2+ (SSG-WG-SFV)	2+ (SFV-WG-RP)	1+ (GP)	
f'' positive – CCU – min	A-G-N	1+ (SSG-SP)			1+ (SP)	1+ (SP)
CCD – max	G-N	1+ (TP-SP)	1+ (SFV-TP)			
CCU – min	G-N	1+ (TP-SP)		1+ (SSG-SFV)		1
$f' = 0, f'' > 0$ – min	G-N				1	
$f' = 0, f'' < 0$ – max	G-N				1	

*A = algebraic, G = graphical, N = numerical, + = (gestures used); SP=static point, SSG=static shape of graph, SFV=static fixed value, TP = tracing point, WG = writing

In the interest of space, we will examine the differences in how Instructors A, B and C used gesture to highlight key information in their linking episodes. Instructor A presented the second derivative test before starting her example (about 3 minutes), and then reiterated it during her example problem (about 2 minutes). Instructors B and C both explained the second derivative test completely within the context of the example and used 2-3 minutes of class time.

Instructor A

During Instructor A's brief overview of the second derivative test, she drew a generic graph with a maximum and minimum value. As she pointed to the maximum value on her graph, she stated "we know we have [a maximum] [SP]," and then she pointed to the minimum on her graph [on the far right in Figure 1] and stated "we know we have [a minimum] [SP], but we don't know this starting

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out." She reviewed what a critical value was and asked the students "what does the second derivative tell us?" The students responded, "concavity." She proceeded to make a static point to the maximum on the graph and asked "[what's the concavity right here?] [SFV-graph on far right of Figure 1]" while making a circular motion with her hand around the maximum on the graph. This led to the conclusion that a concave down graph would have a maximum which was pointed to again. Similarly, the class arrived at the idea that a concave up graph would have a minimum. It was then indicated that they would dig into this idea in the next example. Here, we observe Instructor A making explicit the second key link between the concavity of the graph and the existence of a maximum (or minimum) and that she highlights these features on the graph with static gestures.
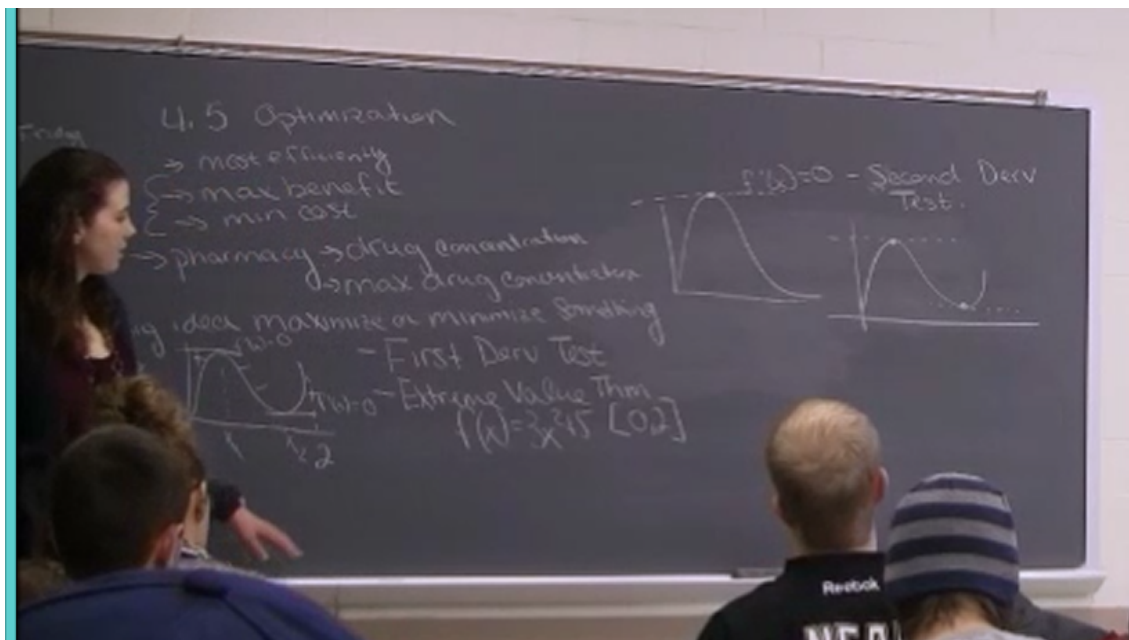
Figure 1. Instructor A's overview of the second derivative test.

The first example of an optimization problem presented to the class immediately following the discussion of the second derivative test was a problem involving a farmer who wishes to maximize a rectangular area along a river with a given amount of fence. Upon getting to the point in the problem where the maximum needs to be verified, the instructor discussed what the domain of the problem was and in doing so, made several tracing point gestures in reference to her diagram. She then stated, "We could use either the first or second derivative test, but since we haven't seen the second derivative test in action, let's do that." She continued, "So, second derivative, I'm going to look [at] [SP -to the first derivative function] the first derivative, right? So, the derivative of fifteen hundred is nothing and that's negative 4 (writes equation for second derivative on the board)." Her next statement included the first hand gestures relating the shape of the graph and the second derivative test. She stated:

A: So, I have either [a peak][SSG-made a small, cupped hand gesture mimicking the shape of a concave down graph] or [a valley][SSG-made a small, upward swooping hand gesture mimicking the shape of a concave up graph] because the derivative is zero. What does the [second derivative negative mean?][SP-to second derivative function] Concave up or concave down? [Down,] [WG-draws downward parabola shape] so it looks like that. Does that give you a max or min?

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Here, we observe Instructor A making explicit both key links with gesture (primarily pointing gestures) in a single linking episode.

Instructor C

The first example of how to solve an optimization problem presented by Instructor C involved finding the dimensions of the largest possible rectangular shaped vegetable garden with a fixed perimeter. As he neared the end of the problem, he introduced the second derivative test. He verbally explained that they computed the value for the second derivative and found it to be negative. At this point, he exclaims, "The second derivative is [ne-ga-tive] [SRF-arms up and out and pointing down with fingers] What does that mean the function is doing? Second derivative. Concave? [Down!] [SRF-points arm and finger down]." Hence, he has made explicit the first key link between the sign of the second derivative and the shape of the graph (concave down).

Next, Instructor C indicated, "When x is 40, the graph is concave down. If you draw just anything that's [concave down] [WG-draws concave down graph on board and makes point at local max] is the point a [maximum point?] [SP-points at the point made on graph]." Here, he made explicit the second key link between the concave down shape of the graph and the function having a maximum value.
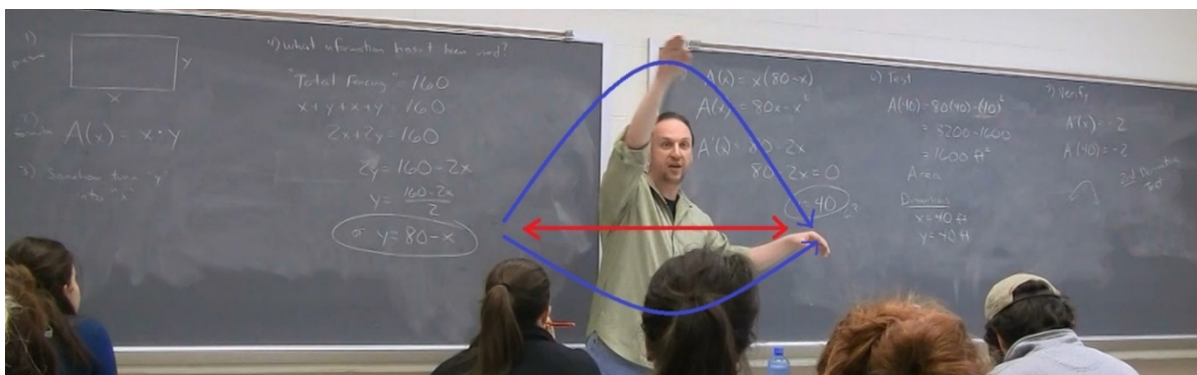
Figure 2. Instructor C's gestures to illustrate the second derivative test.

Instructor C then moved to the center of the classroom and exclaimed, "Watch! Watch! Watch!" and made sure that all eyes were on him before he proceeded to summarize the second derivative test with a sequence of large gestures (Figure 2). He demonstrated that a concave down graph would have a maximum value (blue arching motion above the red line) while a concave up graph (blue arching motion below the red line) must have a minimum value. He related each of these ideas back to the idea of a horizontal tangent line (which he indicated with the horizontal red line motion). This was the only instance we observed where the gestures were made as a deliberate part of the lesson.

Instructor B

Here we focus on the distinctly different way in which Instructor B makes the second key link between the concavity and the optimizing function having a maximum. She began by drawing a concave down graph on the board and making a small, one-handed concave down motion like Instructor C, asking whether there would be a maximum or minimum. She then proceeded to point to the maximum on the graph while stating that it was the maximum of the graph (Figure 3a). She took one step to the side of the graph and reiterated that a concave down graph could have a maximum while making the small, one-handed gesture again. Next, she took another step to the side and made a very large two-handed gesture while stating that since it was concave down everywhere, the value would have to be an absolute maximum (Figure 3b) followed by stepping back to her graph on the board and pointing to the maximum again.

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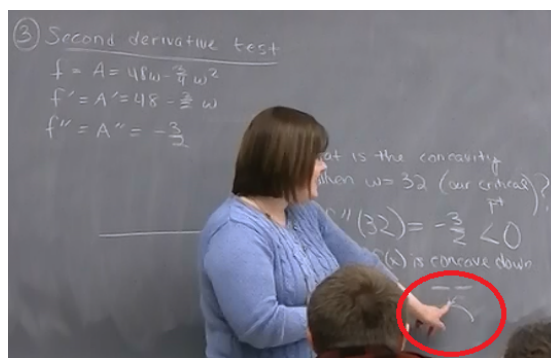
Figure 3a. Pointing to the maximum on a concave down graph.
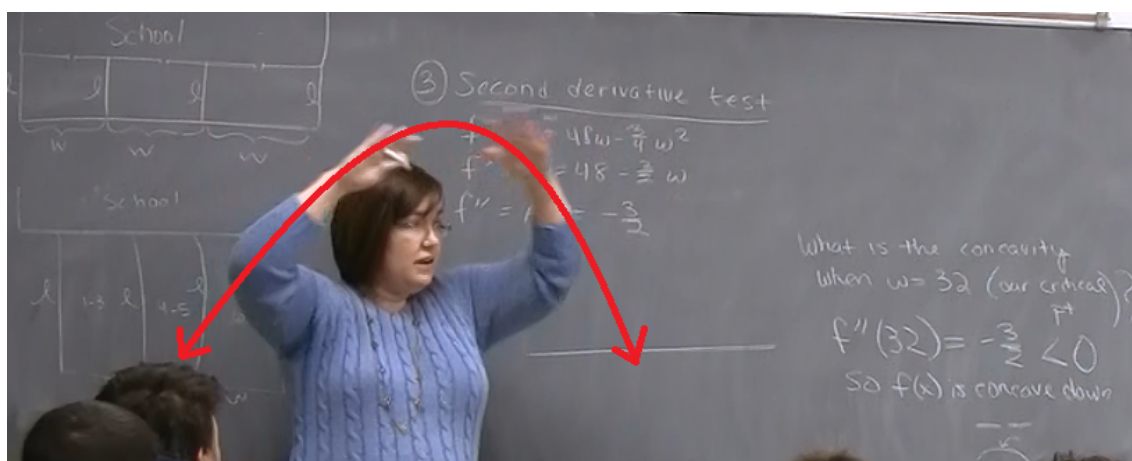
Figure 3b. Large, two-handed gesture for a concave down graph.

While making the second key link, Instructor B was very animated in her use of gesture to make sure students made the connection between the graph being concave down and the function having a maximum value. In fact, during this one linking episode she made the link three times, each of which was multi-modal. Instructor B used a combination of points at the board with large gestures in the space in front her.

Similar to the linking episodes observed by Alibali and colleagues in middle school teachers (Alibali et al., 2014), these instructors were linking several mathematical ideas for their students. Each instructor linked ideas about the second derivative function, the concavity/shapes of graphs, and maximum/minimum values during their lessons. Hence, we conclude that any linking episode about the second derivative test should include the link between the sign of the second derivative and the concavity and the link between the concavity (shape of the graph) and the optimizing function having a maximum or minimum value. However, from Table 1, we know that the manner in which these links are made can vary greatly. Indeed, Instructor D, summarized these key links uni-modally (in speech alone) by saying: "The first derivative test tells us that if f prime is equal to zero and f double prime is positive, we have to have a local min. If f double prime, if f prime is zero, f double prime is negative, we have to have a local max. If they're both equal to zero, we don't know." Note that these verbal statements fail to capture the link between concavity (shape of the graph) and the existence of a max/min.

Conclusion

While each of the instructors used a similar example problem to introduce the second derivative test, there was significant variation in the linking episodes used to convey the key ideas of the

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concept. As other studies have indicated, we observed that instructors frequently used gesture when referring to a graph or diagram to illuminate key features of the object. These findings support theories about connections between diagrams and gesture presented by de Freitas and Sinclair (2011).

As observed by (Alibali et al., 2014), instructors were often making links between the various representations of the concept. Four of the five instructors used a gesture that was depictive of a concave down (up) graph while emphasizing with words that a concave down graph would have a maximum (minimum) value. Some instructors went beyond this, as Instructor B did, and pointed at least once to the maximum on the graph that she had drawn.

Not all instructors used the same quantity or distribution of gestures. However, quantity is not the only measure of gestures usefulness. An appropriately timed gesture of a particular type may be equally important. For example, Instructor C made certain that his students would be paying attention to at least one sequence of gestures he made. Future research will employ a design experiment in which we will examine the impact of particular gestures used during instruction on student learning. We will assess students' learning based on whether they learned from an instructor who deliberately incorporated particular gestures or an instructor who did not purposefully incorporate gesture into the lesson. In time, we expect to create an online catalogue of lessons to showcase how the deliberate incorporation of gestures can enable instructors to foster greater understanding of calculus concepts in their students.

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TEACHERS' INSTRUCTIONAL PRACTICES WITHIN A CONNECTED CLASSROOM TECHNOLOGY ENVIRONMENT TO SUPPORT REPRESENTATIONAL FLUENCY

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The purpose of this study was to examine the ways that teachers use Connected Classroom Technology (CCT) to potentially support achievement on translation problems that require moving between algebraic representations. Four mathematics classrooms were chosen based on their gain scores on pre- and post-test Algebraic translation problems. Two classrooms with the highest and the lowest gain scores were chosen among the classrooms with pre-test scores that were below 50%. This study used video-recorded observational data and found that teachers in effective classrooms created environments wherein students used multiple representations simultaneously and translated between representations through discussion. In contrast, teachers in less effective classrooms fostered environments wherein students used representations independently and missed opportunities to translate representations through discussion.

Keywords: Instructional Activities and Practices, Technology

Representational fluency is a cognitive competence that includes being able to interpret and construct representations as well as translate flexibly between them (Sandoval, Bell, Coleman, Enyedy, & Suthers, 2000). It is considered both a mechanism for supporting the development of deep conceptual understanding (Duncan, 2010; Pape & Tchoshanov, 2001) and a means of assessing conceptual understanding (Suh, Johnston, Jamieson, & Mills, 2008).

Representational fluency not only supports conceptual understanding but is also an essential component for problem solving (Nistal, Van Dooren, Clarebout, Elen, & Verschaffel, 2009). Students are more successful when they possess the ability to translate between representations as well as use multiple and non-symbolic representations (Brenner et al., 1997; Nathan & Kim, 2007). The education community continues to emphasize the importance and need for developing representational fluency (National Governors Association Center for Best Practices, Council of Chief State School Officers [CCSSO], 2010). However, students leave school without attaining representational fluency (Ainsworth, Bibby, & Wood, 2002; Herman, 2007; Knuth, 2000).

There are at least two factors that potentially support the development of representational fluency: communication and technology. Through communication, representational fluency may be supported by active engagement in discussions about interpretation, construction, evaluation, comparison, and generalization of representations (Warner, Schorr, & Davis, 2009). The development of representational fluency may be supported by allowing for quick access to multiple representations (e.g., symbolic, tabular, and graphical) through technology (Bieda & Nathan, 2009). Because evidence suggests that communication and technology may separately support students' developing representational fluency, the present study investigated instruction that is characterized by the use of Connected Classroom Technology (CCT) with the aim of examining the relationship between these instructional strategies and increasing such fluency.

Connected Classroom Technology

CCT are wireless communication systems that connect the teacher's computer and students' handheld technology (Pape et al., 2013, p. 169). These systems are designed to provide greater opportunities to discuss connections among multiple representations. Recent studies emphasize a sociocultural perspective that focuses on the relationship between learning opportunities and students' abilities to take advantage of these opportunities during learning (Gee, 2008). CCT

provides at least two learning opportunities that support the development of students' representational fluency. These opportunities include "the mobility of multiple representations of mathematical objects" and "the ability to flexibly collect, manipulate and display to the whole classroom representationally-rich student constructions, and to broadcast mathematical objects to the class" (Hegedus & Moreno-Armella, 2009, p. 403). Jim Kaput once postulated, "wireless connectivity 'inside' the classroom would change the communicative heart of the mathematics classroom" (Hegedus & Penuel, 2008, p. 171). CCT's progression has potential to enact this transformation.

The Texas Instruments (TI) Navigator system includes two components to support representational fluency: Screen Capture and Activity Center. With Screen Capture, teachers can project a "snapshot" of students' calculators. This feature allows both teachers and students to compare the representations through productive discussion. With Activity Center, the teacher can project a coordinate plane on which students submit points, equations, and graphs. Another affordance is that it promotes examination and analysis of patterns as well as justification of mathematical generalizations (Hegedus & Moreno-Armella, 2009).

These components provide a context for effective classroom discourse because they are designed to publicly display multiple linked representations. The public display of students' mathematical constructions in conjunction with the communication of ideas and strategies fosters representational expressivity (Hegedus & Moreno-Armella, 2009). Also, the activities in the activity center with multiple representations may support translation between representations (Bostic & Pape, 2010), which are distinguishing characteristics of mathematical proficiency (CCSSO, 2010; Kilpatrick, Swafford, & Findell, 2001). Although teachers have found CCT to be an efficient means of instruction, there is little evidence that demonstrates or evaluates its effectiveness in relation to supporting representational fluency (Vahey, Tatar, & Roschelle, 2007).

Method

Teachers who participated in classroom observations during the third year of a four-year project were chosen for the present study. Forty of the 41 teachers' classroom observations served as the initial pool of observations with one being eliminated due to poor audio quality. Data sources consisted of classroom videos (typically two class periods), their verbatim transcripts, and algebra pre- and post-test. The length of each observation was between 48 and 97 minutes. To measure representational fluency, translation problems were extracted from the pre- and post-tests. Translation problems are those in which the initial representation (i.e., input) and the answer's representation (i.e., output) are different (Nathan, Stephens, Masarik, Alibali, & Koedinger, 2002).

There were two criteria for participant selection. First, classroom observations that focused on quadratic equations were considered. Second, classrooms with initial mean pre-test scores below 50% were considered. Among the classroom observations, the two teacher's classrooms with the highest gain (i.e., Ms. BW and Ms. MB) and the two teachers' classrooms with the lowest gain scores (i.e., Ms. MA and Ms. JR) were selected. Gain scores were calculated as the percentage of the maximum possible change (i.e., (Post – Pre)/ (Maximum Score (100) – Pre)). The cases were viewed without knowledge of effectiveness. Selected teachers' demographic information is listed in Table 1.

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Table 1: Teacher Participants Demographic Data

Classroom	Teachers	Year of CCT use	Undergraduate Major	Graduate Major	Years Teaching Experience
Effective	Ms. BW	2	Pre-Vet Med	Ph.D., Animal Feeding/Animal	13
	Ms. MB	3	Communication	MA, Journalism	3
Less Effective	Ms. MA	2	Mathematics	MA, Educational Computing	21
	Ms. JR	2	Mathematics	Curriculum and Instruction	20

This research employed a qualitative study design and analysis (Hatch, 2002) with the intent of providing contrasting or illustrative instances in instructional use of representations. To increase the credibility and validity of the conclusions, *usefulness, the chain of evidence, truthfulness,* and *reporting style* strategies were applied (Gall, Gall, & Borg, 2007).

Results

The teachers' practices in terms of how they supported students to engage with representations were explored in this study. Following the initial coding, the classes' effectiveness status was identified, and cross-case analysis of the two categories of cases were compared. Two major themes that potentially support representational fluency were identified. In each subsection, a theme and how it was practiced in both effective and less effective classrooms is described.

Using Different Translations

Unidirectional and bidirectional translations were observed during these classroom observations. Unidirectional refers to translating between different representations within the same activity (e.g., Symbolic1 → Graphical → Symbolic2) and bidirectional refers to translating between the same representations within the same activity (e.g., Symbolic1 → Graphical → Symbolic1). The students in all four classrooms frequently used different unidirectional translations. In effective classrooms, however, unidirectional translation was observed only once in Ms. MB's classrooms. All unidirectional translation sequences and the longest sequence of translations were observed in Ms. BW's classroom, an effective classroom. Many unidirectional translations were observed in Ms. MA's classroom video recordings, a less effective classroom. Although Ms. JR did not include translations in her first class period, the longest unidirectional translation among less effective classrooms was observed in her classroom. Although many unidirectional translations were observed in less effective classrooms, the students did not generally translate between representations; instead they observed their teacher's translations. Translations used in the unidirectional category are summarized in the Table 2.

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Table 2: Unidirectional Translations

Classroom	# of Translations	Translation Sequence	Teacher Practiced
Effective	One	Graphical → Symbolic	Ms. MB
	Two	Symbolic1 → Graphical → Symbolic2	Ms. BW
		Tabular → Symbolic → Graphical	Ms. BW
	Three	Symbolic1 → Graphical → Symbolic2 → Verbal	Ms. BW
	Four	Symbolic1 → Graphical → Tabular → Symbolic2 → Graphical	Ms. BW
Less Effective	Two	Symbolic1 → Graphical → Symbolic2	Ms. MA
		Symbolic → Tabular → Graphical	Ms. MA
		Pictorial → Tabular → Graphical	Ms. MA
		Tabular → Symbolic → Graphical	Ms. MA
	Three	Symbolic1 → Graphical → Symbolic2 → Graphical	Ms. MA
	Four	Symbolic1 → Graphical → Tabular → Symbolic2 → Graphical	Ms. JR

In addition to unidirectional translations, one of the main features that differentiated the effective and less effective classrooms was the presence of bidirectional translations, which were observed only in the effective classrooms. Using four translations including bidirectional translation in Ms. BW's class might have improved her students' translation abilities because it includes many representations and translations (Figure 1).

(a)   (b)   (c)

(d)   (e)
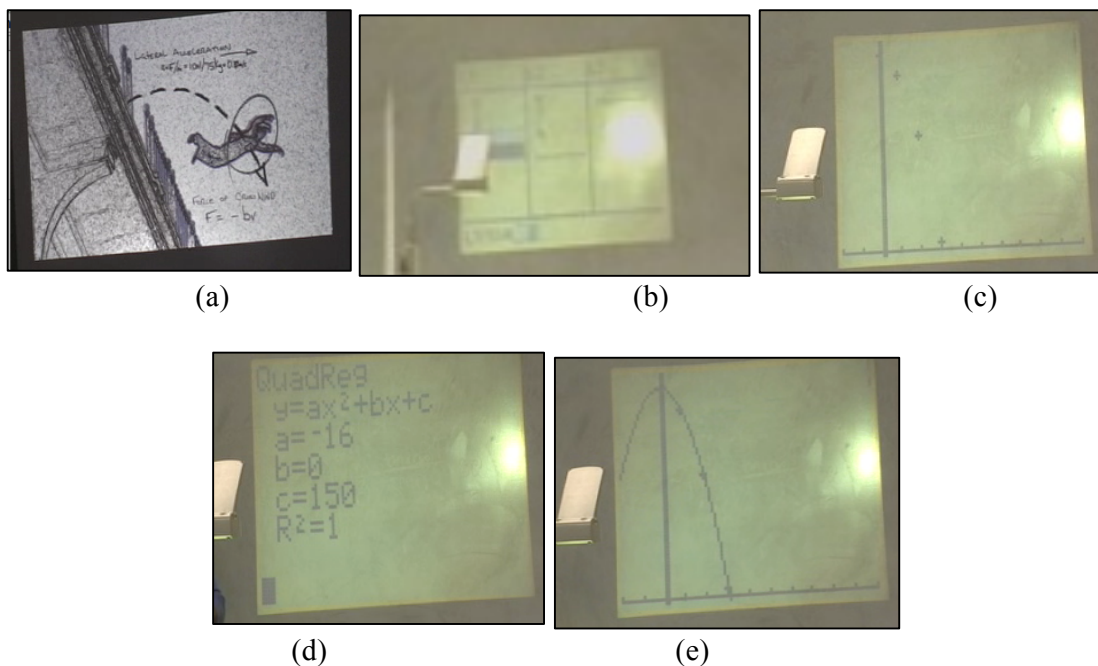
Figure 1. Screenshots from the students' work on the detective problem. This figure illustrates the use of (a) pictorial, (b) tabular, (c) graphical, (d) symbolic, (e) graphical representations, respectively.

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Table 3 displays the sequences of translations in each of the effective classrooms. Although, there was only one bidirectional translation in each of the effective classes, the activities in which these translations were observed occupied a substantial amount of class time.

Table 3: Bidirectional Translations

Classroom	# of Translations	Translation Sequence	Teacher Practiced
Effective	Two	Symbolic1 → Graphical → Symbolic1	Ms. MB
	Four	Pictorial → Tabular → Graphical1 → Symbolic → Graphical1	Ms. BW
Less Effective	None		

In this study cycling translations were observed where students iteratively translated between representations bidirectionally until they reached the correct solution. Both teachers in the effective classrooms provided tasks that allowed their students to view multiple representations simultaneously (Figure 2).
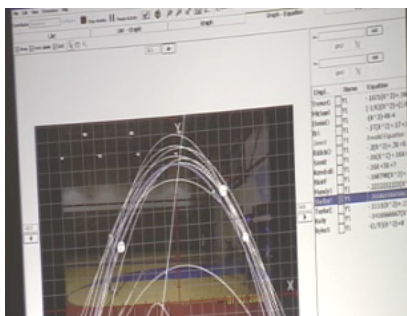
Figure 2. Using dynamic representations simultaneously. This figure demonstrates the cycling translation between graphical and symbolic representations.

These students saw that each modification they made to the coefficients simultaneously changed their graph. They could see the coordinates of a point on the parabola while changing the point's location. Finally, cycling translations were observed twice in effective classrooms: (a) two translations with one cycling, and (b) three translations with one cycling. On the other hand, in the less effective classrooms, a cycling translation (i.e., Symbolic → Graphical ↔ Symbolic) was observed only once in one of Ms. JR's class (Table 4).

Table 4: Cycling Translations

Classroom	# of Translations	Translation Sequence	Teacher Practiced
Effective	At least two	Symbolic → Graphical ↔ Symbolic	Ms. BW
	At least three	Pictorial → Symbolic → Graphical ↔ Symbolic	Ms. MB and Ms. BW
Less Effective	At least two	Symbolic → Graphical ↔ Symbolic	Ms. JR

These translations were not, however, used within a real-world context, and the students did not interpret the representations. Ms. JR also gave many hints during the cycling process, which potentially limited the students' independent thinking. An additional difference was the fact that representations were not generally dynamically linked within the less effective classrooms. The students in these classrooms typically explored representations independent of one another.

Students perform better when they use more and multiple representations (Bostic & Pape, 2010; Herman, 2007; Nathan & Kim, 2007), when they have the ability to translate between representations (Brenner et al., 1997), and when they use non-symbolic representations (Suh & Moyer, 2007). Although students in both the less effective and effective classrooms used multiple representations, only the students in the effective classrooms generally used bidirectional and cycling translations. Since the new version of handheld calculators enables bidirectional translations (Özgün-Koca & Edwards, 2009) and repairing representations is a practice that supports learning (Warner et al., 2009), teachers can create environments where students have the opportunity to modify their representations until they arrive at the most accurate representation of their thinking. The students might be more successful if they can use bi-directional and cycling translations in conjunction with multiple unidirectional translations.

Scaffolding Translation through Teachers' Questioning

The teachers in the effective classrooms, Ms. MB and Ms. BW, asked questions to promote students' translation between representations. Ms. BW usually asked questions requiring short answers. When Ms. MB realized that her students seemed lost or confused or if she needed a student to clarify an answer, she would ask follow-up, open-ended, or hypothetical questions. Ms. MB invited all students to participate. She also encouraged them to share their solutions and opinions while solving the problems.

The teachers in the less effective classrooms, Ms. MA and Ms. JR, missed many opportunities to create discussion-rich environments. Also, by providing hints or asking questions that led to obvious generalizations, both teachers did not adequately challenge their students during problem-solving activities. Thus, the questioning techniques they used did not require the in-depth thinking that would encourage students to make translations between representations. In addition, Ms. MA and Ms. JR did not sufficiently interact with their students when they made mistakes. Instead, they quickly provided explanations. When discussing alternative ways of solving a problem, Ms. MA often started using her method without allowing her students adequate time to think for themselves.

Significance of the Study

This study provides a description of four teachers' practices and provides models for how teachers might construct their classroom to better promote their students' representational fluency abilities. They may offer examples to mathematics teacher educators when they prepare pre-service teachers or provide professional development for in-service teachers.

Teachers should be aware of their students' representational knowledge and seek technological or cognitive tools to visualize their students' thinking. Through the use of CCT such as the TI-Navigator, instructors can monitor and assess their students formatively to adapt their instruction based on their students' needs and misconceptions. Ultimately, teachers should create environments for students to interpret representations by linking them to real-world scenarios. Students should not only be able to see multiple representations on one screen but should also see the changes to a representation as its related representation (i.e., algebraic) is modified. Moreover, students should be provided more opportunities to make judgments about the accuracy of their representations and to change them as appropriate during problem-solving activities, which can be accomplished by including activities that require bi-directional and cycling translations. In addition, teachers should frame questions that facilitate students' developing understanding of representations and translations over time. One way to promote such sustained thinking is to foster discussion-rich environments.

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NOT ALL QUESTIONS ARE ALIKE: CREATING A CATEGORIZATION FOR STUDENT QUESTIONS IN MATHEMATICS LESSONS

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Students ask a variety of questions during mathematics lessons. Some questions focus on procedures or formal notation, while others focus on ideas, connections, or representations. Student questions are important as they provide vastly different learning opportunities for students. In order to support teachers as they help students learn to ask productive questions in mathematics lessons, we need to first understand the types of questions students ask. This paper describes the process of creating a categorization for student questions in middle and high school mathematics classrooms.

Keywords: Classroom Discourse, Middle School Education, High School Education

Question asking is an important part of the learning process. The National Council of Teachers of Mathematics (NCTM) *Principles and Standards for School Mathematics* (2000) states that "students gain insight into their thinking when they formulate a question about something that is puzzling to them." Question asking affords students opportunities to extend their own understanding of lessons as well as to raise personal concerns about topics discussed in class (Daly, Kreier, & Roghaar, 1994). Posing questions not only shapes students' thinking, but also exposes it, which provides opportunities for teachers to learn more about their students' understanding (Marbach-Ad and Sokolove, 2000). Students who ask questions retain material better than those who do not (Marbach-Ad & Sokolove, 2000), and asking questions in mathematics lessons is linked with motivation (Stipek, Salmon, Givven, Kazemi, Saxe, & MacGyvers, 1998).

While it is clearly a useful activity for students to ask questions in classrooms, students often refrain from asking questions because they fear a negative reaction from their teacher (Pearson & West, 1991). Students also avoid asking questions because they "don't want to look stupid" in front of their peers (Kemmerle, 2013). Unfortunately, students who most need the benefits of question asking are often the ones who are afraid to ask questions, and over time, lower-achieving students ask fewer and fewer questions (Good, Slavings, & Mason, 1988). The converse is true as well— confident students who feel empowered in their learning ask more and more questions over time and reap the benefits (Kemmerle, 2013). Pearson & West (1991) found that higher achieving students ask more substantive questions, and lower-achieving students ask more procedural questions.

Research has focused on student question asking in science education (see Marbach-Ad & Sokolove, 2000; Kelling, Polacek, & Ingram, 2009; Rop, 2002 for examples) and in English Language Arts instruction (see Rosenshine, Meister, & Chapman, 1996 for a review), and studies have looked at *teacher* questioning patterns in mathematics classrooms (see Boaler & Brodie, 2004; Boaler & Humphreys, 2005) but there are very few studies that specifically address *student* question asking in *mathematics* classrooms. The importance of the issue as well as the gap in the literature prompts my desire to contribute to this area. In order to better understand the dynamics around student question asking in mathematics lessons, it is helpful to develop understanding of the different kinds of questions students ask while they are learning mathematics. This paper presents analysis and a categorization of such questions. In order to do this, I observed six middle school and high school mathematics classrooms and collected all student questions that arose during my observations. This produced 1,737 student questions which were grouped into 15 distinct categories. Initial inter-rater reliability came in at 74% which caused a further refinement of the codes until full agreement was reached. Details of the categorization follow.

Methods

Data Collection and Analysis

The study focused upon six classrooms chosen to offer a variety of instructional styles. The choice of substantially different learning environments enabled me to compare, contrast, and synthesize student questions across the cases (Miles and Huberman, 1994). See Table 1 for a summary of the six participants.

Table 1: The Six Teachers in this Study

Teacher	Grade	School and Classroom Information
Mr. Cordoba	8th	Small charter school in an urban district; 100% of students eligible for free/reduced lunch; mostly Latino/Hispanic student body; teacher focus on social justice and an "open" approach to mathematics (Boaler, 1998); approximately 50% of class time spent in small groups
Ms. Lyndon	7th	STEM middle school in an urban district with an emphasis on design thinking methodologies; 60% of students eligible for free/reduced lunch; student population is mostly Latino/Hispanic, Asian, and White; approximately 75% of class time spent in small groups
Mr. Ezzo	6th	Affluent suburban middle school where only 4% of students qualify for free/reduced lunch and student population is mostly white; teacher espouses a fairly "traditional" approach to instruction; teacher encourages his students daily to ask many questions.
Ms. Gibson	10th/12th	Medium sized high school in a mostly white but economically diverse school district; teacher uses a non-traditional curriculum designed to blend algebra, geometry, trigonometry, statistics in a four-year mathematics sequence
Ms. Kapoor	9th	Nationally-ranked, large high school in an extremely affluent suburban area; mostly white and Asian student population; classroom instruction consists mostly of lecture and note-taking; school culture is extremely competitive
Ms. Chang	10th	Independent day and boarding school for students who will be the first in their families to go to college; located in a historically underserved neighborhood; teacher blends traditional with innovative methods of instruction and values self-efficacy and sense-making

Data for this study consisted of approximately 45 hours of transcribed videotape, extensive field notes, semi-structured interviews with classroom teachers, and student surveys. I went through several iterations of transcription of the videotapes—during the first pass through the videos, I highlighted all student questions in the dialogue and took notes about each student question, thinking carefully about the function of each student question (Saxe, 1991). After I had worked through all the transcripts once, I went back and open-coded them again, using what I had learned the first time through the transcripts. I started to see patterns and similarities in the student questions and further refined my codes as I worked through more and more transcripts (Glaser and Strauss, 1967). I collaborated with my research group (Engle, Conant, & Greeno, 2007) about possible functions or purposes of student questions. We considered specific student questions that might epitomize a certain kind of question, in other words, a "case" of a kind of question (Shulman, 1992). I used the transcripts from my interviews with each teacher to gain more insight into their thoughts and feelings about how student questions affect mathematics instruction and student learning and to understand

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more about why different types of student questions appear in different classrooms. Lastly, I used short-answer responses from student surveys to hear directly from students about their experiences asking questions in their mathematics lessons and to triangulate my findings.

Results

What types of questions did students ask?

I begin by describing each of the classroom contexts and illustrating the types of questions that were asked in each of these classrooms. Mr. Cordoba believes students should be "active participants in mathematical activity through generating and discussing mathematical ideas in class." He presents his students with tasks that require high cognitive demand (Smith & Stein, 1998), and asks them to collaborate (more than half of class time is spent in small group work). See Table 2 for a small sample of questions Mr. Cordoba's students asked during my observation along with my initial codes for each question. The curiosity in Mr. Cordoba's students' questions and the desire to extend mathematical ideas to slightly different situations is notable. Mr. Cordoba's students also asked questions with the goal of helping each other learn.

Table 2: Questions from Mr. Cordoba's classroom

Student Question	My initial code
Can I rephrase his question?	Helpful to classmate/community building
Can k be an odd number, like 27, and could you still have x and y the same number?	Mathematical curiosity question; also seeking specific information
If k were to be bigger and the k was a negative, where would it be?	True extension question--student is changing a few components of the problem and thinking about what that would mean
What is your equation to figure out y?	Seeking procedural/algorithm information
How did you get k is zero?	Seeking conceptual information
How does it show it in the trend line? Would it ever cross the x line?	Mathematically curious question/Representation question; seeking conceptual information
Does your y have to be negative?	Seeking conceptual information, explanation, meaning
Can your x be bigger, like if your number is 24 or lower, can the number you're multiplying be bigger and multiply by a decimal?	Another true extension question. Also shows mathematical curiosity

Ms. Lyndon's approach to teaching mathematics is to emphasize sense-making, intellectual risk taking, and mathematical communication in her classroom. Ms. Lyndon's students spend the majority of class time collaborating on group-worthy tasks (Cohen & Lotan, 2014). Ms. Lyndon intentionally and consistently teaches her students how to ask each other questions and how to work together to solve problems. See Table 3 for a sample of student questions from Ms. Lyndon's classroom and note that her students ask questions that think critically about various approaches to problems. They seek to make connections from one idea to the next and they want to find the best or most efficient approach to a problem.

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Table 3: Questions from Ms. Lyndon's classroom

Student Question	My initial code
How does this relate to Alyson's method?	A question that connects a new mathematical idea to something already known/ A different approach question
In the corners, should we write $n-1$?	Student is wanting to know what teacher expects for the presentation of the work
So how did you get your picture?	Student wants to understand a visual representation of the mathematics
What's useful about this?	Mathematically curious question; also seeking connection to other mathematics, other subjects, or the real world
How did you get 34?	This could be a question that only seeks for procedural information, but looking closely at the context reveals that the student is trying to understand conceptually
Wait, why did you minus it?	Student trying to understand the mathematical procedure
I'm confused about x. What does x mean?	Asking for clarification
What's good about this method?	Procedure/algorithm question, but it's more than just asking how to do something; it's thinking critically about the merits of the method itself

Mr. Ezzo's approach is what some might call traditional: every day class begins with a scoring of the previous night's homework, followed by a lecture on new material. Mr. Ezzo espouses a "closed" (Boaler, 1998) approach to mathematics in that each student is expected to solve mostly context-free problems each day in the precise way they are taught by their teacher. See Table 4 for a sample of student questions from Mr. Ezzo's classroom and note that in Mr. Ezzo's class, many students are focused on getting the right answer, pleasing the teacher, and making sure they score well on homework and exams.

Table 4: Questions from Mr. Ezzo's classroom

Student Question	My initial code
I did 2 and 75 hundredths, does it count?	Meeting teacher's expectations (about presentation of answer)
What was the answer to 23 again?	Seeking information about a correct answer
Wait, how is it 3 out of 5?	Seeking meaning/conceptual information
Should we copy that down?	Classroom routine/meeting teacher's expectations about note-taking
Could we also just do 4 over 10?	Clarification question—this feels conceptual, knowing that 4/10 is equivalent to 2/5; this question is also focused on form of the answer
Once you find the first one, don't you just have to keep adding it on to find the next one?	Question about a procedural/method
Do you think we'll be able to do some homework in class?	Classroom routine question
Is it okay if I do it the way you just showed us?	Seeking go-ahead for a particular method
Can we do any of these ways on the test?	Assessment question

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Ms. Gibson considers herself a math reformer and uses a non-traditional curriculum that "requires a great deal of collaboration, investigation, experimentation, and question asking." Ms. Gibson expects her students to present their mathematical thinking at the board each day and she emphasizes sense-making and a safe learning environment. See Table 5 for a sample of student questions from her classroom. These questions suggest that Ms. Gibson's students want to understand the mathematics conceptually and also to know how the mathematics is useful.

Table 5: Questions from Ms. Gibson's classroom

Student Question	My Initial Code
Why are you finding the inverse?	Seeking conceptual information
Are all the lines called curves on a graph?	Question about terminology
I don't understand that last part. Why do we need the 225?	Seeking conceptual information, seeking to understand
Don't we need two measures to find the minor arc and don't we only have one?	Clarification, trying to better understand the procedure
Would you put this kind on the test?	Assessment
Do you want me to try to explain?	Community building
I have a question about when I would use a t-table, like at a job.	Application to real world

Ms. Kapoor's classroom follows a traditional format of homework review, new lecture, and new homework. For the most part, Ms. Kapoor students sit quietly and take notes throughout instruction. Ms. Kapoor states that she wants her students to be ready for college and "expects a lot" from her students. Ms. Kapoor's tests are known to be extremely difficult. Table 6 offers a sample of student questions from Ms. Kapoor's classroom. These questions suggest that Ms. Kapoor's students are anxious to please her and to score well on assignments and exams.

Table 6: Questions from Ms. Kapoor's students

Student Questions	My initial Code
When do we get the test back?	Seeking information about assessment, indicates concern for his/her grade
Is this right?	Seeking confirmation of a correct answer or method
Is there a calculus [state] test? What's the last year we have to take it?	Seeking information about an assessment
How many pages are on the test?	Question about assessment
Where do you want us to turn this in?	Classroom routine, meeting teacher expectations
Do you want us to draw it or label all the points?	Note-taking question, shows desire to please teacher
Wouldn't you have to do the angle bisector, inscribe the circle, and then do some nightmarish math?	Seeking confirmation of a method/procedure/approach
"What's the center of the triangle, would that be the orthotriangle? Circumcenter?"	Terminology question
"This just occurred to me, when it says counterclockwise 90 degrees, does that mean around the origin?"	Seeking information about meaning, conceptual information
"So what do I do here?"	Seeking information about procedure

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Finally, Ms. Chang, the youngest of all six teachers, recently earned her teaching credential at a teacher education program that focuses on teaching for understanding. She states that she wants all her students to "understand the math on a deep level." The mathematics department at her school developed the curriculum she uses, and Ms. Chang expects all her students to participate actively each day by taking notes, asking questions, and presenting problems at the board. Table 7 shows student questions from Ms. Chang's classroom. These questions suggest that Ms. Chang's are interested in getting the right answer and understanding and exploring the mathematics.

Table 7: Questions from Ms. Chang's classroom

Student Questions	My Initial Code
"Did I do it right?"	The student seeks confirmation from the teacher that she is correct.
"Is it okay to graph the original one first, and then shift it?"	Seeking information/approval about a method or approach
"Why is the period still 2π?"	Seeking conceptual information
"If the 2 was a ½, would it be a stretch?"	The student is exploring and speculating mathematically. Mathematically curious question; extension question
"Can I explain?"	Community building
"Do you put that in increments of π if you were to label it?"	Representation/notation question
"So, what is a phase shift?"	Terminology question
"How many points are on the quiz?"	Assessment question

As I observed and analyzed the student questions in Ms. Kapoor's and Ms. Chang's classrooms, I realized that no unusual or unexpected student questions emerged, thus indicating that I had reached data saturation (Marshall, 1996; Guest, Bunce, & Johnson, 2006). I thus stopped data collection and focused my efforts on refining, naming, and describing my categories. When determining the boundaries of each student question category, I thought carefully about the specific functions different questions served for the students asking them (Saxe, 1991). After many iterations of sorting and thinking about names for the categories, I produced 15 categories (See Table 8).

Table 8: Not All Questions are Alike—Student Question Categorization

Question Category	Examples from Transcripts
Form of Answer	-Do I have to show it as a repeating decimal, or can I round up?
Correct Answer	-But what is the answer? -I got 3 and 7 tenths, does it count?
Assessment/Grading	-Could I get extra credit for this? -How many points are on the quiz?
Meeting Teacher's Expectations	-Wait, do we copy down the percents bar model too? -Did I do it right?
Seeking Procedural[1] Information	-How did you get 16? -What is your shortcut for filling in your t-table?
Seeking Conceptual[1] Information	-Why did you have to cross out the 40 and the 10? -Why is the period still 2π? -Why are you finding the inverse?
Visual Representation --Conceptual --Informational	-If your x were to get bigger and then the trend line, would your points, can it ever cross the x line? -Where would it be on the graph? -Do you put that in increments of π if you were to label it?
Community Building	-Can I rephrase his question? -Do you want me to explain?
Making Connections	-How does this connect with what Alyson did? -Did you all use the same method to find all of your k's?
Mathematical Curiosity/ Extension	-Can k be an odd number, like 27, and could you still have x and y be the same number? -If k were to be bigger and the k was negative, where would it be? - If the 2 was a ½, would it be a stretch?
Terminology	-So what is a phase shift? -What's the center of the triangle, would that be the orthotriangle? Circumcenter? -Are all the lines called curves on a graph?
Application to Real-World	-I have a question about when I would use a t-table, like at a job. -What's useful about this?
Clarification/Confirmation	-Do you mean like that there is only some in the positive side and none in the negative side?
Notation	-Should I write f(x) or y?
Classroom Routine	-Can we go over 9? -Will there be time at the end to work on homework?

Conclusion

In this paper, I present a categorization schema for student questions within mathematics lessons. Such a categorization helps define and articulate differences between the various types of questions students ask in mathematics lessons. This tool can help researchers, educators, and teachers think more carefully and analytically about the student questions within mathematics lessons. This type of analysis will potentially lead to deeper, more conceptual, more curious mathematical questions from students.

Endnotes

[1]The difference between Seeking Procedural Information and Seeking Conceptual information is difficult without the context. While it is usually possible to identify the other categories with just the question alone, these two categories are often hard to differentiate without the before and after dialogue.

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UNPACKING TEACHERS' PERSPECTIVES ON THE PURPOSE OF ASSESSMENT: BEYOND SUMMATIVE AND FORMATIVE

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The current age of standardization and accountability in education presents a need for educators to be proficient consumers and producers of data. To understand what mathematics teachers see as the purpose of data and assessment, as well as their perceived needs in these areas, we administered a survey to all mathematics teachers in two school districts in the Midwest. We analyzed perceptions of the purpose of assessment using a framework that breaks from the common formative/summative divide and considers the purpose(s) for assessment as being nuanced and potentially multi-faceted. The responding teachers demonstrated a robust working knowledge of assessment focusing on purposes that serve students, teachers, schools, and states. Responses also indicated that there are gaps in connecting the purposes of assessment and their knowledge on how to implement assessments in ways that meet these purposes.

Keywords: Assessment and Evaluation, High School Education, Teacher Knowledge

In the current climate of accountability, school administrators and teachers face unprecedented demands to be assessment-literate and use a wide range of data to inform educational decisions that document and promote student achievement. Accordingly, *data-driven decision-making* (DDDM) has become a central focus for educational policy and practice at all levels as an innovative strategy for school system and instructional reform (Gill, Borden, & Hallgren, 2014; Luo, 2008; Mandinach, 2012). DDDM is broadly defined as "the use of data analysis to inform choices involving policies and procedures" (Gill et al., 2014, p. 338). It is a complex undertaking, even for experienced educators and administrators equipped with quantitative skills (Mandinach, 2012). Developing the data literacy required for effective implementation of DDDM involves systematic experience with a variety of data (Mandinach & Gummer, 2013), including demographic, behavioral, achievement/ performance, attendance, financial, policy, programmatic, and compliance data. The data most under teacher and school control, however, are gained from classroom or school-level assessment. With this study we address a critical need to understand teachers' current assessment practices and current understanding of how to connect instruction, assessment, and related data in a way that can inform teaching practice (Datnow & Hubbard, 2015).

It is estimated that teachers spend up to one half of their classroom instruction time on some form of assessment (Stiggins, 1999). What then, do teachers see as the purpose(s) for devoting teaching time to this activity? In this paper, we share one piece of a larger study of DDDM by focusing on the critical component of assessment. We address the research questions: (a) *What do mathematics teachers see as the main purposes of assessment, and* (b) *what challenges impede teachers' use and understanding of assessment data?*

Background & Framework

Effective DDDM requires adequate *data literacy*, which can be broadly defined as the ability to understand and use data effectively to inform decisions (Mandinach & Gummer, 2013). Data-literate educators are a driving force of student learning, and ensure the continued success and funding of their schools in the era of accountability where data-based evidence plays a prominent role in both instructional and evaluative decision making (Orland, 2015). *Assessment literacy* is a critical component of data literacy, often defined broadly as being able to recognize sound assessment, evaluation, and communication practices to benefit student learning and achievement (Stiggins,

1999). As part of data literacy, educators integrate assessment results with other data reflecting content, context, perception, motivation, process, and behavior (Mandinach & Gummer, 2013).

Assessment, including both high-stakes and classroom assessments, can be one of the most influential activities in education. Research, however, suggests that teachers enter the profession with insufficient practice and knowledge for developing assessments for learning, evaluating student progress, and interpreting data (Bocala & Boudett, 2015; Popham, 2009; Volante & Fazio, 2007). When courses are offered in teacher education, topics tend to focus on the use of assessment for evaluating student outcomes rather than on the use of assessment as part of student learning (DeLuca & Klinger, 2010). Moreover, programs do not prepare preservice teachers to develop assessment skills that are adaptable to diverse student populations (DeLuca & Lam, 2014).

There is a large body of research on assessment, with significant attention given to clarifying the role of school assessment. For example, Griffin (2008) suggests that the most fundamental role of assessment is to help interpret observable behaviors in order to infer learning; "the more skills are observed, the more accurately generalized learning can be inferred" (p. 195). Assessment can be used to provide information to make decisions about student achievement (i.e., assessment *of* learning) and to support ongoing student learning (i.e., assessment *for* learning) (Stiggins, 2005). The distinction between summative and formative assessment, which are often distinguished by purpose, timing, or level of generalization (Black & Wiliam, 1998; 2003; Harlen, 2005; Sadler, 1989), is a prominent feature of the majority of this literature base. The fact that the purpose of assessment can be interpreted in a number of different ways is, according to Newton (2007) one of the most basic points for an educator to appreciate. Newton cautions, however, against reducing the purpose of assessment to two or three categories:

> We give the wrong message when we try to simplify assessment purposes by allocating them to a small number of categories (such as formative, summative and evaluative): we imply that the sub-purposes within those categories are importantly alike. This risks the impression that results which are fit for one sub-purpose within a category will be fit for the other sub-purposes as well…this is contrary to the impression that ought to be given to policy-makers, to ensure that wise decisions are made. (p. 161).

Newton suggests that the dichotomy between summative and formative assessment has been ineffective in understanding the nuanced nature of assessment and may hinder advancements in assessment theory. Bennett (2011) concurs that the use of assessments in support of learning is not limited to a certain kind of assessment such as formative or summative, because more than one type of assessment can contribute to judgments about students' achievement. Making distinctions of some sort, however, is useful because it helps us consider what assessment practices teachers are using and for what purposes.

In our work, we take the perspective of Newton (2007) who claims that, "to avoid getting ourselves confused, and to avoid confusing others, we need to use the language of assessment with greater precision" (p. 157). To this end, we choose to focus on the language that practicing mathematics teachers use to describe their understandings of the main purposes of assessment in their schools with the hopes of breaking the common borders between summative and formative distinctions. Newton shares at least three levels at which the purpose of assessment can be distinguished, including: (a) the *judgment* level—which concerns the technical aim of an assessment event (e.g., the purpose is to derive a standards-referenced judgment, expressed as a grade); (b) the *impact* level—which concerns the intended impacts of running an assessment system (e.g., the purposes are to ensure that students remain motivated, and that all students learn a common core); and (c) the *decision* level—which concerns the use of an assessment judgment, the decision, action or process which it enables. Newton further illustrates a range of purposes that may occupy discourse at

this last level. We use these (shown in Table 1 along with judgment and impact levels) as an initial framework for unpacking secondary mathematics teachers' descriptions of assessment purposes.

Table 1: Purposes of Assessment, Initial List (Newton, 2007; 2010)

Level	Categories	Description
Decision	Social Evaluation	Judge the social or personal value of students' achievements
	Formative	Identify proximal learning needs, guiding subsequent teaching
	Student monitoring	Decide whether students are making sufficient progress in attainment in relation to expectations or targets
	Transfer	Identify educational needs of students who transfer to new schools
	Guidance	Identify the most suitable courses, or vocations for students to pursue, given their aptitudes
	Institution monitoring	Decide whether institutional performance – relating to individual teachers, classes or schools – is rising or falling in relation to expectations or targets
	Resource allocation	Identify institutional needs and allocate resources
	Program Evaluation	Evaluate the success of educational programs or initiatives, nationally or locally
	Placement	Locate students with respect to their position in a learning sequence
Judgment		Derive a standards-referenced judgment, expressed as a grade
Impact		Ensure that students remain motivated, and that all students learn a common core

Methods

We addressed the research questions for this study by conducting a survey with teachers across two school districts in the Midwest.

Participants

School District A is a small district with 2,200 students. It is located in a rural area and it has implemented evidence-based decision making since the 1990s. There is one middle and one high school in District A. The district has a history of high academic performance, maintaining a higher passing rate of state mandated test than the state average for the last eight years. School District B is also rural with 6,700 students. It has two middle schools and one high school. District academic performance has declined somewhat over the last eight years. District B has not yet formulated a strong culture of DDDM. A total of 99 mathematics, science, and language arts teachers at the middle and high schools and 16 administrators participated in the online survey. Of these, 29 mathematics teacher survey results were analyzed for the current investigation.

Data collection and Analysis

The survey was designed by the research team by drawing on current research and assessment/data related reports from the U.S. Department of Education. The survey contained both open-ended and multiple-choice items and was designed to allow us to identify respondents'

perceptions of the purpose(s) of assessment and DDDM. The survey was distributed through Qualtrics®, an online survey tool. We focused our analysis on three survey questions:

1. What do you feel are the three main purposes of assessment in your school? (open-ended)
2. What do you see as your biggest areas of need?
3. What do you see as the biggest challenges related to using data from student assessment?

Survey responses were summarized descriptively for this paper to understand teacher and administrators' understandings of the purpose of assessment, challenges in meeting assessment goals, and teachers' current assessment practices. We used Newton's (2007, 2010) list of purposes (Table 1) to guide the axial coding analysis of open-ended items.

Findings

Purpose of assessments

Coding teachers' responses to the question, "What do you feel are the three main purposes of assessment in your school?" allowed us to expand our framework and further understand the variety of ways in which teachers view assessment use. From the 29 responding teachers, we received 75 distinct responses to this item. Some responses could be broken up into separate statements that fit different codes, resulting in a total of 87 total coded statements in our data. Table 2 provides the summary of the axial coding analysis these 87 statements from the mathematics teachers' responses. After initially coding responses to fit under the decision, impact, or judgment levels described above (Newton, 2007), we identified who seemed to the target of assessment decisions, impacts, or judgments – students, teachers, schools, or the state. We analyzed these codes further to identify where they fit in our framework and created additional codes that we posit to be beneficial in capturing the nuances of teachers' perceptions.

Table 2: Teacher Identified Major Purposes of Assessment

Levels	Focus	Codes	N
Decision Level	Students	Formative - Pre-assessment	2
		Formative - Instruction	9
		Formative - Student mastery	13
		Student monitoring (long term)	13
		Resource allocation - remediation	3
		Placement	2
	Teachers	Institution monitoring - Teacher accountability	12
	School	Institution monitoring - School accountability	9
Impact Level	Students	Encouragement	5
		State Exam Practice	4
Judgment Level	Students	Grading	5
	School	Collect data	6
	State	Meet a requirement	4

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Focus on students. The majority of responses focused on the assessment purposes that targeted students (57 out of 87). These were found at all three levels. Coded responses at the decision level were broken down into three separate categories of what Newton (2010) referred to as formative (where the purpose of assessment is to "identify students' proximal learning needs, guiding subsequent teaching" (p. 392)): (a) to preassess students' prior knowledge, (b) to make decisions during class about whether students had mastered a concept, and (c) to make decisions about how to move forward with instruction. Some examples of responses coded into these categories include:

- "Determine if a student has mastered a concept." (*mastery*)
- "In my classroom I use them to find out students ability to recall and apply concepts." (*mastery*)
- Student scores are then used as a tool to adjust my lessons." (*instruction*)
- "To determine where students currently are with a certain standard before it is taught." (*pre-assessment*)

Student mastery formed the largest group under the formative code (*n*=13). Student monitoring was also coded for 13 responses, and was distinguished from the formative categories if the response implied that assessment served a purpose of determining student learning or growth over time (instead of within a lesson). Some examples include:

- "To measure growth of knowledge."
- "To make sure students are learning the state standards."
- "To see what students learn and retain throughout the year."
- "To determine how much students know after teaching standards."

At the impact level, which concerns the intended impacts of assessment, teachers reported assessment purposes that target students. This student-mindedness was evidenced by identifying assessment as a tool for keeping students motivated and allowing students to practice for high-stake state exams. For example:

- "Assessment gives students clearer goals to help put more emphasis and importance on learning the content; it works well for students as an incentive to grow." (*encouragement*)
- "Prepare students to succeed when taking state standardized tests." (*practice*)

One teacher added that assessments were "almost like a reward in the sense that students are given problems they've been training themselves for and are given a number or letter grade showing how much they know."

Only five teachers made mention of giving grades as a purpose of assessment. These responses were coded at the judgment level, as they described a technical aim of assessment.

Focus on Teacher/School/State. The survey results revealed that teacher, school, or state accountability is a common purpose that teachers see for assessment. At the decision level, these foci appeared as responses indicating that assessments are used to "compare teachers," "Determine if a teacher is an effective teacher", and "to make sure everyone is teaching the same content."   These and similar responses were coded under institutional monitoring, but focused on teacher accountability. Additional responses claimed that assessments were to "compare to other schools," "calculate school performance," and "let the public how the school is doing."   These were coded as institutional monitoring with a focus on school accountability. Additional comments coded as focusing on assessment purposes for the whole school included six responses that explained the

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purpose of assessment in basic terms, "to get data," which is a technical aim of assessment (and thus coded at the judgment level) in their school.

Some negative connotations of assessment were also found at the judgment level, with four teachers indicating that assessment served the purpose of "making the state happy" or meeting state requirements. One teacher added that they served the state by "reporting numbers that don't tell the whole story", indicating frustration with the roles that s/he sees assessment playing at the school or state level.

Areas of Need and Challenges

Table 3 summarizes the areas of needs relative to assessment and data practices that teachers identified in their responses. As shown above, a large number of teachers identified long-term student monitoring as a significant purpose of assessment. However a majority also identified collecting useful growth data as a major area in which they need additional guidance or support. Similarly, teachers identified examining student data to identify which instructional practices work best for which students and analyzing with an equity lens as problematic areas. However, a majority also indicated little to no need with help on adapting instructional activities to meet students' individual needs in this same question. It is possible that teachers see this activity as disconnected from the use of data or assessment. It is also interesting to note that none of the teachers made mention of differentiation of any kind in their descriptions of the purposes of assessment above. It will be important to follow up on this issue in interviews with the teachers. A majority of the mathematics teachers did feel that they were well practiced with designing assessments aligned to state standards, which is not surprising given the current influence of state exams on teaching practices.

Table 3: Teacher Identified Area of Need

Need	Some to Great Need	No to Little Need
Collecting useful growth data.	18	7
Examining student data to identify which instructional practices work best for which students	18	7
Analyzing data through an equity lens.	16	9
Collaborating and sharing ideas with colleagues regarding data inquiry and analysis issues	16	9
Structuring the district organization and practices to support data-driven decision-making.	14	11
Communicating with families about student progress.	12	13
Developing curriculum-embedded formative assessments.	12	13
Interpreting assessment data to identify gaps in student achievement	12	13
Adapting instructional activities to meet students' individual needs	10	15
Designing assessments aligned to standards.	8	17

Additional challenges in using assessment data for instruction were identified in teachers' survey responses, including resource limitations, significantly, time. Teachers also perceive some major challenges in being able to access multiple sources of student data in useful ways to guide decision-

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making, and many suggested that they were not prepared or did not have adequate guidance for making instructional decisions from assessment data.

Conclusions

The teachers who responded to our survey have a working knowledge of multiple purposes for assessment. This runs contrary to the sentiment in current literature that teachers enter the profession with deficient assessment literacy and that this skill deficit remains throughout their professional careers (Bocala & Boudett, 2015; Popham, 2009; Volante & Fazio, 2007). In discussions about what teacher education can do to provide assessment-related professional development, the focus is often on the presumption that teachers do not have a strong knowledge base for assessment literacy. The data presented here does not really support this. Through their professional experiences or training, teachers have well-developed ideas about the varied purpose of assessment and what they want to be able use them for. What there does seem to be is a gap in their ability or understanding of how to enact assessment and data use in a way that serves the purposes they know they can serve.

Our results suggest that the key to developing teachers' assessment literacy is not to inundate pre-service and in-service teachers with information about the technical uses of assessment. Instead, it would be beneficial to use teachers' existing assessment knowledge base to craft professional development opportunities targeted at developing their skills at linking this knowledge base to actionable instructional strategies. To this end, it seems critical to find ways to provide time and incentives for teachers to collaborate with their colleagues in sharing assessment data and share best practices for linking assessment data to instructional adjustments. The more advanced perceptions on the purpose of assessment came from statements that focused on the students as the benefactors of assessment (rather than on teacher or school accountability). Thus is also seems important to help teachers direct their discourse away from a primary focus on state exams as "the" assessment and reflect more directly on daily teaching practices that involve assessment, prompting teachers to continually reflect on the goodness of fit between their assessments and their purpose(s). These strategies are critical components to include when focusing on the development of a school and district-wide culture of DDDM and data sharing.

The teachers' descriptions of purposes of assessment support the need to broaden the distinctions that we make as a field beyond summative and formative. These words did not appear in teachers' descriptions and to not adequately capture the wide variety of roles they see assessment playing in their classrooms and schools. Our use of Newton's work (2007) as an initial framework proved useful in unpacking the teachers' perceptions in a meaningful way. We feel this is a positive step in helping teachers and teacher educators develop more productive discourse around the use of assessment for guiding instruction in classrooms.

There is a paramount need for teachers to be assessment-literate in this age of standardization and accountability. Our survey suggests that teachers have the skills and knowledge to answer this call. However, teachers report that they need support and guidance transforming their working knowledge of assessment into habits of using this data to inform dynamic instruction. As mathematics educators, we have the resources to provide this guidance, creating an opportunity for us to work together with teachers to ensure that their students are given every opportunity to receive quality experiences in the mathematics classroom.

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IMPRECISION IN CLASSROOM MATHEMATICS DISCOURSE

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We theorize about ambiguity in mathematical communication and define a certain subset of ambiguous language usage as imprecise. For us, imprecision in classroom mathematics discourse hinders in-the-moment communication because the instance of imprecision is likely to create inconsistent interpretations of the same statement among individuals. We argue for the importance of attending to such imprecision as a critical aspect of attending to precision. We illustrate various types of imprecision that occur in mathematics classrooms and the ramifications of not addressing this imprecision. Based on our conceptualization of these types and ramifications, we discuss implications for research on classroom mathematics discourse.

Keywords: Classroom Discourse, Instructional Activities and Practices, Standards

The border between effective and ineffective communication of a mathematical idea can be crossed based on the use or misuse of a single word. Communication is necessarily problematic, requiring constant negotiation of meaning (Sfard & Kieran, 2001; Voigt, 1994). We never know exactly what someone else means by what they say; we infer those meanings. Communication in general, and classroom communication in particular, works because of our overall assumptions of shared meaning (Cobb, Yackel, & Wood, 1992)— for most of the words we use we assume that the individuals around us have constructed meanings that are fairly comparable with our own meanings. Though unwarranted inferences cause miscommunication, classroom discourse would come to a standstill if teachers followed every student statement with, "Please explain what you mean." Whether consciously or not, we are constantly judging whether the words others use can stand on their own or seem to require clarification.

Those who look closely at the complexities of communicating in mathematics classrooms see one particular aspect of communication—sometimes referred to as ambiguity (e.g., Barnett-Clarke & Ramirez, 2004; Barwell, 2003)—as both inherent (and thus unavoidable) and as providing opportunities for learning. As Barwell (2003) stated, "It is the potential for ambiguity inherent in all language that allows students to investigate what it is possible to do with mathematical language, and so to explore mathematics itself" (p. 5). There is a subset of ambiguous situations, however, that we see as a barrier to mathematical communication, as hindering the negotiation of mathematical meaning in the moment. We have come to refer to such situations as imprecise. We see attention to imprecision as a critical, but possibly overlooked, aspect of the mathematical practice of attending to precision (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). The practice of attending to precision often focuses on mathematical precision, like when a student uses equal in place of congruent, or when a student incorrectly strings together expressions using equal signs. Although these situations call for improved mathematical precision, we can typically infer what the student means—there is not a need to improve mathematical precision for communication to continue. We, instead, focus on language precision in the context of mathematics—when clarifying what has been said will increase the likelihood that all members of the classroom community can successfully and reliably make sense of the mathematics at hand. In this paper we define this *imprecision*, then describe and provide examples of different types of

imprecision. We conclude by theorizing about issues related to not addressing imprecision and ways to productively address it. We see this work as critical to our ongoing research to understand the teaching practice of productive use of student mathematical thinking.

Defining Imprecision

We now discuss an example derived from an excerpt of classroom mathematics discourse in order to motivate our definition of imprecision. In a class where students have been studying data about a group of bikers on a multi-day trip, they are examining a graph where distance is measured by the distance from a given city (see Figure 1). In a discussion about Figure 1, the class has interpreted the plotted points at times 1.5 and 2 as an indication that the bikers are stopped on the interval between 1.5 and 2 hours. A student then volunteers, "And then they went up." The teacher asks, "What do you mean, 'They went up?'" to which a number of students respond by making hand gestures, raising their hands up as they move from left to right.
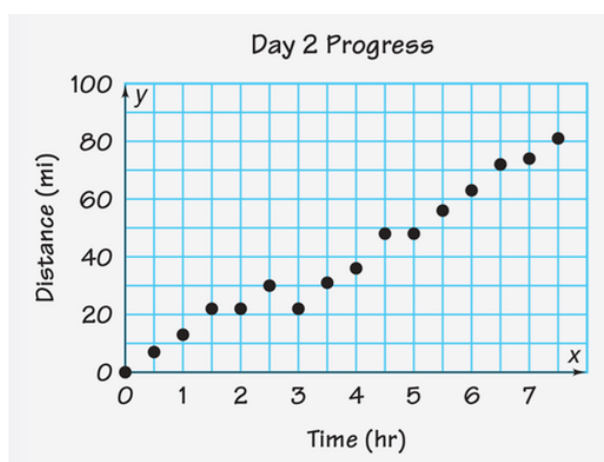
Figure 1. Bikers' progress (from Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006, p. 12).

We see the student statement, "And then they went up," as imprecise because it is unclear to what the student is referring by *they*. In this context *they* could easily refer to either the bikers on the road or the dots on the graph, and these two interpretations mean very different things in this context. The teacher seems to recognize a communication problem and pushes for clarification. Notice, however, that the students' responding gestures seem to indicate their understanding of the word *up,* but the ambiguity remains. What might be missing from the teacher's effort to clarify? We believe the ambiguity persists because the students do not know which part of the phrase *they went up* the teacher is referring to—*they* or *up*. This excerpt illustrates how imprecise language can hinder mathematical communication, creating a situation where individuals are talking past each other because of how they have interpreted the word *they*. We believe the teacher's response could have been more productive had the request for clarification zeroed in on the specific part of the student sentence that was unclear. Had the teacher responded to the student statement with a more-specific question such as, "When you say, 'they went up,' what do you mean by *they*?" students would have been better positioned to clarify their meaning and the discussion could have proceeded accordingly.

Ambiguity as illustrated here creates a moment when clarification is needed in order to continue the sense-making activity, and when lack of clarification will likely cause a breakdown in the negotiation of meaning. We have come to view this type of ambiguity in the following way: If a part of speech is used such that it can be interpreted in multiple viable ways, and the existence of those interpretations causes the overall meaning of the statement in which it occurs to be unclear, then we refer to both the part of speech and the statement in which it occurs as *imprecise*. We use viable

interpretations to mean interpretations some students in the class might reasonably infer given the context in which the part of speech is used. We do not consider viable extreme or outlying interpretations that are unlikely to exist in the given classroom context. Thus, for us, imprecision in classroom mathematics discourse hinders in-the-moment communication because the instance of imprecision is likely to create inconsistent interpretations of the same statement among individuals.

Types of Imprecision

In this paper, we present a theory that both researchers and practitioners can use to analyze mathematics classroom discourse. Although this is a theoretical paper, it was both prompted and informed by empirical work conducted as part of the National Science Foundation (NSF)-funded *Leveraging MOSTs* project. A central component of that project was the conceptualization of instances of student thinking teachers should take advantage of in the moment they occur during classroom discourse, what we called MOSTs—Mathematically significant pedagogical Opportunities to build on Student Thinking (Leatham, Peterson, Stockero, & Van Zoest, 2015). The associated MOST Analytic Framework provides characteristics and criteria for identifying MOSTs and for articulating why some instances fall short of being prime in-the-moment opportunities to build on student mathematical thinking. The foundational characteristic of a MOST is student mathematical thinking, and the first criterion of this characteristic is that *student mathematics* must be inferable. Thus, as we analyze student contributions to classroom mathematics discourse, we first attempt to articulate the *student mathematics* of each student's turn, or in other words, restate their comment in complete sentences or thoughts and replace pronouns and gestures with their referents when possible. In applying this framework, we often found ourselves in situations where we could not infer the student mathematics of an instance. Although at times this inability to infer the student mathematics stemmed from incomplete or inaudible statements, there were many times when student statements could be heard and seemed to be complete, but wherein we still could not make an inference. It was in this context that we came to realize that our inability to infer the student mathematics was due to language ambiguity—there were multiple viable interpretations of what the student had said. We began to wonder about these ambiguities and their effects on classroom discourse. This paper is a result of the theoretical work that followed. The examples in this paper are based on examples of imprecision we observed in middle school mathematics lessons that were analyzed for the *Leveraging MOSTs* project.

Because of the centrality of "parts of speech" to our definition, we categorize types of imprecision according to these parts of speech. The four basic parts of speech that occurred most frequently in our instances of imprecision are subject, object, adjective, and verb, and are thus our four main types of imprecision. That "parts of speech" ended up having the explanatory power that it did as we theorized about classroom imprecision was a surprisingly straightforward way of capturing what was a complex problem for us, one with which we grappled for a long time. The related grammatical phenomenon of unclear referents is a typical topic in textbooks on grammar, but not so common in research related to oral communication and learning to write and speak in general. When research does attend explicitly to unclear referents it tends to be research on those who are acquiring a second language (Block, 1992) or who have learning challenges such as delayed development (e.g., Eigsti, de Marchena, Schuh, & Kelley, 2011) or dementia (e.g., Almor, Kempler, MacDonald, Andersen, & Tyler, 1999). The theory presented in this paper contributes to the literature by extending a focus on language imprecision to typical classroom mathematics discourse and the complexity of orchestrating that discourse.

Subject

The subject of a sentence is the person, place, or thing that is doing or being something. Subjects are either nouns or pronouns and can be imprecise if it is not clear which person, place, or thing is

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being referenced. We first share an example of how the use of a pronoun as the subject of a sentence can be imprecise in a mathematics classroom and then an example of an imprecise demonstrative pronoun.

Pronoun. In the previous example – "And then they went up." – the subject of the sentence is they and they is imprecise because of the ambiguity of its referent. As mentioned before, in this context, they could be referring to the dots on the graph or to the bikers on the road. These alternate possibilities for the subject of the sentence result in very different mathematical interpretation for the overall meaning of the sentence. But, if we know whether they is referring to bikers or dots, it is easier to infer the meaning of went up and thus the meaning of the entire sentence.

Demonstrative pronoun. Demonstrative pronouns are pronouns such as *this, that, these,* and *those* that point to specific things. In a class discussion about slopes of linear equations, a student says, "That has a positive slope." The precision of this statement depends on the context in which it occurs. If there is a single linear equation on the board or being talked about, then it is likely clear what subject is being referenced and there is no imprecision. If, however, there are multiple linear equations on the board and only one of them has a positive slope, then the student statement is imprecise because the demonstrative pronoun *that* could be referring to either of those equations and thus there are multiple viable interpretations of the statement—the meaning behind the student statement is very different depending on which equation they are referencing. Were the teacher to seek clarification here, awareness that the imprecision lies with the demonstrative pronoun that is the subject of the sentence might lead to a question such as, "What has a positive slope?"

Object

Similar to the subject of a sentence, objects are either nouns or pronouns and are imprecise when it is not clear which person, place or thing is being referenced. If the object is a pronoun or demonstrative pronoun, the imprecision can occur in the same way it occurs in the subject of the sentence, and clarification is best if it hones in on the imprecise object itself. In the case of an object, however, another type of imprecision can occur, one that generally does not occur with subjects (in the English language): the object of a sentence can be implied. Such imprecision can occur with objects of a verb and with objects of an adjective.

Implied object of verb. Consider the teacher-student interchange when a teacher says, "What about unit rate? Could we use unit rate to solve this proportion [6/4=f/10]?" and a student responds, "Yes, by dividing." From the context we can infer that the student is saying, "We can use unit rate to solve the proportion 6/4=f/10 by dividing." The latter part of the sentence, however, is incomplete; the verb *divide* has an implied object and therefore there is no indication of which numbers are to be divided. There are several legitimate possibilities for these numbers, not to mention several others that might reveal misconceptions about the "unit rate" strategy or about proportions in general. We thus see the statement *by dividing* as an example of imprecision because of an unclear implied object of a verb. A teacher could zero in on the part of speech that created the imprecision by asking, "Which numbers would you divide?" thus acknowledging the unclear part of the statement and pushing for an articulation of the object.

Implied object of adjective. In an 8th grade algebra class, students were learning about the composition of functions and were given two equations: $P=2.50V - 500$ and $V=600 - 500R$, where profit (P) is related to the number of visitors (V) to an amusement park, and the number of visitors (V) is related to the probability of rain (R). Students were first asked to determine the profit when the probability of rain is 25% and then to find the probability of rain when the expected profit is $625 (from Lappan et al. 2006b, p. 25). After students had worked on these problems, the teacher said, "So, when they tell you a value of a variable, you substituted that variable with the value they told you. So, you guys were okay with that part. Why do you think the second part was kind of hard?" A student responded, "Because you had to do the opposite." In this statement the word *opposite* is an

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adjective and we know what it means. What we do not know is what object this adjective is describing—the opposite operation, the opposite order or the opposite process. Any of these objects are viable interpretations of what the student has said. Thus the statement is imprecise because the implied object of the adjective *opposite* is imprecise.

Adjective

Adjectives, words that describe nouns, are sometimes the culprit of the imprecision. For example, a teacher might ask, "Could we still use *that* strategy?" in a context where multiple strategies have been discussed. For students to answer this question they need to know which strategy is under consideration—the demonstrative adjective *that* is imprecise. Or suppose a class is discussing a graph displaying a dozen points and a student says, "Find the distance between those points." In order to pursue this line of reasoning, and for the class to follow along, the class needs to know which of *those* points the student is considering.

The adjectives in these examples are demonstrative adjectives—in essence they are pronouns being used as adjectives. Note how slight variations on these examples change the part of speech that is imprecise. Compare "could we still use *that*" with "could we still use *that* strategy". In the first case the pronoun *that* is the object of the sentence. In the second case *that* is a demonstrative adjective. The second sentence is "less imprecise" in that we at least know that the object of the sentence is a strategy, we just do not know which strategy. This distinction matters because, whereas in the first sentence one would seek broader clarification of the object of the sentence with a question such as, "Could still use what?", in the second sentence the clarification question could be much more precise—"Could still use *which* strategy?". The more one can hone in on the part of speech that is imprecise the more precise one can be in seeking clarification.

Verb

To understand a sentence, one must understand the subject's action, or the *verb*. This understanding is particularly important in mathematics because of the many carefully defined mathematical verbs like *add*, *divide*, *solve*, *invert*, and *integrate* that are used to communicate specific actions. A common type of imprecise verb use is the use of generic, colloquial action verbs such as *work*, *do*, and *make*, which do not have precise meanings in a mathematical context and thus can often be interpreted in more than one way. In a sense, these generic action verbs are used like pronouns to replace more precise mathematical verbs.

The following example illustrates how use of generic action verbs can create imprecision. It comes from the same 8th grade algebra class mentioned previously where students were using the equations $P=2.50V - 500$ and $V=600 - 500R$. Students were able to solve for P given R relatively easily, but many struggled when asked to solve for R given P. During a conversation about that struggle, a student said, "[In the first case you] just do the equation instead of doing multiple step equations." Here the student used the verb *do* in a general, colloquial way; it is not clear what she meant by "do the equation" or "doing multiple step equations," and whether each use of *do* is the same. She may have meant solve the equation(s), evaluate the equation(s), substitute something within the equation(s), or manipulate the equation(s). Because there are multiple viable options for what was meant by *do*, the student statement was imprecise. This student seemed to have something valuable to contribute to the mathematical conversation about students' struggles with this task, but the imprecision hindered her communication. To clarify this imprecision, the teacher could ask the student to clarify what she means by *do* when she says, *"do the equation."* This teacher response would hone in on the verb imprecision while simultaneously legitimizing the student statement as an important contribution to the mathematical conversation.

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Commonality Across the Types

Across these various types of imprecision, one particular commonality stands out. Multiple viable interpretations occur when generic or implied words are used in place of more specific words. Pronouns are a wonderful tool for streamlining communication, but when their referents are unclear from context, imprecision occurs. Further complicating matters, the English language allows for specific subjects and objects to be completely absent, creating an even deeper layer of inference and associated possibilities for imprecision. Our examples of imprecise verbs also fit this pattern, as noted earlier, as these verbs are used in a generic way, almost like a pronoun.

Ramifications of Not Addressing Imprecision

Imprecision could cause a student to think they do not understand something when, in reality, there is just a breakdown in the negotiation of meaning because of an imprecise statement. When an imprecise statement is not explicitly addressed, students are likely left unaware that what has been said is imprecise. If the teacher implicitly infers the meaning of an imprecise statement, students are likely not aware of the sense making that the teacher has engaged in, so have no idea that their interpretation of the statement is different from the teacher or other students in the class. In addition, there seems to be a norm in classrooms that a teacher moving on implies that what was said was clear or true. If a student cannot make sense of an imprecise statement, they may not know whether they lack understanding or there was a problem with what was said. In other words, they may think their own understanding is flawed because it does not reconcile with the imprecise statement when, in fact, the imprecise statement is where the flaw lies.

Furthermore, a number of instances of imprecision we have observed have led to the creation of simultaneous, yet parallel inconsistent conversations, wherein various participants proceeded with differing interpretations of an imprecise statement, in essence talking past each other. Because imprecision occurs when a word, statement or action has more than one viable interpretation, communication is hampered when part of the class adopts one of those interpretations and another part of the class adopts a different interpretation.

These two main ramifications of imprecision—student confusion and parallel conversations—have serious implications regarding the teacher's and students' experience in the classroom. First, students might disengage from the class discourse because of their inability to make sense of an imprecise statement. When imprecision occurs and students are confused or when their understanding does not align with the teacher's, some proactive students might push on the issue until the imprecision is cleared up and the confusion is resolved. Unfortunately, this is likely the exception as many students are unwilling to challenge a teacher or stall progressing discourse. These students are likely to remain confused, ultimately causing them to disengage from the discourse because of their inability to make sense of the ensuing conversation, all caused by unaddressed imprecision. Second, these ramifications may cause the teacher to miss opportunities to better understand a student's thinking, and thus miss opportunities to further that student's and the class's understanding of the mathematics at hand. Since we began to think about imprecision as we observed it during the MOST project, we particularly emphasize this implication. When a student's utterance is imprecise and when the teacher does not address that imprecision, the teacher is not able to articulate the *student mathematics* of that student's statement with confidence. Without that understanding, they are not able to effectively further that student's thinking about the mathematics at hand, nor are they able to use that student's comment to further the rest of the class's understanding. Third, there could be repercussions for students' mathematical understanding. For instance, in the "they went up" example, failure to explicitly address the imprecision could cause or reinforce the misconception that a graph is a picture of the physical situation.

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Explicitly Addressing Imprecision

Student mathematical thinking should be at the heart of classroom mathematics discourse (e.g., National Council of Teachers of Mathematics, 2014). In order for students and teachers alike to fully benefit from students making their thinking public, teachers need to recognize and then attend to roadblocks that hinder the effective communication of the intended ideas. To do this effectively, a teacher needs to attempt to internally make sense of student comments in order to recognize instances of imprecision; then it is critical to push for clarification when imprecision occurs to allow others in the classroom to also make sense of what is being said. Although it is unwise to ask for clarification about every student statement, it is equally unwise to never seek such clarification. Some teachers, particularly novice teachers, may be reluctant to push for clarification from their students because they feel such requests may come across as a lack of mathematical understanding on their part. Members of a classroom community should recognize that a push for clarification is not an indication of weak mathematical understanding, but rather an acknowledgement of the importance of clear communication and evidence of the centrality of students sharing their thinking to mathematics teaching and learning.

As we saw in the example of "they went up," even when teachers seem to recognize that something is amiss, their requests for clarification may miss the mark and thus not solve the problem. If teachers can learn to attend to precision by attuning themselves to the specific cause of imprecision (i.e., the particular part of speech that is imprecise), they can ask for clarification that hones in on just what it is the student needs to clarify. Such clarification specificity has at least three advantages. First, this specificity helps the student to know what aspect of their communication was problematic, providing guidance for them as they seek to clarify their ideas. Second, this specificity scaffolds the entire class as they try to negotiate the meaning of what has been said. Third, and perhaps most important, it sends the message that most of what a student has said has been taken as understood—as understandable and meaningful. Such messages play an important role in helping students to gain confidence in their abilities to contribute legitimate, useful mathematical thinking. By explicitly addressing an instance of imprecision, teachers legitimize all students' efforts to make sense of others' ideas; they also model the importance of attending to precision.

In conclusion, we see attending to imprecision as a critical and possibly overlooked aspect of the study of the productive use of student mathematical thinking in classroom discourse. Future research could use this conceptualization of imprecision as a tool that could help us better understand the barriers to effective classroom discourse. We believe that this tool is also readily accessible to teachers in their in-the-moment analysis of classroom discourse, thus blurring the border between research and practice.

Acknowledgements

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TEACHER ATTRIBUTES AND SCHOOL CONTEXT: WHAT ARE THE BARRIERS TO DISCOURSE IN ELEMENTARY MATHEMATICS?

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This study utilized a multilevel model to examine the relationship between teacher attributes, school context, and the characteristics of the discourse within novice elementary mathematics lessons. MKT and teacher beliefs were significant predictors of the level of student explanation within teachers' lessons while school SES and perceived levels of support accounted for variance in the overall mathematics discourse community.

Keywords: Classroom Discourse, Elementary School Education, Mathematical Knowledge for Teaching, Teacher Beliefs

Introduction

Research has shown that elementary teachers are often anxious about teaching mathematics due to their prior experiences (Stevenson & Stigler, 1994), hence the work of teacher preparation programs to reverse this trend among their preservice teachers (PTs). Preparation programs strive to build PTs' confidence in teaching mathematics by improving their specialized content and pedagogical knowledge. One element of mathematics teaching and learning on which teacher preparation programs typically focus is the use of mathematical discourse to support students' justification and explanation of ideas. The importance of mathematical discourse has been supported by research (e.g., Michaels & O'Connor, 2015), emphasized in standards (National Governors Association Center for Best Practices, Council of Chief State Officers [NGA & CCSSO], 2010; National Council of Teachers of Mathematics [NCTM], 2000), and echoed in the recently published *Principles to Actions* (NCTM, 2014).

Despite the emphasis on the importance of discourse, research has shown that high-quality mathematical discourse is not the norm in U.S. classrooms (Michaels & O'Connor, 2015); therefore, more research is needed to understand the borders that promote and hinder its actualization in elementary classrooms, particularly for novice teachers who recently graduated from programs advocating its use. This study aims to fill that void by examining the relationships among teacher attributes, school contextual factors, and mathematical discourse. By studying novice teachers, we attempt to investigate the bridge between teacher preparation and induction into the profession and the borders that are productive or problematic for the implementation of high-quality discourse.

Theoretical Perspective

To study the co-construction of discourse within the mathematics classroom, this study employs a situative approach to learning (Lave & Wenger, 1991; Cobb & Yackel, 1996). A situated perspective allows one to conceptualize the teacher as an influential "old timer" in the classroom community of practice where students learn the expected norms through their interactions with the teacher and fellow classmates (Lave & Wenger, 1991, p. 29). Further, broader school contextual factors are often considered when attempting to fully situate learning.

This study attempts to gain insight into the mathematical communities established in novice teachers' classrooms by examining the classroom using various units of analysis. We first focus on the novice teacher's "distinct way of being" (Skott, 2009, p. 31) by attending to attributes measured at the individual level. Next, we explore the school context in terms of teachers' perception of support and school demographics, which shed light on the sociocultural influences. Lastly, using a participatory lens, we explore the nature of the classroom community of practice through an

examination of the type of discourse that occurs within mathematics lessons. As a result of various units of analysis within the community of practice, we are able to explore associations between teacher attributes, context, and mathematical discourse.

Related Research

Discourse

From a situative perspective, communication is essential to the development of mathematical concepts and therefore is a necessary component of quality mathematics instruction (Cobb & Bowers, 1999). If students are to engage in authentic practices of mathematicians, then they need opportunities to discuss mathematical ideas and justify their reasoning (Lave & Wenger, 1991).

When communicating ideas, students challenge and reinforce the learning within the classroom (Sfard, 2006). In order for this co-construction of knowledge to be possible, students must feel comfortable and supported in their efforts to dialogue with one another and the teacher. The sociomathematical norms are developed through interactions between the teacher and the students (Cobb & Yackel, 1996). The types of questions posed, opportunities for student-to-student discourse, and solicitation of students' ideas are ways that the teacher explicitly and implicitly sets the norms and builds the discourse community.

In addition to the overall participation of students in the community discourse, the level to which students explain and justify their reasoning depends upon the teacher's actions. As teachers press for further explanation, they can simultaneously model the "sophistication" expected within mathematical responses (Cobb & Yackel, 1996). Michaels, O'Connor, and Resnick (2008) state that teachers support "accountable talk" by holding students accountable to their reasoning and knowledge.

Teacher Attributes

The teacher is influential in the classroom community of practice (Lave & Wenger, 1991) by drawing upon his/her knowledge and beliefs when making instructional decisions, and in turn cultivating the community norms. First, teachers need specialized content and pedagogical knowledge, often referred to as mathematical knowledge for teaching (MKT), to effectively facilitate mathematics learning (Ball & Bass, 2003). Research has shown that teachers' MKT influences the use of standards-based instructional practices (Hill et al., 2008) and student achievement (Hill, Rowan, & Ball, 2005). This study takes a closer look at the discourse within the community of practice to better understand the relationship to teachers' MKT.

In addition to knowledge, research has indicated that teachers' instructional practices are related to their beliefs about mathematics (Stipek, Givvin, Salmon, & MacGyvers, 2001), beliefs about teaching and learning (Walshaw & Anthony, 2008), and beliefs about their own ability to teach (Beard, Hoy, &Woolfolk Hoy, 2010). The mathematics education field continues to sort out definitions within the beliefs literature (Philipp, 2002), but this study addresses teachers' epistemological beliefs (beliefs about the nature of mathematics and learning) and personal mathematics teaching efficacy (PMTE; Enoch, Smith & Huinker, 2000).

School Context

The school culture in which a classroom community of practice is situated has impacts on actualization of the teachers' goals for his/her classroom (Skott, 2009). Teachers specifically cite administrative and team-level support as influential in their instructional choices (Cobb, McClain, de Silva Lamberg, & Dean, 2003). This study accounts for teachers' feelings of support, specifically in regards to mathematics instruction.

Well documented within research is the relationship between socioeconomic status (SES) and students' mathematics achievement (e.g., Ma & Klinger, 2000). However, studies analyzing the associations between mathematics instruction and SES are often using broad categories such as procedural or conceptual instruction; there has been little attention to the interactions within the classroom. This study examines the discourse community and level of explanation and justification and their relationships to the overall SES of students.

Methodology

Participants

Participants in this study were 118 novice elementary teachers in their second year of teaching. Participants, with an average age of 23, taught in various elementary schools across one southeastern state and also completed their undergraduate teacher preparation program at public universities within the state. Propensity score matching (Fan & Nowell, 2011) was used to create a comparable sample based on college entrance characteristics including SAT/ACT and high school grade point averages. From this sampling technique, teachers were identified and recruited to participate. Overall, the sample was 98% Female and 85% Caucasian which is representative of alumni of the preparation programs. Descriptive analysis of school level information showed that 70% of teachers in the sample taught at schools that were classified as Title 1 schools and had over 50% of student receiving free and reduced price lunch.

The participants attended a one-day summer session before their second year of teaching. The session included: training on how to use an instructional log (not a focus of this study); training on how to effectively video record instruction; and data collection on a variety of surveys and assessments. The surveys and assessments utilized in this study will be described in the measures section. The training on video recording prepared the participants for recording three mathematics lessons (a focus of the current study) across their second year of teaching.

Nine teachers submitted 1 video, 18 teachers submitted two lessons, and 89 teachers submitted three or more lessons. Teachers with less than three lessons were retained in the analysis because past research using the same observational measure found that less than one percent of the variance in scores on the M-Scan measure (described below) was attributable to the number of observations (Walkowiak, Berry, Meyer, Rimm-Kaufman, & McCracken, 2014).

Measures

The level of Mathematics Discourse Community (MDC) and level of Explanation/Justification (EJ) are the two outcome variables. Dependent variables include teacher-level attributes related to knowledge and beliefs: MKT, PMTE, and EB. Also, school contextual variables are dependent variables and include school SES (percent free and reduced price lunch; FRPL), grade level, and perceptions of the school support.

Mathematics Scan (M-Scan) Observational Measure. The M-Scan is an observational protocol that measures the presence and extent of standards-based mathematics teaching practices (Berry et al., 2010). Of the measure's ten dimensions, this study specifically examines two: Mathematics Discourse Community (MDC) and Explanation and Justification (EJ). The indicators for MDC are *teacher's role in discourse, sense of mathematics community through student talk, and questions*. The indicators for EJ include *presence of explanation/justification* and *depth* of *explanation/ justification*. A trained observer rates a mathematics lessons on these dimensions using a 7-point scale that is divided into three levels: low (1-2); medium (3-5); and high (6-7). There are specific descriptions for each level that are thoroughly defined in the coding protocol. The M-Scan development team gathered sources of evidence of validity and score reliability (Walkowiak et al., 2014).

Mathematical Knowledge for Teaching (MKT) Measure in Number and Operation (N&O). The MKT-N&O measure consists of 26 items designed to measure teachers' mathematical knowledge for teaching number and operations (Hill, Schilling, & Ball, 2004). This particular form of the MKT was selected because a majority of K-5 content is focused on number and operations. Item response theory models (IRT) were used with the data and the IRT reliabilities for the different domains of the MKT-N&O were good to excellent, ranging from .71 to .84. Participants completed the MKT-N&O via an online interface; they were allowed to use paper and pencil. Scores were recorded in a master database as an IRT score relative to the national sample.

Mathematics Teaching Efficacy Beliefs Instrument (MTEBI). The PMTE subscale of the MTEBI measure (Enochs, Smith, & Huinker, 2000) is comprised of 13 items and uses a 5- point Likert scale (strongly disagree, disagree, uncertain, agree, and strongly agree) with "strongly agree" denoted by a 5. The PMTE subscale measures a teacher's belief in his or her own ability to effectively teach mathematics. In the developers' work, reliability analysis produced an alpha coefficient of 0.88 (N = 324) (Enochs, Smith, & Huinker, 2000).

Mathematics Experiences and Conceptions Surveys (MECS). The MECS (Jong, Hodges, Royal, & Welder, 2015) was designed to measure pre-service and novice teachers' attitudes, beliefs, and dispositions toward mathematics teaching and learning. This study utilizes the beliefs subscale from the MECS-Y1 version, which is designed for teachers in their first three years of teaching. Validation work (Jong et al., 2015) reported relatively high reliability of this subscale (α= .78). To provide a more precise categorization of epistemological beliefs (EB), an exploratory factor analysis was conducted and revealed three factors within the beliefs domain, which are classified as epistemological beliefs about the nature of mathematics (Nature, α= .60), epistemological beliefs about the use of calculations (Calculations, α= .70), and epistemic value of mathematics (Value, α= .81). These three subgroups are used in the analysis.

Teachers' perceptions of school level support for mathematics instruction (TPSS). Participants also responded to 6 items that gauged their perceptions of barriers within their school to quality mathematics using the *scale significant problem*, *somewhat of a problem*, or *not a significant problem*. An index of TPSS (possible range 6-18) was created from these items to provide a measure of the level of support the teacher felt in regards to mathematics instruction.

School contextual variables. A measure of the school's SES was created using the percentage of students receiving free or reduced price lunch (FRPL). The grade level for each teacher was recoded to create a grade band variable. Kindergarten, first, and second grades were considered primary grades, while third, fourth, and fifth grades were considered intermediate grades.

Analysis

The first step in the quantitative analyses was to measure the discourse present in each lesson using the M-Scan observational measure. A team of five coders was trained on the M- Scan observational protocol during a 12-hour training facilitated by a co-developer of the measure. Each coder established 80% within-one in comparison to the master coder on each of the 10 dimensions. The team met every other week for drift check meetings.

Next, to allow for more meaningful interpretation of results, z-scores were calculated for participants' MKT scores. Other dependent variables including PMTE, EB subscales, FRPL, and TPSS were grand-mean centered to provide a meaningful zero. That is, zero for each variable is interpreted as the sample mean and allows results to be interpreted in reference to this sample.

Central to the study, a series of multilevel means-as-outcomes analyses were performed using SAS software to examine the proportion of variance in the classroom discourse dimensions, MDC and EJ, that is explained by teacher attributes and school context. By examining the variability in MDC and EJ across multiple lessons, multilevel modeling (MLM) is a powerful and flexible approach compared to techniques, such as multiple regression, because estimates of both between-

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person and within-person variability are possible (Lee & Bryk, 1989). Additionally, multilevel modeling uses all available data from each participant and can effectively manage unequal data (Raudenbush & Bryk, 2002).

MDC and EJ were entered into the model as Level-1 variables. Each teacher's PMTE score, EB subscales, and MKT score were entered as Level-2 variables because they describe attributes of the teacher. School contextual variables, FRPL, TPSS, and grade level, were entered as Level-2 variables due to the fact that TPSS represents teachers' perceptions of their school support, grade-level is viewed a classroom characteristic, and school level variables are associated with teacher due to the fact that teachers are not at the same school.

A fully unconditional model/null model (Raudenbush & Bryk, 2002) was conducted to determine the variability within teachers (Level 1) and the variability between teachers (Level 2) on the variables of MDC and EJ. The null model for each dependent variable, MDC and EJ, would serve as the reference to explain variation within the sample. This model was conducted to ensure that there was sufficient variability at Level 2 to warrant continuation with analyses.

Next, a sequential Means-as-outcomes Regression was conducted to test the main effect of predictors at Level 2 (i.e., MKT, PMTE, EB subscales, and TPSS) on teachers' MDC and EJ, separately. First MKT was entered into the model, followed by teacher attributes (PMTE, EB subscales), and lastly school contextual factors (FRPL, TPSS, grade level). For each model, significant main effects, significant interactions, and percentage of variance explained between teachers were examined.

Results

Results from the full unconditional or null model indicated that 37% of the variability in the level of MDC was between teacher (τ_{00}= .51, z=4.57, p<.001) and 63% was within lessons for an individual teacher (σ^2=.88, z=10.50, p<.001). The fully unconditional model for EJ indicated that 33% of the variability in the level of EJ was between teacher (τ_{00}= .75, z=4.25, p<.001) and 67% was within lessons for an individual teacher (σ^2=1.51, z=10.46, p<.001). These results indicated sufficient variability for further analyses.

Mathematical Discourse Community (MDC)

Table 1 displays the results of the multilevel analyses for MDC. When MKT was entered into the model by itself (Model 2), it was marginally significant (p=.06). However, when controlling for efficacy and the EB subscales, MKT was not a significant predictor of the level of MDC (Model 3). The Nature subscale was a significant predictor of the level of MDC (p<.05). That is, teachers with more sophisticated/standards-based beliefs about the nature of mathematics also tended to have higher levels of MDC within their lessons on average, when controlling for levels of MKT and PMTE. For each scale point increase in the *Nature* score, the MDC score on average would increase .08 points. In comparison to the null model, Model 3 accounted for 24% of 37% variance between teachers.

In the next step (Model 4), school context variables were added. When controlling for all other variables, *Nature* (p<.05), FRPL (p<.05), and TPSS (p<.05) accounted for a significant amount of the variance between teachers in their scores on MDC. For each unit increase in the *Nature* score, one would expect the average MDC score to be .13 units higher while for each unit increase in the FRPL percentage, one would expect an average decrease of .80 units on MDC. Finally, for each unit increase in TPSS, the expectation is a .06 unit increase in MDC. Overall, Model 4 accounts for 24% of 37% variance between teachers on their MDC scores.

Table 1: Estimated Effects of Teacher Attributes and School Context on the Level of Mathematical Discourse Community

Fixed Effects, MDC, β_0		Model 1 DF=118	Model 2 DF=115	Model 3 DF=109	Model 4 DF=100
		(SE)	(SE)	(SE)	(SE)
Intercept, γ_{00}		3.78**(.08)	3.79** (.08)	3.79** (.08)	3.79**(.11)
Teacher Attributes					
MKT, γ_{01}			0.16 (.08)	.11 (.09)	.14 (.09)
PMTE, γ_{02}				-.001 (.02)	.01 (.02)
Nature, γ_{03}				.08* (.04)	.13* (.04)
Calculations, γ_{04}				.03 (.05)	.03 (.04)
Value, γ_{05}				.03 (.05)	.08 (.05)
Teaching Context					
FRPL, γ_{06}					-.80* (.38)
Grade band, γ_{07}					-.01 (.18)
TPSS, γ_{08}					.06* (.03)
Random Effects					
MDC (τ_{00})		.51** (.11)	.48** (.11)	.47**(.11)	.38** (.10)
Within-teacher variation (σ^2)		.88** (.08)	.89** (.09)	.89** (.09)	.89**(.09)

$**p < .001, *p < .05$

Explanation & Justification (EJ)

Subsequently, models were fit to examine the relationships between all dependent variables and EJ (see Table 2). MKT was entered into the model alone (Model 2), and it was a significant predictor of EJ ($p<.05$). In the next step, controlling for PMTE and the EB subscales, MKT remained a significant predictor of the level of EJ ($p<.05$), and the *Nature* subscale was significant as well ($p<.05$). That is, for each unit increase in MKT score, EJ was expected to increase .22 units. Also, teachers with more sophisticated/standards-based beliefs about the nature of mathematics tended to have higher levels of EJ within their lessons, and for each scale point increase in the Nature score, the EJ score on average would increase .13 points. In comparison to the null model, this model accounted for 14% of 33% variance between teachers on their EJ scores.

In the final step, school context variables were added. When controlling for all other variables, MKT ($p<.05$), and *Nature* ($p<.05$) remained significant predictors of EJ, while no school contextual variables were significantly related to the level of EJ. When controlling for the other variables, for each unit increase in a teacher's MKT score, one would expect the average EJ score to be .24 units higher. For each unit increase in the *Nature* score, an increase of .17 on EJ is expected. This model accounted for 23% of the variance between teachers on EJ scores.

Discussion

The purpose of this study was to examine teacher attributes and school context in regard to their relationship to the overall mathematical discourse community and the level of student explanation and justification in novice teachers' mathematics lessons. MKT had a significant positive relationship with the level of EJ in a teacher's lesson, but did not impact the MDC. That is, a teacher's level of MKT does not seem to influence their likelihood to solicit student ideas or allow opportunities for student-to-student talk, but it does impact the level of questioning posed to promote students' explanation of ideas. This aligns with previous studies that have shown that teachers with higher MKT are more likely to respond to students' thinking and provide rich opportunities with the mathematics (Hill et al., 2008). This finding supports programmatic efforts within teacher preparation programs that strive to increase PTs' MKT.

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Table 2: Estimated Effects of Teacher Attributes and School Context on the Level of Explanation and Justification

		Model 1 DF=118	Model 2 DF=115	Model 3 DF=109	Model 4 DF=101
Fixed Effects, EJ, β₀		(SE)	(SE)	(SE)	(SE)
Intercept, γ_{00}		3.19** (.11)	3.20** (.10)	3.21**(.10)	3.16** (.14)
Teacher Attributes					
MKT, $\gamma 01$			.27* (.10)	.22*(.11)	.24* (.15)
PMTE, $\gamma 02$				-.01 (.02)	-.004 (.02)
Nature, $\gamma 03$				.13* (.05)	.17* (.05)
Calculations, $\gamma 04$				.04 (.06)	.05 (.05)
Value, $\gamma 05$				-.01 (.06)	.05 (.06)
Teaching Context					
FRPL, $\gamma 06$					-.76 (.48)
Grade band, $\gamma 07$					.09 (.04)
TPSS, $\gamma 08$					.07 (.04)
Random Effects					
EJ ($\tau 00$)		.75** (.18)	.70** (.17)	.65** (.17)	.58**(.17)
Within Teacher variance ($\sigma 2$)		1.51** (.14)	1.50**(.15)	1.53**(.15)	1.50** (.15)

$**p < .001, *p < .05$

The field continues to diverge in their conceptions of epistemological beliefs. This study teases apart distinct elements of epistemological beliefs and found that teachers' beliefs related to the nature of mathematics were predictive of the MDC and the level of EJ within their classrooms. It seems plausible that a teacher must believe that mathematics is an interactive discipline before employing discourse-based teaching strategies. This finding confirms the importance of attending to and explicitly addressing PTs' beliefs because beliefs may be a border preventing some teachers from constructing opportunities for classroom discourse.

School contextual variables, FRPL and TPSS, also accounted for differences between teachers' MDC. Teachers within schools with higher FRPL had lower levels of MDC, meaning that on average students of low SES had less opportunity for sharing their ideas with their teacher and fellow classmates. This finding supports previous research that students of SES receive more procedural instruction (Desimone & Long, 2010) in that procedural instruction is less likely to include discussions of ideas. Also, teachers reporting higher levels of school support in regards to mathematics instruction were also more likely to solicit student ideas and allow time for students to discuss mathematical ideas with one another. This positive relationship between the school culture and classroom practice has implications for administrators and instructional support personnel (e.g., coaches) as they work to promote quality instruction. Messages need to align between the mathematics education field and the school culture to advance discourse-rich classrooms. School context is influential; next steps for this work will include a more in-depth examination of the discussions in classrooms of a variety of SES. SES should not be a border that keeps some students out of mathematical conversations because it is those conversations that provide access to the mathematics.

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TRANSCENDING TRADITIONAL/REFORM DICHOTOMIES IN MATHEMATICS EDUCATION

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Here, we report on the development of a theoretical framework for mathematics teaching that looks aside from common traditional / reform dichotomies. Attention is focused on managing the amount of new information students must attend to, while structuring that information in a manner that allows discernment of key features and continuous extension of meaning. Drawing on and combining ideas based on mastery learning and on the variation theory of learning, we propose an alternative where fluency and emergent knowing are inseparable and mutually reinforcing, and motivation is based on success and the challenge of pressing the boundaries of knowing. Gains in student achievement associated with this framework have been significantly faster than national norms.

Introduction and Purpose

Recent calls to find a balance between allegedly opposite instructional approaches—commonly dichotomized as "traditional" and "reform" approaches—have at times posited these extreme approaches as complementary, suggesting that the two may productively interact. For instance, Ansari (2015) claimed that "[I]t is time to heed the empirical evidence coming from multiple scientific disciplines that clearly shows that math instruction is effective when different approaches are combined in developmentally appropriate ways" (para. 14). In this paper, we argue that such seemingly contradictory approaches are in fact very similar in significant ways and propose a 'third way' that addresses important features not stressed by either approach.

The work reported here developed from our work with the Math Minds initiative, which is a five-year partnership between a large school district, a mathematical charity, a children's support group, a university education faculty, and a funder. It is aimed at improving mathematics teaching and learning at the elementary level. More specifically, it aims to deepen understanding of relationships between teachers' knowledge, curricular resources, professional development, and students' performance, a combination not commonly addressed in the literature. In particular, we are working to identify features of resources that can support the development of teachers' mathematical knowledge for teaching. Here, we outline key principles that we have identified as significant for teaching mathematics and then contrast these principles with other approaches. While our primary aim here is theoretical, we also provide a brief statement regarding impact on student achievement.

Theoretical Perspectives

While there are widely varying approaches to teaching mathematics, most approaches may be placed somewhere along a continuum with respect to the degree of emphasis they place on (A) mastery of fixed algorithms as a means of achieving procedural fluency and (B) conceptual understanding and mathematical process (*cf*. Star, 2005). They may similarly be placed along a second continuum according to the degree of (A) teacher guidance or (B) student responsibility for developing their own strategies and procedures (*cf*. Chazan & Ball, 1999). While both A's are often

associated with traditional approaches to teaching mathematics and both B's are often associated with reform approaches (*cf.* National Council of Teachers of Mathematics [NCTM], 1989), even attempts to balance (A) and (B) must by definition buy into one or both dichotomies (*cf.* Ansari, 2015; NCTM, 2006; Common Core State Standards Initiative [CCSSI], 2015). What we propose is not a compromise, but something that shifts attention to features seldom addressed by either: In this light, A's and B's emphases turn out to have some surprising similarities. Perhaps most notably, both A and B (and points between) typically require students to attend simultaneously to multiple pieces of new information.

Although the notion of cognitive load has been primarily used to critique Type (B) (*cf.* Kirschner, Sweller, & Clark, 2007), it can also lead to difficulty in Type (A) approaches that present complex procedures that require simultaneous attention to multiple pieces of new information. Here, Marton's (2015) Variation Theory of Learning provides a helpful framework for analysis. Marton argued that "we can only find a new meaning through the difference between meanings" and that "the secret of learning is to be found in the pattern of variance and invariance experienced by learners" (p. xi). More specifically, these patterns of variation are divided into three main categories: contrast, generalization (or induction), and fusion. Contrast allows separation or discernment of a critical feature by varying only the thing one wants to draw attention to; generalization separates the ways a previously discerned feature or object can vary, and fusion recombines features that have been separated. When variation is not carefully structured, learners may overlook significant discernments critical to the intended object of learning. In addition, individual items may be perceived as either unique and difficult or as boring and repetitive, with no further meaning to be potentially gleaned from the juxtaposition of different items or from their combination in increasingly complex arrangements.

Type (B) approaches typically aim to engage students in mathematical contexts that help them make sense of mathematical relationships through processes such as problem solving, reasoning, proof, communication, representation, and making connections (*cf.* CCSSI, 2015; Kilpatrick, Swafford, & Findell, 2001; NCTM, 2000; Western and Northern Canadian Protocol, 2006). While it is possible to attend to these features in conjunction with mastery and careful variation, the importance of doing so is typically not made explicit. For example, in our own work outside of this project, we have noted that with an emphasis on multiple strategies, students have often remained unaware of the connections between them. Problems too complex for students to unpack on their own have prompted teachers to do so *for* them. Further, some students have engaged in ways that allowed them to bypass key learning objectives.

In documenting the difficulties teachers sometimes face when transitioning away from transmission-based models of math instruction (Type A), Swan, Peadman, Doorman, and Mooldjik (2013) noted that teachers may "at first withdraw support from students and then recognise the need to redefine their own role in the classroom" (p. 951). In our work, we have not asked teachers to begin with complex problems nor to withdraw key supports but have instead focused directly on what *different* supports might look like, and how these might contribute to both sense-making and continuous extension of mathematical knowing.

A key motivating principle of this study is the conviction that virtually all students are capable of learning challenging material (in our case, mathematics) if given appropriate supports (Bruner, 1960). From the outset, we discussed with teachers the importance of nurturing a growth mindset (Blackwell, Trzesniewski, & Dweck, 2007), which includes the belief that mathematical ability is learnable rather than innate. We explicitly advocated a mastery approach to learning (Guskey, 2010), with an emphasis on parsing material into manageable pieces, assessing continuously (also see Wiliam, 2011), moving forward as students demonstrate independent mastery, and extending as needed to ensure challenge for all. We further distinguish our approach through use of what we call *micro*-variation, in which we treat variations that might otherwise be seen as trivial as legitimate

obstacles (Metz et al., 2015). By structuring variation in a responsive manner, we have worked to nurture classroom environments where all can succeed and be challenged. Motivation is thus based on success and continuous growth (Malone & Lepper, 1987; Pink, 2007).

Building on the Marton's Variation Theory of Learning (*cf.* Gu, Huang, & Marton, 2004; Kullberg, Runesson, & Mårtensson, 2014; Marton, 2015; Park, 2006; Runesson, 2005; 2006; Watson & Mason, 2005; 2006), we have worked to avoid the fragmentation that can happen when curricula are parsed into tiny pieces. By using systematic variation (Park, 2006) to draw attention to key ideas that are often overlooked and by considering how these may change within and eventually between topics, we have aimed to support the development of rich webs of interconnected understanding (for both teachers and students). These webs then allow the continued emergence of new understanding (Davis & Renert, 2014).

Mode of Inquiry

We began the study with the intent of exploring the impact of a mastery approach to learning with an explicit focus on structured variation. The study draws on multiple sources of data (classroom observation, video-taped classes, teacher and student interviews, informal conversations with teachers, and standardized tests) to inform next steps. The principles that currently guide our work have been both informed by and used to inform a supporting resource and an associated professional development program for participant teachers. The primary study site is a small K-6 elementary school with a history of low achievement. Our work is consistent with Cobb, Confrey, diSessa, Lehrer, and Schauble's (2003) description of design-based research in that it involves "both 'engineering' particular forms of learning and systematically studying those forms of learning within the context defined by the means of supporting them" (p. 9). Also consistent with a design focus, our work has been subject to ongoing test and revision, continuously informed by classroom observations and interactions with teachers and students. It has also developed in response to insights gained through regular meetings among members of the research team and meetings involving the research team, school leaders, and a representative from the teaching resource used to support the initiative. Relationships among initiative partners are deeply reciprocal: School leaders contribute their expertise and inform the other partners of school-based needs as well as learn from the expertise of the others; the person who represents the resource offers support to both teachers and to the research team in accessing key features of the resource and maximizing their potential, while also gathering feedback that will be used to improve the resource. The research team relies on the insights and expertise of the school-based leaders and the resource representative, while offering feedback based on research observations.

As part of the initiative, teachers were provided with guides, resource materials, and student materials that support our emphases on growth mindset, mastery learning, continuous assessment, and careful attention to variation. With these in place, ongoing professional development has emphasized the importance of appropriate parsing and pacing of instruction, continuous assessment and feedback, and strategies for extending work beyond that provided in the resources. In working with teachers as they attempt to use these ideas and materials, we have found it useful to clarify distinctions between this approach and the more familiar (A) and (B) approaches described earlier as dichotomies; while perhaps overgeneralized if taken all together and taken in their extreme forms, these have provided useful points of contrast for what we here describe as approach (C).

Distinguishing a Third Way

We now offer a summary of our current thinking on these matters. Here, we refer to (A) and (B) to elaborate the dichotomies we presented earlier, while (C) represents our own current thinking. Importantly, we see (C) not as a balance but as something significantly different from either (A) or (B). Perhaps most notably, (C) *asks students to attend to one new idea at a time, where both (A) and*

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(B) tend to require simultaneous attention to multiple topics, whether those be (A) elements of complex procedures or (B) multiple topics / strategies. By systematically varying one thing at a time, (C) invites continuous extension and elaboration of meaning rather than the articulation of meaning found in given applications (as is common in A) or broad contexts (as in B). Therefore, in (C), fluency and understanding are mutually supporting rather than competing. Considerations of the role of practice follow closely from these views, with (A) using repetitive practice to build fluency, (B) using embedded practice presumably made meaningful through context, and (C) offering practice through continuous extension. Positions (A) and (B) tend to result in a wide range of achievement: In (A), some students successfully master content, while others do not, where (B) intentionally allows multiple entry points and open-ended paths. Position (C) supports the mastery of a common base of understanding that may be personalized through extension.

In working with teachers, we have found it important to draw attention to the manner in which selected resources offer variation, both so that teachers can draw student attention to such variation and so that they can adapt given examples in ways that support struggling learners and those who require extension. In Tables 2 and 3, we offer examples to clarify the distinctions we are attempting to make; here, Task Sets A, B, and C correspond to the same distinctions we have described in terms of a traditional (A) / reform (B) dichotomy and a proposed alternative (C).

In Table 1, Task Set A offers focused practice with grouping coins to make particular values, but both the values and the numbers of coins vary from item to item. As a result, there is limited value in looking for connections between individual items: The sequence does little to scaffold such relational thinking. For instance, a student fluent in calculating differences might note that moving from $0.30 to $0.45 with two additional coins would require an additional dime and nickel. However, a similar transition makes less sense in moving from $0.45 to $0.55 with three additional coins. In addition to regrouping coins, working effectively with Set B requires a systematic approach to finding *all* combinations. If both re-grouping money values and exploring combinations are new to students, this question likely varies too much at once; importantly, only students who work systematically are exposed to meaningful variation among items, while others may choose a more random approach to finding possible coin values. In Set C, only the number of coins varies, and students are not (*yet*) responsible for *creating* the variation that prompts attention to relationships between subsequent items (as in Set B).

Table 1: Three ways to vary coin-grouping tasks

A	B	C
Using quarters, dimes, and / or nickels: Make $0.30 with 3 coins. Make $0.45 with 5 coins. Make $0.55 with 8 coins. Make $0.60 with 6 coins.	Find all the ways to make $0.45 with quarters, nickels, and dimes.	Using quarters, dimes, and nickels: Make $0.45 with 3 coins. Can you do it with 4 coins? Can you do it with 5 coins? Can you do it with 7 coins?

In Table 2, the distinctions between the sets are perhaps even more pronounced. In Set A, the numbers were chosen to allow variation in the size of number and number of factors (though of course bigger numbers do not always have more factors); while this may offer practice with creating factor trees, it does little to direct attention to ways that prime factors can make particular number structures visible, as too many features change from item to item. To successfully complete Set B, students must discern many relationships between prime factors and factors, not the least of which is that a particular pattern of unique and non-unique prime factors will always yield the same number of factors (e.g. 2 x 2 x 3 has the same number of factors as 3 x 3 x 5, as both follow an a x a x b pattern); here, the onus is on students to organize their work in a manner that allows exploration of

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particular patterns of variation. If they fail to do so, the variation they experience will be largely random. By carefully controlling how much is changing, Set C1 can be used to draw attention to the fact that regardless of how a number is factored, the prime factors are always the same (which is not obvious to many students). Similarly, C2 draws attention to patterns in the structure of numbers; teachers and students alike are sometimes surprised to discover that doubling a number does not double the number of prime factors.

Table 2: Three ways to explore prime factors

A	B	C1	C2
Make factor trees for each of the following: 15 18 40 60	What is the smallest number with 14 divisors? Use patterns in prime factors to help you explore this problem.	Make a factor tree for 36. Can you do it another way? Another? Another? What is the same/different about your solutions?	Make a factor tree for each of the following. In each set, what changes from one to the next? 8, 16, 32, 64 5, 25, 125, 625 10, 100 1000, 10,000 25, 50, 75, 100

In sessions for professional development, we have observed that teachers who worked through these examples noted that the tasks in the (C) sets made it easier for them to recognize important ideas as well as to create their own extensions (something that most have struggled with): They, too, were prompted to think differently by this arrangement of content. In other words, when a resource offers clearly structured variation, both teachers and students engage differently with the content. This is not to say that the tasks in A and B have *no* place if properly contextualized within an appropriate sequence; rather, our key point is that *C serves a critical purpose that is often overlooked.*

While our aim here is primarily theoretical, we note that evidence based on weekly classroom observations shows students who previously struggled have become more willing to take part, and many students have become excited to keep pushing their understanding to new levels. We have also noticed a consistent improvement in mathematics basic skills. The school's total math scores on the Canadian Test of Basic Skills (Nelson, 2014) improved over a two-year period at a rate that was significantly faster than the national normed population gains [$F(2,70)=6.977$, $p=.002$]. Overall, the model indicates that the mean total math score increased from an estimated average score of 43.5 (national average=50) to that of 47.0 [$t(60.42)=3.732$, $p<.001$], with roughly equal gains over the national normed population in each year (year one = +1.7%, year 2 = +1.8%).

Conclusions & Significance

The approach briefly described in this paper resembles current research emerging in the UK that has emphasized combining mastery learning with structured variation (*cf.* National Centre for Excellence in Teaching Mathematics, 2014; Schripp, 2015). Two features that set our work apart from these efforts are (a) the emphasis on what we have been referring to as *micro-variation* and (b) attention to intrinsic motivation by using micro-variation to continuously elaborate understanding

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and increase levels of challenge in ways that are accessible to all. We argue that it is not enough to combine instructional approaches; rather, it is necessary to transcend the borders of restrictive dichotomies. As we suggested in the beginning, we see (C) as an alternative to approaches that, when considered in terms of how content is structured, turn out to be more similar than typical traditional vs. reform dichotomies suggest. Based on results from the first three years of the program, we propose that combining mastery learning and structured variation in a context that attends closely to continuous assessment, intrinsic motivation, and emergent knowing holds great promise for improving student achievement in mathematics and for supporting teachers in achieving these goals.

Considering this third way opens up new avenues for research. It will be important to investigate strategies to support teachers who are using this approach. We are interested in the interaction between educational curricular material and teachers' knowledge.

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EQUITABLE PARTICIPATION IN A MATHEMATICS CLASSROOM FROM A QUANTITATIVE PERSPECTIVE

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Inequity is a pressing concern in the mathematics education community. Recent research shows how inequity in everyday classroom interaction can shape student participation in subtle ways. This paper focuses on a tool, EQUIP, which uses a quantitative approach to illuminate aspects of such inequities. EQUIP cross-tabulates relatively low-inference indicators of classroom interaction with demographics (e.g., gender, race), in order to highlight inequities in participation across different groups of students in a given classroom. We present analyses of whole-class discussions in an elementary mathematics classroom taught by an experienced teacher with strong commitments to equity. Findings show that even though in most ways participation was distributed equally by gender and race, an intersectional analysis revealed statistically significant inequities for Latin@ male student participation.

Keywords: Equity and Diversity, Classroom Discourse, Gender, Research Methods

Equity is a theme that cuts across seminal policy documents in mathematics education (National Council of Teachers of Mathematics, 1989, 2014; National Research Council, 1989). The fundamental idea is that mathematics should be accessible to all students, rather than to only the privileged few (Ernest, 1991; Gutiérrez, 2002). And yet, while there has been some progress on this front, research shows that mathematics remains relatively inaccessible for children of color, young women, and students living in economically marginalized communities (Oakes, 2015).

In researching equity at the classroom level, prior research has conceptualized equity in terms of students' access to opportunities to participate in mathematical discourse and classroom activities (Esmonde, 2009; Herbel-Eisenmann, Choppin, Wagner, & Pimm, 2011; Langer-Osuna, 2011). Grounded in sociocultural perspectives that view participation and learning as fundamentally linked (Lave & Wenger, 1991; Nasir, 2002), much of this work has utilized qualitative methodologies. Indeed, our own work to date on equity in mathematics classrooms has been qualitative in nature (see Shah, 2009, 2013). However, the resource-intensive nature of high quality, in-depth qualitative analysis can make it difficult to conduct such research on a large scale. In this paper, we ask: what are the potential affordances (and limitations) of a quantitative approach to analyzing equity in classroom interaction, and how might quantitative approaches complement extant qualitative methodologies?

This paper introduces a classroom observation tool for analyzing equity-related patterns in classrooms called EQUIP (**E**quity **QU**antified **I**n **P**articipation). EQUIP offers a way of gathering and analyzing quantitative data on relatively low-inference dimensions of classroom participation. To illustrate our approach, we present an analysis of participation patterns in an elementary mathematics classroom taught by a highly experienced teacher with strong commitments to equity. Findings show that while in most ways participation was distributed equally by gender and race, an intersectional analysis (i.e., cross-referencing gender and race) revealed significantly lower levels of participation for the Latin@ males in the class. We argue that a quantitative methodological approach made it

possible to identifying this inequity for Latin@ males, which otherwise may have been too subtle for even an equity-minded teacher to notice. We conclude by reflecting on how quantitative and qualitative methodologies for studying equity in mathematics classrooms might complement each other.

Conceptualizing Equity in Terms of Participation

Equity and equality are distinct concepts and should not be conflated. Whereas equality means that all students are treated in an identical manner, equity means that students should be treated in a fair manner (Gutiérrez, 2002; Secada, 1989). Achieving equity might involve one group of students being treated differently than another group of students. However, in certain situations, equality can be thought of as a baseline, or minimum requirement, for equity.

From a theoretical standpoint, we follow other researchers in conceptualizing equity in terms of student participation in classroom practices (see Boaler, 2008; Esmonde, 2009; Langer-Osuna, 2011). All students should have opportunities to participate in the disciplinary practices constitutive of the learning process, such as sharing ideas, asking questions, and justifying one's reasoning (Bransford, Brown, & Cocking, 2000). In order to provide students such opportunities, teachers should allot students ample time to engage in classroom discourse (Ball, 1993; Cazden, 1988;).

Equitable participation is about both *who* participates and *how* students get to participate (Wager, 2014). On a basic level, an equitable classroom can operationally be defined as one in which the amount of participation proportionally aligns with the demographics of the class. For example, if Black students make up 21% of the students in the class, then approximately 21% of class participation should be by Black students. However, beyond the issue of *who* participates, opportunities to participate should also be of a cognitively demanding nature (cf. Stein, Grover, & Henningsen, 1996). A classroom where all students participate—but only in low-level, Initiate-Response-Evaluate sequences—would be considered less equitable because students are not afforded opportunities to act as sense-makers.

Intersectionality

Building upon decades of work by activists, Kimberlé Crenshaw coined the term *intersectionality* to highlight complexities in the lived experiences of women of color that cannot be captured by examining race or gender alone (Crenshaw, 1991). Taking up intersectionality in Black feminist studies, Patricia Hill Collins defines intersectionality as "particular forms of intersecting oppressions," noting that "intersectional paradigms remind us that oppression cannot be reduced to one fundamental type, and that oppressions work together in producing injustice (Collins, 1999, p. 18). Mathematics education researchers have also leveraged intersectional perspectives to address issues of equity that affect traditionally and historically marginalized students in schools (e.g., Esmonde & Langer-Osuna, 2013; Gutiérrez, 2013). As we will show, our work is informed by the concept of intersectionality.

Methods

Video data were collected during a two-week, university-based elementary mathematics summer program. The program consisted of 30 hours of instructional time (3 hours per day, for 10 days), which focused on fraction representations and how to participate in mathematical discourse. A typical class session involved small-group work, individual work, and whole-class discussions (WCDs). The course instructor had over 30 years teaching experience and a strong disposition towards equity. During the program, the instructor collaborated with various colleagues in the mathematics education community to discuss pedagogical issues, such as how to manage equitable student participation.

The program consisted of 30 rising 5th grade students, who had been identified as struggling at

their home schools by coordinators of the summer program. Racial demographics were as follows: 21 (70%) Black Students, 5 (17%) Latin@ students, and 4 (13%) White students. Gender demographics were evenly distributed between male and female students within each racial category. The forthcoming analysis focuses on classroom interactions in WCDs, which are a centerpiece of mathematics classrooms at all grade levels, and thus represent an important context for study.

Analytical Approach

The field has produced observation tools for analyzing mathematics classroom activity, several of which account for issues of equity (e.g., Schoenfeld, 2014; University of Michigan, 2006). However, a limitation of these tools—perhaps because equity was not their primary conceptual focus—is that they do not illuminate how opportunities to learn can become inequitably distributed across markers of difference (e.g., gender, race, language proficiency). The EQUIP analytical approach is predicated on three basic equity-related questions: 1) who gets to participate; 2) what is the nature of that participation; and 3) how are different forms of participation distributed across the students in the class? That is, do all students participate in cognitively rich ways, or are those opportunities only made available, for example, to the White and Asian students, or male students, or English dominant students in the class?

EQUIP uses "participation sequences" as a fundamental unit of analysis. A participation sequence is a consecutive sequence of verbal turns between a single student and a teacher. When another student speaks, a new participation sequence begins. Each participation sequence is coded using the following relatively low-inference dimensions of participation: *WCD type, solicitation method, student wait time, talk length, type of student talk, teacher solicitation, and explicit evaluation*. These dimensions concern: whether or not the discussion was mathematical (*WCD type*); whether and how the teacher solicited the student's participation (*solicitation method*); the level of mathematical explanation requested of the student (*teacher solicitation*); the time allotted by the teacher prior to the first verbal turn in the participation sequence (*wait time*); the level of mathematical explanation provided by the student (*type of student talk*); the most number of words uttered by a student in a single uninterrupted talking turn (*talk length*); and whether or not the teacher verbally assessed a student's response (*explicit evaluation*).

Findings

Overall, the data show considerable evidence of equitable classroom activity. The interactive style of WCDs in this particular classroom provided students opportunities to participate in rich mathematical discussions. This is evidenced by: the length of student talk, long wait time, and minimal explicit evaluation by the instructor. With respect to the length of student talk, students contributed on average 31 words per participation sequence, which indicates participation that goes beyond abbreviated or simple contributions. Further, wait time exceeded three seconds in 71% of participation sequences, which indicates that students were given time to think before responding to teacher solicitations. Finally, the teacher did not explicitly evaluate student responses in 80% of participation sequences.

In the next sections, we complexify these initial findings by disaggregating the data by gender and race, as well as by conducting an intersectional analysis by gender and race simultaneously.

Disaggregation by Gender

There were generally equal levels of participation for male and female students. We used a chi-squared test to determine whether there were any statistically significant differences in the quantity and quality of participation. Table 1 focuses on three dimensions measured by EQUIP that provide information about student participation: participation sequences, WCD type, and the type of student talk. For participation sequences, there was no significant difference by gender. For WCD Type, a

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chi-squared analysis also showed no difference by gender in student participation across mathematical and non-mathematical WCDs, $X^2 = 0.2193$, $p > 0.05$. For type of student talk, a chi-squared analysis also showed no difference by gender across four different types of student talk, $X^2 = 4.1227$, $p > 0.05$. While three dimensions related to teacher moves did show significant differences by gender, the effect sizes (Cramer's V) were "small" or "very small."

Table 1: Participation Sequences for WCD Type and Student Talk by Gender

	Participation Sequence ($n = 1343$)	WCD: Math ($n = 1067$)	WCD: Non-math ($n = 276$)	St. Talk: What ($n = 393$)	St. Talk: How ($n = 40$)	St. Talk: Why ($n = 321$)	St. Talk: Other ($n = 586$)
Female	647 (48%)	518 (80%)	129 (20%)	205 (32%)	21 (3%)	147 (23%)	273 (42%)
Male	696 (52%)	549 (79%)	147 (21%)	188 (27%)	19 (3%)	174 (25%)	313 (45%)

Disaggregation by Race

Compared with gender, analysis showed greater disparities in student participation by race. There were significant differences in student participation by race in nearly all of the dimensions. For the sake of continuity, this section focuses on the same three dimensions as in the previous section: participation sequences, WCD type, and the type of student talk (see Table 2).

For participation sequences, a chi-squared test indicated a significant difference by race, $X^2 = 18.581$, $p < 0.05$. The effect size (Cramer's V = 0.083) was small. As shown in Table 2, Black students were more likely to participate, while Latin@ students were less likely to participate. For WCD type, a chi-squared test indicated no significant difference by race, $X^2 = 0.1038$, $p > 0.05$. For type of student talk, a chi-squared test showed a significant difference by race, $X^2 = 11.1139$, $p < 0.05$. The effect size was very small (Cramer's V = 0.0644).

Table 2: Participation Sequences for WCD Type and Student Talk by Race

	Participation Sequence ($n = 1340$)	WCD: Math ($n = 1067$)	WCD: Non-math ($n = 276$)	St. Talk: What ($n = 393$)	St. Talk: How ($n = 40$)	St. Talk: Why ($n = 321$)	St. Talk: Other ($n = 586$)
Black	1018 (76%)	810 (80%)	212 (21%)	301 (30%)	32 (3%)	230 (23%)	458 (45%)
Latin@	150 (11%)	119 (79%)	30 (20%)	50 (33%)	5 (3%)	35 (23%)	57 (38%)
White	172 (13%)	138 (80%)	34 (20%)	42 (24%)	3 (2%)	56 (33%)	71 (41%)

Intersectional Analysis by Gender and Race

When considered independently, analyses indicated that, in general, participation by gender was more equally distributed, but that participation by race was somewhat inequitable. An intersectional analysis of gender and race revealed additional nuances in the equity dynamics of this classroom (see

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Table 3). For WCD type, a chi-squared test indicated a significant relationship, $X^2 = 30.2288$, $p < 10^{-7}$. Using a Fisher's exact test, we found that the effect size was small for mathematical WCDs (Cramer's V = 0.168) and medium for non-mathematical WCDs (Cramer's V = 0.292).

In addition to inequities in *who* participated in WCDs, we found inequities in the quality of student participation. A chi-squared test examining the relationship between the types of student talk and the intersection of race and gender showed that the relationship between the variables was significant for Why-level student talk, $X^2 = 15.5739$, $p < 0.001$. The effect size was small to medium (Cramer's V = 0.220).

A key finding here was that Latin@ male students in the class were much less likely to participate in Why-level Student Talk than any other group of students. From the previous discussion about racial patterns (Table 2), the data show that Latin@ students overall were less likely to participate in Why-level talk compared with Black and White students. However, the intersectional analysis reveals a gender disparity between Latin@ male and female students. In fact, in terms of the overall participation, the number of female Latin@ students' participation sequences was 2.75 times the number of male Latin@ students' participation sequences. This is striking in relation to the more modest differences between genders among both Black and White students.

Table 3: Participation Sequences for WCD Type and Student Talk by Race and Gender

	Participation Sequence (n = 1340)	WCD: Math (n = 1067)	WCD: Non-math (n = 276)	St. Talk: What (n = 393)	St. Talk: How (n = 40)	St. Talk: Why (n = 321)	St. Talk: Other (n = 586)
Black Female	447 (33%)	362 (81%)	85 (19%)	139 (31%)	16 (4%)	97 (22%)	194 (43%)
Black Male	575 (43%)	448 (78%)	127 (22%)	162 (28%)	16 (3%)	133 (23%)	264 (46%)
Latin@ Female	111 (8%)	85 (77%)	26 (23%)	38 (34%)	3 (3%)	27 (24%)	43 (39%)
Latin@ Male	38 (3%)	34 (89%)	4 (11%)	12 (32%)	2 (5%)	8 (21%)	14 (37%)
White Female	89 (7%)	71 (80%)	18 (20%)	28 (31%)	2 (2%)	23 (26%)	36 (40%)
White Male	83 (6%)	67 (81%)	16 (19%)	14 (17%)	1 (1%)	33 (40%)	35 (42%)

Discussion

A key takeaway from this study is that even in the classroom of an experienced, equity-minded teacher, inequities can emerge. In a sense, this is an important but unsurprising finding. Classrooms are complex spaces, and noticing subtle inequities can be difficult for even the most highly trained, well-intentioned teachers. Still, tools like EQUIP can support both researchers and practitioners in identifying inequities in classrooms. In our view, the methodological approach described here represents a step in that direction.

Overall, the data suggest that this was a classroom characterized by rich opportunities for students to participate in whole-class discussions. When disaggregated by gender and race, the data show minimal differences in participation. This is remarkable considering that there were over 1,000 participation sequences in the data set. However, an intersectional analysis revealed inequity in Latin@ male participation. Disaggregated analyses of this kind at increasing levels of granularity that account for intersectional subjectivities can provide important nuance to discussions of equity. This quantitative approach emphasizing relatively low-inference (but high leverage) dimensions of

classroom activity has the potential to facilitate research on equity at scale.

To be clear, in highlighting the affordances of this approach, we do not mean to suggest that equity can be reduced to a mere "technical" concern (Secada, 1989). Equity—as distinct from equality—revolves around deep philosophical questions of fairness and justice that cannot be fully quantified. Students' subjective experiences around issues of equity are also critically important (Martin, 2006). That is, independent of the numbers, students must also *perceive* classrooms to be equitable spaces. Thus, in the spirit of looking across methodological borders, our goal is to consider how quantitative and qualitative methodologies might complement each other in mixed methods approaches to researching issues of equity in classrooms.

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FROM SPEAKING TO WRITING: THE ROLE OF THE REVERSAL POETIC STRUCTURE IN PROBLEM-SOLVING

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Speakers in conversation typically repeat and modify earlier comments. In mathematics conversations, these repetitions, or poetic structures, can facilitate the collaborative discovery of mathematical relationships. A close analysis of 90 turns of an algebraic problem-solving conversation reveals eight types of poetic structures. This report summarizes the general results of the analysis and highlights the role of a particular type of poetic structure, the reversal. In a reversal poetic structure, two elements switch places syntactically, for example, subject and predicate or adjective and noun switch roles. Students working with no teacher intervention used the reversal poetic structure as they rearranged variables and coefficients in their verbal method, and as they transitioned from a spoken method to written mathematical notation. This analysis highlights the ways in which speaking supports mathematical thinking.

Keywords: Classroom Discourse, Problem Solving, Algebra and Algebraic Thinking

Introduction

Speakers of all languages repeat each other. Because repetition is pervasive in daily speech (Du Bois, 2014), it can contribute to mathematical problem-solving conversations. This paper highlights ways in which particular types of repetition—poetic structures—facilitate students' mathematical learning. Poetic structures occur when speakers repeat the grammatical structures of phrases spoken before, perhaps changing words or small aspects of grammar. This paper reports on the role of a particular poetic structure, the reversal, in an algebraic problem-solving session. In this mathematical conversation, the reversal poetic structure was associated closely with the transition from verbal problem-solving to written mathematical notation.

This study contributes to research on language as a resource for mathematical learning, which developed from studies of multilingual classrooms (e.g. Barwell, 2015; Planas & Setati-Phakeng, 2014). Language-as-resource research is concerned with issues such as code-switching, the influences of educational policy on classroom communicative practice, and language as a resource in formal vs. informal mathematics discourse. Collaborative learning pedagogies rely fundamentally on students' linguistic interaction. It seems intuitively correct that small changes to previous statements can account for collective mathematical constructions. Poetic structure analysis grounds this intuition in observable conversational actions. Because repetition is so commonplace, its analysis can deepen our understanding of the ways in which language facilitates learning in multilingual classrooms, or in monolingual classrooms in any language.

Theoretical Foundation

Dialogic syntax, an emerging research focus in linguistics, forms the theoretical foundation for this analysis (Du Bois, 2014; Sakita, 2006). Dialogic syntax recognizes that as speakers repeat prior statements—their own or those of others—they reproduce syntactic arrangements that create meaningful relationships across sentences and across speakers. Hearers decode and respond to the meanings that are created at these structural levels beyond the sentence. For example, in the hexagon task described in this paper, Sheila's *minus 2…times 2* is recast in Joseph's clarifying question:

78 *S:* So number of hexagons would be 4 times 6 minus n minus 2. So 4 times 6 would be 24. Number of hexagons would be 1, 2, 3, 4. 4, uh, times 2.

79 *J:* Times 2 or minus 2?

The verbs *minus* and *times* shift within Sheila's and Joseph's comments separately, and are repeated across their comments, while retaining focus on the direct object of *2*. This example highlights the ways in which speakers use repetition to negotiate mathematical conjectures. Dialogic syntax proposes this coordination as a "new, higher order linguistic structure…the coupled components recontextualize each other, generating new affordances for meaning" (Du Bois, 2014, p. 360). The field of dialogic syntax holds this exchange as a new type of grammar, in which linguistic structure is no longer confined to the sentence or to a single speaker.

Du Bois (2014) provides a useful review of the theoretical antecedents of dialogic syntax, which draw from a wide range of fields, including linguistics, anthropology, literary theory, and cognitive science. He identifies four foundational themes, some of which resonate with prior research in mathematics education. The first theme, parallelism, refers to the concrete repetitions within nearby utterances. In the example above, Joseph's *minus 2* is parallel to his *times 2*, and both are repetitions of the endings of Sheila's sentences. Staats (2008) highlights ways in which these parallel, poetic structures can express both inductive and deductive mathematical reasoning. Oslund (2012) uses poetic structures in teachers' narratives to trace their shifts across fraction metaphors that were familiar or unfamiliar to their students.

Underlying grammatical parallelism is the principal of indexicality, or the capacity of language to refer to or point to other words and to elements of the situational context. Indexical words like *this*, *that*, and variable names like *n* have been associated with mathematical activities such as generalization and collaborative learning (Barwell, 2015; Radford, 2003). Parallelism occurs when units larger than a word—*times 2*—"point to" corresponding units like *minus 2*, creating bundles of indexicality.

Du Bois' second theme, analogy, refers to the meanings created through manipulation of similar units. For example, *times* and *minus* are alternatives within the frame of mathematical operations. The third theme, priming, is the experimentally-measured tendency to repeat lexical or syntactic units (see Staats & Branigan, 2014 for a discussion of applications and limitations within mathematics education research).

The fourth theme, dialogicality, has received slightly more attention in mathematics education research. Barwell (2015) following Bakhtin (1981), discusses three orientations of dialogicality: multivoicedness, multidiscursivity, and linguistic diversity. The first of these, multivoicedness, recognizes that all speech has a history. Speakers recast words and meanings from their past interactions each time they talk. This paper provides an analysis of multivoicedness in a mathematical problem-solving session. Overall, then, dialogic syntax is a new framework for mathematical education research, but through its interdisciplinary character, it shares theoretical antecedents with research on language as a resource for mathematical learning.

Participants and Task

Sheila and Joseph are undergraduate students who participated in a paid problem-solving session outside of class that was audio- and video-recorded. They had recently completed a university class in precalculus. Their task was to find an equation for the perimeter of a string of *n* adjacent hexagons, arranged so that pairs of interior sides are removed from the perimeter. They worked for about 40 minutes without any teacher intervention; about nine minutes of the conversation are analysed here. The task includes diagrams for hexagon strings for n = 1 to n = 4 hexagons, shown in Figure 1. The task also includes a table of values for n = 1 to n = 5 hexagons and the corresponding perimeter. A correct answer is *p = 4n + 2*. The task was based very closely on a proposed measure of readiness for undergraduate study (Wilmot, Schoenfeld, Wilson, Champney, & Zahner, 2011, p. 287).

Figure 1. Task diagrams for *n* = 1 to *n* = 4 hexagons.

Methods for Identifying Poetic Structures

The first 90 turns of the conversation were coded using a spreadsheet to note the ways in which a phrase formed a poetic structure with a previously spoken phrase. It was necessary to develop a coding protocol because a phrase can repeat elements of several previous phrases. The coding approach relied on a combination of close attention to the syntax of poetic structures and grounded theory coding to iteratively improve the choices about what phrases counted as repetitions of prior statements (Charmaz, 2006). The resulting system was comparatively conservative. In Gries (2005), for example, any repetition of syntax counts as repetition, even if all the words change. The phrase *3 times 2* would be considered a repetition of the phrase *4 minus 1*, because both involve a subject-verb-object construction. However, mathematics education audiences are concerned with language that facilitates mathematical learning. To better focus on continuity of mathematical topic, then, two phrases had to share syntax and at least one word in order to be considered a repetition. When multiple previous utterances could have been the foundation of a repetition, I chose the most recent one. Overall, then, this method undercounts poetic structures in comparison with related linguistics research, because it prioritizes continuity of mathematical topic.

Each repetition was classified as either *Internal* or *Across*. An internal repetition refers to repetitions within a single turn at talk by a single speaker. An across repetition refers to repetition involving two distinct turns at talk. An across comment could refer to a comment that the same speaker said in one of her previous turns at talk, or to a comment from the other speaker. I recorded the most recent turn in which the phrase occurred, even if this was within the same speaker's conversational turn; what the earlier phrase was; whether there was a change in speaker; and whether the phrase was a nearly-perfect duplicate of the previous line or a transformation of it.

I separated the conversation into four episodes, each representing a mathematical insight that the students achieved together. In episode 1, turns 1- 28, Sheila and Joseph filled a table of values on the task sheet for n = 1 to 5 hexagons and the corresponding perimeter. In episode 2, turns 29-58, they determined that they should calculate perimeter rather than area. In episode 3, turns 59-71, they initiated the idea that the shared interior sides of the hexagon strings required them to subtract two, but they did not resolve how many times to subtract two. In episode 4, turns 72-90, they expressed a

correct verbal method, and wrote a formula in which both H and N stand for the number of hexagons, #H(6) – 2(N – 1) = .

Results: Types of Poetic Speech

A close analysis of 90 turns at talk shows that poetic structures occurred very frequently. Coding produced just over 50 across repetitions and just over 55 internal repetitions. Out of 90 turns at talk, about 30 were very short comments, such as *Okay*, *Yup* or *Oh, okay, there we go*, which were too short or non-mathematical to produce poetic structures under the coding protocol. On average, then, there were about 1.75 poetic structures per substantial conversational turn.

The analysis revealed eight types of repetitions that contributed to the discovery of mathematical relationships. There were in addition poetic structures of that didn't fall into a clear type. The types were: List, Echo, Comparison, Contrast, Interposed List, Consolidation, Expansion and Reversal. Elsewhere, I describe the ways in which these eight poetic structures allowed Sheila and Joseph to gradually construct a method for solving the hexagon problem through repetitions, with small changes, of recent statements (Staats, 2016). Here, I briefly describe each type of poetic structure and I discuss in more detail the role of the reversal poetic structure.

In Episode 1, echoes and lists predominated. Sheila and Joseph began their exploration by counting the sides of the n = 1 to n = 4 hexagon diagrams. As Sheila counted the perimeter of the n =1 diagram with: *1, 2, 3, 4, 5, 6. 6,* she used a list—from 1 to 6—and an echo in the form of the second "6." At turn 15, Joseph also used echoes and lists: *So, this would be 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. 10.* The students began to fill the table for number of hexagons and the perimeter. An *Interposed List* at line 24, allowed Sheila to coordinate the independent and dependent variables: *so we're just putting in the 1 to 6, 2 to 10, 3 to 14.* At line 27, Joseph used a *Comparison* poetic structure, *22 for 5*, to extend the interposed list and to help complete the table of values.

In episode 2, Joseph suggested drawing additional interior line segments so that they could use the geometry of triangles and the formula P = 2L + 2W. Sheila suggested focusing on the exterior perimeter instead, and comments in line 52:

> 52 *S:* Complete the table showing the number of hexagons in 1 chain along with the perimeter. So then we're counting all the sides, so it'd be 6L. For 2 it'd be 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. 10L.

This is a *Consolidation* poetic structure, because a list from a previous turn, *1, 2, 3, 4, 5, 6, 7, 8, 9, 10*, has been fit inside a previous comment from turn 24: *2 to 10*. Consolidation is similar to another poetic structure in this conversation, *Expansion*, which first occurs are line 62:

> 62 *S:* Uh, so this would be 6L. 6. And then this would be 10L minus 2. Minus 2. This would be 2, 4 minus 4. This would be 6. 18L. So the total number of sides minus 2 on this side. So it'd be, uh, 6. 6, and then this would be, uh, 12 minus 2. So.

Here, there is a poetic structure repetition of the form /*this would be 6L*/t*his would be 10L…*/*This would be 2, 4 minus 4*/ *This would be 6. 18*/. Here, the short list *2, 4* and a moment later, *6*, were the first attempts to count the interior sides. This is an expansion of the comment /*this would be 6L*/t*his would be 10L…*/ with a new list. Expansion differs from consolidation because expansion introduces a new element that had not been spoken before.

At turn 79, the *Contrast* poetic structure occurred, which we saw above, when Joseph clarified, *Times 2 or minus 2?*, The final type of poetic structure for this conversation, the reversal, is discussed more thoroughly below.

It is important to note that each of the eight poetic structure types is a discursive move that could easily occur in a non-mathematical conversation. Contrast could occur, for example, as *steamed rice or fried rice?* A comment *Mark has some advice for you* could prompt the reversal: *Well, I have*

some advice for Mark! Because these poetic structures are all general discursive options, when they occur in mathematics conversation, they help us identify moments when language is a resource for mathematical learning.

Discussion: The Role of the Reversal Poetic Structure

Different types of poetic structures emerged and were prominent in different episodes of the hexagon conversation. In Episode 1, for example, many of the poetic structures were simple ones— echoes or lists. The reversal poetic structure emerged in Episodes 3 and 4. In a reversal poetic structure, two elements switch places syntactically. For example, subject and predicate or adjective and noun switch roles, as in *10 minus 12* versus *12 minus 10* or *negative two* versus *two negatives*. Reversal occurred in six turns, 62, 75, 76, 78, 81, and 90. Sheila and Joseph tended to use the reversal poetic structure as they arranged variables and coefficients in their verbal method, and as they transitioned from a spoken method to written mathematical notation.

Reversal first occurred as Sheila's internal repetition in turn 62, when she began to count the interior sides of the hexagon diagrams. This was the first time either student mentioned "minus" or subtraction.

62 *S:* Uh, so this would be 6L. 6. And then this would be 10L minus 2. Minus 2. This would be 2, 4 minus 4. This would be 6. 18L. So the total number of sides minus 2 on this side. So it'd be, uh, 6. 6, and then this would be, uh, 12 minus 2. So.

Sheila mentioned the perimeter first for the n = 1 and 2 diagrams: *6L*, and then *10L minus 2.* Then she mentioned the interior sides for the n = 3 diagram: *This would be 2, 4 minus 4.* The reversal occurred next, at the n = 4 diagram, when she switched the order of the interior sides and the perimeter: *This would be 6. 18L.* This reversal of the previous comment seems to represent Sheila's shift from a quantity that she understands well—perimeter—towards the quantity that she needs to understand better. The reversal seems to facilitate her focused attention on a new pattern in the diagram.

In turn 75, there is an across reversal from turn 72. In turn 72, Sheila commented on *total number of hexagons times six.* In turn 75, Joseph switched the order to place the 6 first: *So it'd be like 6 times x per se number of hexagons.* The phrase *number of hexagons* had been spoken before, but only while reading from the task page. Sheila's *total number of hexagons* at 72 was the first time that the *number of hexagons* was used as an independent phrase for conjecturing. Joseph's reversal at turn 75 was notable because he proposed shifting to a more standard way of speaking and writing, in which coefficients precede variables. As he said this, he wrote on the same paper as Sheila was working on: 6(x) —.

At turn 76, Sheila does not take up Joseph's suggestion immediately, but she does use reversals in turns 76 and 78 as she grapples with the n = 4 hexagon case.

76 *S:* Uh, divided, um, um, 4. 4 minus, minus n 2. Uh, like n would be the total number. 1, 2, 3, 4, number of cases. So, like in statistics the number of cases would be n and that would be your number minus 2. So 4 h, 4 h would be your number of hexagons. [At turn 77, Joseph responds: Yup.]
78 *S:* So number of hexagons would be 4 times 6 minus n minus 2. [Here, she wrote: 4(6) – N – 2]. So 4 times 6 would be 24. Number of hexagons would be 1, 2, 3, 4. 4, uh, times 2.

Sheila uses an internal reversal at 76 by switching the variable *n* with its description, number of cases: *n…would be the total number…number of cases would be n.* Her final comment in turn 76: *4 h, 4 h would be your number of hexagons* was reversed at 78 so that it began with hexagons: *So number of hexagons would be 4.* A moment later, but still at 78, Sheila incorporated Joseph's *6 times* from line 75, which is an across repetition: *So number of hexagons would be 4 times 6…*and she

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began to build the subtraction of the interior sides, though inaccurately at this point: *So number of hexagons would be 4 times 6 minus n minus 2.*

In turns 75 to 78, there are reversals across the word *times*, and there are reversals of the variable *n* and the description of the variable. A possible interpretation of these reversals is that they represent some tension between topic of focus of each speaker. Joseph's reversal helped him propose the beginning of a formula that was written in a standard format, but that leaves the variable, *number of hexagons* in the terminal position. In contrast, Sheila is working on how to start with the number of hexagons—at this moment, the n = 4 case—and perform calculations on n = 4 in order to verify the perimeter of 18. As a verbal representation, saying *number of hexagons* first stabilizes the quantity for further conjecturing and calculation. Sheila used a couple of reversals to get the 4 into a convenient position to calculate with it. Brian was moving towards a standard form of writing mathematics, but Sheila was still focused on working out the algebraic relationships regarding the interior sides.

In this passage, then, we can hear the transition from reading the task (the phrase *number of hexagons),* to proposing a written form; (*6 times x per se number of hexagons*); to writing it as 6(x) —; to recasting it in an easier verbal form for conjecturing (*So number of hexagons would be 4 times 6 minus n minus 2*). Analysis of poetic structures, and especially the reversal poetic structures, helps to highlight these transitions across several ways of experiencing and representing mathematics.

The reversal poetic structure was also associated with writing in turn 81. Sheila and Joseph were still working on the n = 4 hexagon case. They knew that six times four yields 24, that they must subtract pairs of interior sides, and they must get 18, but they didn't know how many twos to subtract. Here, Sheila subtracted 8 instead of 6 to account for the interior sides.

80 *S:* Negative 2 would be negative 8. So 24 minus 8 is how much?
81 *J:* Wouldn't it be 24 plus 8 because there's these two negatives? I'm sorry. There's these two negatives.

The reversal in line 81 is that Joseph's *two negatives* reverses Sheila's *Negative 2* in turn 80. In this phase of the conversations, the students were working on the written statement, where both H and N stand for the number of hexagons: 4H — N — 2.   There is some ambiguity because there were several erasures and the angle of the video recorder did not capture each moment well. However, it is clear that Joseph used a reversal to focus Sheila's attention on a problem with the written notation.

The final reversal occurred in line 90. In line 90, Sheila first articulates a correct method for the task. By the end of line 90, Sheila had written: 24 — 2(4 — 1) = 18.

90 *J:* So this would be 2, 4, 6. So that would be 1, 2, 3, 4, 6. So let's see, 1, 2, 3. So number of insides, 4, so 4 minus 1 times, uh, 4 minus 1, so this would be 2 into 4 minus 1 equals, right? So that would be 3. 3 times 2 would be 6. 6 from 24 is 18, right?

As Sheila said the phrase *2 into 4 minus 1 equals*, she wrote 24 — 2(4 — 1), so it seems reasonable to consider that for Sheila, the word *into* meant *times.* The reversal poetic structure occurred when Sheila's phrase *so 4 minus 1 times* was reversed so that *4 minus 1* took the second position after the operation *times* or *into*: as *2 into 4 minus 1*. The *4 minus 1* reversed position from beginning to end of its phrase. This reversal poetic structure co-occurs with Sheila's decision to write the two before the parentheses of (4 — 1), and so this is a third case in which reversal facilitated the transition of a verbal method into a standard written format.

Conclusion

Each time we speak, we cross a border. We acknowledge the common ground of previous statements and offer, through shifts in our sentences, a pathway forward. Applying this perspective to

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mathematical conversation can highlight the small collaborations—equally linguistic and mathematical— that must take place as mathematical insight develops. In this conversation, poetic structure reversals helped students shift from verbal and computational methods into standard ways of writing mathematics. The reversal poetic structure may prove to be concerned with positioning important mathematical entities, and therefore, it may facilitate movement between different mathematical representations, ones that are read, spoken and written. The theoretical framework for this analysis, dialogic syntax, is an interdisciplinary, border-crossing perspective that provides several rich directions for future research in mathematical discourse. Dialogic syntax draws attention to the subtle precision of informal language in supporting mathematical discovery.

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CROSSING "THE PROBLEM OF THE COLOR LINE": WHITE MATHEMATICS TEACHERS AND BLACK STUDENTS

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In this paper, the authors explore—within an eclectic theoretical framework of critical theory, critical race theory, and Whiteness studies—the life experiences of four White high school mathematics teachers who were "successful" with Black students. The data were collected through three, semi-structured interviews, conducted over a 5-month time period. Through a cross-case analysis of the data, three commonalities among the teachers were identified as being significant contributors to their success in teaching Black students. Two commonalities the participants themselves felt strongly about, and a third became apparent during the cross-case analysis: (a) forming meaningful relationships with students, (b) engaging students in racial conversations, and (c) reflecting both individually and collectively with colleagues on issues of race and racism. Implications for classroom practice and teacher education are discussed.

Keywords: Equity and Diversity, High School Education, Teacher Beliefs

Prologue

One of the most often cited passages from W. E. B. Du Bois's classic text *The Souls of Black Folk* (1903/1989) is the statement regarding the capital "T" problem of the twentieth century: "The problem of the twentieth century is the problem of the color-line,—the relations of the darker to the lighter races of men in Asia and Africa, in America and the islands of the sea" (p. 10). World and U.S. history make available numerous incidents that have transpired that validate this prophetic statement and illustrate how the color line was problematic for the twentieth century, and has continued to be problematic for the twenty-first century (Stinson, 2013). How might mathematics educators become knowledgeable of the ways in which they are implicated (or not) in (re)producing and regulating the problem of the color line? How might mathematics educators learn to reduce (if not eliminate) the problem of the color line, at least in mathematics classrooms (Stinson, 2013)?

Introduction

Jacqueline Jordan Irvine's 1990 book *Black Students and School Failure: Policies, Practices, and Prescriptions* has become somewhat of a classic for those who research issues of Black children and schooling. In many ways, Irvine's book can be thought of as a modern-day version of Carter G. Woodson's 1933 classic *The Mis-Education of the Negro*. What Irvine provided in her book is one of the initial counter-analyses—or counter-narratives, if you will—to the emergence and proliferation of the Black–White "achievement gap" analyses found in both the scholarly and the popular presses of the 1970s and 1980s (see, e.g., Bradley & Bradley, 1977; Maeroff, 1985).

One component of Irvine's (1990) analysis is the importance of "cultural synchronization" between Black children and teachers (and administrators) to Black students' school success. The concept of cultural synchronization, for Irvine, is based on anthropological and historical research that suggest "black Americans have a distinct culture founded on identifiable norms, language, behaviors, and attitudes from Africa" (p. 23; see also Hilliard, 1992). Irvine also provided an analysis of how a lack of cultural synchronization between Black children and teachers can contribute to Black students' school failure. Hilliard (1992) argued that cultural misunderstandings—or a lack of cultural synchronization, if you will—between Black children and (White) teachers

has been shown to lead to errors in the estimation of a student's or cultural group's: (1) intellectual potential (the consequences of which—mislabeling, misplacement, and mistreatment of children—are enormous); (2) learned abilities or achievement in academic subjects such as reading; and (3) language abilities. (p. 372)

In many ways, the enormous errors (and oftentimes harm) brought about by a lack of cultural synchronization begs the question: Do Black children need separate schools (i.e., schools with Black teachers, teaching Black children)? Du Bois (1935), more than 80 years ago, asked this very question. He concluded: With all things being equal, "the mixed school is the broader, more natural basis for education for all youth" (p. 335). But given that things are seldom (if ever) equal, the "Sympathy, Knowledge, and the Truth [about Black children's lives and academic abilities found in the segregated school] outweigh all that the mixed school can offer" (p. 335).

Within the context of mathematics education, Martin (2007) asked a similar question: Who should teach mathematics to Black children? Through a cautioning of missionaries and cannibals, Martin argued for an "experience lens" when thinking about who should teach mathematics to Black children. His experience lens argument—"that achievement outcomes among [Black] students are indictors of the way that they experience mathematics learning and participation as [Black students]" (p. 15)—is similar to Irvine's (1990) cultural synchronization. In that, the experience lens necessitates, among other things, that effective teachers of Black children possess (or have the desire to develop) in-depth knowledge of Black children's socio-cultural and -political life experiences and take seriously their role in positively shaping Black children's racial, academic, and mathematical identities. What might be learned from successful Black mathematics teachers, teaching Black children was the focus of Chazan, Brantlinger, Clark, and Edwards's (2013) extensive research project. Specifically, they challenged the taken-for-granted notions of the knowledge base and resources needed to be an effective mathematics teacher of Black children (p. 2). Similar to Martin, Chazan and colleagues (see also Johnson, Nyamekye, Chazan, & Rosenthal, 2013; Birky, Chazan, & Farlow Morris, 2013; Clarke, Badertscher, & Napp, 2013) suggested that it might be the in-depth knowledge of Black culture and the unwavering belief of the brilliance of Black children (see Leonard & Martin, 2012) that determine who is (or is not) an effective teacher of Black children.

Problem Statement and Research Questions

Altogether, what Du Bois, Woodson, Irvine, Hilliard, Martin, and Chazan and colleagues have argued is the importance of having Black teachers, teach Black children. Or, more generally, they argued for teachers who are synced or responsive (see Gay, 2010) to Black children's culture and life experiences. They did not, however, argue that only Black teachers are effective with Black children. In fact, Irvine (1990) explicitly noted that the justifying of cultural synchronization does not "ignore the fact that some white teachers are excellent teachers of black children or that some black teachers are ineffective with black children, treating them with disdain and hostility" (p. 61). But these "some white teachers" who are "excellent teachers of black children," who are they? What makes them excellent? How are they different from other White teachers? Similar to Chazan and colleagues (2013), the purpose of this study was to determine what might be learned from well-respected mathematics teachers of Black children. But here, unlike Chazan and colleagues, our focus was on well-respected or "successful" White teachers. Three questions guided the study:

1. How do the life histories of successful White mathematics teachers of Black children influence their decision to teach Black children?
2. How do these life histories influence their pedagogical practices as successful White teachers of Black children?
3. How do successful White mathematics teachers of Black children view the role of their Whiteness in their teaching?

Theoretical Framework

This project was framed within an eclectic theoretical framework (Stinson, 2009). Critical theory (e.g., Bronner, 2011) served as the overarching framework, while critical race theory (CRT; e.g., Tate, 1997) and Whiteness studies (e.g., Leonardo, 2002) brought the complexities of race and racism to the forefront (see Figure 1). In effect, we borrowed theoretical concepts and methodological procedures from different theoretical paradigms, which we used side by side throughout the research project. Similarly to Koro-Ljungberg (2004), we do not view qualitative research projects that contain elements from more than one theoretical paradigm as an ontological and epistemological failure, but rather as representing "a planned mixture of theoretical and philosophical assumptions, fluxing commonalties, and complicating rhizomatic (see Deleuze & Guattari, 1987) intersections of theoretical understandings" (p. 618).

Critical theory kept us focused on what it might mean to conduct "'good' education research" (Hostetler, 2005) in the public interest (Ladson-Billings & Tate, 2006); it made us think not about "how things [are] but how they might be and should be" (Bronner, 2011, p. 2). CRT and Whiteness studies allowed the participating teachers (and us) to challenge racial hierarchies; to reflect on and discuss the roles of race, racism, and White supremacy in schools and in society at large; to examine the racial dynamics between student and teacher; and to begin to understand how the teachers' racial experiences may have influenced their classroom practices. No study on race, particularly one focusing on the dynamic relationships formed between White teachers and Black students, should be absent of these theories. Given that we believe that racism and White supremacy are two sides of the same coin, an analysis of race and racism without a critical examination of the hegemony of White supremacy would be dangerously incomplete (see Hilliard, 2001).

Figure 1. Three theories intertwined.

Methods

Four teachers—Caroline, Carrie, Oliver, and Patt (pseudonyms)—were the participants of the study. Each teacher self-identified as White, taught high school mathematics (ranging from a low of 4 years to a high of 23 years), and each had experienced success in teaching Black children. (For complete details of participant selection and methods, see Bidwell, 2010). The data were collected through three, semi-structured interviews conducted and transcribed by the first author, over a 5-month time period. The first interview allowed participants the opportunity to (re)tell their stories, beginning with childhood and continuing on to the present as mathematics teachers of Black students. The conversations documented during the second round of interviews centered on Gary Howard's (2006) book *We Can't Teach What We Don't Know: White Teachers, Multiracial Schools*, which the participants were asked to read and reflect on prior to the second interview. (Engaging

participants in literature prior to interviews has been proven beneficial in other studies; see Jett, 2009; Shildneck, 2009; Stinson, 2008.)

Reading Howard's (2006) book acted as a catalyst for engaging in complex discussions on race, White hegemony, color blindness, and the development of an anti-racist White identity. Howard's prose became somewhat of an invisible but participating third person during the interview. This invisible person, so to speak, assisted in making the often-difficult task of discussing race approachable, and gave both the participants and the interviewer a language to engage in "race talk" that permitted richer and more descriptive conversations. The third and final interview was followed by an opportunity for participants to critically read and provide feedback on the transcripts of their previous two interviews; it also served as a follow up to those conversations, a method of "member checking," so to speak. (This method of member checking has been effectively used in other studies with practicing mathematics teachers; see Wilson, Cooney, & Stinson, 2005.)

Findings

As the four participants shared their life stories, they often included intimate and uncomfortable details of their lives. They, however, led the discussions. They chose which stories to share and which would remain unspoken. Although most of these teachers had limited exposure to or knowledge about the theories that framed the study and guided data analysis, in their storytelling they all demonstrated tenets of critical pedagogy (see, e.g., Freire, 1970/2000, 1998). They were willing to be disruptive and were extremely reflective about every topic discussed. Not one of these teachers particpated in "dysconscious racism" (King, 1991); that is, they did not accept poor performance from their students as "just the way it is." They were so comfortable in their teaching, that they were willing to change a lesson while it was being taught for the benefit of their students. They questioned their own teaching as it happened just as they questioned the ideas discussed during the interviews. And as was somewhat hypothesized at the onset of the study, all four teachers employed, to some degree, culturally responsive (Gay, 2010) or relevant (Ladson-Billings, 1994) pedagogy in their classrooms. Similar to Ms. Rossi, the sixth-grade mathematics teacher in Ladson-Billings's (1994, 1995) study, these teachers (a) believed that all their students could succeed, (b) saw themselves as part of the community, (c) built strong relationships with students that often extended beyond the classroom, (d) encouraged students to work together collaboratively, and (e) were passionate about mathematics and were willing to scaffold content for students when gaps in their knowledge became apparent.

Through a cross-case analysis of the data, three commonalities among the teachers were identified as being significant contributors to their success in teaching Black children. Two commonalities the participants themselves felt strongly about, and a third became apparent during the cross-case analysis: (a) forming meaningful relationships with students, (b) engaging students in racial conversations, and (c) reflecting both individually and collectively with colleagues on issues of race and racism. Given the limitation of space, each commonality is discussed only briefly here.

Forming meaningful relationships with students. Caroline, Carrie, Oliver, and Patt all developed strong relationships with their students and viewed those teacher–student relationships as an essential part of their jobs. These relationships, however, were not superficial. Rather, they were genuine relationships developed out of these teachers' capacity to empathize with their students. All four teachers in this study subscribed to Ladson-Billings's (1997) belief that "we must come to develop caring and compassionate relationships with students—relationships born of informed empathy, not sympathy" (p. 706).

All four participants spoke extensively about their personal experiences in relationship building and reasons why they believed these relationships to be necessary. Carrie asserted that forming relationships (or not) with students could "make or break the deal" (Interview 1) in being able to connect with students. She emphasized the importance, for her, of extending those relationships

beyond the classroom by attending extra-curricular events in which her students participate. Oliver desired to strengthen his relationships with students by living in the community with them rather than residing in a distant suburb. Caroline recognized the potential for her (especially as a White woman) to be a roadblock to her students' learning if she failed to connect with each of them individually. She acknowledged that her students already have many socio-cultural and -political barriers that can keep them from being successful but, as she stated, "If a student has an issue with the teacher, then nothing's going to be learned" (Interview 1). And Patt, who was known in her community for "taking in" troubled students in times of crisis, claimed, "I think addressing a child's affective domain is the most important thing to be able to teach them" (Interview 2). Unfortunately, as both Patt and Martin (2007) pointed out, there is widespread failure to recognize that there are different skills needed to be effective with non-White children and that one of those important skills is for a teacher to be able to connect with her (or his) individual students.

Engaging in racial conversations

Caroline, Patt, and Oliver strongly believed that their willingness and ability to engage in racial conversations with students allowed them to seem more "real" to their students and played a vital role in their overall success in teaching Black children. In fact, for these three teachers, avoiding race talk was simply not an option:

> If you like [your students], you'll find that it is your place to bring [race] up. I mean, if you're a White teacher, and especially in my situation where most of the kids live very insular lives and they don't deal with a lot of White people—it's their chance to know you. (Patt, Interview 2)

It is important to note, however, that in these teachers' classrooms, race was not discussed merely as a means of building relationships. They believed racial differences between teacher and student, as well as differences among students themselves, should not be ignored and that discussions about race, in general, were healthy discussions in which to engage. As Irvine (1990) claimed, "lack of synchronization *increases*, not decreases, when teachers and administrators pretend that they don't notice students' racial membership" (p. 26, emphasis added). Although race talk often naturally ensued from conversations among students, these teachers themselves, at times, initiated discussions about race and incorporated them into their mathematics lessons. These racial discussions provided not only an opportunity for students to grow and learn about racial differences but also allowed the teachers to grow as they moved forward (and sometimes backward) on a continuum of racial understandings (see G. Howard, 2006).

Reflecting both individually and collectively with colleagues on issues of race and racism

Through the course of the three interviews with each of the participants, Caroline, Carrie, Oliver, and Patt were all extremely reflective. Caroline, who was working in a racially and ethnically diverse urban school, shared that the faculty openly talked about race and discussed ways to facilitate these conversations with students. Carrie, on the other hand, taught in a similar school where race was never discussed among faculty; yet, she still reflected a great deal about the interactions among students and those between students and faculty of different races. All four teachers seemed to naturally take on the role of reflective practitioner (see Schön, 1983) and were willing to be disruptive with any issue that concerned the welfare of their students (see Freire, 1998).

The participants became particularly reflective during the conversations about G. Howard's (2006) book. On the topic of White dominance, Carrie reflected, "How do you ever break it and how does society ever break it when some people won't even admit that it exists or matters" (Interview 2)? And Caroline, reflecting on her own childhood where she was taught not to "see" color, admitted that she subscribed to a colorblind ideology when she began teaching. But during Interview 2, she shared, "I think it's just important not to think about them [students] as all being the same and also

not to classify them in a certain way, but just to try to appreciate their cultures which I think is something that I'm still working on as a teacher." Three of the participants identified as being situated in the immersion/emersion stage of racial identity development (G. Howard, 2006), one stage prior to the autonomy stage, which G. Howard describes as a time when "we are engaged in activities to resist the many manifestations of oppression" (p.97). Caroline, Carrie, and Oliver all shared that they each do not do enough on a daily basis to fight racism. As Oliver stated, "It's not enough to just teach in this school, but I've got to be active or I'm not going to disrupt anything" (Interview 2).

Discussion and Closing Thoughts

In the study described here, the lived experiences of four successful White mathematics teachers were examined within an eclectic theoretical framework—one that allowed these teachers to think critically about their own life histories, their pedagogical practices, and the role their own Whiteness plays in their teaching. The three commonalities that were identified during data collection and analysis assisted in painting a picture of what might be envisioned as a successful White mathematics teacher of Black children; one who crosses the problem of the color line so that she (or he) might be more culturally synced with her students. But where does this lead us, as educators, going forward? The answer lies in the practices of teacher educators charged with the initial preparation and the ongoing professional development of mathematics teachers, and in the many district and state leaders who support teachers' professional growth throughout their careers.

The traditional method of teaching pre- and in-service teachers about non-White cultures—too often a mere one semester "required" multicultural course—must be extinguished. Teacher educators must go beyond surface exposure of "other people's" (Delpit, 2006) cultures, what Ladson-Billings called a "foods-and-festivals" approach to culture (1994). They must delve deep beneath the surface to explore the complexities of both their own and their students' cultures through (continuous) multicultural, anti-racist education (Gay & T. Howard, 2000; G. Howard, 2006; Ladson-Billings, 1994; McIntyre, 1997). Pre- and in-service teachers, especially those who are White (i.e., White folks too often avoid race talk), need safe spaces in which to discuss race, challenge White hegemony, and confront assumptions about their own and other's cultures. As G. Howard (2006) explained, "We cannot help our students overcome the negative repercussions of past and present racial dominance if we have not unraveled the remnants of dominance that still lingers in our minds, hearts, and habits" (p. 6). In these safe spaces, pre- and in-service teachers need to learn and engage in dialogue about social injustices while being provided "opportunities to critique the system in ways that will help them choose a role as either agent of change or defender of the status quo" (Ladson-Billings, 1994, p. 133). And lastly, pre- and in-service mathematics teachers need to learn to "care with awareness" (Bartell, 2011) as they learn about and engage with culturally relevant mathematics pedagogy (e.g., Gutstein, Lipman, Hernandez, & de los Reyes, 1997; Waddell, 2014; Tate, 1995).

Many of the experiences and conversations that should take place in teacher preparation programs should also continue throughout a teacher's career. In multi-ethnic, -racial, and -cultural school settings, ongoing professional development needs to be provided to: (a) support teachers in understanding the importance of strong teacher–student relationships and teach strategies to help build those relationships; (b) assist teachers in making connections to the surrounding communities; (c) engage teachers in multicultural, anti-racist education; and (d) develop teachers as reflective practitioners. G. Howard (2006) asserted: "An unexamined life on the part of a White teacher is a danger to every student" (p. 127). But teachers cannot be expected to examine their own lives without some direction; they need safe spaces in which to dialogue with fellow teachers about race and White hegemony. It is the responsibility of building-level administrators to create these spaces and devote time for teachers to engage in racial dialogue. And in the end, it is the responsibility of *all* who support teachers to be committed to improving education for Black children by providing

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meaningful experiences that allow us *all* to examine ourselves, our children, and the racialized society in which we teach and learn.

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CONCEPTUALIZING THE TEACHING PRACTICE OF BUILDING ON STUDENT MATHEMATICAL THINKING

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An important aspect of effective teaching is taking advantage of in-the-moment expressions of student thinking that, by becoming the object of class discussion, can help students better understand important mathematical ideas. We call these high-potential instances of student thinking MOSTs and the productive use of them building. The purpose of this paper is to conceptualize the teaching practice of building on MOSTs as a first step toward developing a common language for and an understanding of productive use of high-potential instances of student thinking. We situate this work in the existing literature, introduce core principles that underlie our conception of building, and present a prototype of the teaching practice of building on MOSTs that includes four sub-practices. We conclude by discussing the need for future research and our research agenda for studying the building prototype.

Keywords: Classroom Discourse, Teacher Education-Inservice/Professional Development, Teacher Education-Preservice

Mathematics education researchers recognize the important role student mathematical thinking plays in crafting and carrying out quality mathematics instruction (e.g., Fennema et al., 1996; Stein & Lane, 1996). The field has begun to understand how to effectively use written records of student work to support mathematics learning (e.g., Smith & Stein, 2011), but much less is known about how to effectively use in-the-moment student thinking that emerges during whole-class discourse, often fertile ground for valuable student mathematical thinking (Van Zoest et al., 2015a, 2015b). In fact, research has documented that many teachers fail to notice or act on opportunities to capitalize on such thinking to further students' mathematical understanding (Peterson & Leatham, 2009; Stockero, Van Zoest, & Taylor, 2010).

Although research in mathematics teacher education suggests the benefits of instruction that uses student thinking (e.g., Franke & Kazemi, 2001), what it means to "use student thinking" is not well defined. For example, our interviews with secondary school mathematics teachers about productive use of student thinking revealed a range of perceptions. Some teachers viewed validation of student participation as productive use, others felt a discussion of student errors was not productive because it would confuse students, and still others saw productive use occurring when student thinking (correct or incorrect) was made the object of consideration for other students in the class (Leatham, Van Zoest, Stockero, & Peterson, 2014). These results highlight a need to develop a common understanding of and vocabulary for talking about productive use of student thinking in order for the field to better communicate about this important aspect of effective teaching. These interviews also revealed that some teachers felt differently about productive use of student thinking depending on the grade and ability level of their students; these teachers felt that it was possible to make student thinking the object of consideration for their advanced classes, but not for their beginning or remedial classes. Given what we know about the benefits of engaging students with each other's thinking, this perspective creates an inappropriate restriction on students' opportunities to engage in considering

each other's thinking. A challenge to the field is to prompt teachers to question such artificial borders and to provide them with tools that support them to productively use the thinking of all their students.

Not all student thinking warrants the same consideration by the class, however, since it is not all about mathematical ideas, nor does it always provide leverage for accomplishing mathematical goals. Leatham, Peterson, Stockero, and Van Zoest (2015) described a framework to identify those instances of student thinking—MOSTs—that provide such leverage. To move work related to the teaching practice of using student thinking forward, it is critical that teachers and teacher educators develop an understanding of productive use of high-leverage instances of student mathematical thinking—what Leatham et al. (2015) called *building* on MOSTs. As a first step toward developing this understanding and providing a common language for the field, the purpose of this paper is to conceptualize *the teaching practice of building* on MOSTs.

Theoretical Framework

Our theorizing about the teaching practice of *building* on student thinking takes as its foundation the framework for identifying student thinking worth building on developed by the MOST research group (Leatham et al., 2015; Stockero, Peterson, Leatham, & Van Zoest, 2014; Van Zoest, Leatham, Peterson, & Stockero, 2013). We defined MOSTs—**M**athematically Significant Pedagogical **O**pportunities to Build on **S**tudent **T**hinking—as occurring in the intersection of three critical characteristics of classroom instances: student mathematical thinking, significant mathematics, and pedagogical opportunities. For each characteristic, two criteria were provided to determine whether an instance of student thinking embodies that characteristic. For student mathematical thinking the criteria are: "(a) one can observe student action that provides sufficient evidence to make reasonable inferences about *student mathematics* and (b) one can articulate a mathematical idea that is closely related to the student mathematics of the instance—what we call a *mathematical point*" (p. 92). The criteria for significant mathematics are: "(a) the mathematical point is *appropriate* for the mathematical development level of the students and (b) the mathematical point is *central* to mathematical goals for their learning" (p. 96). Finally, "an instance embodies a pedagogical opportunity when it meets two key criteria: (a) the student thinking of the instance creates an *opening* to build on that thinking toward the mathematical point of the instance and (b) the *timing* is right to take advantage of the opening at the moment the thinking surfaces during the lesson" (p. 99). When an instance satisfies all six criteria, it embodies the three requisite characteristics and is a MOST.

MOSTs are instances of student thinking worth *building* on—that is, "student thinking worth making the object of consideration by the class in order to engage the class in making sense of that thinking to better understand an important mathematical idea" (Van Zoest et al., 2015b, p. 4). Such use encapsulates the core ideas of current thinking about effective teaching and learning of mathematics. Thus, building on MOSTs is a particularly productive way for teachers to engage students in meaningful mathematical learning. After discussing related literature, we share our current conceptualization of the teaching practice of building on MOSTs.

Related Literature

We see our work as connecting and contributing to research both on professional noticing—attending to, interpreting, and deciding how to respond to student mathematical thinking (Jacobs, Lamb, & Philipp, 2010)—and on teaching practices that enact and coordinate these decisions. This section elaborates on these two areas of contribution.

We see studies of professional noticing as generally falling into two categories: (a) noticing *within* an instance of student thinking, and (b) noticing *among* instances (Stockero, Leatham, Van Zoest, & Peterson, in press). Noticing *within* studies include interventions in which teachers (or prospective teachers) are given a specific instance of student thinking that they are asked to analyze, using media such as one-on-one student interviews (e.g., Schack et al., 2013) or student written work

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(e.g., Fernández, Llinares, & Valls, 2013). In such studies, the task of the teacher is not to identify which instances to analyze, but instead, to notice what is happening *within* the instance of student thinking they are provided. Noticing *among* studies typically use classroom video (e.g., Sherin & van Es, 2009; van Es, 2011) as a medium for teachers to select the instances they deem important, and to write about or explain why those particular instances were important or interesting. Most of these studies, however, do not assign value to any particular instances, beyond the broad category of student thinking. We contend that, in order to productively use student mathematical thinking, teachers need to learn how to coordinate the skills of noticing *within* and noticing *among* instances of student mathematical thinking. The MOST Analytic Framework (Leatham et al., 2015) provides a tool for attuning teachers and researchers to this coordination and our current conceptualization of productive use of student thinking provides a theoretical characterization of what this coordination might look like in practice.

The theorizing we are doing about the teaching practice of building on MOSTs has the potential to contribute in substantial ways to understanding the third component of noticing, *deciding how to respond* (Jacobs et al., 2010). Most studies of noticing, particularly those in the noticing *within* category, ask teachers to propose a "next move." There is not yet a common understanding in the field, however, of what makes those next moves more or less productive, particularly during a whole-class discussion. In the case of one-on-one student interviews, asking a question or proposing a follow-up task to the student that is based on evidence of the student's understanding would generally be considered a productive move (see, e.g., Jacobs et al., 2010). In a whole-class setting, however, where the teacher needs to work toward engaging many students in making sense of the mathematics on the table, what makes a next move productive is more complicated. In fact, our work suggests that this practice requires a series of teacher moves. Unpacking and clearly articulating this collection of moves, of which our theorizing is a first step, is requisite for helping teachers to improve their use of student thinking in the classroom.

Prior work related to teacher responses to student mathematical thinking has also influenced our efforts to theorize the teaching practice of building on MOSTs. Research has identified a number of patterns in mathematics teachers' responses to student thinking. Mehan (1979) coined *IRE*—Initiation, Response, Evaluation—to describe a common pattern of classroom interaction where the teacher's main response to elicited student thinking is to evaluate it. An IRE interaction is an example of what Wood (1998) referred to as *funneling,* where the teacher's response is intended to corral students' thinking within predetermined and often narrowly-defined parameters. By contrast, Wood characterized certain other teacher responses as *focusing*, in which, for example, a teacher might ask "clarifying questions to keep attention focused on the discriminating aspects of the solution" (p. 175). Clarifying questions are just one example of the many types of productive questions teachers might ask in response to student mathematical thinking. Unfortunately, Boaler and Brodie (2004) found that questions geared toward "gathering information, leading students through a method" (p. 777)—funneling questions—characterized the vast majority of questions in US classrooms using a traditional mathematics curriculum. Although 60-75% of the questions asked by teachers using an NSF-funded reform curriculum were also of this type, the remaining 25-40% covered a broader range of more productive questions. Boaler and Brodie's (2004) findings highlight the additional learning opportunities afforded students who are exposed to a broader range of questions. These findings also point toward the value in teaching practices that engage with and capitalize on student mathematical thinking.

As we have argued elsewhere (Leatham et al., 2015), work on the cognitive demand of tasks (e.g., Stein, Smith, Henningsen, & Silver, 2009) and on orchestrating classroom discussions around rich mathematical tasks (e.g., Smith & Stein, 2011) has also been influential in conceptualizing the productive use of student mathematical thinking in classrooms. The MOST framework and our theorizing about the teaching practice of building on MOSTs is related to Smith and Stein's (2011)

practices of monitoring, selecting and planning how to sequence student thinking that is observed as they work on a task, but extends Stein and Smith's work by focusing on *recognizing* and *responding* to potentially productive student thinking in the moment that it occurs. There are two particularly important differences between the MOST work and that of Stein, Smith, and colleagues. First, the MOST work focuses a broader range of contexts in which student thinking can emerge, including student questions that arise during a lecture or student comments that emerge during a discussion of homework. Although high cognitive demand tasks certainly create opportunities for student thinking to occur, we have also found high-potential instances of student mathematical thinking in classrooms that lack rich tasks. The MOST framework applies to the broad range of instructional situations in which student thinking might emerge in mathematics classrooms. Second, our work on building focuses on responding to student thinking *at the moment in which it occurs* during a lesson, rather than on monitoring and selecting student work and then purposefully sequencing the presentation of that work later in the lesson (see Smith & Stein, 2011). Although student thinking can be valuable to use at a later point in a lesson, our work has convinced us that there are certain instances of student thinking that lose their instructional value if they are not acted on immediately and in particular ways. The purpose of this paper is to conceptualize a productive response to MOSTs—the teaching practice of building.

Our Current Conceptualization of The Teaching Practice of Building

We base our conception of *building* on core principles of quality mathematics instruction that we distilled from current research and calls for reform. The NCTM, for example, in their *Principles to Actions* document (2014), states that students "construct knowledge socially, through discourse, activity, and interaction related to meaningful problems" (p. 9), and that "effective teaching of mathematics facilitates discourse among students to build shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments" (p. 10). We see embedded in these statements four core principles of quality mathematics instruction: mathematics is at the forefront, students are positioned as legitimate mathematical thinkers, students are engaged in sense-making, and students work collaboratively. We have applied these principles to the productive use of MOSTs in Figure 1. An important aspect of this application is that in the first principle, the mathematics that is at the forefront is the mathematics of the MOST—mathematics closely connected to the student thinking under consideration. We use these core principles to determine whether a given use of a MOST is productive.

Principles Underlying our Conception of Productive Use of MOSTs
1. The mathematics of the MOST is at the forefront.
2. Students are positioned as legitimate mathematical thinkers.
3. Students are engaged in sense making.
4. Students are working collaboratively.

Figure 1. Principles underlying our conception of productive use of MOSTs**.**

In our initial work identifying productive use of MOSTs, we focused on the productivity of a single teacher move that followed a MOST, but we quickly realized that this approach was insufficient. Because teaching is a complex system (Stiger & Hiebert, 1999), one needs to look beyond single actions, such as inviting students to share solutions at the board, to characterize effective teaching. Similarly, trying to ascertain productive use of MOSTs by only focusing on discrete teacher moves misses the real purpose of those moves. To evaluate productive use, one needs to consider the combining and coordinating of teacher moves. We thus conceptualize the teaching practice of *building* as several teacher moves woven together to engage students in the

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intellectual work of making connections between ideas and abstracting mathematical concepts from consideration of their peers' mathematical thinking.

To highlight the distinction between the teaching practice of building and the moves that may or may not be part of the practice, consider a teacher who invites two students to share their different, but both correct, solution strategies for a problem. The initial move of inviting the students to share their thinking could be the start of building because the teacher is inviting the whole class to consider the two students' mathematical thinking. Consider two different moves the teacher could make once the strategies have been shared: (a) the teacher says to the class, "See, there are many correct ways to solve problems like this and you can use whichever method makes the most sense to you," and moves on to the next problem; and (b) the teacher asks the class, "What similarities and differences do you notice in these two strategies?" and engages the students in a discussion about those noticings. Although the initial setup move was the same (making the two student solutions public), we see here that the follow-up moves vary significantly in their potential to accomplish the building goal of students coming to understand important mathematical ideas. Specifically, variations in follow-up moves might cause the resulting practice to deviate from any of the four principles underlying our conception of productive use of MOSTs listed in Figure 1: (1) the mathematics of the instance could be abandoned, (2) the teacher could trivialize students' contributions, (3) the overall practice (regardless of the actor) could focus on recall of facts or on procedural steps rather than on making sense of the underlying structure of the mathematics, or (4) the teacher could limit or eliminate engagement with the idea beyond the individual who contributed the instance. Thus, although a move that makes student thinking public *may* be part of a broader building practice, such a move does not imply engagement in the practice—building is more than a single move.

As we have begun to think about what collection of teacher actions meet the requirements of building, we have theorized that there are four sequential sub-practices of building, each consisting of a move or collection of moves, as well as some prerequisites. Before teachers can build, they must have completed two prerequisite actions: (1) invited or allowed students to share their mathematical thinking; that is, elicited student mathematical thinking, and (2) recognized that an instance of student thinking is a MOST—a high-potential instance of student thinking. In addition, the success of a teacher's enactment of the building practice is influenced by the norms present in the classroom. One such prerequisite norm is that students listen to and make sense of each other's thinking. The absence of that norm would greatly inhibit successful building.

Once these prerequisites are satisfied, we hypothesize that there are four sub-practices of the teaching practice of building; our current building prototype is outlined in Figure 2. The first sub-practice of building is to ensure that the student mathematics of the MOST—the object of consideration—is clear. We say the teacher should *make precise* what it is that students are meant to consider. Sometimes a MOST has been communicated in such a way that both the object and the need to engage with it are obvious and no further action is needed, but often the teacher must focus the class on what student thinking is to become the object of consideration. The second sub-practice turns the object of consideration—the student mathematical thinking—over to the students. We use the term *grapple toss* because it captures two key aspects of this sub-practice—the teacher must "toss" the student thinking of the MOST over to the students to be considered, and they must do so in such a way that the students are positioned to "grapple" with the object of consideration in order to make sense of it. The third sub-practice involves orchestrating the students' process of making sense of the MOST. We use *orchestrate* to mean, "arrange or direct the elements of (a situation) to produce a desired effect, especially surreptitiously" ("Orchestrate," n.d.). Although this orchestration could require only a few teacher moves, this sub-practice could easily consist of a large and complex collection of moves. The fourth sub-practice is to facilitate the extraction and articulation of the important mathematical idea from the discussion; that is, to *make explicit* that idea.

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> **Sequence of Sub-Practices of the Teaching Practice of Building on MOSTs**
> 1. Make the object of consideration clear (make precise)
> 2. Turn the object of consideration over to the students with parameters that put them in a sense-making situation (grapple toss)
> 3. Orchestrate a whole-class discussion in which students collaboratively make sense of the object of consideration (orchestrate)
> 4. Facilitate the extraction and articulation of the mathematical point of the object of consideration (make explicit)

Figure 2. Building prototype: Our current conception of the teaching practice of building**.**

Given that our understanding of this practice is primarily theoretical at this point, further research is needed to study the building prototype. Our plans for future research include studying the prototype by (1) analyzing current teacher responses to MOSTs to see the extent to which those responses coordinate our core principles; and (2) generating instantiations of the building prototype and engaging in a similar analysis of these responses. Specifically, this direction for future research will allow us to refine our building prototype by addressing three primary research questions: (1) What teaching practice(s) coordinate the core principles underlying productive use of MOSTs? (2) How do teachers' responses to MOSTs align with the core principles underlying productive use of MOSTs? and (3) In what ways, if any, do teachers' responses to MOSTs empower or disenfranchise students (particularly those from traditionally underrepresented populations) with respect to mathematics? Once the teaching practice of building on MOSTs is better understood, it will be possible to design professional development to support teachers in improving their abilities to build on MOSTs.

Conclusion

Conceptualizing the teaching practice of building is a first step towards achieving the goal of productively using students' mathematical thinking during instruction—a central tenet of effective teaching (e.g., NCTM, 2014). There is little research about the complex but essential practice of responding to student thinking *in the moment*, yet this is something that teachers face every day. Our conceptualization of building contributes to a common understanding of and vocabulary for talking about productive use of student thinking that will support the field in communicating about this important aspect of effective teaching. Better understanding the in-the-moment practice of building on MOSTs—particularly opportune instances of student thinking—has the potential to significantly impact mathematics instruction for all students. Focusing attention on student thinking and how it can be built on supports teachers in looking for the mathematics present in instances of student thinking, and thus helps to avoid deficit thinking (e.g., Frade, Acioly-Régnier, & Jun, 2013) and making judgments based on student characteristics rather than the content of their thinking. Having a systematic way to interpret student mathematical thinking (i.e., the MOST Analytic Framework, Leatham et al., 2015) and a mechanism for responding to MOSTs (the teaching practice of building) positions teachers to question artificial borders that prevent them from engaging all their students—no matter what their ability or experience level—in these important learning opportunities.

Acknowledgements

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SOCIOMATHEMATICAL NORMS AND INCLUSION IN MIDDLE SCHOOL

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This paper presents initial findings from a project investigating sociomathematical norms in two grade five classes. Here I focus on the different sociomathematical norms perceived by students in the two classrooms and how these norms contributed to students' sense of inclusion.

Keywords: Instructional Activities and Practices, Equity and Diversity, Middle School Education

Introduction

Why should we be concerned about sociomathematical norms in middle school in particular? A compelling argument rests in Sullivan, Tobias, & McDonough (2006, p. 81), who assert that "the very students who have most to gain from active participation in schooling are those who are most difficult to engage" and note that students' sense of exclusion from school mathematics frequently begins in the upper primary years (grades 5-7). Much has been written on students' feelings of exclusion from school mathematics (e.g. Nardi & Steward, 2003), often focusing on either the circumstances of individual students, or on broader systemic problems or societal constructs that are beyond the influence of the classroom teacher. Here I argue that teachers – especially middle school teachers – may wish to attend to sociomathematical norms, as these norms are one of the few elements largely within the control of the teacher that can contribute to students' sense of inclusion. This work contributes to the conference's stated theme of questioning borders within mathematics education, as students' sense of inclusion is a key factor in the literature on social justice and equity in mathematics education (e.g. Skovsmose, 2007).

Theoretical Framework: Sociomathematical Norms

Yackel & Cobb's work in sociomathematical norms "aims to account for how students develop specific mathematical beliefs and values" (1996, p. 459). They differentiate between social and sociomathematical norms; for example, taking class time to discus multiple, different solutions would be a social norm, while sociomathematical norms convey what is valued mathematically, e.g. what counts as a mathematically "different" solution. Fukawa-Connelly (2012) reviews the literature on sociomathematical norms and confirms that "norms and beliefs influence each other and as one changes, so does the other" (p. 403), making norms an appropriate lens through which to view students' beliefs about math and whether these beliefs lead them to feel included or excluded from school mathematics.

Students play a significant role in creating and maintaining sociomathematical norms, but the process is guided by the teacher, who represents the discipline of mathematics in the classroom, (Bowers, Cobb, & McClain, 1999). In addition to explicit statements teachers make about their mathematical expectations, there are many implicit ways teachers endorse sociomathematical norms. For example, teachers' reactions to students' questions and solutions demonstrate what mathematics is valued, as "the teacher acts as a participant who can legitimize certain aspects of the children's mathematical activity and implicitly sanction others" (Yackel & Cobb, p. 466).

Levenson, Tirosh & Tsamir (2009) note that most studies regarding sociomathematical norms focus either on the teacher's perspective or on observable actions in the classroom, but typically do not address "what students view as being normative in their classrooms" (p. 171). They identify "three aspects of sociomathematical norms: teachers' endorsed norms, teachers' and students' enacted norms, and students' perceived norms" (p. 173). This paper focuses on students' perceived

norms, exploring these perceptions primarily through students' responses to interview questions regarding mathematics, including "What does it mean to be good at math?"

Research Questions & Methodology

The study was situated in two grade five classes at the same private school. Each class had a single teacher who taught mathematics as well as other core subjects. The two teachers, Wendy and Frank, were both experienced teachers, but fairly new to teaching grade five. (It was Wendy's first year and Frank's third.) The classes were similar in makeup (by gender, race, and ability levels). Both classes were small, with 12 and 13 students.

This paper explores the question: *What is the relationship between students' perceived sociomathematical norms and their sense of inclusion in school mathematics?* This question is addressed mainly through interviews with the students, and is infused by classroom data collected during the researcher's visits to the two classrooms. The researcher spent three weeks in each class, documenting the environment through video recordings and field notes in order to develop a rich understanding of students' experiences of the mathematics learning environment. In subsequent interviews, students discussed their perceptions of their math classes and the topic of mathematics in general. Student answers to interview questions were coded according to themes that emerged; for example, for the question "What does it mean to be good at math?" the categories that emerged were speed, understanding, job/success, and teaching/sharing.

Results/Discussion

Wendy's math classes generally followed the same format each day, with students gathering for a Smart Board lesson from the prepared materials provided by the school's math program. After the lesson, students returned to their desks to complete the work pages associated with the lesson. Certain students were allowed to do "bypass", completing only the first two and last two questions, after which they could work on an extension task or help their classmates.

Frank's math classes rarely followed such a pattern. Frank held many mini math classes as questions came up in other subject areas that could be linked to mathematics, and the time slot ascribed to math class was often used for work that tied in another subject as well. Frank held an explicitly stated teaching philosophy, shared with his students, that "Everything is connected to everything" and felt that the division between mathematics and other subject areas was artificial.

What mathematical thinking is valued?

Key in the creation of sociomathematical norms is the teacher's ability to legitimize or sanction certain lines of mathematical thinking (Yackel & Cobb, p. 446). In Wendy's class, while discussing what types of things can be divided into halves, Oliver has an idea. Oliver's ideas are often unconventional. As he approaches the board, chalk in hand, Wendy pauses.

Wendy: Is this nonsense, or is this worthwhile math?
Oliver: If you have twelve people… I mean, if you have eleven people…
Wendy: I'm going to ask you again. Is this nonsense, or is this mathematically worthwhile?
Oliver: Well… it's mathematical.
Wendy: But is it worthwhile?

Oliver accepts Wendy's words as gentle teasing, not as criticism; Wendy met students' less conventional ideas with amusement, but it was clear that a diversion would not be made to accommodate them. Her description of questions that veered off the lesson plan as "nonsense" or not "worthwhile" contributed to the perceived sociomathematical norms of Wendy's classroom.

In Frank's class, lesson plans were more fluid. When the class began a unit on ancient civilizations, students spent one class visiting stations that included artifacts from various cultures

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(pictures of pyramids, a Mayan calendar wheel, etc.). Students then met in a circle to discuss what mathematics they saw in the various artifacts, and their conversation spanned angles, ratios, perimeters, circles, pi, and more. During the research period, Frank would often modify his teaching based on student questions, and an interesting question could lead to a lengthy class discussion. At one point, Paul, a high-achieving student, took over the class, reorganizing the furniture and arranging his classmates in order to illustrate his thinking.

The researcher speculated that this contrast in how the teachers dealt with questions that fell outside their lesson plan could shape students' perceptions of the sociomathematical norms in the classes, and this was borne out by the interview data.

Interviews

Since student comfort was paramount, selection processes for student interviews were modelled on the work of Nardi & Steward (2003) on "quiet disaffection" in mathematics, with students choosing whether to be interviewed individually or in small groups. Most students in Wendy's class preferred to be interviewed individually, while nearly all students in Frank's class chose groups. This could suggest that the norms (endorsed or perceived) in Wendy's class led students to see mathematics as an individual activity, whereas the norms in Frank's class placed more emphasis on the collective. Other patterns were apparent, particularly in which students felt excluded from the class discourse. The higher-achieving students in Wendy's class expressed frustration at having to sit through Smart Board lessons and not being allowed to move ahead or go at their own pace. Still, when asked to recall classroom interactions where they asked questions that were labeled "nonsense" or "a technicality", these students tended to agree with Wendy's implication that their questions were essentially a distraction. This speaks to the role of students in establishing sociomathematical norms, suggesting that these students endorsed the sociomathematical norms they perceived in Wendy's class, even while pushing against them.

In Frank's class, a different group of students felt excluded from the conversation. Here, Charlotte, Ben, and Abbie admit to feeling left out when the class gathers in a circle, and explain why the part of math class they like best is "math blast", where students complete work sheets:

Ben: It's easy. Easier than listening...
Allan: It's easier to do the questions on your own than to sit in a group and talk about them?
Ben: Yes, because it sometimes gets confusing in your head when you're in a group and you can't write it down. You've got to think it all in your head when he asks a question.
Abbie: When other people are talking about different answers, and then I'm thinking of something else, then it kind of mixes me up.
Charlotte: Sometimes if it's hard, it's hard to think it out in your head, you need paper.
Ben: Then if there's only one answer and someone says it... [*shrugs*]

Later, Liam recalls the time when Paul took over the class, enlisting his classmates in moving tables to help him illustrate an idea. Liam sees Paul as shaping the discourse of Frank's class in a way that Liam can't, endorsed by the sociomathematical norms of their shared classroom.

In addition to students' explicit descriptions of their mathematics classes, their perceptions of sociomathematical norms can be inferred from their responses to some interview questions, including "What does it mean to be good at math?" In their responses, Wendy's students stressed speed and accuracy, while Frank's students focused on connection and understanding.

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Table 1: Student responses to "What does it mean to be good at math?"

Class	Sample responses	Frequency
Wendy	It means to be quick, to know the answer quickly It means that you can figure out problems really quickly It means that you can do it frequently and with minimal mistakes If someone asks you a question, you'll get it in a couple of seconds.	speed: 4 understanding: 1 job/success: 3 teaching/sharing: 0
Frank	You probably understand it pretty well You notice math all the time, when you're everywhere else You understand…and want to go beyond the expectations [It] means to be able to share it with other people, to teach	speed: 0 understanding: 5 job/success: 1 teaching/sharing: 2

Conclusion

My focus here is on students' perceived norms, which is why I look to student interviews. Among the sociomathematical norms perceived by Wendy's students were the great value placed on speed, organization, and accuracy. These norms were enacted explicitly through Wendy telling her students their workbooks would be graded on "organization and correctness of answers" and implicitly through the obvious status assigned to students who were allowed to do "bypass", as well as the dismissal of questions that fell outside the planned outcomes of the lesson. Among the sociomathematical norms perceived by Frank's students were the value placed on originality, finding connections, and asking questions that pushed the boundaries of the mathematics being studied. There were also relevant social norms that were unique to Frank's class, including what he described as the "culture of risk taking" that he tried to foster.

Each class included a range of abilities, but there were patterns in which students felt included or excluded by the sociomathematical norms they perceived. In Wendy's class, higher-achieving students struggled with the constraints of Smart Board lessons and the valuing of speed and accuracy over creativity; meanwhile, in Frank's class, some lower-achieving students perceived that a few of their classmates shaped the classroom discourse in ways they were unable to, and felt excluded by the value placed on originality, connection, and risk taking.

Students' sense of exclusion can often be attributed to factors outside the classroom, leaving teachers unsure how they can intercede. Though no single classroom could fit every student, I suggest here that teachers may wish to attend to sociomathematical norms, as these norms are one of the elements largely within the control of the teacher that can contribute to students' feelings of inclusion in middle school mathematics.

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WRITING MATHEMATICAL EXPLANATIONS IN LINGUISTICALLY DIVERSE ELEMENTARY CLASSROOMS

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New standards and assessments place increasing pressure on teachers to help students write mathematically, a task few are prepared for. How teachers interpret the push for "explanations" will impact all students, especially those learning math in their second language. This study engaged two 5th grade teachers in professional development over one year exploring features of mathematical explanation and designing lessons that highlight rhetorical moves of this new genre. Quantitative and qualitative data expose the most valued features of written explanations, the importance of attention to audience, a possible trajectory of students' writing development, and ways to move language learners beyond set phrases. This work sheds light on how teachers and students learn what it means to write an explanation, with implications for professional development and assessment practices.

Keywords: Elementary School Education, Design Experiments, Classroom Discourse

Purposes

This study highlights findings from a year-long professional development program with two 5th grade teachers and their 58 linguistically diverse learners (LDLs) as they implement classroom practices designed to support students in writing mathematical explanations. New assessments require that students "write to explain" on a variety math tasks, with little clarification of what constitutes a "good" versus "poor" explanation. Math educators define "explanation" in varied and competing ways (e.g. Bicknell, 1999; Kazemi & Stipek, 2001), and although several studies document classroom practices that promote development of *oral* explanations, few studies emphasize *written* explanations. Moreover, recent surveys indicate writing instruction is strikingly rare in elementary math lessons (Banilower et al., 2013).

Rather than expecting students to develop desired forms of communication in mathematics through exposure alone, it may be necessary to explicitly teach them, drawing students attention to ways language is used to communicate particular types of meaning (e.g. Schleppegrell, 2007). However, there may be danger in emphasizing correct language forms to the detriment of deep mathematical thinking (Moschkovich, 2012), requiring teachers to carefully consider the tension between scaffolding too much or too little, and between narrowing or broadening what will be accepted as valid explanations in their classrooms. This study aims to increase our understanding of (a.) what counts as a "good" explanation, and (b.) how students' writing develops over the course of a year in project classrooms.

Data Sources

As part of a larger program of research, this study features data from two second-year teachers who both recently received their master's degrees and demonstrated special interest in LDLs and mathematical communication. These teachers volunteered to take part in professional development, meeting with the researcher and two other teachers at least once a month for 3-hour workshops over one year, from June 2015 to June 2016. The first four workshops consisted of discussion prompted by readings, teachers writing their own "ideal" explanations, listing and ranking features in order of importance. Then, a rubric including the features deemed most important to project teachers was co-designed, piloted, and refined. Subsequent workshops focused on analyzing students' written work and creating lessons to support students in developing written explanations.

Data sources at the time of analysis include transcriptions and artefacts from 7 monthly teacher workshops and the following data collected from each class: video-taped interviews with five focal students (10 total), three informal observations (6 total), three video recorded project-developed lessons (6 total), and 7 explanation writing prompts (406 total). After substantial side by side scoring practice, students' written explanations were scored by their teachers using the project-developed rubric. Twenty percent, selected randomly from each class, were double by the researcher and a 90% inter-rater agreement was obtained, with agreement defined as within 1 point of overall score (out of 16). A coding system was developed for qualitative analysis of students' written explanations and teachers' reflections and discussion during workshops. The coding process included repeated readings and open coding with iterative refinement.

Ms. Velasquez's 5th Grade Class

Ms. Velasquez is a bilingual teacher in a two-way immersion program. She teaches in a small school with 75% of students economically disadvantaged. She alternates between languages, teaching one math unit in English and the next in Spanish. Half of her 27 students are Latina/o, ten white, and 3 African American. Fourteen students are native Spanish speakers; seven are currently identified as ELs. Additionally, 13 students are native English speakers learning Spanish, i.e. Spanish learners (SLs). Ms. Velasquez describes her class as "high in mathematics" and ready for the challenge of learning how to share their mathematical thinking via writing. Data for this class include students' writing, lessons, and interviews in both Spanish and English.

Ms. Holt's 5th Grade Class

Ms. Holt teaches in a high-poverty urban elementary school with 98% of students economically disadvantaged. Seventy percent of students are Latina/o, 11% Asian, 8% African American, and 4% white. Twenty-one of Ms. Holt's 31 students have a native language other than English, with 15 currently classified as ELs. She describes her class as "low mathematically" and although many students still struggle with basic math facts, she emphasizes "learning how to show evidence of *why* an answer is correct is really important." However, Ms. Holt also expressed concern that many of her students have better developed speaking skills than writing skills, and thus their content knowledge may not be apparent on assessment tasks that require writing.

Results

What Counts as a "Good" Explanation?

Both teachers agreed from the beginning that the most important features of an explanation are that it *shows reasoning* and *provides convincing evidence,* reflecting the belief that "full explanations" include more than procedural narration of the steps taken. In discussion during the first workshop, Ms. Holt reflected, "There's procedural recounting where they say the steps how they did the math, but the *explanation* is explaining why, the reasoning, the justification."

Teachers differed in their values of several features of written explanations. While Ms. Velasquez placed high value on the feature *direct answer to question*, Ms. Holt ranked this as one of the least important features, reasoning that she could usually infer her students' final answers from the work they show. Additionally, Ms. Velasquez ranked the feature *appropriate mathematics vocabulary* as one of the least important features, explaining that students can communicate their mathematical ideas in many ways even when they don't know the proper terminology. In contrast, Ms. Holt consistently ranked it in the middle, noting that developing students' academic vocabulary was heavily emphasized by administration.

Teachers' ranking of features of written explanations in July, October, and January, show some shifts in their notions of an ideal explanation. Both teachers' comments in workshop discussions

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demonstrate a shift toward placing higher value on *use of drawings and diagrams*, and on *multiple solution strategies*. Both of these features were found to support students in communicating their reasoning and providing evidence that their ideas make sense mathematically. During the January meeting, teachers were asked to look through their class's writing samples and choose the "best" explanations. All five explanations selected included a clear written description of both what steps the student took to solve and why they took those steps. Further, four of the explanations were written with the structure "I [multiplied] because [reason]," and three included more than one way to solve.

Audience and Purpose Matter

Students were given two different explanation prompts with very similar mathematical tasks involving place value. One was a short prompt, asking students to replace a digit in a number and explain how the value of the number changes, similar to items on many standardized tests. The other was written as a letter from a character we named Puzzled Peter. It explains a situation in which he is buying cupcakes for the whole school and misread one of the digits in the number of students. He thinks he only needs 5 more cupcakes, but his friend Sara thinks he needs 500 more cupcakes. The letter concludes asking students to help him figure out who is correct and to use convincing evidence so he knows he won't waste his money. Though both tasks contained the same mathematical concept and both asked for an explanation, Ms. Holt's students wrote an average of 6 more words on the letter prompt than the short prompt, and Ms. Velasquez's students wrote an average of 4 more words. Several students wrote more than 20 additional words when the prompt was written as a letter with a clear audience and a purpose for producing an explanation. In addition to simply writing more, students produced higher quality explanations on the letter prompt, with rubric scores averaging 1.6 and 2.5 points higher (out of 16) than on the short prompt.

Developmental Trajectories: Building on Procedural Narratives

The first explanations students wrote were primarily procedural narrations. In our January workshop Ms. Holt reflects on her students' beginning-of-the-year writing, "At first my students started sequentially, 'first I did this, then I did this, finally I got my answer.'" She describes how engaging students in lessons that build their understanding of what it means to write a rich mathematical explanation has increased her faith in students' capacity to write mathematically: "When we worked on the shared writing activity, they had great ideas to put on the poster to make it convincing." She elaborates, "Now that I have seen what my students have come up with, I know that they can provide those components of an explanation."

After a lesson in which students practiced scoring sample explanations with a simplified rubric and self-assessed their own writing, we asked them to reflect on ways they could improve their explanations. Approximately 25% said they would tell *why* they decided to solve the way they did (e.g. "to make my score bigger I had to say why I did 10x1.5"). Other common responses include adding visuals, writing more, and solving more than one way. Moreover, Ms. Velasquez noted a key shift in her students' writing after their self-assessment with the rubric: "I saw that students began to understand the difference between an answer and an explanation." Students began, not only to write more but to include many features of a rich explanation that go far beyond procedural narration, including presenting solid supporting evidence and explaining visuals in writing so they may be interpreted by a distant audience.

Language Learners: Moving Beyond Set Phrases

At the beginning of the school year, several language learners used fixed lexical phrases that were the same across several of their explanations, despite changes in the mathematical task. Moreover, these phrases often did not match the way the students actually solved the problem. For

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example, one EL used the phrase, "I use my drawing and the model," when it was clear from her work shown that she did not use a drawing or a model to solve the problem. This phrase showed up in four of her written explanations between August and November, and only once does it appear she may have actually used a drawing to solve. Similarly, several Spanish learners in Ms. Velasquez's class used set phrases when writing in Spanish, but not when writing in English, their dominant language. For example, "Pienso que la respesta es ____ por que hice las matematicas" (I think the answer is ___ because I did the math). Because these "chunks" of language can be retrieved whole from memory, they contribute to students' fluency, accuracy, and cohesion (Lewis, 1997). Importantly, use of set phrases may allow students to produce language associated with many different functions while they are acquiring the language of instruction. However, after the third project-developed lesson in which students analyzed mentor texts by highlighting phrases used to introduce, sequence, and reason, students' explanations included a wider variety of phrases and syntactic structures. Thus, as students engaged in meaningful experiences reading and writing mathematical explanations, their linguistic repertoires may have begun to expand.

Discussion and Significance

This study helps shed light on the features of written mathematical explanations valued by project teachers. Showing reasoning and providing convincing evidence were valued most highly, with visuals playing an important role in communicating students' reasoning. However, these features present a very different picture of "mathematical explanations" than those emphasized on current national assessments. The text box format of standardized assessments may severely limit the resources LDL students can draw upon to communicate their understanding and call into question what we want students to learn about what counts as mathematical communication. Developing writing tasks that more closely mirror the ways mathematical writing is used outside the classroom has the potential to broaden the resources and repertoires LDL students may draw upon to communicate mathematically. Being explicit about our expectations, the features of explanation we value, and the language used to achieve them, is like "letting them in on the secret" (Brisk & Zisselsberger, 2010).

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IN THE PROCESS OF CHANGING INSTRUCTION, A COMMUNITY OF PRACTICE LOST SIGHT OF THE MATHEMATICS

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Few studies of post-secondary mathematics instruction have investigated the mathematics that is offered by instructors in College Algebra courses, which historically have had high failure rates. Using Speer and colleagues' framework for analyzing post-secondary mathematics teaching practice, the study described here aimed to understand the differences in the mathematics provided by instructors for the concept of solving rational equations. Four instructors' lessons were video recorded and analyzed using this framework. Findings include that the mathematics varied widely and was often problematic. Recommendations for further analysis and professional development for mathematics instructors are provided.

Keywords: Post-Secondary Education, Instructional Activities and Practices

Few studies have investigated mathematics instruction in post-secondary contexts, and, as a result, post-secondary mathematics instruction is not well understood (Speer, Smith, & Horvath, 2010). Most explorations of post-secondary mathematics instruction have focused on upper-division courses such as Real Analysis or Abstract Algebra. The small number of investigations of mathematics instruction in lower-division courses have focused mostly on pedagogical techniques (e.g., Mesa, Celis, & Lande, 2013). In order to gain an understanding of the *mathematics* that is taught to students in lower-division courses, the study described here provides insight into the mathematical content that is presented by instructors in College Algebra courses. Such a study is important in understanding undergraduates' progress in mathematics, given that approximately half of the 700,000 students taking College Algebra in the U.S each year fail to complete the course (Haver et al., 2007).

Background and Theoretical Framework

Although students have multiple resources such as lectures, textbooks, tutors, peers, and video tutorials to rely on for their learning of mathematics, the mathematical content offered by instructors during class time remains a significant component of what students learn; as stated by Porter (2002), "students are more likely to learn the content that they are taught" (p. 3), and Weinberg, Wiesner, and Fukawa-Connelly (2014) agree that "what students take away from lecture is closely linked to what they learn" (p. 168). Thus, investigating the mathematics that is offered to students by their instructors is important to understanding what they learn in a course. Moreover, Porter (2002) acknowledged that content is often taken for granted in investigations of mathematics instruction. Knowing that content, an influential component of student learning, is often overlooked, it becomes crucial to investigate the quality and presence of mathematical content provided by instructors during class time. The selection of mathematical concepts taught and the methods used to present them can have detrimental effects on student learning. For example, students will not write down mathematical ideas provided during lecture if they cannot see how they are connected (Weinberg et al., 2014), implying that students are potentially missing important mathematical notions. Nardi, Jaworski, and Hegedus (2005) provide another example of how the quality of mathematics offered by instructors has a negative impact on students' understanding, noting that "compartmentalization" of mathematics can have a damaging effect on students' ability to "transfer tools across topics" (p. 297). Further, Bergqvist and Lithner (2012) found that students whose instructors provided algorithms without meaning or motivation came away with different understandings of mathematics than the

instructor intended. Therefore, providing motivation for topics and making connections between concepts is essential for promoting effective student learning in mathematics.

Observations of mathematics instruction have been fairly common in K-12 settings for several years, but corresponding techniques in post-secondary contexts are fairly new. In both settings, with a few exceptions, lesson content has been largely ignored for an observational focus on pedagogical moves (Porter, 2002). However, Mesa, Celis, and Lande (2013) state that "when attending to what teachers say and do in the classroom, the quality of the mathematics becomes crucial" (p. 121). Recently, Speer et al. (2010) offered a preliminary framework for analyzing post-secondary mathematics teaching practice. This framework describes seven elements of teaching practice including time allocation, sequencing of content, posing questions and using wait time, preparing and evaluating, representations and relationships, and motivating content. This preliminary framework provides a foundation for content-focused observations in undergraduate mathematics classrooms.

Methodology

Due to historically high failure rates nationally and locally, the lecture-based College Algebra course at a large, public university was redesigned in order to increase student engagement in mathematics during class time. Class sizes were reduced from 180 to 60 students, and class time was reorganized into four one-hour sessions per week, with the second and fourth sessions designated as time for students to work on small-group activities. A community of instructors worked together to choose content and develop the course schedule, activities, and exams that would be used across all sections of the course. The instructors, chosen for their backgrounds in mathematics and prior experience teaching the course multiple times in its traditional form, each provided significant input in the development of the redesigned course. Moreover, to cultivate a community of practice (Wenger, 1999) that would support the modification of their instructional methods from lecturing to assisting of groups of students in their learning, the instructors were offered two days of pre-term professional development. In addition, the instructors continued to meet weekly throughout the year (Edwards, Haven, & Small, 2011) that the redesigned College Algebra course was offered.

At different points during the year of implementation, four instructors' classrooms were video recorded; this data was initially collected with the goal of determining whether the instructors were enacting practices that would support the student engagement model. However, based on observing the wide variation in the mathematics presented to students, it became clear that a more immediate focus should be about how instructors presented mathematical topics and whether those presentations would provide guidance for students' work during group activities. Thus, the recordings were analyzed using two aspects of Speer et al.'s (2010) framework for analyzing teaching practice: how the instructor motivates specific content and how the instructor represents mathematical concepts and relationships (p. 107). In particular, video clips featuring each instructor teaching one mathematical concept, namely solving rational equations, were identified and analyzed through the lens of this framework.

For each video clip, questions chosen to correspond to the two components of the framework were answered: How is the concept introduced? What does the concept mean, and how does the instructor explain this meaning? Do instructors relate the concept to any other mathematical ideas that the students have already seen? If so, what is this connection like? The following questions were also considered to provide additional information about the mathematics present in each the clip: How is the process explained? Is it procedural, conceptual, or both? Is the mathematics provided rigorous and correct? The answers to these questions allow for direct comparison of the instruction provided in each video clip. Furthermore, the video clips were viewed independently by three researchers who met to discuss their findings and revisit the videos to confirm their observations and conclusions.

Findings

Given their similar backgrounds in mathematics and the close work of the instructors, it was assumed that the mathematics offered to students would be similar, if not identical. However, significant differences were found not in the pedagogical practices but instead in the mathematics that was offered to students. Below we present one particular case that illustrates these differences -- the case of rational equations. We chose to focus on the instructors teaching this topic because of its importance in future courses such as Differential Calculus.

The Case of Rational Equations

After learning about rational functions and their properties in College Algebra, students are introduced to rational equations; both of these topics are important to subsequent mathematics courses. Rational equation problems often involve an arithmetic combination of rational expressions and require the student to find one or more solutions to the equation. One way to solve rational equations consists of determining a common denominator which is then applied to the equation to simplify it. However, this application varied between instructors without mention of the other possible solution methods, as described below.

We observed variation in how the concept of solving rational equations was motivated by the instructors as well as in the procedure for solving rational equations. For example, two of the instructors motivated the process of solving rational equations by first providing examples of adding rational numbers with unlike denominators by obtaining common denominators. These two instructors then related this process to an example of a rational equation, with correspondences between the two examples (rational numbers and rational equations) made explicit. However, these two instructors diverged in how they solved the equations, with one instructor rewriting all rational expressions in an equation so that they had a common denominator, combining the rational expressions, and then solving by ignoring the denominator. The other instructor solved the rational equations by multiplying each rational expression by the common denominator in order to eliminate it from the denominators from the equation.

In contrast, one instructor motivated the concept of solving rational equations solely by discussing a fractional equation $?/7 = 0$. In this case, the instructor focused on what the numerator must be in order to make the equation true and then related this thought process to a rational equation example. This method of motivating the process for solving rational expressions represents a significant departure from not only the other instructors but also the textbook and assignments that were presented. However, like the other instructors, this instructor also used the idea of rewriting all expressions with a common denominator. All of the instructors discussed the need to determine whether the solutions satisfied the original rational equation. In one case, an instructor related this notion to one similar in the context of solving radical equations. Another instructor addressed the problem of a solution causing division by zero. One of the instructors explained the need to identify valid solutions, but did not offer motivation or connections to other mathematical ideas. Finally, in watching the video clips, we observed only one instructor providing both a symbolic representation and a pictorial representation of rational expressions. In particular, this instructor first represented rational numbers by drawing bars and circles divided into equal parts, and shading of those parts. The instructor used the equal sizes of shaded shapes as a way to motivate the necessity of a common denominator. This representation and idea was then connected to the need for a common denominator for rational expressions.

Discussion

Holding these findings up against the research literature on what students learn and take away from mathematics classes, we observed that the instruction on solving rational equations was often compartmentalized and did not make connections to other mathematical concepts or ideas.

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Algorithms for solving the equations were often provided without meaning. In teaching mathematics in these ways, the students in these instructors' courses might come away with incomplete ideas about mathematics and without the ability to "transfer across topics" (Nardi et al., 2005, p. 297). Porter (2002) noted that "the content of instruction is an essential variable in research on factors affecting student achievement" (p. 3). In a course with such large failure rates, continued investigation of the content presented by instructors in this course will help to uncover these factors. Speer and colleagues' framework (2010) allowed us to focus on particular aspects of mathematics instruction and provided a valuable tool for investigating how mathematical topics were motivated and represented. However, the framework did not help to capture other mathematical features of instruction. As one example, the instructors also used varied mathematical language and some language was imprecise (e.g., expression versus equation) and incorrect (e.g., use of word undefined rather than indeterminate; which is an important distinction for students who continue on to calculus courses). Thus, future analysis of the mathematics instruction these video clips will aim to develop a more fine-grained and mathematics-focused lens. Finally, reflecting on the professional development that was offered to these instructors, it seems that little time was spent addressing the *mathematics*. Based on our observations in this study, we recommend that professional developers who offer training for changing instructional practice should also allow a significant amount of time to address what mathematics is offered, which problem-solving methods are emphasized, and what and how instructors will teach mathematics.

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CHALLENGING A SOCIAL NORM TO ESTABLISH EFFECTIVE SOCIOMATHEMATICAL NORMS IN AN ELEMENTARY CLASSROOM

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This qualitative study documents the establishment of new social norms in a second grade classroom in which the teacher allowed students to speak directly to one another without having to raise their hands during whole group mathematics instruction. The students initially viewed mathematics as a set of rules to follow and exhibited the role of passive recipients of information. This changed as students were provided opportunities to participate in discussions in a different way.

Keywords: Classroom Discourse, Elementary Education, Instructional Activities and Practices

Introduction

Research has provided ways in which teachers can engage their students in mathematical discourse, but has not addressed the issue of how established social norms impact student discourse. Numerous studies have established the merit of focusing on students constructing their own knowledge during mathematics instruction (Brownell, 1945; Carpenter, Fennema, & Franke, 1996, Cobb, Hodge, & Gresalfi, 2011). These researchers have provided a wealth of data to support the need for teachers to allow and highlight student discourse during instruction.

Framework/Literature

The intersection of social constructivism and sociolinguistics provides a framework for this study. The work of Vygotsky and Luria (1930) created a foundation of theory that establishes the necessity of engaging students in discourse. After conducting a series of experiments with young children, they determined that children use speech as they make sense of tasks in which they are involved. They found that as tasks became increasingly more challenging, the speech of the children also increased.

Mathematics reform efforts stress the importance of dialogue in the classroom. Students are expected to participate in a dynamic classroom in which they explain, justify, and question solutions (Cobb, Yackel, & Wood, 1992). When students participate in this type of environment, they build on their understanding as they engage with their peers. Recent efforts have proven successful for changing the expectations for student discourse during mathematics instruction (Chapin, Anderson, & O'Connor, 2003; Kazemi & Hintz, 2014).

Methods

To develop a better understanding of the negotiation of social norms, an ethnographic study was conducted with a second grade teacher, "Mr. Sharp" and his students. The goal was to describe the development of new social norms when students were no longer required to raise their hands to speak. Within this context, it was possible to learn more about how mathematical discourse patterns develop.

During the study, pre and post interviews were video recorded with the teacher and students. Video and audio recordings were also used to capture mathematics lessons. These were later analyzed with field notes taken during mathematics instruction. The teacher provided "member checks" of the information compiled during five weeks of mathematics instruction.

Analysis

Social and sociomathematical norms exist in all classrooms. Understanding these norms may provide an understanding of changes that have the potential to enhance students' experiences with mathematics. This section will describe qualities of social and sociomathematical norms in existence prior to the study and at the conclusion of the study. Social norms can be viewed as established expectations for behavior in a group setting. They can be explicitly stated or implicitly understood. Sociomathematical norms are "Normative aspects of mathematical discussions that are specific to students' mathematical activity" (Yackel & Cobb, 1996, p. 458). Effective sociomathematical norms support a negotiation of meaning. This entails establishing what constitutes an acceptable mathematical explanation and a legitimate challenge. There is also a focus on solution methods being 1) different from others and 2) sophisticated (Cobb, Hodge, & Gresalfi, 2011).

Typical Instruction

At the beginning of the study "Mr. Sharp" shared that he valued student talk and felt that he could learn about his students' mathematical conceptions and misconceptions by listening to them. However, there was a mismatch between what he valued and what he enacted during his typical mathematics lessons. While he stated the importance of having students explain their reasoning, there was no evidence of him actually doing so during instruction. His practices were not aligned to the tenets of what he espoused.

Polly and Hannafin (2011) suggest that teachers have faulty assumptions related to their enactment of student-focused instruction. Mr. Sharp's initial patterns were consistent with this research. The pre-existing sociomathematical norms were focused on procedural rather than conceptual understanding. The initial social norms, including the norm of raising hands to speak, contributed to the development of traditional sociomathematical norms that focus primarily on the actions and verbalizations of the teacher.

Student Interviews

Interviews with the students provided insight into their thinking in regard to the pre-existing social and sociomathematical norms in their classroom. When the students were asked about talking during mathematics, their responses conveyed the value of silence during math lessons. The questions were as follows: Do you talk a lot during math class? Why/Why not? Tell me about the kind of talking you do during math. The student responses are listed in Table 1.

Table 1: Sample of Beginning and Ending Interview Responses About Talking

Student	Beginning Interview Response	Ending Interview Response
Sam	No, unless it's to answer a question because most the times I know the answer. Most of the times when I raise my hand, I really don't get picked sometimes. Basically, I think in my head. Sometimes I think about the questions.	Sometimes. When we talk about shapes and how much sides and vertices they have I might talk to tell how much sides and vertices and how they make the shapes.
Charlese	Not really because I'm kind of shy. Sometimes I might get the wrong answer and I feel shy. I tell the teacher the answers like when he asks me, what is something like fifty times two, it's one hundred.	Sometimes. I'm shy. Well, I talk about what we're focused on like being an active listener and trying to do that.
Emma	No, we don't talk a lot in math because we are learning. We usually talk to the person next to us about what could the answer to the question be.	Yes, because we have to discuss the answer. Like we disagree or agree and we repeat what they said and we tell our answer. We talk about what the answer might be.
Bay	Maybe a little. Maybe talking to my friends about I got my tooth pulled this weekend…	No, because you can only talk to your friends about math.

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At the beginning of the study, the sociomathematical norms perpetuated an expectation that students were recipients of information during mathematical discussions. Student talk was limited to answering the questions provided by the teacher. It was interesting that the students equated listening with learning. Student responses indicated a focus on talking about answers to questions. Notably absent were descriptions of engaging in mathematical discussions related to the process of the mathematical tasks.

Marcus made an interesting comment about talking a lot in reading. It was his understanding that math class had a different set of expectations for talking. His impression was that talking occurred during reading class but not in math class.

Bay, on the other hand, equated the talking that she did during math class with getting in trouble. When asked if she talked a lot during math class, she assumed that meant the kind of talking that got her in trouble with the teacher.

As indicated by the interview responses at the end of the study, the students in the class developed a different mindset towards discussions during mathematics lessons. They no longer viewed their role as silent recipients of information. It was evident through the interviews with the students that new sociomathematical norms were established. Students, through their direct discussions, sought mathematical agreement and accuracy. Mathematical misunderstandings were viewed as opportunities for conversation and growth. When students shared their mistakes, it helped others in the class determine the source of the mistake and in turn provided further opportunities for understanding.

Modified Instruction

The qualities of discussions that occurred when students could speak without raising their hands were markedly different than the discussions that occurred at the beginning of the study. Rather than answering questions, the new expectation for students was that they would explain their reasoning and bring clarity to their justifications. Another quality was that of making sense of the thinking of others.

Sociomathematical norms were established concurrently with the new social norms. As students were given the opportunity to speak openly and join the conversation, their comments were geared toward bringing accuracy and meaning to the discussion.

Discussion

Figure 1 summarizes which of Mr. Sharp's actions and verbalizations supported or undermined the establishment of more effective sociomathematical norms during his mathematics lessons.

Teacher Actions and Verbalizations That Support Effective Norms
• Stating expectations
• Bringing attention to student discourse
• Modeling
• Providing challenge
• Maintaining complexity

Teacher Actions and Verbalizations That Undermine Effective Norms
• Highlighting student-teacher discourse
• Interrupting
• Demanding participation
• Requesting single word or choral responses

Figure 4. Actions and verbalizations that support or undermine effective norms

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Making the change of allowing students to talk directly to each other without raising their hands proved to be a complicated task. It was often a matter of taking two steps forward and one step back. Negotiating social and sociomathematical norms is a lengthy process that is likely to look differently in different classroom settings.

Our society needs a generation of students that have the capability to be independent thinkers. Our teaching strategies in elementary mathematics classrooms have the potential to teach students to persevere and thrive when they encounter academic challenges. Elementary students have something to gain when they learn to take ownership of their ideas and can openly share their ideas during mathematics instruction.

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FRONTERAS DE INTERACCIÓN ENTRE LA INVESTIGACIÓN Y LA ENSEÑANZA: PARADIGMAS EN TRANSICIÓN

INTERACTION AND BORDER-CROSSING BETWEEN RESEARCH AND TEACHING: PARADIGM SHIFT

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En este trabajo hacemos un análisis teórico sobre la forma como los hallazgos de Investigación en Educación Matemática (IEM) se encuentran relacionados con indicadores para la enseñanza a través de ciertos paradigmas de investigación los cuales están presentes en las reformas educativas y estos sufren de un proceso que llamamos de adelgazamiento natural al transformarse en indicadores para los maestros, cruzando así fronteras. Hemos reconocido la existencia de dos tipos de interacciones sustanciales en esta transición de paradigmas: 1. Existen tres actores en el tránsito de los paradigmas: (a) La comunidad de IEM, (b) Los diseñadores educativos y (c) La comunidad de profesores y 2. Algunas reformas dejan huellas profundas, contribuyendo a la formación de una memoria histórica. Constataremos la memoria histórica y personal de los profesores en la etapa empírica.

Palabras clave: Teorías del aprendizaje, Estándares (en sentido amplio), Análisis del currículo.

Introducción y Problemática

En este trabajo nos proponemos entender el proceso que los resultados de investigación sufren desde que son formulados por la IEM hasta llegar al salón de clase, por lo que es de gran importancia enfocar en detalle las diversas perspectivas implicadas en el fenómeno de manera que podamos reconocer las interacciones que le dan cuerpo.

En este reporte hablaremos de dos primeros núcleos de tránsito, ya que consideramos que el proceso completo consta de tres actores principales: (a) La comunidad de IEM, (b) Los diseñadores educativos y (c) La comunidad de profesores.

Hemos llevado a cabo una investigación documental exploratoria y explicativa, con base en una revisión bibliográfica y hemerográfica en el terreno de la investigación educativa en matemáticas.

Consideraciones Desde una Primera Frontera, La Investigación

En esta primera parte mostraremos algunos de los supuestos de importancia que hemos encontrado relativos a la forma como se relacionan los resultados de investigación con los diseños educativos de la comunidad de la educación matemática, desde la perspectiva de los avances teóricos y sus supuestos centrales o paradigmas de investigación.

Los resultados de la IEM, Ciencia de Diseño, su validez y alcances

En primer lugar consideramos que la IEM es una disciplina científica (Godino, 2010), y más específicamente una Ciencia de Diseño (Lesh & Sriraman, 2010) lo que hace de ésta una disciplina muy amplia en lo relativo a sus principales problemáticas y objetos específicos de estudio. Suponemos que existen dos tipos de resultados de acuerdo a los dos campos de desarrollo de la disciplina: (1) Los que aportan al desarrollo teórico de la disciplina (teóricos) y (2) Los que aportan a las aplicaciones para la mejora de la educación matemática (prácticos) (Sierpinska & Kilpatrick, 1998). Aunque los primeros puedan inducir a los segundos.

Además, la validez de los resultados o hallazgos de investigación depende principalmente de la categoría de *pertinencia* en su contexto de uso, y es de gran importancia que los hallazgos de

investigación puedan soportar situaciones de *reproducibilidad* mediante las cuales sea posible controlar tanto las condiciones iniciales como las propiedades emergentes de las interacciones entre los (tres) sistemas complejos implicados, según Lesh & Sriraman (2010) estos son: (a) Sistemas de la vida real, (b) Sistemas complejos que desarrollan los humanos para modelar o comprender los anteriores y (c) Los modelos que desarrollan los investigadores para describir y explicar las habilidades de los primeros modelos –De acuerdo con Lesh & Sriraman (2010), los modelos son diseños que permiten considerar situaciones realistamente complejas y se apoyan típicamente en una variedad de teorías–.

Para los fines de este trabajo, debido a la amplitud del tema en cuestión, consideraremos solamente algunos de los hallazgos que han trascendido notoriamente, tanto dentro de la propia teoría educativa como en el contexto de la aplicación para la enseñanza y aprendizaje de las matemáticas a través de los últimos tiempos.

La Conformación de Paradigmas en la Teoría y las Propuestas Educativas

Desde nuestro punto de vista y de acuerdo con Kuhn (1962), "Los paradigmas son logros científicos universalmente aceptados que durante algún tiempo suministran modelos de problemas y soluciones a una comunidad de profesionales" (p.94) de tal forma que le permiten a la comunidad en cuestión afinar los resultados obtenidos y entonces surgen novedades que pueden incluso provocar cambios sustanciales en las teorías.

Dicho lo anterior, pensamos que los paradigmas de investigación en educación matemática influyen fuertemente en el terreno práctico del quehacer de la misma y son indicadores que en última instancia se consideran para proponer líneas de actuación que se concretan, con base en ciertos intereses, como actividades y expectativas específicas para los profesores.

En un esfuerzo por conjuntar las aportaciones más representativas de la IEM en determinado periodo, Wegerif (2002, p.10) reconoce cuatro orientaciones hacia el aprendizaje: 1.Conductismo, 2.Cognitivismo/Constructivismo, 3.Humanismo y 4.Participacionismo. Este es un ejemplo de la producción de diferentes paradigmas en la IEM, estas orientaciones además se pueden caracterizar de acuerdo con cinco cuestiones fundamentales, tres de índole filosófica: (1) Idea del proceso de aprendizaje, (2) Localización del aprendizaje, (3) Propósito en la educación; y dos más de índole práctico: (4) Vías de transferencia y (5) Rol del educador.

Para este trabajo partimos de que son dos las formulaciones (o paradigmas) básicas ampliamente aceptadas presentes en casi todo marco teórico actual del campo: (a) La corriente constructivista y (b) Las teorías socioculturales, a partir de ellas podemos detectar supuestos generales que sustentan y apoyan a la IEM y estas son consideradas al perfilar las prácticas de Enseñanza y Aprendizaje (EyA), de la siguiente forma:

Tabla 1: Dos paradigmas básicos para la IEM. Fuente Principal: Gutiérrez & Boero (2006)

Paradigma	Propuesta filosófica epistemológica *Perspectiva de EyA* →	Interpretación educativa *Intervención didáctica* →	Elementos de prácticas educativas
Constructivista	La génesis de la cognición se apoya sobre todo en un proceso constructivo	El conocimiento es el resultado de la actividad del que aprende	Uso de manipulables, conflicto cognitivo y resolución de problemas
Socio-Cultural	El desarrollo intelectual es una transición desde la unidad social hacia la identidad individual a través de una compleja acción mediada	La capacidad de aprendizaje depende en gran medida de las oportunidades que ofrece el contexto	Zona óptima de intervención, influencias interpersonales y sistemas semióticos de representación

Para la transición de los paradigmas teóricos a las propuestas educativas, consideramos como determinantes los objetivos de difusión y el contexto educativo, lo que provoca un proceso natural de adelgazamiento de los hallazgos teóricos que en principio aportan las teorías. Sólo de esta manera pueden éstos ser usados por los diseñadores educativos quienes deben establecer los objetos de

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estudio, los objetivos y la forma cómo estos se presentan en la clase, por ello decimos que los paradigmas teóricos se transforman en este tránsito en *paradigmas de uso o populares* al interior de las distintas comunidades participantes del fenómeno de enseñanza-aprendizaje.

Aportaciones Desde una Segunda Frontera, Los Diseños Educativos

En esta segunda parte del análisis, describiremos los tipos de interacciones entre propuestas de investigación que han provocado los cambios de paradigmas, y las propuestas educativas desarrolladas con base en ellas, de manera que se reconoce un primer nivel de interpretación y uso de resultados de investigación para la práctica educativa.

Para mostrar un escenario de transición entre las propuestas teóricas para la educación y la tarea de los diseñadores para proponer planes y programas de estudio en matemáticas, vamos a abordar a La reforma de la Matemática Moderna (MM) como una reforma global representativa.

Implementación de una Propuesta Educativa, el Caso de la Matemática Moderna

Antes de la reforma de la MM existía un paradigma dominante: el conductismo, y dicha reforma dio paso a un cambio de postura, la que podemos observar a partir de dos propuestas básicas, como se observa en la figura 1.

Observamos que la reforma de la MM implementó novedades para la educación matemática que, aunque finalmente resultó inadecuada de manera global, aportó importantes aspectos que se retomaron en las posteriores transformaciones educativas, específicamente: (a) Buscar la unificación de las prácticas de enseñanza, (b) Modernizar los contenidos incluyendo estructuras simples de la matemática e (c) Integrar prácticas matemáticas como la deducción, la abstracción y la actividad de resolver problemas.

Antes de la reforma:
Actividades educativas apoyadas casi exclusivamente en la memorización. Énfasis fundamental en aspectos pedagógicos (de tipo conductista). Las tareas presentadas a estudiantes son esencialmente teóricas, alejadas de contextos de aplicación. Se presenta discontinuidad entre dos estrategias de enseñanza relevantes: la procedimental y la deductiva.

Después de la reforma:
(Desde Francia, 1970) Se retoma la preocupación por enseñar otros contenidos y estudiar matemáticas funcionales en áreas profesionales, enfocándose en la comprensión de la relación entre las matemáticas y la realidad del mundo. (1980) Se puso el acento en las estrategias de resolución de problemas, elementos de aplicación y una enseñanza que permitiera la experimentación, el razonamiento y la argumentación

Figura 1. Posturas educativas antes y después de la MM. Fuentes: Kline (1976) y Gispert (2014).

Las propuestas de la MM, o las propuestas planteadas a través de grandes reformas, no se abandonan del todo, porque son retomados elementos que integraran posteriores diseños educativos debido a que resultaron de utilidad, lo que apoya la idea de la construcción de una memoria histórica en lo relativo a la trascendencia de los paradigmas de investigación a través de distintas reformas educativas, que si bien está sujeta a cambios también muestra rasgos de permanencia entre las comunidades respectivas.

Observaciones Finales

Reconocer el carácter científico de la IEM nos da oportunidad de: (1) Hablar de tipos de resultados ampliamente reconocidos y (2) Establecer de qué forma pueden éstos ser útiles y válidos para la educación matemática. Además, pensar en esta ciencia como de Diseño permite: (1) Enfatizar

el papel de los intereses iniciales de esta actividad *vs* las necesidades de su uso práctico en la educación, (2) Establecer lo relativo a la validez o *pertinencia* de los hallazgos de investigación e (3) Indicar su necesario carácter consustancial de *reproducibilidad* para su implementación en los modelos educativos.

Los resultados de investigación dentro de la comunidad de IEM toman cuerpo dentro de grandes estructuras teóricas y sus fundamentos se observan a través de lo que hemos llamado *paradigmas de investigación*, éstas sustentan supuestos ontológicos y epistemológicos, y contribuyen con indicadores para el desarrollo de la práctica educativa en matemáticas.

Actualmente, desde nuestro punto de vista, es posible reconocer la existencia de dos paradigmas básicos aceptados ampliamente como válidos para la IEM, los que distinguimos como *El Paradigma Constructivista* y *El Paradigma Socio-Cultural*, que aportan diferentes perspectivas de enseñanza y aprendizaje y herramientas para la intervención didáctica.

Observamos que dichos paradigmas sufren naturalmente un proceso de simplificación o *adelgazamiento* en aras de su divulgación, pero aun siendo esta la base de apoyo, los diseñadores de planes y programas de estudio toman también en consideración otros intereses específicos, lo que posiciona a dicha comunidad, por su tarea de diseño, como un primer nivel de tránsito para la interpretación de los resultados de investigación para propuestas prácticas.

Las comunidades de usuarios de los resultados de investigación necesitan reformular los paradigmas teóricos de manera que se privilegien algunos indicadores para su uso, lo cual provoca que se formen los que estaremos llamando *paradigmas populares*.

Con base en la idea de la formación de una *memoria histórica* en la esfera institucional tenemos el paso por dos fronteras: (1) Los paradigmas de investigación y (2) Los paradigmas de uso o populares, los primeros son producto de la investigación y los segundos de la terea de los diseñadores educativos.

Lo anterior nos permite contar con categorías de análisis para rastrear los resultados de investigación hasta el salón de clase, esto es, a través de una *memoria histórica personal* en el profesor, fundamentada por la eficiencia de sus prácticas docentes y de su experiencia a través de distintas reformas.

In this paper we make a theoretical analysis on how the findings of Research in Mathematics Education (RME) are related to indicators for teaching trough certain research paradigms, which are present in educational reforms and these suffer a process that we call of natural thinning, transforming into indicators for professors, and thus crossing boundaries. We have recognized the existence of two types of substantial interactions in this process of paradigm shift: 1. There are three actors involved in the paradigm shift: (a) The community of RME, (b) Educational designers and (c) The community of professors; and 2. Some educational reforms leave profound marks, contributing to the formation of a historical memory. We will verify the existence of a historical and personal memory in mathematics professors through an empirical study.

Keywords: Learning Theory, Standards (broadly defined), Curriculum Analysis

Introduction and Subject Matter

In this paper we aim to understand the process suffered by research results from their formulation posed by the RME to their reach to the classroom, so it is very important to approach in detail the various perspectives involved in such phenomenon in order to recognize the interactions that shape it.

In this report we discuss two initial transit cores, as we believe that the entire process consists of three main actors: (a) The RME community, (b) Educational designers and (c) The community of professors.

We have carried out an exploratory and explanatory documentary research, based on a selected bibliography review in the field of research in mathematics education.

Considerations from a First Frontier, Research Endeavour

In this first part we show some of the main assumptions we identify about how research results relate to educational designs of mathematics education community, from the perspective of theoretical advances and its core assumptions or research paradigms.

The Results of the RME, a Design Science, Validity and Scope

First of all we consider that RME is a scientific discipline (Godino, 2010), and more specifically a Design Science (Lesh & Sriraman, 2010) which makes this a very broad discipline with regard to their main problems and specific objects of study. We assume that there are two kinds of results according to two general fields of development of the discipline: (1) Those which contribute to the theoretical development of the discipline (the theoretical ones) and (2) Those that contribute on the applications for the improvement of mathematics education (the practical ones) (Sierpinska & Kilpatrick, 1998), while the first may lead to the latter.

In addition, the validity of results or research findings mainly depends on the category of *relevance* in its context of use, and is of great importance that research findings can withstand *reproducibility* situations by which it is possible to control both the initial conditions and emergent properties of interactions between the (three) complex systems involved; according to Lesh & Sriraman (2010) these are: (a) real life systems, (b) complex systems developed by humans to model or understand the first ones, and (c) Models developed by researchers to describe and explain the skills of the previous models –According to Lesh & Sriraman (2010) models are designs that allow consider realistically complex situations and typically rely on a variety of theories–.

For the purposes of this paper, because of the breadth of the subject matter, we consider only some of the findings that have transcended markedly, both within the educational theory itself and in the context of the applications for teaching and learning of mathematics through recent times.

Formation of Paradigms within Educational Theory and Proposals

From our point of view and according to Kuhn (1962), "Paradigms are universally accepted scientific achievements that for some time provide models of problems and solutions to a community of professionals" (p. 94) so as to allow the community concerned refine the results obtained and then novelties emerge that may even lead to substantial changes in theories.

That stated, we think that the research paradigms in mathematics education strongly influence its field of practical endeavor and are indicators that ultimately are considered to propose courses of action which materialize, based on certain interests, as activities and specific expectations for professors' practice.

In an effort to bring together the most representative contributions of the RME over a certain period of time, Wegerif (2002, p. 10) recognizes four orientations towards learning within the discipline: 1. Behaviorist, 2. Cognitivist/Constructivist, 3. Humanist, and 4. Participatory. This is an example of the production of various different paradigms in the RME; these guidelines may be also characterized according to five fundamental issues, three of them are of philosophical nature: (1) View of the learning process, (2) Locus of learning and (3) Purpose in education; and two more of practical nature: (4) Transference pathways and (5) Educator's role.

For this study we draw from the premise that there are two basic formulations (or paradigms) widely accepted present in almost every current theoretical framework of the field: (a) The constructivist school and (b) the socio-cultural theories, departing form them we may detect general assumptions that sustain and support the RME and these two formulations are taken into account when profiling Teaching and Learning (T&L) practices, as follows:

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Table 1: Two basic paradigms for the RME. Main source: Gutiérrez & Boero (2006)

Paradigm	Philosophical and epistemological proposal	Educational Interpretation	Elements of educational practices
Constructivist	The genesis of cognition is based primarily on a constructive process	Knowledge is the result of the learner's actions	Use of manipulatives, cognitive conflict, and problem solving
Socio-Cultural	Intellectual development is a transition from the social unit to the individual identity through a complex mediated action	Learning capability depends largely on the opportunities offered by the context	Zone of proximal development, interpersonal influences and semiotic systems of representation

(*T&L perspective* → between columns 2 and 3; *Didactical approach* → between columns 3 and 4)

For the theoretical paradigm shift into educational proposals we regard as determining the objectives of dissemination and the educational context, which causes a natural thinning process of theoretical findings in principle provided by theories. Only in this way can they be used by instructional designers who must establish the objects of study, the objectives and the way these are set out in class, thus we say that the theoretical paradigms are transformed over this transit in *paradigms of use or popular paradigms* within different communities participating in the phenomenon of teaching and learning mathematics.

Contributions from a second border, Educational Designs

In this second part of the analysis, we will describe types of interactions between research proposals caused by paradigm shift, and the educational proposals developed based on them, in a way that a first level of interpretation and use of research results for educational practice may be recognized.

In order to show a shift scenario among theoretical proposals for education and designers' role in proposing plans and programs of study in mathematics, we will address the educational reform of Modern Mathematics (MM) as a representative global reform.

Implementation of an Educational Proposal, the Modern Mathematics Case

Before the reform of Modern Mathematics (MM) there was a dominant paradigm: behaviorism and this reform led to a shift of approach, which can be noticed from two basic proposals, as shown in Figure 1.

We notice that the reform of MM implemented innovations for mathematics education which, even though ultimately this reform proved inadequate globally, it contributed important aspects that were taken up in subsequent educational restructurings, specifically: (a) Pursue the unification of teaching practices, (b) Modernize contents including simple mathematical structures and (c) Integrate mathematical practices such as deduction, abstraction, and problem solving activities.

Before the reform:
Educational activities supported almost exclusively on memorization. Fundamental emphasis on pedagogical aspects (behaviorist type). The tasks presented to students are essentially theoretical, away from application contexts. There is discontinuity between two relevant teaching strategies.

After the reform:
(From France, 1970) Concern about teaching other contents and studying functional mathematics in professional areas is resumed, focusing mainly on understanding the relationship between mathematics and the real world. (1980) The emphasis was placed on strategies of problem solving, application elements and an instruction that would allow experimentation, reasoning and argumentation.

Figure 1. Educational approaches before and after MM. Information sources: Kline (1976) & Gispert (2014).

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The MM proposals, or in general the proposals posed through major reforms, are not abandoned altogether since some elements are retaken to integrate into further educational designs because they turned out to be useful, which supports the idea of the building of a *historical memory* with regard to the transcendence of research paradigms through various educational reforms, albeit it is subject to change it also shows traits of continuity between the respective communities.

Concluding Remarks

Recognize the scientific disposition of the RME gives us the opportunity to: (1) Talk about types of widely recognized results and (2) establish how they can be useful and valid for mathematics education. Also, thinking about it as a design science allows us to: (1) Emphasize the role of this activity's initial interests *vs* the needs of practical use in education, (2) Establish aspects related to the validity or *relevance* of research findings, and (3) Indicate the essentially inherent character of *reproducibility* needed to implement the findings into the educational models.

Research results within the RME community take shape within large theoretical structures and its foundations are observed through what we call *research paradigms*, they support ontological and epistemological assumptions, and contribute with indicators for the development of educational practice in mathematics.

Currently, from our point of view, it is possible to recognize the existence of two basic paradigms widely accepted as valid for the RME, which we distinguish as *The Constructivist Paradigm* and *The Socio-Cultural Paradigm*, which provide different perspectives of teaching and learning, and tools for didactic intervention.

We note that these paradigms naturally undergo a process of simplification or *thinning* in the interest of outreach, but even this being the base of support, designers of study programs and curricula also take into account other specific interests, which position the design community, because of its design endeavor, as a first level of transit for interpretation of research results towards practical proposals.

Communities of users of research results need to reformulate the theoretical paradigms in a way that some indicators for their use are privileged, which causes to form what we call *popular paradigms*.

Based on the idea of building a *historical memory* in the institutional sphere we pass through two borders: (1) research paradigms and (2) paradigms of use or popular paradigms; the former are the result of research while the latter result from the educational designers endeavor.

This allows us to have categories of analysis to track the research results to the classroom; that is, through a *personal historical memory* in the professor, based on the efficiency of their teaching practices and their experience through various reforms.

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TEACHERS' MOTIVATIONS AND CONCEPTUALIZATIONS OF FLIPPED MATHEMATICS INSTRUCTION

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Although an increasing number of teachers report flipping a lesson or their entire class, little is known about what leads teachers to this decision or how they implement flipped instruction. Understanding these factors is important because it can allow teacher educators and researchers to investigate ways to support teachers in enacting practices that can achieve their goals and improve student learning. In this study we examine two teachers' motivations for and conceptualizations of flipped instruction. We discuss the ways in which these motivations aligned, or not, with their enactments and suggest implications for research and practice.

Keywords: Instructional Activities and Practices, High School Education, Technology, Curriculum

Although a majority of teachers report flipping (Smith, 2014), research has not kept pace. Further, studies of flipped instruction tend to treat it as a uniform approach in which teachers assign lecture videos for homework and do problem sets in class (e.g., DeSantis et al., 2015). Our study, however, has revealed great variation among flipped classes in terms of both the homework assigned and the work conducted in class. It is also likely that teachers have different motivations for implementing flipped instruction. We examined two mathematics teachers' motivations for and conceptualizations of flipped instruction in order to better understand varying perspectives of this instructional model.

Background

Teachers have long been central to innovations and reforms in mathematics education and their perceptions of such efforts are important to consider. Smith (1996) revealed that teachers' feelings of self-efficacy shifted with reform efforts because high-quality teaching was no longer equated with giving clear demonstrations. Similarly, the rise of flipped instruction may entail a new set of skills associated with high-quality teaching, such as creating compelling videos for homework or the ability to facilitate discussions in class that build on video homework.

Adopting flipped instruction may also be analogous to adopting new curricular materials. Lloyd (2008) found that such changes take time for teachers as they have to manage both the students' expectations of how to participate and their own comfort with the new curricular vision. Although flipped instruction is not a curriculum, it still requires teachers to grapple with the purposes of homework and in-class activities and to manage students' expectations of their role in their own mathematics learning. Some literature is emerging (e.g., Moore, Gillett, & Steele, 2014; Palmer, 2015) with first-hand accounts of instructors' experiences flipping and we add to this work by answering the following questions: What were the teachers' motivations for flipping their mathematics classes? How did they intend to enact flipped instruction and what did they perceive to be the outcomes of doing so? This work can help inform the development of support for teachers who are flipping their classes and direct future research on flipped learning.

Method

Two teachers participated in this qualitative study. Ms. Schaefer was a mathematics instructor at a community college and taught a flipped college algebra course. Though she had flipped lessons previously, the course we observed was the first she had fully flipped. In contrast, Ms. Temple, an eighth grade teacher, had been flipping her mathematics course for five years.

We conducted three cycles of observation in Ms. Schaefer's class. For each cycle we conducted audio-recorded pre- and post-lesson interviews to allow Mrs. Schaefer to describe the lesson we observed and her instructional decisions related to the flipped format. We also video recorded the three lessons. Additionally, we collected copies of Ms. Schaefer's videos and any tasks she used in class. Ms. Temple's data was collected from a distance and so we conducted a single initial interview with her. We conducted two observations in Ms. Temple's class, complete with video recordings and copies of her instructional materials. We also collected survey data from the teachers, which consisted of 48 item (26 multiple choice, 22 open-ended) in order to identify their implementation and perspective about flipped instruction. For the present study, the primary data sources include the interviews, surveys, and course materials.

With regard to analysis, the research team first identified sections of the interview and survey data in which the teachers discussed flipped instruction. The team then open-coded (Strauss & Corbin, 1990) these excerpts using analytic memos focused on the particular aspects of instruction the teachers discussed. From this initial coding, we met to collapse codes into the following themes: flipped experience, flipped benefits, flipped struggles, flipped in-class work, flipped at-home work, views of students, video use in class, vision of flipped instruction, students' needs, resources for flipped, student experiences, differences between flipped and traditional, and student views of flipped. These themes were used to examine each teachers' perspectives in relation to the research questions.

Findings

In this section we discuss the teachers' motivations for flipping their instruction, then discuss the ways in which the teachers conceptualized their enactments and the perceived outcomes from it.

Motivation for Flipping

Ms. Temple began flipping her class five years ago after learning about other teachers who were flipping. She discussed a specific encounter with a colleague as an impetus for flipping:

> He just said, "Hey, I think you should try this. It's really cool how much time you get with your kids in class. I think you would love it." After he bugged me for a couple months, I said, "Alright, what the heck, I'll try it." So I tried it with my advanced class and I was like "Oh my gosh, this is so cool!" … I can talk to these kids and physically see what they are doing.

Her colleague's encouragement led to Ms. Temple's implementation of several flipped lessons. However, she did not fully flip an entire course until her student teacher worked with her to develop a set of video lessons. Once she had created the videos and implemented the lessons, Ms. Temple said that her students' engagement served as encouragement to continue flipping.

Similar to Ms. Temple, Ms. Schaefer began flipping her class after hearing about the benefits of flipping from other teachers. Ms. Schaefer's motivation centered on her desire to increase interaction with students and she was drawn to the prospect of helping students develop deeper mathematical understandings. "I liked the idea of having time to deepen the understanding of math and make the real world connections that you typically do not get to do."

Enactments of Flipped Instruction

In addition to the similarities driving their initial forays into flipped instruction, both teachers had nearly identical ways of defining flipped instruction. In her survey response, Ms. Schaefer defined flipped instruction by describing the students' experiences, "Students watch prepared videos or reading assignments prior to class. Then we use class time to really dive into the material and get a better understanding of the concepts." Ms. Temple provided a similar definition of flipped instruction in her survey response, "I see flipped instruction as a method to deliver the basic content outside of

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class and then work together and dig deeper in class together." The teachers viewed the underlying structure of this format as one in which teachers assign instructional videos for homework and practice problems in class. Despite their agreement that flipped instruction provides teachers the opportunity to provide students with a deeper understanding of mathematics, the teachers' visions for the day-to-day implementations differed.

At-home activity. Students in both teachers' classes were assigned instructional videos to watch at home. Rather than select ready-made videos such as those from Kahn Academy, the teachers created their own videos. Ms. Schaefer's videos were created using an iPad application that allowed her to capture her explanations as she wrote on her iPad. Ms. Schaefer expected her students to watch the videos and take notes as though they were attending a typical mathematics lecture. Although Ms. Temple also created lecture videos, her students' homework included those videos in conjunction with mathematics problems aligned with the videos' content. This was possible because Ms. Temple used iBooks to develop her homework. The iBooks format allowed her to include text, video, and interactive questions in a single file.

Although both teachers' homework assignments were meant to instruct students on the mathematical concepts, their instructional videos highlighted differences in their instructional practice. Ms. Schaefer presented information to the students and included a number of examples. She also pointed out a number of common pitfalls students should avoid. In contrast, Ms. Temple's homework included many anticipatory questions or activities followed by examples and explanations of a particular concept. This instruction was then followed by further opportunity for students to complete related problems. More drastic differences were evident in the teachers' conceptualization and enactment of in-class activity, as described below.

In-class activity. Ms. Schaefer and Ms. Temple had different conceptions for how to utilize in their class time with students. When deciding how to utilize her in-class time, Ms. Schaefer developed a routine for her flipped mathematics class. She referred to her format as "30-30-30." This meant that for each 90-minute class, she would spend the first 30 minutes answering students' questions on the homework video, the second 30 minutes on worksheets that she had created to reinforce the ideas from the video, and the final 30 minutes for students to complete practice problems and ask questions. Because the students tended to work at their own pace and there were no planned whole-class discussions during these segments; the students tended to work independently or with partners throughout the entire class. During this time, Ms. Schaefer walked around to provide assistance as needed.

Ms. Temple's in-class implementation differed substantially from Ms. Schaefer's. Ms. Temple did not have a fixed format but she did have several typical routines she followed.

> When they get to class they sit in groups of three or four facing each other. Typically, there is a math starter when they walk in … And then normally they go into their team discussions … then they practice in class. And it's kind of old school math practice. It might be textbook problems. It might be a handout … And then they do a quick check … Then the process starts all over again. And they usually go home with "You're going do these two pages in your notes. Here is the reading part and the video that goes along with it." And then we just repeat the cycle. And not every day is just practice. But that is just kind of more typical.

Ms. Temple's class included more purposeful grouping of students and several points at which she checked in with the whole class. Thus, while students in both classes were given some flexibility in terms of the pace at which they worked, Ms. Temple's students were typically at a similar point at the end of class because of the routine check-ins whereas Ms. Schaefer's students tended to have more variation in how far they progressed by the conclusion of a class meeting.

Perceived Outcomes from Flipped Instruction

Both teachers' discussions of the benefits of flipped instruction centered on students' collaboration in class. Ms. Temple described the defining difference between her flipped and traditional classes as student engagement, saying, "In my traditional class they (observers) wouldn't hear kids talk as much… In my flipped class, kids are talking all the time about math… and helping each other all the time." Ms. Schaefer similarly reported collaboration as the big difference between her traditional and flipped classroom. Thus, both teachers thought that because instruction was delivered via homework, the flipped model freed up class time for students to work together on mathematics.

Although the teachers agreed that the additional class time was beneficial, they reported different impacts of flipped instruction on their relationships with students. Ms. Temple said flipped instruction allowed her to develop deeper relationships with her students. She said, "I think I know my students better than I ever have, both mathematically and not mathematically." Ms. Schaefer did not feel the same way and discussed that she found it harder to build relationships with students because she only interacted with them when they had questions.

Conclusion

Ms. Schaefer and Ms. Temple had similar motivations for flipping their instruction but there was variation in the degree to which the motivations aligned with their enactments. Both teachers desired to enhance opportunities for students to develop conceptual understanding but the teachers differed in the ways in which they structured class time and activity for students to develop these understandings. For both teachers, the vast majority of their preparation time was spent on creating the video homework but they recognized the in-class time was perhaps even more vital with regard to the students' mathematical learning. Future research in this area should attend to both the at-home and the in-class portions of flipped instruction and should consider the potentially new skills and challenges that may arise from enacting flipped instruction.

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FEELING THE SQUEEZE: FACTORS CONTRIBUTING TO EXPERIENCING A LACK OF TIME IN COLLEGE CALCULUS

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Calculus is an important course for STEM-intending students, yet has been shown to dissuade students from pursuing STEM due to poor teaching and being overstuffed with material. This report examines factors related to students' reporting a lack of time spent in class for students to understand difficult ideas. This work is part of a large, national study on college calculus, and provides an ideal landscape to examine these questions on a large scale. We find a number of factors related to experiencing negative opportunities to learn for students, such as student gender, the lack of previous calculus experience, and reports of poor and non student-centered instruction. These results point to a number of ways to support students in class.

Keywords: Curriculum, Instructional Activities and Practices, Post-Secondary Education, Data Analysis and Statistics

Calculus is an important course for students who intend to pursue STEM, and yet has been shown to dissuade students from pursuing STEM due to poor teaching and being overstuffed with material (Ellis, Kelton, & Rasmussen, 2014; Seymour & Hewitt, 1997). Because it serves a variety of STEM-related client disciplines, each with their own applications of calculus concepts, a lot of content is often expected to be covered. This has lead to interesting debates about both the breadth of and depth at which topics should be covered (e.g., Yoshinobu & Jones, 2012; Wu, 1999). How instructors perceive and respond to the high amount of content they are expected to cover is likely to affect students' experiences in Calculus I. In this paper, we explore what factors are related to a student's perception that there is enough time in class to learn difficult ideas. This analysis gives insight into the variability in perceived pacing of the course.

Introduction

Presumably, the more content taught in a course, the more content a student can learn in a course. In this way, the amount of material covered is directly related to the students' *opportunities to learn* (OTL) in a course. However, the sheer number of topics in a course is just one facet of OTL. As summarized by Reeves, Carnoy, and Addy (2013), the OTL construct comprises three components: (a) content coverage (list of topics and subtopics covered), (b) content exposure (amount of time devoted to instruction and time-on-task), and (c) content emphasis (which topics are selected for emphasis). There are many factors that influence content coverage, exposure, and emphasis, including interactions between the instructor and the students, student preparation (and the perception of preparation), and the results of formative and summative assessments.

While instructors ultimately dictate the pacing of the course, they may experience internal or external pressures that influence how they manage their course. For instance, 99% of Calculus I students are not math majors, with 30% pursuing engineering degrees (Bressoud, Carlson, Mesa, & Rasmussen, 2013). As a result, Calculus I is primarily a service course, where the client disciplines (engineering, physics, chemistry, etc.) have a strong influence over what topics are taught. Additionally, because of the large number of sections taught at a university in any given semester, Calculus I courses are very often coordinated (Rasumussen & Ellis, 2015). In this paper, we seek to understand what factors influence the perceived OTL from the student's perspective. Specifically, we seek to investigate characteristics of students who reported a negative OTL experience in Calculus I. Examples of negative OTL experiences include an inability for students to realize a provided OTL

(e.g., content was presented but the student did not have adequate time, preparation, or support to learn the material) and limited OTL (e.g., there was little time-on- task or an insufficient amount of material covered).

Methods

Project Background

Data for this project comes from the Characteristics of Successful Programs in College Calculus (CSPCC) project (NSF DRL # 0910240). CSPCC is a national study designed to investigate Calculus I in the United States, and involved online-surveys sent to a stratified random sample of Calculus I students at the beginning and end of the fall 2010 term. These surveys sought to investigate a number of different questions, and provided a comprehensive understanding about the landscape of Calculus I nationally, as well as providing further insight into student experiences in these courses. We have complete data (related to all questions that we address in these analyses) for 2,562 students.

Data collection and analysis

For this analysis, we use the following question as a proxy for perceived OTL: "My Calculus instructor allowed time for me to understand difficult ideas" with answer options (1) strongly disagree, (2) disagree, (3) slightly disagree, (4) slightly agree, (5) agree, and (6) strongly agree. Answers 1-3 were grouped together to indicate that the students reported that there was not enough time in class for them to understand difficult ideas, and answers 4-6 were grouped together to indicate that the students reported that there was enough time in class for them to understand difficult ideas.

What does it mean to report that you do not have enough time to understand difficult ideas? There are three components embedded within this question: what it means to have enough time, what it means to understand, and what are difficult ideas. In this paper, we are interested in student's ability to realize the OTL present in the class. Therefore, we are most interested in attending to whether or not students thought there was enough time for them to learn what was presented in the course. However, students in the same class may respond differently to this question not only because they perceived the pacing differently, but rather because they have different perspectives on what it means to understand an idea or different perspectives on what ideas are difficult. Thus, reporting that there was not enough time may indicate that the student (a) felt that instructor taught too fast for them to understand difficult ideas, (b) the instructor's teaching style did not help the student to understand difficult ideas, (c) the student was not prepared for class and so, perhaps regardless of instruction, she would not feel that there was enough time to understand difficult ideas, or (d) the student may not believe that she could understand the difficult ideas of calculus.

To begin to disentangle the multitude of factors related to the perceptions on the pace of the classroom, we conduct a regression model to understand what factors predict students' perceptions of the amount of time to understand difficult ideas. We use the binary responses to the "time for difficult ideas" question as the outcome and a number of questions from the beginning- and end-of-term surveys related to the potential explanatory factors.

Table 1 lists the factors that were hypothesized to be related to perceptions on pacing and the survey questions used in the analysis to investigate each factor. One may expect that marginalized or vulnerable populations would have less agency and influence over what happens in class, and so that marginalized or vulnerable populations would report negative OTL experiences. We specifically attend to gender because multiple studies indicate that women are more likely to leave their STEM intentions after their experience in introductory STEM courses such as Calculus I (e.g., Ellis, Fosdick, & Rasmussen, 2015; Hill, Corbett, St. Rose, 2010). We investigate the impact of classroom features as characterized by two aggregate variables of student reports of "Instructor Quality" and

"Student Centered Instruction." One may expect that reports of poor instructor quality and less student centered practices would be related to negative OTL experiences. We include previous calculus experience, standardized math test score, and the level to which the student agreed with the statement "I believe I have the knowledge and ability to succeed in this course." The first two variables characterize actual student preparation, and one may expect that less preparation would be related to negative OTL because if students are not prepared they may be less able to actualize OTLs and thus report negative OTLs. The third variable is related to students' perception of their ability to actualize OTLs rather than their actual ability. The final factor that we investigate is related to what it means to succeed in calculus. Students were asked if student success in calculus is more reliant on students' ability to solve specific kinds of problems or to form logical arguments. These questions target perceptions of what it means to actualize OTLs and provide insight into how students conceptualized "difficult ideas" in Calculus I.

Table 1: Factors and variables used in analysis

Factor	Variables used in analysis
Traditionally marginalized or vulnerable populations	• Gender
Classroom features	• Instructor • Student reports of "Instructor Quality" • Student reports of "Student Centered Instruction"
Student preparation and perception of student preparation	• Previous calculus experience (none, HS, college) • SAT/ACT mathematics score • AbilityToSucceed: "I believe I have the knowledge and ability to succeed in this course" (agree versus disagree)
What it means to "succeed" in calculus	• SuccessPerception: "My success PRIMARILY relies on my ability to": 1,2,3=1 ("solve specific kinds of problems"), 4,5,6=2 ("make connections & form logical arguments")

We fit a logistic mixed-effects regression model containing all variables in Table 1. Instructor was included as a random effect to account for unmeasured characteristics of the instructor that may contribute to pacing perceptions.

Results and Discussion

Two variables are associated with reporting negative OTL: identifying as a women versus identifying as a man, and having no previous calculus experience versus having had taken calculus in high school. This means that students who are women and students who have not taken calculus before are more likely to report negative OTL compared to men and compared to students who took calculus in high school. Three variables are associated with reporting positive OTL: perception of having the ability to succeed, reporting more student-centered practices, and reporting higher instructor quality. Of all associations, the strongest is from instructor quality. This indicates that the biggest factor related to reporting positive OTL is the perception of a "good" instructor – one who listens to your questions, makes time outside of class, encourages students to continue studying mathematics, and is approachable (among other qualities). Finally, three variables were not statistically related to reporting negative OTL: reporting that success is related to making connections and forming logical arguments versus ones ability to solve specific kinds of problems, standardized test percentile, and having had taken calculus in college versus previously taking calculus in high school. This analysis offers some insight into how we might create positive OTL in our own classrooms. This includes preparing students before they enter calculus so they feel confident in their

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abilities and engaging in "good teaching" practices in the classroom like listening to students' questions, making time outside of class, and encouraging students to continue studying mathematics. This also calls attention to the disparity between experiences of similar students who differ in gender, with female students equally prepared to male students reporting less time to understand difficult ideas in class. This finding calls in to question how equitably we are providing opportunities in class, not just between men and women but also between students from underrepresented minorities and low socioeconomic status, and encourages more work to be done in this area.

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CLASSROOM MATHEMATICS DISCOURSE IN A KINDERGARTEN CLASSROOM

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This study provides an analysis of classroom discourse at the kindergarten level. We illustrate how young learners can participate in mathematics discourse and how the teacher can support their engagement. The teacher in our study played a crucial role in supporting students' talk through practices that promoted collaboration around important mathematical ideas, and the provision of resources that allowed students to communicate about their reasoning with others.

Keywords: Classroom Discourse, Instructional Activities and Practices

Classroom discourse is integral to mathematics instruction at all levels. The expectation that all students will engage in mathematics discourse is central to the Common Core State Standards, which emphasize practices like conjecturing, justifying, and reconciling different ideas to analyze a problem situation (CCSSI, 2012). This work is challenging at all levels, but perhaps especially for very young learners who are still developing their general oral language skills while at the same time beginning to acquire academic language.

In mathematics, in particular, research suggests that children as early as kindergarten can consider alternative strategies and are capable of sophisticated mathematical thinking (Carpenter, Fennema, Franke, Levi, & Empson, 2014). Learning mathematics in the early years, however, is tied to students' development of language and communication skills (Greenes, Ginsburg, & Balfanz, 2004). Classroom mathematics discourse requires not only that students have the language facility to participate in general discussions, but more specifically that they are able to use language to communicate reasoned academic mathematical messages that others can understand and respond to. The teacher has a critical role in supporting student engagement in such practices, and research in mathematics education provides helpful insights into the work that teachers do in such a role. However, there is very little research on how students as young as kindergartners engage in mathematics discourse and how teachers support them in the process given their level of mathematical and linguistic development. Our work in this study aims to address this point. We use the case study of a kindergarten classroom to examine the questions "How can kindergartners participate in mathematics discourse? How can teachers support kindergartners' mathematical discourse?" Our findings highlight the way kindergartners can draw on resources available to them as young learners to engage in making mathematical claims and providing explanations. We highlight two main aspects of the teacher's work in supporting the students' discourse: her use of resources that facilitate mathematical communication, and her promotion of collaboration among the kindergartners around key mathematical ideas.

Theoretical Framework

A socio-constructivist perspective on mathematics teaching and learning frames our views of discourse and its development in the classroom (Cobb, 1994). Students' collaborative engagement in mathematics discourse supports their construction of an understanding of mathematical concepts and fosters their disposition to see mathematics as sensible and worthwhile (Michaels, O'Connor, & Resnick, 2007). Developing this level of intellectual work by young children requires intentional work on the part of the teacher, who among other things needs to "provide a range of experiences, opportunities, resources and contexts that will provoke, stimulate, and support children's innate intellectual dispositions" (Katz, 2015).

Our interest in the central role of the teacher in supporting students' development of discourse is

grounded in three teaching commitments, identified by Ball and Bass (2003) as: the *integrity of the mathematics*, the *collective as an intellectual community*, and *taking individual student's thinking seriously*. Teachers attend to the integrity of mathematics through the promotion of mathematical ways of reasoning when they press students for conceptual thinking that emphasizes not only the sharing of strategies but also explanations and justifications. Attending to the work of the community as a collective rests on the assumption that mathematical knowledge is socially constructed and validated. Helping students to engage in collective mathematical activity requires supporting their abilities to listen to and represent others' ideas, even those with which they disagree. Teachers convey their attention to student thinking in the way they hear or see the mathematical details of students' strategies, and highlight them for other students using various representations.

Methods

The study focuses on Ms. Sanders (pseudonym), a kindergarten teacher who participated in a larger study of K-5 mathematics teachers and their enactment of ambitious teaching practices. About 65% of the students at Ms. Sanders' school are economically disadvantaged, and about 6% of the students are English Language Learners. Ms. Sanders' students are representative of the demographics of her school. The ambitious nature of discourse that was evident in her class motivated us to conduct a more in-depth analysis of her practice.

We created video transcripts of five mathematics lessons from Ms. Sanders' classroom collected over the course of one academic year. Focusing mainly on the whole class discussion part of the lessons, we used grounded theory (Glaser & Strauss, 2009) to thematically code the transcripts, noting patterns in students' mathematical communicative practices and the teacher's work in supporting their discourse. We grouped these patterns into categories that reflected the students' participation structures and Ms. Sanders' instructional approaches in supporting their mathematics discourse in ways that attended to the three commitments in our theoretical framework. We each individually examined the data for themes, and then met as a research group to confirm or disconfirm our findings, and negotiate our disagreements.

Findings

Our analysis revealed that the students in Ms. Sanders' classroom consistently engaged in mathematics discourse during whole-group instruction. The following practices characterized the participation structure: (1) Students consistently explained their thinking using various resources around the classroom; (2) Students followed established classroom norms, which promoted engagement in productive mathematics discourse. These norms included sharing ideas, attending to errors respectfully, and listening and responding to one another's thinking; (3) In response to Ms. Sanders' questions, students shared solutions to a problem, explained the reasoning behind their solutions, and worked collaboratively to increase the clarity and sophistication of their explanations with the help of both the teacher and other students.

Our analysis of the ways Ms. Sanders supported her students' participation in mathematics discourse revealed that she leveraged two main categories of work: (1) using - and orienting students to - resources that facilitate mathematical communication, and (2) promoting collaborative work around the mathematics in a way that is responsive to student thinking. Ms. Sanders' facilitation of whole group discussion also followed a consistent pattern in which she solicited various student solutions to a problem, pressed them for explanations, and oriented them to each other's ideas by asking them to restate someone's idea or explain someone's reasoning. In the process she used verbal and written representations to support students' explanations and reasoning, and infused more precise use of language into the discourse. In the following sections we describe the role of both the teacher and students in the use of resources to communicate ideas and in collaborative work on mathematics.

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Using Resources

Ms. Sanders and her students consistently drew on a variety of material and conceptual resources in communicating their thinking. Material resources include displays (e.g., number lines), pictorial representations, and manipulatives. Conceptual resources consist of common language (e.g., math words and symbols) and a taken-as-shared knowledge base built around students' ideas. In our analysis, we noted three specific ways Ms. Sanders supported her students' mathematics discourse with resources: (1) She made resources available around the classroom and physically within the students' reach, (2) She established the norm of moving freely around the room to seek resources to support one's thinking; and (3) She frequently oriented students to resources they could use in supporting their explanations.

In several instances of our data, we noted how students used resources in their explanations both as a direct response to a teacher's suggestion and through their own spontaneous initiative. As an example, in one lesson, the students were sitting in front of a pocket chart where the number of days the class has been in school was represented with bundled sticks (100s, 10s, 1s). They had just counted collectively 9 sticks in the ones' pocket. Ms. Sanders asked them "how many days until we make another group of 10?" A student, Aiden, replied "one." When asked to explain, Aiden stood up and pointed to the 9 on a hundred's chart that was hanging on the side of the board. He stated, " Because this is when we start, [then pointing to the 10 on the hundred's chart] and this is when we put 10." Here Aiden did not use the hundred's chart per the teacher's request, but readily reached for it to support his thinking. Our findings suggest that such behavior was typical of other students' use of resources in this class while explaining their thinking. Ms. Sanders' use of the phrase "group of ten" is an example of the way she oriented students to available conceptual resources. Later in this discussion (as we will illustrate next), Aiden uses this language to more clearly explain his thinking.

Collaboration Around the Mathematics

Ms. Sanders facilitated students' participation in discussions by promoting expectations for sharing one's reasoning. For example, Ms. Sanders would often say to the students, "Be ready to share with us how you knew that your answer was correct." Furthermore, she facilitated discussion among the entire classroom community by consistently orienting students to each other's thinking, asking students to evaluate their classmates' assertions and apply their reasoning to others' ideas. She used discourse structures such as "turn and talk," which gave students an opportunity to practice communicating about an idea with a partner before sharing with the whole group. Furthermore, to support the kindergartners' engagement with each other's ideas, we found that Ms. Sanders revoiced student responses in particular ways: Following a student's explanation, she often *narrated* student thinking to make key aspects of it visible to the class and to model the use of academic language. She also consistently *inserted specific mathematical language* during her narration to teach these young learners how to communicate reasoned mathematical messages.

The following excerpt illustrates Ms. Sanders's use of the practices described above. The excerpt continues from the example provided earlier, where after Aiden explained his reasoning in relation to "how many days till we make a ten?" by pointing to the hundred's chart, the teacher moved the chart to the center of the board and pressed Aiden to clarify his thinking.

> Ms. Sanders: What do you mean this is where we start, Aiden? Why do you start at 9?
> Aiden: [Pointing to the ones' place of the pocket chart with bundled sticks] That's our one.
> Ms. Sanders: So Aiden is looking at the ones' place and he found that we have 9 ones today. And then what did you say next, Aiden?
> Aiden: You take one more hop and you make a group of ten.

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Ms. Sanders: So he says, "one more hop, and we make a group of ten." [Looking at the class while pointing to the 10 on the hundred's chart] How does Aiden know that we're making a group of ten? Isabel, what about that number tells us that we are making a group of ten there?

Isabel: Because we are having one more hop.

Ms. Sanders: Ok, can someone add on? She said, "We've got one more hop." How do we know we've got a group of ten here?

The excerpt illustrates the way Ms. Sanders used Aiden's thinking as a resource for helping the class attend to the idea of place value. By pushing the hundred's chart the center of the board, she legitimized his use of such representation to explain his thinking. She also supported Aiden's explanation by revoicing it with more precise language, inserting the notion of "the ones' place" and repeating the key idea of "a group of ten" intentionally during the exchange. Her press to have Aiden clarify his reasoning around these ideas supported him in being able to provide a reasoned mathematical message using more precise language. One can note, for example, how Aiden responded to Ms. Sanders above by saying, "You take one more hop and you make a group of ten." Then, she repositioned Aiden's contribution as the focus of the classroom community, engaging other students in practicing mathematical explanations using precise language that she reinforced in the process. This is evident in Ms. Sanders' response to Isabel, where she inserted the notion of "a group of ten" in inviting others to add to her explanation.

Discussion and Conclusion

The case of Ms. Sanders provides an existence proof that kindergartners are capable of engaging in sophisticated discourse, but to do so, they need specific supports that build on their developing knowledge and communication skills as young learners, and the resources available at their disposal. Ms. Sanders acted intentionally to gently guide discussions toward more precise language and mathematical ideas. Her use of narration and repetition of key mathematical points were aspects of her work that we found to be supportive of kindergartners' participation in mathematics discourse. They helped reposition contributions for the work of the collective.

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SMALL GROUP LEARNING ENVIRONMENTS IN MIDDLE GRADES CLASSROOMS

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This study of middle grades classrooms addresses small group learning environments through an integration of three frameworks for examining mathematics tasks, peer cultures, and mathematics discourse. Naturalistic observations of classrooms provide illustrative episodes of limited group functioning when aspects of each framework are problematic and of high group functioning when aspects of all three frameworks are of high quality and work in concert.

Keywords: Classroom Discourse, Instructional Activities and Practices, Middle School Education

Small group learning environments promote opportunities for conceptual learning and powerful mathematical work (e.g., Esmonde & Langer-Osuna, 2013). Facilitating students' group work presents challenges for teachers because their influence on what transpires in groups is indirect. Effective group work depends on tasks that are challenging and appropriate for group efforts, peer interactions that support collaboration, and group discourse that promotes engagement and meaning-making (Cohen, 1994). This paper reports on foundational work of a research study of small group learning environments in middle grades mathematics classrooms, addressing the research question: How is the functioning of small group learning environments characterized by aspects of mathematics tasks, peer cultures, and mathematics discourse?

Conceptual Background

The operational definition for our central construct is: High functioning group work occurs when students are individually engaged and collegially communicating, seeking and receiving the help they need from one another, to explain, justify, and make meaning of mathematical ideas. This definition draws on three frameworks that we integrate to analyze mathematics tasks, peer cultures of effort and achievement, and mathematics discourse in small group learning environments. We use these frameworks to describe group functioning, to understand challenges teachers face, and to craft ways to help teachers deliver on the promise of small group work.

Mathematics Tasks

Students' work with mathematics content is operationalized through tasks. The structure and expectations of written tasks denote a set of demands for students, such as providing an answer, showing work, justifying a method, representing, and making meaning. In the context of group work, the Mathematics Task Framework depicts how tasks evolve at three critical transition points. First, when the teacher assigns a task, expectations are communicated by the written task. Second, the teacher sets up and structures the task, providing further expectations or constraints. Third, as students engage with a task, their own expectations and propensities influence the nature and quality of mathematical work they do (Stein, Grover, & Henningsen, 1996).

Peer Cultures of Effort and Achievement

Peer Cultures of Effort and Achievement reflect the activities, routines, and norms that students develop in interaction with one another, which communicate the acceptability, desirability, and value of effort and achievement. Teachers can influence peer cultures by setting expectations for students' social behaviors, providing opportunities for social interaction among students, and scaffolding the kinds of social interactions they would like students to have with classmates (Hamm & Hoffman, 2016).

Mathematics Discourse

Implicit in the assignment of tasks to student groups is the expectation that students will communicate with one another. Tasks demanding conceptual mathematical thinking must be matched by group discourse that supports a focus on reasoning, justification, and connections. Teachers can make expectations for group discourse purposeful and explicit via strategies that engage students in productive discussion (Webb et al., 2014). The Mathematics Discourse Matrix (Malzahn, 2013) specifies four key dimensions: (a) *explaining* ideas, representations, solutions, or justification; (b) *questioning* for an answer, explanation, justification, meaning, or connection; (c) *listening* to another's ideas for understanding, interpreting, evaluating, or comparing; and (d) using formal or informal, verbal or non-verbal *modes of communication*.

Naturalistic Study of Small Groups in Mathematics Classrooms

Initial data collection and analysis for the study of small group learning environments involved observations in classrooms of 11 teachers (in 2 school districts, one municipal, one rural; in 4 middle schools and 1 high school; spanning grades 6-9). Two researchers observed each classroom for two block periods, or three single periods, keeping field notes as data. We also audio-recorded the work of small groups ranging from pairs to groups of four, and five to fourteen student groups per classroom. The observed lessons were conducted just as the teacher planned, with no intervention or expectation other than the inclusion of small group work.

Our analysis in this phase considered the functioning of each small group in light of the three frameworks underlying high functioning group work. Our intent was to qualitatively characterize the work of student groups using the main concepts of the three frameworks. The results presented below illustrate an example of group work that is high functioning when the key aspects of tasks, discourse, and peer culture worked in concert to support students' conceptual engagement with mathematics. We also share examples from our results that illustrate episodes of group work that are lacking on key aspects of each framework. These examples show specific challenges teachers and students encounter when the task, discourse, or peer culture do not support students' engagement with conceptual mathematics. Key evidence is underlined.

Results: Illustrative Episodes of Small Group Functioning

In the episode below, group functioning is limited by the students' actions that reduce the demands of the *mathematics task* of analyzing relationships among lines in graph and equation form. Within this group, the students are actively avoiding challenge and meaning making the teacher seeks. The group is "on task," but it is not the conceptual task that was intended.

T: Tell me something about the lines. I'm talking about the lines.
S2: They're the same lines, just that one is in a different spot, because the y-intercept and …
T: What do you mean by …The same line usually means one on top of the other.
S1: The same line means they have the …
S2: Identical lines. Identical. That doesn't mean on top of each other, it means they're visually the same. There you go.
T: Well, then all lines are the same, they're just rotated and moved. All lines are the same.
S1: <u>Are you nitpicking on purpose?</u>
T: No, I'm just trying to get some …
S2: <u>You're doing it on purpose.</u> *S1:* Yeah.
T: Some other differences or comparisons.
S2: <u>It's entirely on purpose. You're just trying to make us think harder than we have to.</u>

The *peer culture* of the group in the next episode features a devaluing and dismissal of one student's attempted contributions toward a problem of finding surface area, leading to limited group

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functioning. Although S3 attempts to share ideas relevant to the group's work, the other group members refuse to engage with this student's efforts. In this case, two of the students shut down what otherwise might have been a more inclusive and productive group effort.

> *S4:* What other boxes did you guys want to try? Or do you not trust me on the surface area?
> *S5:* I want to try the stack.
> *S6:* (*to S4*) I don't trust you.
> *S5:* (*to S4*) Well, since you didn't write any of your work down to show us.
> *S4:* Look, I wrote it down.
> *S5:* Ok, that is like in a very small …
> *S4:* (*Points to work to show groupmates*) Is that a 19 or a 190?
> *S6:* (*Flatly*) It's a 19.
> *S4:* (*Frustratedly*) It's not a 190? How is that not a 190?

Although engaged with the task of finding surface area of a cylinder, the group's *mathematics discourse* in the following episode remains superficial. The discussion is confined to the procedures of the task, without addressing justification or meaning for their work. The students may have deeper understandings of the task, or may have conceptual questions about the mathematics content, but the conversation provides little evidence of these understandings or questions, and consequently, minimal opportunity for the group to engage with them.

> *S7:* Did you get 12.6? Not for the answer, for the first [part].
> *S8:* For the first set, yeah.
> *S9:* Wait, I got 1102.14.
> *S8:* That's what …
> *S7:* No, that's what I got [for the final answer]. Mine was for the first step.
> *S8:* High five. You are awful at high fiving. So then, times 2 right?
> *S9:* You don't … no. No, it's done.
> *S7:* You have to find the whole surface area. S9: Nawww.
> *S8:* No, see you have to do this times two, because there's two. Then you have to find the circumference.
> *S7:* You have to find the circumference.
> *S8:* No, then you do the circumference of the whole thing, and then times it by 100.
> *S9:* Wait, are we done?
> *S8:* (*Sarcastically*) Nooooo.
> *S9:* We still have more?

In contrast to the other episodes, the students in the group episode below are grappling with the demands of a task to find the surface area of a tube (a cylinder with a concentric cylinder cut out). They show clear concern for justification and meaning of their work. The peer culture supports an inclusive process in which students are comfortable sharing ideas, and seeking help to understand which student's idea is appropriate for the task. The group's discourse includes evidence of explanations, questions, listening, and use of multiple modes of communication geared toward developing a solution that reflects shared understanding of the mathematics.

> *S10:* But we have to like … don't we have to subtract the bigger circle from the smaller circle? Right? To be able to do … ok.
> *S11:* No, because we count the inside circle and the outside of the circle.
> *S10:* No, but like the outside circle, since there's nothing there, so we don't count that as the surface area. So we have to subtract the …
> *S11:* No, we don't subtract.

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S10: We have to subtract the area from the smaller circle.

S11: We add it.

S10: Why would you add it?

S11: Because, you want to find the whole entire surface area. So that counts the outside and the inside. So this is surface area (*outside face*), and this is surface area (*inside face*).

S10: Yeah, but I'm talking about the empty circle that's there. You don't add that in.

S11: It's surface area; it's not volume.

S10: Oh … no. But like, the empty circle. There's nothing there to like … I mean …

S11: See, this is counted as surface area, and around the shape is counted as surface area.

Conclusion

Small group work integrates a mathematical task, a peer culture, and student discourse. Naturalistic data from middle school mathematics classrooms demonstrates that when these three dimensions are of high quality, small group work can support deep engagement with meaningful mathematics, and peer support for learning. Yet, other episodes reveal that when any single dimension is out of balance or of poor quality, group work suffers. Key next steps involve investigation of elements of lesson design and teaching moves that strengthen quality and support balance across the three dimensions to support higher group functioning.

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ENACTING MATHEMATICAL ARGUMENTATION: MARILYN'S STORY

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Changing mathematics instruction to meet the expectations of the Common Core State Standards is challenging. This case study examines one teacher, Marilyn, who endeavored to promote mathematical argumentation as described in the CCSS. The results demonstrate that this requires an understanding of what mathematical arguments are as well as using effective pedagogical moves. A teacher may have a strong vision, but developing the kinds of questions and prompts to move to that vision is challenging. We can learn a great deal from understanding Marilyn's successes and challenges as she worked to enhance her practice.

Keywords: Instructional Activities and Practices, Classroom Discourse

Introduction

The Common Core State Standards in Mathematics (CCSSM)(NGA & CCSO, 2010) require significant changes in the way mathematics is taught in the United States. The eight Standards for Mathematical Practice (SMP) delineated in the CCSSM describe the mathematics proficiencies and habits of mind necessary for *all* students to be competitive in post-secondary education and in the 21st-century workplace.

To realize these outcomes, teachers need support in understanding these proficiencies and in implementing new instructional strategies. This case study is the story of one 6th-grade intervention teacher, Marilyn, who enacted learning from a professional development program that targeted the understanding and implementation of SMP #3. Marilyn's goal was to help her students improve their ability to develop mathematical arguments and critique the reasoning of others, both verbally and in writing. Individual stories such as this capture the complexities of implementing change, a slow and – at times – inefficient process. Specifically, the research question asks: *What are some of the successes and challenges teachers encounter when working with students to develop valid arguments and critique the reasoning of others?*

This case study was conducted as part of a larger 15-month professional development project – Bridging Practices Among Connecticut Mathematics Educators (BPCME). Forty teachers in grades 2-11 participated in BPCME, which involved 80 hours of workshops and monthly meetings throughout the school year. An important focus of the PD was developing a deep understanding of argumentation by analyzing student work, viewing classroom videotapes, and participating in a variety of interactive activities to establish a collective understanding of mathematical argument. Based on the Toulmin model (Toulmin, 2003) the Project established that an argument needs a stated claim, mathematical evidence, and a warrant; a good argument logically connects the evidence to the claim in a way that is accessible and understood by the community. The participants collaborated throughout the school year to support enactment.

Prior Literature

This study is grounded in situated learning theory (Lave & Wenger, 1991), focusing on learning as embedded in a sociocultural context. The learner does not gain "a discrete body of abstract knowledge which (s)he will then transport and reapply in later context. Instead, (s)he acquires the skill to perform by actually engaging in the process" (p. 14). In teacher learning, the classroom is viewed as the primary site for instructional development (Cobb & Bowers, 1999).

Marilyn participated in a community of practice with other teachers during a professional development program and worked to improve her own instruction by developing a community of

practice in her classroom. However, expecting elementary school children to support each other's mathematical understanding in a community of practice is problematic. For example, when Nathan and Knuth (2003) studied a 6th-grade teacher as she attempted to enact reform math teaching strategies, they found that during her first year she mostly provided analytic scaffolding of mathematical ideas. This reinforced her role as the "expert" and didn't give students the chance to construct their own learning. To encourage her students take more responsibility, she used more social scaffolding of norms for social behavior and discourse expectations in the second year. However, without the teacher guidance, student-led discussions lacked the mathematical precision and depth previously provided by the teacher. Other researchers (e.g. Cobb, Wood & Yackel, 1993; Forman, Larreamendy-Joerns, Stein, & Brown, 1998) also documented challenges in the enactment of reform mathematics. It is critical to share individual teacher's experiences, since each teacher, classroom, and school is unique. More research is needed concerning the day-to-day reality of enacting new mathematics practices.

Context: The Case Study Teacher

Marilyn was a math interventionist at Belmont School, located in a semi-urban district in the northeast. Of the 406 students enrolled at Belmont, 252 are students of color (62%). Marilyn is an African-American female who taught elementary school for 14 years in the district before coming to Belmont. As an interventionist, she co-taught and provided small group instruction to students identified as needing extra support. She sought to create a supportive environment so that her students felt safe trying different strategies to solve problems. Marilyn had high expectations for her students as evidenced by her expectation that even though they needed extra math support, they were capable of developing mathematical arguments.

As a case study teacher, Marilyn collaborated closely with me during an intervention class of six students that met for 45 minutes weekly for ten weeks. Her goal was to apply her learning from BPCME to improve students' confidence about mathematics, get them to explain their thinking, and eventually develop mathematical arguments both verbally and in writing.

Methodology

During our collaboration, Marilyn was interviewed three times (pre, mid- and post-interviews), lessons were observed and videotaped eight times, and student work was collected. In addition, the data collected from the summer workshops were available: pre- and post- Learning Mathematics for Teaching (LMT) assessments (Hill, Schilling & Ball, 2004), confidence surveys, teacher analyses of student responses to argumentation questions, teacher solutions to math content problems, reflections, and exit slips. Because this paper focuses on Marilyn's challenges and success as she implemented her new learning, the analysis draws heavily on the interviews and classroom observations.

The first step of the data analysis process was coding transcriptions of the interviews and eight lessons to better understand Marilyn's successes and challenges in promoting mathematical argumentation. I developed *initial codes* (Saldana, 2012) to document the strategies she used to support discourse: Positive Reinforcement, Sharing Mathematical Solutions, and Prompts to Explaining Reasoning, Communicate Mathematically, and Explain Others' Solutions. In her own way, Marilyn had *deconstructed* (Grossman & Shahan, 2005) argumentation into smaller components, or sub-practices, that supported discourse, though not specifically argumentation.

The next step was to develop three *second-order codes* that targeted the specific sub-practices of argumentation that were discussed in BPCME. I re-coded the transcripts to identify Marilyn's prompts and questions that (1) pressed for the development of a logical chain of mathematical reasoning; (2) pressed for student explanations of *why* mathematics supported the claim rather than telling the step-by-step procedures; and (3) required students to evaluate other students' arguments.

At the same time, I developed several *time-ordered matrices* (Miles & Huberman, 1994) to track changes in quantity and quality of Marilyn's questions and in student discourse.

Results and Discussion

Marilyn took what she learned in BPCME and developed manageable practices around discourse. However, teaching in ways that resonated with the BPCME vision of mathematics was not easy for her. During the ten-week observation period, she experienced several successes and faced several challenges. In the end, she did not move her students to meet the CCSSM standard of constructing viable arguments and critiquing reasoning of others.

Marilyn's greatest success was that she developed a trusting environment in which her students were comfortable expressing their mathematical thinking. Considering that her students were identified as needing intervention, this is no small accomplishment. Based on the initial codes, over time there were more instances of students volunteering to share without needing prompting and fewer instances of Marilyn providing positive reinforcement. By the end of the ten weeks, all but one (Tina) regularly presented his or her work publicly. Most striking was the change in Corey, who was very quiet in the beginning but became one of the primary presenters.

Through her prompting, Marilyn also made students aware of several mathematical practices: finding more than one method to solve a math problem; communicating clearly so that others understand; and being aware that stating a step-by-step procedures is not an explanation of why. Perhaps, if given more time, some of these practices would have become part of the students' repertoire.

Marilyn also faced several *challenges*. Thinking of the appropriate questions and prompts to get students to develop logical arguments and be astute enough to critique others is challenging. In the second-order coding I looked for class argumentation and commented on what was present and what was missing. I found instances where Marilyn tried, but didn't ask questions that really pressed students deeply. As demonstrated in one observation towards the end (5/18), Marilyn was still unable to get students to critique each other's logical reasoning.

> *Marilyn*: So how many of you solved your problem like Corey?
> *Jonathan*: I did 4 X 4.
> *Marilyn*: Did you do like Gerry did?
> *Jonathan*: I guess it's like Gerry's but without fractions.
> *Marilyn*: Can you show us yours then? (Jonathan shows his work.)
> *Marilyn*: So is there more than one way to solve this problem?
> *Students(chorus)*: Yes.

This is a typical example of Marilyn prompting for logical reasoning, but then being satisfied having students present different solution strategies. Developing questions to prompt for mathematical argumentation is complex. One factor that might have made it particularly difficult for Marilyn is that, as stated in her first interview, she struggled with math as a child. Although her MKT score increased as a result of her participation in BPCME, perhaps she was uncomfortable pressing students since her own depth of understanding was fragile. Another potential factor was that she had previously taught literacy and tended to incorporate literacy strategies into teaching math. For example, the time-ordered matrix demonstrated that throughout the ten-week study, 62% of Marilyn's prompts used social scaffolding (Nathan & Knuth, 2003), which successfully promoted student discourse. However, social scaffolding does not necessarily promote the kind of discourse necessary for justification of mathematical arguments.

Moreover, Marilyn's teaching environment presented logistical challenges. The study was brief—changing teacher practices cannot reasonably be expected to occur in ten weeks. Also, there were multiple conflicting initiatives pressuring teachers at Belmont School, making it difficult to

schedule PLCs. Consequently, Marilyn had little opportunity for collegial sharing with other BPCME teachers that may have provided her with further support.

The combination of the complexity of the kind of change Marilyn was attempting and the outside factors that existed made it difficult for her to fully accomplish her goals. However, during her final interview, she reported that the project was a valid learning experience that she felt made her a better teacher. She understood what a mathematical argument was and knew where she wanted her students to be, even if she couldn't quite get them there yet.

Implications and Conclusions

In most classrooms, students are not challenged to justify their thinking, develop arguments, and critique the reasoning of others. Few teachers are prepared to support their students to develop this practice. More research is needed that describes real classrooms when teachers undertake this daunting task. Understanding argumentation is necessary but not sufficient; teacher educators need to be mindful of questioning skills to press for logical reasoning. The more we know about individual teacher's experiences, the better equipped teacher educators will be to create significant learning experiences that will lead to enactment.

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USING A VIRTUAL ENVIRONMENT TO UNCOVER BIASES, ASSUMPTIONS, AND BELIEFS AND IMPLICATIONS FOR MATHEMATICS TEACHING

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As the population of the United States becomes more diverse, there is a pressing need to build teacher education courses to better prepare elementary teachers to teach students from diverse backgrounds in equitable and culturally responsive ways. In this pilot study, we examine how activities designed for a course within a virtual world environment for elementary education pre-service teachers supports pre-service teachers to uncover their biases, assumptions, and beliefs about students from different social identities. Our findings reveal the pre-service teachers were very receptive to learning in the virtual world about diverse learners, and they openly discussed their biases and assumptions about students.

Keywords: Teacher Education-Preservice, Instructional Activities and Practices, Equity and Diversity

Introduction

As the population of the United States becomes more diverse, there is a pressing need to build teacher education courses to better prepare elementary teachers to teach students from diverse backgrounds in equitable and culturally responsive ways. To address this need, we designed a course for elementary education majors that uses a virtual environment populated by students of diverse social identities to support pre-service teachers' (PSTs) understanding of individual differences and diverse cultures and communities. Although research indicates that teaching can be deeply resistant to change (Kennedy, 1999), recent research suggests that practice-based strategies can support change in teachers' pedagogies (Lampert, Beasley, Ghousseini, Kazemi, & Franke, 2010; Windschitl, Thompson, & Braaten, 2011). According to the InTASC standards, "Teachers must have a deeper understanding of their own frames of reference (e.g., culture, gender, language, abilities, ways of knowing), the potential biases in these frames, and their impact on expectations for and relationships with learners and their families" (p. 3). Therefore, within the course, PSTs are provided opportunities to (1) develop awareness of their hidden biases and stereotypes of who can and cannot do mathematics and science and (2) rehearse alternative responses and practices that serve students of diverse social identities more equitably. Prior to fully implementing the course with our pre-service teachers, we piloted activities we designed to examine how the activities help unmask PSTs' hidden biases. Consequently, the purpose of this study is to examine how the virtual environment can be used to support PSTs to uncover their hidden biases, assumptions and beliefs about diverse students from different social identities.

Theoretical perspective

In the quest to not only integrate equity issues into the course curriculum but frame it as a vital component of teaching mathematics, we drew on Villegas and Lucas' (2002) vision of culturally responsive teachers. They describe culturally responsive teachers as: (a) being socioculturally conscious, (b) having affirming views of students from diverse backgrounds, (c) seeing themselves as responsible for and capable of bringing about change to make schools more equitable, (d)

understanding how learners construct knowledge and are capable of promoting knowledge construction, (e) knowing about the lives of their students, and (f) designing instruction that builds on what their students already know while stretching them beyond the familiar. While the course incorporated each of these components, for the purpose of this study, we focused on how PSTs develop affirming views of students from diverse backgrounds. Teachers who have affirming views of students from diverse backgrounds "acknowledge the existence and validity of a plurality of ways of thinking, talking, behaving and learning" (Villegas & Lucas, 2002, p. 23). Teachers' views of students affect the learning opportunities of students and affirming teachers positively affect student achievement.

Course Context

The researchers developed a course that integrated the virtual world and traditional environment to prepare PSTs to teach mathematics and science in culturally responsive ways. Through combining virtual and traditional environments, the course "authentically" engages PSTs in practice-based rehearsals with diverse learners in ways that are typically not possible in traditional courses. The virtual school environment was constructed using OpenSim, a free, open source multi-platform 3D virtual world. For the course, the researchers developed twenty student avatars (visual representations of a person in a virtual environment) and corresponding background case profiles, which represented a diverse set of students' social identities. The PSTs had the opportunity to assume the "identity" of their assigned student avatar in the course. In addition, the PSTs designed their own "teacher" avatar, which was a virtual version of themselves, which reflected the social identities most salient to them as teachers. Throughout the course, the PSTs (1) unpack their own biases, behaviors, and assumptions related to students' mathematics and science identities; (2) analyze how their instructional decisions support or hinder mathematics and science learning for students of all social identities; (3) establish classroom norms that encourage mathematics and science participation for students of all social identities; (4) incorporate students' everyday understandings and experiences in ways that support their mathematics and science identity construction; and (5) use discursive practices such as eliciting student thinking and drawing on students' funds of knowledge that have been linked with increased learning among students from non-dominant groups. By participating in the course, PSTs will develop a deeper understanding of and be better prepared to meet the needs of all the students in their classrooms.

Methods

This preliminary research took place during the Fall 2015 semester at a university located in the Midwest region of the United States. The researchers recruited three undergraduate students who were majoring in elementary education and seeking either a mathematics or science endorsement. Prior to participating in the study, the undergraduate students completed an elementary mathematics methods course and a course in multicultural education. The three participants were white females who would student teach the following semester. For this study, we piloted activities and assignments that will be used in a course we developed for elementary PSTs.

Data Collection and Analysis

The undergraduate students (henceforth referred to as PSTs) participated in five two-hour video-recorded sessions. During the sessions, the undergraduate students (1) learned to navigate their avatar in the virtual world, (2) designed their teacher avatar in the virtual environment, (3) assumed the "identity" of an assigned student avatar, and (4) participated in a classroom teaching scenario as their student avatar in the virtual world. After each session, we had a debriefing session with the PSTs where they reflected on the activities and shared their thoughts and ideas on the assignments and ways to improve it. Two stationary cameras captured how the PSTs engaged in the activities and

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their dialogue as they engaged in the activities. In addition, the PSTs wore camera glasses to individually capture their interactions in the virtual world.

The PSTs were assigned a student avatar and were provided a case profile of their avatar. The case profile included the student's (1) name, (2) age, (3) race, (4) ethnicity, (5) sex, (6) socio-economic status, (7) faith/religion, (8) (dis)ability, (9) parental employment, (10) first/home language, (11) family structure, (12) household information, (13) student quotes about schooling/living experiences, (14) family/community cultural practices, (15) family or community history, (16) educational history (IEPs, test scores, experiences with other teachers, etc.), (17) parental involvement, (18) information about previous mathematics and science learning, (19) former teacher gossip, (20) interests and experiences outside of school, and (21) personality. The PSTs read through their case profiles, logged into the virtual world where they saw the representation of their student avatar, and discussed their initial thoughts and perceptions of their new identity. The PSTs used Jing within the virtual world to record a 3 - 5 minute autobiography of themselves as the student avatar. Within their autobiography, the PSTs addressed the following prompts: (1) Who are you?; (2) What three words describe you and why?, (3) What is a typical weekday in your home?; (4) What is a typical weekend?; (5) How do you feel about mathematics and science and why?; (6) What do you find easy about learning mathematics and science, and what do you find challenging about learning mathematics and science?; and (7) What do you want me to know about you?

Two of the researchers individually analyzed the video recorded sessions and recorded autobiographies. We individually wrote analytic memos (Maxwell, 2005) based on the data. We then identified patterns within the data related to the PSTs' biases, assumptions, and beliefs about students with different social identities.

Results

Prior to receiving the case profile on their student avatar, the PSTs had an opportunity to discuss their initial assumptions and beliefs about the student based solely on the student's name and what the student looked like. For example, when the PST saw Alejandro as an avatar, she was initially confused because she did not expect him "to be a white boy with red hair." She assumed Alejandro's name sounded Hispanic, and she had a preconceived image of what Alejandro would look like as a Hispanic boy, which was not representative of the avatar. Moreover, the PST noticed Alejandro was dressed in overalls. She was perplexed that a kid who looked Caucasian with red hair wore overalls. Furthermore, she thought only farmers and African Americans wore overalls. But, she soon refuted the notion that Alejandro was a farmer because he only wore one strap, and "farmers would not necessarily wear one strap on their overalls because they are wearing them for a purpose." In actuality, Alejandro was a biracial (African American and Filipino) nine year-old from a middle class family who lives in an apartment near the elementary school. Both of his parents were college graduates, and his mother was a trained nurse from the Philippines.

After the PST received and read the case profile of Lupe, she researched Lupe's cultural heritage to provide additional information to her autobiography. She knew that Lupe was an eight-year old Hispanic Latina whose primary language was Spanish. Her family was from Mexico, and her father was undocumented. Her family's socio-economic status was in the lower income bracket. She liked mathematics, but she struggled with it; and she was not confident in her problem solving skills. In Lupe's autobiography, the PST explained that it was customary for her and people from her culture to be fashionably late. She commented,

> In my research I focused on her Latina and Mexican culture so the site I found said not being on time was socially acceptable, which I never really thought about it, but I work with someone who is and we do something and she always show up 20 minutes late.

Since the PST experienced this assumption, she generalized it to be true for the entire culture.

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As researchers, we noticed particular items Lupe did not discuss in her autobiography. Similarly, we noticed how the PST who assumed the identity of Melissa did not include specific details in her autobiography, which may factor into the PST's biases and assumptions about Melissa. Melissa was a ten-year old Native American whose ethnicity was Meskwaki. Her family was very close, and her extended family lived on the Meskwaki Settlement. She attended tribal school on the Settlement during kindergarten - second grade. When she transferred to her current school, she was tested and identified for special education services. Her teacher said she was a "good kid, but very quiet and a slow learner." After reflecting on her autobiography, the PST realized Melissa's case profile did not say she "did not like this new school." The PST acknowledged that it was "something I assumed because she has been struggling and she has not made any new friends at the new school." In addition, the PST discussed her (Melissa's) religion of Christianity in great detail, but the case profile also mentioned the Meskwaki religion was extremely important. This was not discussed in the autobiography.

Discussion

Although the PSTs were familiar with playing video games like Sims, they were not accustomed to seeing characters/avatars that were non-white or avatars in wheelchairs or on crutches. One PST stated, "It [avatars of color and avatars in wheelchairs] was something I definitely did not expect to see." Through engaging in the pilot activities the PSTs openly shared their biases, assumptions, and beliefs about students from various social identities. These biases, assumptions, and beliefs influence how teachers and PSTs instruct mathematics to their students, which ultimately impacts students' learning of mathematical content. It is important that we, as mathematics teacher educators, take the time and engage our PSTs in tasks and activities that give them the opportunity to expose their biases and beliefs so it does not negatively affect their teaching practices. Once their biases and assumptions are revealed, and the PSTs acknowledge their biases, the PSTs will then be able to develop an affirming view of students from diverse backgrounds. This in turn, will help PSTs develop deeper understandings of and be better prepared to meet the needs of all students in their mathematics classrooms.

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STRUCTURAL CHANGES IN COLLECTIVE ARGUMENTATION IN A COMMUNITY OF MATHEMATICAL INQUIRY

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This report describes an inquiry into the process of argumentation in a community of mathematical inquiry format. The study explores changes in the structure of argumentation with logical-mathematical problems in a dialogical group setting with 21 fifth grade students. It concludes that the process exhibits an alignment of group members' participation in the collective argumentation process, suggesting a process of internalization, which the author illustrates through an analysis of the argumentation macrostructure and its changes over the course of one year.

Keywords: Classroom Discourse, Instructional Activities and Practices, Reasoning and Proof

Background

Argumentation, here denotes a social activity in which subjects engage each other with the goal of coordinating different perspectives through justification of their assertions. An important assumption is that argumentation implies a dialogical dimension, for dialogue is the condition for the emergence of new perspectives within and between the participants (Forman, Larreamendy-Joerns, Stein, & Brown, 1998). From a dialogical point of view, engagement in an argumentation discourse obliges the participants either to justify their positions and to examine their claims in the light of the claims of others, or to balance both justification and negotiation processes in a third way (Leitão, 2000). Both processes prompt awareness of oppositions between the views or beliefs of participants, and reflection on the information they are provided with.

Numerous studies suggest that the experience of being exposed to conflicting views in a context of argumentation leads to significant restructuring of participants' understanding of a topic (Forman et al., 1998; Leitão, 2000; Van Eemeren & Grootendorst, 1994; Krummheuer, 1995; Resnik, Salmon, Zeitz, & Wathen, 1993; Pontecorvo, 1993). Other researchers note that examining opposite sides of an argument does not always lead the participants to a change of views, but rather to further polarization (Stein & Miller, 1993; Perkins, Allen, Hafner, 1983; Kuhn, 1991). Toulmin (1969) offers something of an explanation of this discrepancy by emphasizing the importance of developing rational canons and proper habits of inference, which can serve as stepping-stones for argument-building mechanisms.

Attempts have been made to analyze the different patterns and mechanisms of intellectual exchange that occur during argumentation. Toulmin (1969) has proposed that the "layout" of an argument—data, warrant, backing, conclusion--represents an ideal model of substantive argumentation, and provides a tool for the analysis of argument structures. However, some studies suggest complex relationships between the structure of an argument and the functional aspects of argumentation (Krummheuer, 1995), between argument structure and the nature of the situation in which the collective argumentation happens (Resnick et al., 1993), and between argument structure and the distributed thinking among the participants, who set the shared framework for interpreting an argument (Forman et al., 1998; Inagaki, Hatano, & Morita, 1998). Reflection upon these complex issues poses a methodological question not only as to how to set up a systematic and meaningful unit for the analysis of argumentation itself and changes in its structure.

Most of the research done on students' reasoning and argumentation (e.g. Pontecorvo & Girardet, 1993; Resnick et al., 1993) tends to restrict its sphere of examination to so-called argumentative operations—namely justifications, concessions, oppositions, and counter-oppositions, all based on an

argumentation model offered by Toulmin (1969). This study operates under the assumptions that argumentative operations, although they are the major constructive elements of reasoning, are not the only critical moves involved in the process of argumentation, and that in fact other critical moves contribute to the process on an essential level. The study suggests that in the course of the group inquiry throughout the year, participants begin to align their individual arguments and to use auxiliary operations that help build more complex argument structures.

Research

The research study here described was conducted in a self-contained fifth grade classroom in a suburban public elementary school in northern New Jersey over the course of one year. The 21 students involved in this project were 10 and 11 years old. The study was intended as primarily qualitative and naturalistic in nature. More specifically, I adopted a "grounded theory" approach (Glaser & Strauss, 1967) in order to analyze the structure of argumentation process, and trace how it evolves in and through argumentation.

In keeping with grounded theory, I examined the argumentation process—comprised of a sequence of arguments--and took the argument itself as a unit of analysis in order to explore patterns of change. An argumentation process may consist of one or more single arguments. This complex unity of individual arguments may be thought of as the *macrostructure* of the process. Nor are arguments the only building blocks of the larger structure—they are accompanied by other functional elements such as explanations, clarifications, reformulations, summarizations, questions, etc., that facilitate the formation of the single arguments and their sequences. My research question was concerned with the analysis of and changes in the macrostructure of argumentation in a community of mathematical inquiry. The analysis focuses on the macroscopic argumentation structure, i.e. the pattern of organization and the sequence of the single arguments that comprised the argumentation process.

The exploration of the argumentation macrostructure is based on a comparison of two samples of transcripts. The first four transcripts from the whole series of conversations were selected to comprise the first sample, and the second sample is a collection of the last four transcripts. The categories developed through an open coding of the transcript yielded a picture of the changes in the argumentation structure as progressing from argumentation with a mostly chaotic structure of disconnected arguments towards a structure exhibiting argument alignment schemas, i.e. individual arguments aligned through various links. These argumentation schemas were further "saturated" by searching through the two samples of transcripts—that is, by looking for instances representing the respective schemas.

Analysis

The macrostructure of the first sample displayed a notable tendency toward dissociation in the sequencing of arguments. The argumentation sequence is marked by strings of isolated claims--most of which are enthymemes, which are not connected—i.e. either derived from or referred to—to any of the others. There were no links between them--the macrostructure was characterized by, not a logical sequence, but a chaotic structure of disconnected arguments.

Overall, the profile of the first sample of transcripts displays a macrostructure composed of a series of single arguments, some of which are distinctly dissociated from their ancestors and predecessors. A significant number of consecutive arguments lack any explicit or implicit links between them, which rendered the argumentation sequence chaotic and difficult for the inquirers to follow. It makes the argumentation dependent on the "translation" and "bridging" function of the facilitator (Kennedy, 2009). The chaotic argumentation structure is isomorphic with the chaotic functioning of the group inquiry process, triggering attempts by the facilitator to impose order on the argumentation process.

The macrostructure of the second sample of transcripts exhibits quite a different organization of the argument sequence. Close scrutiny of the latter discloses a new pattern in student responses—that is, aligning their arguments with those preceding them. This new pattern is broadly reminiscent of what Pelerman & Olbrechts-Tyteca's (1969) call "argumentative association," which refers to the identification of an associative relation between two elements that were previously regarded as disconnected. In this case an associative intervention of this type will be identified as an *argument alignment schema*. Two types of argument alignment schemas can be identified in the data. The first type pertains to what will be referred to as *alignment with agreement* with the previously presented argument.

The argument alignment schema can display different structures. First, a process of argument alignment takes place—or a linkage of the argument to be advanced with an already proposed argument. Second, this alignment might be further coupled with a restatement, a clarification (unpacking), or an amplification of the already proposed argument. Below is a brief example of *alignment with clarification* found in Sarah's response (albeit incorrect) in the following excerpt from a discussion about a chiming clock[1].

> *Rashaad*: If you just do 6 times 2 is 12….if you just double….and five times is 10, so…uh, it will be 10 seconds.
> *Sarah*: I agree with Rashaad. Here are 5 fingers and only 4 spaces. [She uses her fingers] How many spaces are there between 6 fingers? [She holds her fingers wide open] That's 5. If you have another 6 [strokes], that will be 5 more spaces, so there will be 10 spaces altogether.

We found that the use of these different schemas contributed to building a serial argument which was in turn composed of several subarguments (or "stages"), in which the conclusion of one served as a premise for the subsequent subargument. It appears then, that the use of the alignment argument schema in its different modalities--the simple, the clarification, and the amplification schema-- may function as a springboard for the argumentation building process.

Another type of alignment argument schema identified in the data will be referred to as *alignment with disagreement*. On such occasions, the student most often signalled her alignment with the phrase "I disagree with." This alignment with disagreement schema differs from alignment with agreement in that it usually bridges two or more often contradictory or divergent arguments--the argument presented by the speaker and the one already presented—and a new conclusion is often juxtaposed with the previous conclusion. The new argument may be presented as counter to the previous argument, but it functions as an intersection that forces the group to make a decision as to which argument is sound. The next example shows an argument with alignment with disagreement that follows soon after Sarah's intervention.

> *Vincent*: I disagree. Some people were saying that it takes 10 sec, but they say 5 sec. here and then another 5 sec. here *separately*, …and don't count the second in between. If you count it … it would be 11 seconds.

Conclusion

In summary, it would appear that the "ideal" structure of both types of alignment argument schemas includes: 1) the signaling of alignment by "agreement" or "disagreement" by a key phrase; 2) a restatement which may include clarification of all or part of the previous argument; 3) the presentation of a new argument consistent with the previous one, which may a) strengthen, amplify, or extend the previously stated conclusion, b) the advancement of a counter argument which leads either to a refutation or a modification of the previous argument, or c) a synthesis of a) and b).

The study's findings suggest the presence of an "invisible," spontaneous at first, and inherently educable collective argumentation structure in group deliberation of this kind, and identifies patterns of change in that structure.

Endnotes

[1]The Chiming Clock problem reads as follows: A clock strikes 6 times in 5 seconds. How long would it take the clock to strike 12 times?

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PRESERVICE ELEMENTARY MATHEMATICS TEACHERS' DECISION MAKING: THE QUESTIONS THEY ASK AND THE TASKS THEY SELECT

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Preservice elementary mathematics teachers' choice of questioning sequence in a hypothetical representation of practice was examined in comparison with their choice of follow-up task. Findings indicate that preservice teachers' task choice, and their rationales for task choice, has a statistically significant influence on their choice of questioning sequence.

Keywords: Classroom Discourse, Teacher Education-Preservice

Background and Overview

Teachers' questioning strategies during mathematical discussion strongly influence students' engagement in and understanding of mathematics (e.g., Franke et al., 2009; Hufferd-Ackles et al., 2004). Most recommendations designed for teachers encourage the use of probing questions and sequences that press students to make mathematical meaning. However, Kosko (2014) found that elementary preservice teachers (PSTs) have a stronger preference for prompts that generate discussion (i.e., soliciting statements and descriptions from other students). Although the amount of teaching experience and associated knowledge are potential factors of incluence, recent study of inservice teachers' mathematical questioning suggests a complex interaction between instructional context, pedagogical disposition, and mathematical knowledge for teaching (Kosko, in press). It is probable that a similarly complex interaction occurs for PSTs. The present study seeks to examine a portion of this system by focusing on how variation in mathematical task interacts with the pedagogical decision of question sequence.

The present study positions itself in Herbst and Chazan's (2012) theory of practical rationality. The theory suggests that teachers can justify their pedagogical decisions by appealing to aspects of their individual disposition (i.e., pedagogical knowledge, content knowledge, beliefs) or to aspects that the teacher is socially nested in (i.e., instructional norms, professional obligations). Specifically, teachers of mathematics use their practical rationality to justify their pedagogical decisions. In the case of facilitating mathematical discussions, an elementary teacher must negotiate a social space that includes multiple students with individual needs, but must also take into account the mathematical knowledge at stake. Herbst and Chazan's (2012) adaptation of the instructional triangle offers a useful visualization of part of this interaction with regards to practical rationality (see Figure 1). Teachers may interact with the posing of the mathematical task, but may also operate more directly by supporting the milieu of the task in which the student interacts: where milieu serves as the environment that students interact with to engage in the task. This latter 'support of the milieu' often serves as scaffolding and includes, among other things, teachers' use of questioning. Thus, Herbst and Chazan's (2012) adaptation of the instructional triangle suggests that a teachers' questioning can directly affect the milieu the student interacts with in working on the mathematical task. Yet, the mathematics at stake influences both how the teacher may consider such questions and how the student may interact with the milieu the teacher is scaffolding.
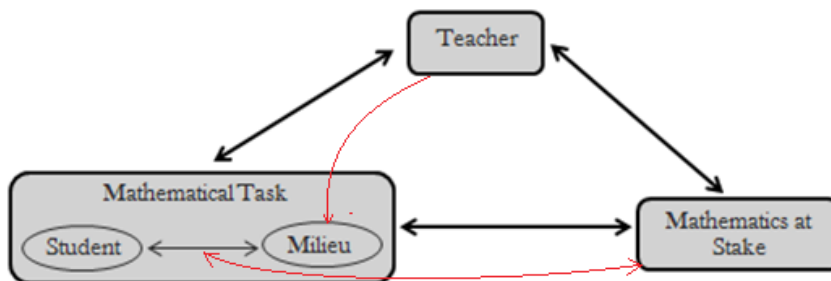
Figure 1. Herbst and Chazan's (2012) adaptation of Cohen et al.'s (2003) instructional triangle.

The study of mathematics teachers' practical rationality for instruction necessarily requires attending to the instructional context. Herbst et al. (2011) describe the use of comic-based representations of practice as one means of providing a common instructional scenario in which multiple teachers may interact. Describing the use of the LessonSketch platform, Herbst et al. (2011) suggest that "comics with cartoon characters offer a semiotic resource that enables a high degree of control…over the possibility of creating alternatives, highlighting issues of tactical decision-making" (p. 94). In the context of the present study, this affordance allows for the presentation of different potential mathematical questioning sequences a teacher may use for a given instructional scenario, as well as the tasks they may pose throughout a lesson. Specifically, the present study presented PSTs with two completed questioning sequences to select following an initial framing of the instructional scenario. PSTs were then presented with the same two options for the mathematical task to follow the questioning sequence choice. Recalling Herbst and Chazan's (2012) adaptation of the instructional triangle (see Figure 1), this approach allows for the examination of how PSTs consider the mathematics at stake through their choice of questioning and rationale for the task they select. Specifically, a PST's choice of a follow-up task in an instructional scenario should logically follow from the mathematics at stake in the lesson at-large, and also be informed by the interaction between the student and milieu of the prior task. Therefore, the choice of questioning sequence used to scaffold the milieu of the prior task should also align with the mathematics considered at stake by the teacher.

Methods

Data were collected from 63 PSTs enrolled in three sections of an elementary mathematics methods course between January and December 2015. As part of the methods course, participating PSTs completed an online experience via the LessonSketch platform. The experience included three components. In the first part of the experience, PSTs reviewed the stem of an instructional scenario in which the depicted second-grade teacher, Mrs. Cox, asks one of her students, Jasmine, to explain why 8+17=10+15 is true. Jasmine replies that "It's true because it's all 50." Prompted for further explanation, Jasmine states that "8 plus 17 is 25. 25 plus 10 is 35. 35 plus 15 is 50." PSTs are then presented with two multi-slide question sequences, along with student responses, summarized in the purpose statements below. PSTs were asked to select the sequence most appropriate to improve students' mathematics learning:

- "The teacher needs to encourage additional student participation to have other strategies for how the problem was solved and compare those strategies."
- "The teacher needs to press Jasmine to explain her thinking more clearly until Jasmine recognizes something is not working with her explanation."

Each above statement was followed by three slides depicting the teacher's prompts and student responses. Following the choice of questioning sequence, PSTs were then asked to select one of two

tasks (see Figure 2). Finally, PSTs participated in a forum in which they shared their rationales for their choices and discussed the pedagogical "pros and cons" of the different options available.

Task 1:
  "Using Cuisenaire rods along a double-sided number line, have students create a number train to show 8+17 and another train to show 10+15."
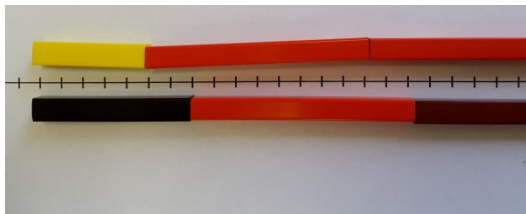
Task 2:
  "Using a number balance, have students represent 8+17 on one side of the balance and 10+15 on the other side."

Figure 2. Descriptions of mathematical tasks following decision of questioning sequence.

PSTs' choice of questioning sequence and their choice of manipulative-based task were compared using a Chi-Square analysis. Table 1 presents the resulting contingency table with observed counts in normal text and expected-by-chance counts in italicized text. Results suggest that PSTs' choice of questioning sequence and manipulative-based task was not independent from chance ($\chi^2(df=3)=6.00$, $p = .014$). Examination of Table 1 indicates that PSTs who chose the "press for meaning" sequence were relatively more likely to choose the double-sided number line than expected by chance, and less likely to select the number balance than expected by chance. Likewise, PSTs who chose the "generate discussion" sequence were relatively more likely than expected by chance to choose the number balance, and less likely to choose the double-sided number line.

Table 1: Contingency table for tasks and question sequences

	Number Balance	Double-Sided Number Line with Cuisenaire Rods	Total
Generate Discussion *multiple strategies*	40 *36.5*	6 *9.5*	46
Press for Meaning *single strategy*	10 *13.5*	7 *3.5*	17
Total	50	13	63

A Systemic Functional Linguistics approach (SFL) was used to examine PSTs' rationales for their choices ($n=18$). Following Chieu et al. (2015), PSTs' rationales for decisions were identified following logical indicators for conjugation (because, so, if-then). Rationales for task choice and for questioning sequence were both examined. Regarding rationales for task choice, four themes emerged from analysis across posts: the choice of task aligns with the mathematical meaning for the equals sign ($n=17$); the choice of task accounts for composing/decomposing numbers ($n=4$); the choice of task provides a visual/concrete representation of the mathematics ($n=8$); and the choice of task accounts for numeric value ($n=4$). Next, data were quantified to examine the statistical effect of PSTs' stated rationales on the choice of task. A lambda statistic was calculated for each rationale. Only one lambda statistic was found to be statistically significant. Specifically, the presence of rationales to *account for the composing and decomposing of numbers in the equations* reduces the

error in predicting the choice of selecting a double-sided number line with Cuisenaire rods by 57% (L=.57, p=.023). Thus, PSTs who identified the mathematics involved in recognizing number combinations were more likely to choose the double-sided number line. For example, one PST stated that having students use Cuisenaire rods requires students "to break the numbers further apart (ex: 17= one 10 rod, and one 7 rod) and that would require background knowledge of breaking a whole number apart into smaller sums to make up the number."

Rationales for questioning sequence were examined with two common themes emerging: the choice of question sequence facilitated Jasmine's individual needs, or the choice of question sequence facilitates the social space of the mathematics discourse. Interestingly, these rationales were found to be statistically similar across question choice. Thus, in the examination of PSTs' rationales for their choices, the only rationales with predictive power were those related to the mathematics pedagogical features of the manipulative-based tasks, and the mathematics each afforded. Confirmatory evidence for this finding comes in the form of four PSTs' explicit change of choice in the forum discussions to change from using the number balance to using the Cuisenaire rod activity. Specifically, conversation in the forum suggested that four these PSTS, composing/decomposing number was an important facet for understanding equivalence. By contrast, no PSTs identified a change from using Cuisenaire rod activity to the number balance activity.

Discussion and Conclusion

The findings from the present study suggest that certain mathematics considered by PSTs to be at stake directly affects the tasks posed. Further, the mathematics at stake may indirectly, but significantly, affects the questions posed by PSTs. In the case of the present study, PSTs who considered composing/decomposing of number were proportionally more likely to select a probing sequence instead of a generate discussion sequence. However, findings suggest that other justifications, both of the mathematics and the questioning sequence, were statistically similar across PSTs' decisions. Thus, PSTs' mathematical questions may be predicted, and facilitated, by the mathematics considered at stake in instructional scenarios, but only in regards to certain mathematics concepts at stake.

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THE DISCOURSE OF BILINGUAL MATHEMATICS TEACHING AND LEARNING

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Mathematics teachers in bilingual contexts have reported not being prepared to teach both mathematics and language. As a response, research efforts have recommended instructional strategies that, although desirable in any context, are particularly important in bilingual classrooms. In this study I report the initial stage of a larger study involving a second grade classroom in a public Spanish immersion school in the US, and a fourth grade classroom in a private English immersion school in Colombia. I interpret data through the lens of definitions of Discourse and situated perspectives on bilingual mathematics learners. Data suggest that teachers and students create a Discourse particular to bilingual mathematics classrooms. Such conceptualization will support the ongoing exploration of nuanced, contextualized instructional strategies and other ways of being that are unique to bilingual mathematics classrooms.

Keywords: Classroom Discourse, Instructional Activities and Practices, Equity and Diversity

In spite of the increase in the number of bilingual students in American public schools, many teachers report not being prepared to teach mathematics in bilingual contexts (Tan, 2011). This dichotomous view has led to teaching tensions, such as how much attention to give to mathematics teaching and how much attention to give to language teaching. Previous studies have responded by recommending instructional strategies (for an example and list of references see Chval, Pinnow & Thomas, 2015). Among the recommended strategies are scaffolding, using multiple representations, providing several terms to refer to the same concept, suspending language-related feedback to focus on students' mathematical ideas, and attending to gestures as a way of communicating mathematical ideas. Although these strategies seem necessary to teach mathematics any context, the strategies seem to be particularly important in bilingual classrooms.

In this study, I present the overall characterization of a Discourse that is particular to bilingual mathematics classrooms. This characterization is the initial part of a larger ongoing study. I analyze two different bilingual contexts, exploring examples of classroom episodes and teachers' insights. I provide examples of characteristics of a distinct Discourse that both bilingual mathematics contexts seem to share. Finally, I discuss implications for future research, including the author's ongoing research: supporting an exploration of nuanced and contextualized instructional strategies and other ways of being that are unique to bilingual mathematics settings.

Theoretical Framework

Gee's (2011) definition of Discourse, and Moschkovich's (2002) characterization of situated perspectives on bilingual mathematics learners support this study's characterization of a Discourse that is particular to bilingual mathematics teaching and learning contexts.

Big D Discourse

Gee (2011) defined big D Discourse as

Distinctive ways of speaking/listening and often, too, writing/reading coupled with distinctive ways of acting, interacting, valuing, feeling, dressing, thinking, believing, with other people and with various objects, tools, and technologies, so as to enact specific socially recognizable identities. (p. 37).

Based on this definition, Discourses are ways of being based on what and how socially recognizable communities act. For the purpose of this paper, for teachers and students to participate in particular Discourses implies to communicate and interact in particular ways. Students and teachers' participation in a particular Discourse also implies engaging with specific tasks, in distinctive ways, using certain resources. Those ways of being are particular to the identities of being teachers and being students, in a schooling setting. Those ways of being are socially discernible ways of being.

Gee (2011) defines three components of a Discourse. First, **social language** is "a specific variety of language" (p. 39). Social language determines what syntax participants use, and the contextualized meanings of certain words. For the purpose of this study, teachers and students use language in a particular way during the schooling experience. Certain ways to give directions or to address each other using spoken language are particular to teachers and students' interactions during schooling situations. Second, **figured worlds** are what a person considers to be typical or normal in a particular context. A figured world might be taken for granted, subconscious, and unquestioned. A part of teachers and students' figured worlds might be that the teacher is the one who gives directions, and distributes turns during class discussions. Third, **social practices** are observable activities that people in a particular community do together. Discourse participants might be unaware of taken for granted, unquestioned social practices. A social practice in which teachers and students engage might be for students to sit in a circle at a carpet at the front of a classroom, while the teacher stands by the board asking questions.

Situated perspectives on bilingual learners

Moschkovich (2002) extended some of Gee's earlier ideas about Discourse to the context of bilingual mathematics learners. Moschkovich (2002) proposed the notion of learning as participating in mathematics discourse practices. Examples of those mathematics practices are precision, brevity, and generalizing. She also argued that understanding the nuances and intricacies of mathematics learning in bilingual contexts implies attending to meanings of words in particular situations, the resources participants use, and the practices in which students engage.

Based on Gee's (2011) definitions of Discourse introduced above, I extend Moschkovich's (2002) conceptualization of mathematics learning in bilingual contexts as participation. Instead of regarding mathematics teaching and learning in bilingual contexts as participation in a Discourse of mathematics, I use empirical data to characterize mathematics teaching and learning in bilingual context as a distinctive Discourse in and of itself.

Methods

Here I report the initial data that guides subsequent stages of a larger study. The larger study is an ongoing exploration of classroom discourse in two different bilingual mathematics classrooms. Current parts of the study, not reported here, expand data collection to conduct discourse analysis of interactions in both classrooms over a more extended period of time.

Sites and Participants

One site was a second grade classroom in which all 23 students were English native speakers, studying mathematics in a Spanish immersion program. This public school is located in a Midwest state in the US, and all students came from low-income families. The teacher is an American female, English native speaker who had taught for 7 years at the elementary school level. She had taught in bilingual contexts for 5 years. The other site was a fourth grade classroom in which all 24 students were Spanish native speakers, studying mathematics in an English immersion program. This was a middle class, private school located in Colombia, South America. The teacher is a Colombian

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female, Spanish native speaker who had taught for 10 years. All her teaching experience had been in bilingual settings at the elementary school level.

Data Collection

Data included class video recordings, field notes, and transcripts of interviews with each one of the teachers. During the last week of the 2014-2015 school year, I observed three consecutive days of mathematics classes, video recording during the third day of observation. The video camera focused on the teacher, following interactions between teacher and students. Observational and analytic field notes (Glesne, 2006) complemented video recordings. One of the interviews with the teachers took place prior to the observations. A second interview took place after the observations, observing and discussing the teacher's video.

Data analysis

I followed Powell, Francisco, Maher's (2003) framework for video analysis: (1) repeated attentive viewing; (2) using descriptive annotations referenced to specific time markers; (3) identifying critical events. I considered critical events those that could potentially exemplify these two components of Discourse, as defined by Gee (2011): social practices, and social language. I transcribed critical events, including spoken language and non-verbal communication. I then analyzed the transcripts, coding social practices, and social language. Social practices codes included the resources participants use, cooperation among students, and the use of multiple representations to communicate mathematics ideas. Social language codes include the language used (English, Spanish, or both), intonation, volume, and variation in speed of speech. Subsequently, I shared these examples with each one of the teachers, as teacher and researcher watched the video of their respective class. The teachers and I attempted to unveil teachers' figured world about teaching mathematics in bilingual contexts. To make teachers' figured worlds explicit, I elicit awareness of some of the social practices.

Findings

Data suggests that research sites share social practices, social language, and figured worlds.

Social practices

The teachers identified in their videos example of several social practices. The teachers related some of those social practices to bilingualism, including: teachers and students' use of non-verbal communication to support spoken language; displaying and referencing mathematical ideas on the board and poster; and students' communicating and explaining mathematical ideas. Although not captured in video, both teachers mentioned that they interact with their colleagues to support their language and their mathematics understandings. Both teachers parallel that practice to students' cooperation to support each other's mathematics and language.

Social language

Teacher identified three main characteristics of how teachers and students use language. First, participants use of both languages, Spanish and English, at different moments. Second, teachers noticed frequently modulating the speed of their speech based on students' reactions. Third, both teachers and students intentionally used the language that more clearly expresses any given mathematical concept. For example, in the Spanish immersion school, the class was discussing the clues that would help children discover a mystery number. A student who was mainly using English (his native language) joined the teacher in using the word *impar* (Spanish word for odd to refer to numbers). The Spanish word for even is *par*, and the Spanish word for odd is *impar*. Knowing the root *par*, and that the suffix *im* implied negation, helped the student categorize numbers as even or

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"not even." The student used of the Spanish word *impar*, emphasizing the syllable *im*, to indicate that the number he was looking for was not even.

Teachers' figured worlds

During a preliminary interview teachers described a variety of instructional strategies of which they were aware, including: scaffolding, focusing on multiple connotations of particular words, and providing sentence stems to support students' production of spoken language. While observing their respective video during the second interview, however, teachers did not refer to their instructional moves as strategies. Instead, they referred to those moves as normal ways to teach their class. When asked about the instructional strategies enacted in the video we were watching, the teacher in the Colombian English immersion school answered "No sabría decirte, porque yo actúo así en clase siempre (risas). No sé." She added, "[in that episode] estaba dando una instrucción. No sabría cómo más [dar la instrucción] ¡Normal!" In several occasions, instead of referring to her instructional moves as strategies, the teacher described the episodes as normal and usual: "No sé... Porque pues, para mí esto es la instrucción de clase normal."

Discussion

The teachers and students in this analysis seem to go beyond participating in the discourse of mathematics with English as social language, and participating in the discourse of mathematics with Spanish as social language. Instead, they seem to establish a Discourse particular to bilingual mathematics teaching and learning. As such, that Discourse involves specific figured worlds–visions of what teachers and students consider normal in their context–, and particular social practices that teachers engage with but of which they are unaware. That Discourse of bilingual mathematics teaching and learning overlaps but differs from a general mathematics discourse, as well as from a mathematics discourse in which English is the mean of communication, or one in which Spanish is the mean of communication. It overlaps but is different from participating in Discourses considered examples of "good math teaching" in general. That particular Discourse provides more affordances to participants than its constituent parts–English math discourse, and Spanish math discourse–. Exploration of bilingual mathematics learning contexts as a distinct Discourse can help understand teaching as a way of being in a particular context. This conceptualization could explain some instructional strategies as situated and responsive, as opposed to discrete, prescriptive, and universal. It also could inform teacher education toward modeling and fostering particular ways of being, as opposed to recommended decontextualized blanket statement strategies. This conceptualization also highlights multiple resources that are unique to bilingual mathematics learning contexts. Thus, it contributes to shift attention from perceived challenges to the powerful affordances of teaching mathematics in bilingual contexts. My purpose is not to generalize or make claims about all bilingual mathematics classrooms. Instead, this initial exploration of particular characteristics of bilingual mathematics classrooms guides the ongoing more detailed analysis of Discourse.

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PARENTS' PERCEIVED COMMUNICATION ABOUT MATH CURRICULUM

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Parents' growing concerns about the "new math" are drawing public attention. Rather than dismiss such concerns, understanding parent perceptions and garnering their support is essential to ongoing curriculum transformation and children's success. In this paper, we present the preliminary findings of a phenomenographic study examining parent perceptions of children's mathematics learning. We focus specifically on the sources of curriculum related communication identified by parents: (a) homework, learning tasks, and other products the children brought home, (b) direct reports from teachers and schools, (c) the child's problem solving demonstrated outside of school, and (d) public media coverage (e.g., newspaper).

Keywords: Elementary School Education, Curriculum, Policy Matters

Purpose of Study

I don't know what the new math curriculum that my kids are learning is all about. There has been no serious explanation and communication whatsoever why they changed something that had worked perfectly fine into something that vague. I don't know enough about the new math; but I don't feel right to compete with what they are being taught in school. So I try to learn with them but here comes my frustration: the new way doesn't work for me. (Parent)

The comment above describes the tension expressed by many parents regarding the perceived lack of communication about changes to mathematics teaching and learning today. In Canada, the tug-of-war between "new math" and "back to the basics" is being played out with much media attention. Decades of research consistently demonstrate the advantages of conceptually-based approaches to mathematics learning in multiple areas including problem solving, modelling, disposition, inclusivity, and equity (e.g., Baroody, 1999; Boaler, 2002; Henry & Brown, 2008; Russell & Chernoff, 2013; Thompson, et al., 2013). However, communication of reforms and advantages has not reached parents—even those potentially open to change. Unconvinced and frustrated parents have joined coalitions (e.g., WISE Math) and launched petitions (e.g., Tran-Davies, 2013) in an effort to oppose curriculum reforms and return to basic skills, standard algorithms, and mastery through memorization.

Recognizing the challenge to communicate reforms, our research aims to identify fruitful ways to engage in productive conversation about mathematics curriculum reform with parents. In this paper, we present preliminary findings from a study aimed at understanding parents' experiences with and perceptions of mathematics curriculum change. In particular, we focus on the sources of curriculum-related communication described by parents as a way to identify useful avenues to communicate current expectations for mathematics learning.

Perspective

Parental involvement in children's educational experiences has far-reaching benefits such as improving achievement, increasing motivation, and reducing anxiety (Pattall, Cooper & Robinson, 2008). Parents are viewed as an "untapped resource for improving mathematics performance" (Hyde

et al., 2006, p. 136). Yet, parental involvement, particularly in mathematics, is frequently viewed to be ineffective and potentially detrimental to learning (Bartlo & Sitomer, 2008). We seek to avoid a deficit framework of parent involvement by drawing on a common theme in current literature that attempts to describe and shape parent perception and involvement (Lightfoot, 2004; Civil & Bernier, 2006; Marshall & Swan, 2010; Peressini, 1998). Furthermore, meaningful parent involvement includes not only communication about their children's performance, but also accessing explanatory sources of the curriculum rationale and discourse, especially in the context of curriculum reform (Remillard & Jackson, 2005). In this study, we build on our previous work of reframing public opposition into collective concerns (McGarvey & McFeetors, 2015a, 2015b) by gaining in-depth understanding of the existing communication approaches as perceived by parents in order to assist in knowing how to dialogue with and empower parents to fully participate in their children's learning. Phenomenography offers the conceptual framework and methodological approach to achieve our project objective.

Research Methodology

We used phenomenography (Marton, 1986; Marton & Booth, 1997) to inquire into the range of parents' perspectives on mathematics curriculum change and how it has been explained to them. It affords portrayal of a broad range of perspectives. Researchers draw phenomena into participants' awareness to characterize and understand their perspectives. Qualitative data consisting of participants' explanations of their views are collected and then organized into categories using emergent criteria based on similarities and differences (Akerlind, 2012). The categories are substantiated through rich description of data excerpts, emphasizing the meaning of participants' perspectives through the relationships among categories to provide a mapping of the field of inquiry. The variation in participants' perceptions, in turn, aids researchers in interpreting the phenomenon under study.

Data Collection

Forty parents from urban and rural communities in a Western Canadian province participated. They completed a demographic questionnaire and participated in one of ten focus groups taking place in their respective communities. Focus groups, each approximately two hours long, were used as generative sites of data collection with the knowledge that differing parent perspectives required participants to explain their perspective to others, allowing us to notice differences in their concerns for their children's mathematical learning and in their experiences of the mathematics curriculum. We sought to develop a finely nuanced understanding of the dimension of variation within parents' perspectives. Therefore, focus groups were structured to prompt parents to address four key components: 1) parents' observations of their child's learning and the curriculum; 2) parents' recollection of their own mathematics learning; 3) parents' effort and interactions when helping their child with mathematics at home; and, 4) parents' expectations about their child's mathematics learning. Individual follow-up interviews were conducted with fifteen parents to elicit further specific examples for the four key components and to seek clarification in understanding perspectives shared in the focus group.

We used a constant comparative approach in sorting statements made by the parents in the focus groups and individual interviews to identify qualitatively different perceptions expressed. The sorting process was comprised of an initial individual reading and a second group comparison. Generated categories were analyzed for their portrayal of the widest range of perspectives and were further substantiated through rich data excerpt descriptions. In this paper we focus on parent descriptions of the ways in which information about their child's learning and curriculum reforms were communicated to them. In listening to parents' perspectives of hearing about mathematics curriculum

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reforms, we found that there were existing forms of communication that we might leverage in order to elicit parents' full participation in their children's mathematical learning.

Results

Across the four components guiding data collection, parents articulated some characteristics in their children's mathematics learning that they attributed to current reforms such as knowing multiple strategies for computation, being able to explain their thinking, and using manipulatives and technology. Many parents also expressed a perceived lack of practice needed to develop fluency in computation.

Parents acquired knowledge of their children's progress in math mainly through four venues: **(a) homework, learning tasks, and other products their children bring home to finish:** *"The biggest thing to know how my kids are doing is homework. That's something I can actually do with them, to know exactly what they're working on."*

(b) direct reports from teachers and schools: *"Once in a while my kid comes home with a piece of test paper with some math questions and a mark he gets. He's supposed to get it signed and returned back to the teacher." "I get email from the teacher about my kid's weekly progress. Then I would ask her about it. It gives you a starting point to actually talk about math, less like pulling teeth. This provides me the ability to be more engaged."*

(c) the child's command of math concepts as demonstrated in their problem solving outside of school context: *"I pulled out the multiplication cards and he had them all right and he using his fingers for some of the computation. He's like, 'Okay, that's a 3 and I can double it' and I could hear him working it out loud. I can tell perhaps there was some mathematics teaching occurring as opposed to just sheer memorization."* and

(d) public media coverage, such as television, newspaper, and the Internet: *"I heard the term 'Discovery Math' from the news mentioning a petition against it. That's the only time when I caught onto the fact that 'Hey, the kids are doing something different than what I thought they were doing.'"*

Parents expressed dissatisfaction with the frequency and volume of the products reported back home, which were considered the primary source of communication. The observations in everyday life and media coverage as indirect sources often issued confusing and even conflicting information. Overall, parents indicated limitations in understanding their children's mathematics learning and a lack of communication around the rationale and pedagogy of the mathematics curriculum. In our presentation, we will further illustrate each category with excerpts from the parent-participants' statements.

Discussion

Our study provides a new perspective to address the "math wars" by specifying sources parents acquire information of the children's mathematics learning and curriculum reform. The research results offered insights into parents' interpretations of curriculum as well as sources shaping their interpretations. Inconsistencies were found between parents' direct contact with mathematics teaching and comprehension of the curriculum rationale, which lead parents to generate presumed understandings about mathematics curriculum reform. However, despite various assumptions, parents indicated a strong desire for communication about current approaches to mathematics and curricular expectations and genuine interest in supporting their children. As we question borders erected between parents and their children's mathematics learning in school, we can begin by enhancing the systems already used to communicate mathematics learning with parents. The research findings, therefore, will open up further possibilities of dialogue with public audiences and engender their support for curriculum reforms addressing the diversity of children learning mathematics.

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OPPORTUNITIES FOR MATHEMATICAL INTERACTION THROUGH PRESCHOOL ROUTINES

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This report presents the findings of the first phase of a professional development project with preschool teachers working alongside mathematics educators. Six preschool teachers were interviewed to identify specific examples of mathematical activity within regular classroom routines such as arrival, snack time, circle time, centres, and clean up. From the interviews the teachers identified a wide range of examples that took place through purposeful planning for circle time and centres. Fewer activities were identified by individual teachers during snack and arrival routines, but collectively they shared several opportunities including sorting, patterning, counting, and measuring. The range of examples demonstrates promise for the collaborative professional development in the next phase of the study.

A preschool classroom is a busy place. Children are often engaged in many different activities simultaneously: filling containers at the sand table, building towers in the blocks area, or arranging furniture in the dollhouse. All of these activities offer rich opportunities for mathematical thinking, but the teacher plays a critical role in making the mathematics explicit through comments, questions, and contributions to activities. The importance of mathematics learning in the preschool years has become prominent in research and policy (English & Mulligan, 2013). Yet, mathematical thinking, beyond counting and shape labelling, has not been part of the educational experiences of many preschool teachers. As a result, opportunities for mathematical interactions may go unnoticed (Ginsburg, Lee, & Boyd, 2008).

In this paper, we report on the preliminary findings of a professional development project with preschool teachers working alongside mathematics education researchers. In the first phase of the study, six preschool teachers were interviewed to establish their current practices with regard to integrating mathematical opportunities into classroom activities. These practices form the baseline for the professional development phase of the study that took place in spring, 2016. The project is based on the following research questions: What do preschool teachers notice as opportunities for teacher-child mathematical interaction in the classroom? And in what ways might collaborative professional development expand our capacity for noticing and articulating opportunities for mathematical interaction in a preschool classroom?

Conceptual Framework

In recent years there have been advances in the field of professional development of preschool teachers (Sheridan, Edwards, Marvin, & Knoche, 2009); however, research that honours the knowledge and expertise of these teachers is needed. Our interest is in contributing to the transformation of pedagogical practice by working alongside preschool teachers to advance the teaching of mathematics in the early years. The study takes into account the goals of professional development articulated by National Council of Teachers of Mathematics (NCTM, 2010) including building teacher mathematical knowledge, making connections to practice, improving dispositions, strengthening ongoing collegial relationships, and fostering "teachers' capacity to notice, analyze and respond to student thinking." With these features in mind, the methodological framework for our study is conceptualized within the professional development literature of "teacher noticing" (Sherin,

Jacobs, & Philipp, 2011). According to Mason (2002), "every act of teaching depends on noticing" (p. 7). Yet, what one notices is a function of a person's experiences, dispositions, educational philosophies, and cultural backgrounds. Becoming sensitized to previously unnoticed opportunities may mean articulating and challenging personal perspectives allowing teachers to move beyond habitual responses in familiar situations.

Methods

The six participants teach children ages 3-5 in three different preschool sites and in four different classrooms. One setting is a stand-alone junior kindergarten while the other two settings are pre-kindergarten programs housed in elementary schools. All participants hold bachelor's degrees—five in education and one in music—and two participants also hold master's degrees. Most of the teachers are relatively new to teaching preschool and range in experience from three to eight years. The five participants who hold bachelor's degrees in education taught in other elementary grades before choosing to teach in preschool.

The data reported in this paper is based on a 30-minute semi-structured interview with each of the six participating teachers. The interviews took place in the participants' school in or near their classroom and consisted of (1) demographic information of their teaching experience; (2) memories of and comfort with mathematics; (3) prompts to recall recent examples of mathematical interactions and activities in classroom routines such as arrival/greeting, circle time, centre time (e.g., house, blocks, sand/water, arts and crafts), snack time, clean up, and others; and (4) challenges or concerns with regard to addressing mathematics beyond what they were currently doing in the classroom.

Findings

Close reading of the interview transcripts revealed that all of the teachers could recall many specific and recent examples of mathematical activities taking place in their classroom. Many of the activities, particularly with regard to centres and circle time, were purposefully planned to include mathematical content such as counting, measuring, sorting, and patterning. Collectively, the teachers could also recall a range of mathematical ideas embedded into some of their other regular routines and they added examples from other routines or components of the program such as music class and lining up activities (see Table 1 next page). The teachers clearly recognized many opportunities for mathematical interaction in their daily activities in the classroom.

When asked what curricula or resources they used to plan for mathematical activity, they all mentioned that they did not have a specified program or curriculum at the preschool level, but four of the six teachers said they referred to the kindergarten curriculum for direction and occasionally used kindergarten resources. During one teacher's interview, she shared a large planning book that listed materials and manipulatives that existed in her classroom. She used the book to plan for centres and while leafing through the book could articulate the mathematical potential that existed in materials. For example, she noted that the measuring cups, bottles, scoops, and medicine droppers that could be used to measure and compare amounts at the sand and water table. She also noted that many of the games (e.g., Candy Land, Connect 4, Bug Bingo) and fine motor manipulatives (e.g., dominoes, bristle blocks, wooden shapes) lent themselves to mathematical ideas involving counting, shapes, and patterns. Even though she said she was aware of the mathematics, she was not sure how often she interacted with the children to make those ideas explicit.

Several of the teachers referred to the developmental appropriateness of some of their activities. For example, although some of the teachers questioned whether calendar activities were suitable for young children, they felt the need to balance the appropriateness with children's desire to know about upcoming events. One teacher said that the children asked incessant questions about whether the next day was a home day or school day; how many more days until Halloween; or when their birthdays would be. Rather than use a standard calendar, she created a visual alternative by placed small

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squares in long rows across the bulletin board. Each square represented a day; special events and home days were shown, but dates, months, and days of the week were not included (see Figure 1 below).

Table 1: Mathematical Activity During Preschool Classroom Routines

Classroom Routine	Preschool Teachers' Examples of Mathematical Activity
Arrival/greeting	• Children ask about the numbers of students present/absent based on sign in grid. • Children find matching shoes from the shoe bucket
Circle time	• Count the number of days on the calendar and use popsicle sticks to represent each day to 100 school days. • Taking attendance and counting the number of children present/absent
Centre time (i.e. house - imaginary play, blocks, sand/water, arts/craft/painting, playdough)	• Board and card games • House – buying groceries and exchanging money • In-class hopscotch • Blocks: balance, how many in a tower • Pattern blocks – making puzzles • Number bingo • Fine motor centres include many sorting activities • Crafts involved making snowman with Styrofoam and comparing heights • Filling buckets with snow – full empty, half full, almost empty
Snack time	• Comparing the size of water bottles
Clean up time	• Object and manipulative sorting • Stacking chairs – maximum of three in a stack
Lining up to go outside or to bathroom	• Line up and count how many boys and girls; creates a concrete bar graph • Numbers on the bathroom stalls
Music	• Patterns with actions to music; "jump, jump, jump" three times. • Songs with counting such as "10 in the Bed" and "5 Little Monkeys" • One-to-one correspondence – one word for one beat., on clothing
Getting children's attention	• Action patterns e.g., "head, tummy, head, tummy …" as the children move to the carpet.
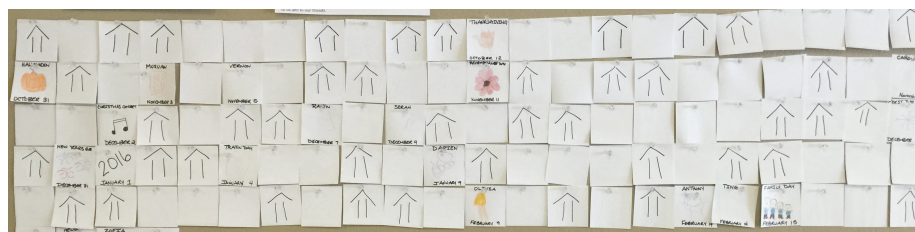
Figure 1. Alternative Calendar.

In addition to purposeful planning based on student interests, the teachers identified some opportunities for mathematical interactions during other routines such as arrival/greeting, snack, and clean up. For example, during arrival/greeting in one class, children had to find their matching pair of shoes from the shoe bucket. At snack time, another teacher noticed a pair of children comparing the heights of their water bottles. As a final example, several teachers indicated that students were using sorting rules (e.g., same type of toy or colour) at clean up.

Conclusion

From these initial interviews, the teachers were able to identify a range of mathematical activities. Of interest were the range of opportunities noticed by the teachers. For example, one teacher appeared to emphasize counting and number in many daily activities, but another referred frequently to patterning and measurement. Also, while they purposefully planned for mathematical

activities during circle and centre time, teachers were looking forward to learning about ways to integrate mathematical thinking into other routines.

The next phase in the project to be completed by spring 2016, will be to engage in professional development sessions through collaborative video viewing with the preschool teachers. Through these sessions, we hope to increase participants' and researchers' attention to and awareness of different possibilities for action within preschool routines, and to further consider ways the teacher may make the mathematics explicit through purposeful interactions.

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MAKING RESPONSIVENESS EXPLICIT: CONFERRING IN THE ELEMENTARY MATHEMATICS CLASSROOM

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Responsiveness is critical to math learning. This study examines how elementary mathematics teachers use a discourse structure adapted from literacy instruction - the conference - to engage in responsive teaching with students as they grapple with mathematical tasks. Teacher-student interactions from nine lessons in two fourth grade math classrooms were analyzed to define the structure of a math conference. An emergent typology of the pivotal conferring teaching move, the nudge, is presented.

Keywords: Classroom Discourse, Elementary School Education, Instructional Activities and Practices

Responsive teaching is critical to math learning. When teachers use their understanding of student thinking to craft instruction, students can learn with meaning (Carpenter, Fennema, & Franke, 1996). Interactions between teachers and students are one crucial way teachers learn about and respond to student thinking. However, Franke et al. (2009) have shown that while teachers readily learn to ask initial questions of student thinking, asking appropriate follow-up questions and planning the next move is much more challenging. Notably, many elementary teachers, who typically teach multiple disciplines, have a responsive discourse structure in literacy instruction, the *conference* (Calkins, 1986). In this study I examine the practice of conferring when it crosses the disciplinary boundary from literacy to mathematics and how the conference structure can support responsive teaching in mathematics.

Conceptual Framework

Calkins (1986) has defined the architecture of literacy conferences in elementary classrooms as *research-decide-teach*. The teacher's role is to determine what the student needs – moves called *research* and *decide* - and provide differentiated, scaffolded instruction in the model of cognitive apprenticeship, or *teach* (Calkins, 1986). To adapt this structure to mathematics instruction I draw on the *professional noticing of children's mathematical thinking*, defined by Jacobs, Lamb, and Philipp (2010) as three distinct but intertwined processes: *attending to*, *interpreting*, and *deciding how to respond to* student thinking. In conceptualizing what it means to *teach* in the mathematics conference, disciplinary differences suggest that students need opportunities for productive investigation rather than cognitive apprenticeship. In previous work, I termed the *nudge* as the set of moves that teachers use to teach in the math conference and which suggest a productive pathway without dictating a series of steps to follow.

Drawing on these concepts I offer the following framework for the teacher's process for conferring with students, (see Figure 1):

1. Attending to students at work, including student thinking and collaborative behaviors,
2. Eliciting student thinking or the status of the collaboration,
3. Interpreting student actions, words, and representations,
4. Deciding on a key instructional focus and how to shift student attention to this focus,
5. Nudging student thinking forward in partnership with students, and
6. Moving fluidly and iteratively through this process as dialogue unfolds until students are on a productive trajectory.
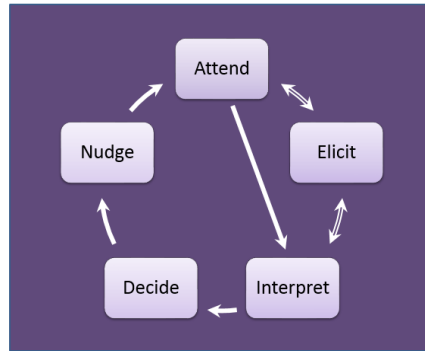
Figure 1. The teacher's conferring process, beginning with *attending*.

In this study I ask: What is the structure of the discourse in a mathematics conference? What types of mathematics conferences might teachers conduct?

Methods

Context and Data Sources

The study was conducted at a semi-rural elementary school in the southern U.S., which had dedicated efforts to mathematics professional development (PD) over the previous five years. The student population is primarily white (54%) and Latino/a (43%), and the state classifies over 60% of students as "economically disadvantaged." Two fourth grade teachers participated; both had taken part in the ongoing PD, which concentrated on conferring, in the year the study took place. I collected audio recordings of all teacher-student interactions during mathematics work time from nine lessons (six lessons from one teacher and three from the other) for analysis.

Analysis

Interactions (n=330), defined as a string of talk turns between the teacher and a given student(s), were classified by their function (e.g., managing behavior, monitoring, conferring) and conferences (n=97) were identified based on the presence of eliciting student thinking, rather than simple checks on workflow or answers. Conferences which included a nudge and conferences that represented variation in the sample were selected and transcribed for coding. Interactions were segmented in episodes, which were defined as a series of talk turns with a shared discursive purpose (e.g., eliciting, probing). The structure of these episodes within each conference was analyzed to determine shared features and categories.

Analysis then focused on a single episode, the nudge (n=37). The nudges were analyzed to determine their characteristics and develop an emergent typology.

Findings

Conference Structure

Within the conferences identified, two groups emerged. Those with a nudge were coded as *complete conferences* (n=32) and those without a nudge were coded as *approximate conferences* (n=65). Conferences as conceptualized include an instructional push on student thinking or practices, and as such I refer to them as *complete conferences*. Complete conferences shared a common structure with some variation across the sample. The transcript in Figure 2, in which students were grappling with how to move between inches and feet, illustrates this structure. Conferences begin with an initiation, which in this case serves to elicit student thinking (lines 1-4), a nudge (lines 10-17) and a closing (line 18). This conference also includes probing student reasoning (lines 5-9), which 10 of the 32 complete conferences included.

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1    Teacher:    Okay. So what kind of ideas have y'all come up with?
2    Student 1:  First, first 'cause there's 12 inches in each foot, I would do, like, 6 feet times 12 inches
3        in each foot would give you 72 inches. Then you add the leftover 5 inches and get 77 inches
4        total.
5    Teacher:   Okay. So what made you think that? How did you know to do that?
6    S1:   I, I was thinking of equal groups of, like, 12 in-, 12, equal groups of 12.
7    Teacher:   Okay… and how come you just added that, uh, 5 in there…        - at the end?
8    S1:                                                                                                                            -
9        Because-              - 6 feet, 5 inches. 5 inches is not a foot, so you have to add that in. It's
10       left over from the 6 feet.
11   Teacher:   Okay. And how would you go and explain that to somebody else? Is there a way to
12       draw a picture or explain it in a way for somebody else to understand?
13   Student 2:   I guess we could draw a picture…somehow. Like we, instead of -
14   S1:    - Oh, yeah, 6 circles [with 12 inches in them…plus the] remainder of 5.
15   S2:                                     [Yeah, yeah, that's what I was thinking!] ...        –
16       Yeah, you could do that -
17   Teacher:   Mmm…
18   S2:                   - To explain it.
     Teacher:   Very interesting. I'm going to come back and check that out.
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Figure 2. Structural example of a complete conference, with eliciting and nudge indicated. Some conferences also contain probing student reasoning. The final line is a closing.

This example illustrates four key characteristics of a nudge. First, a nudge is initiated by the teacher to advance mathematical thinking, engagement in math practices, or collaboration. In this example, note that advancement of students' thinking occurs in the nudge when the teacher asks students how they might communicate or represent their thinking, two math practices. Second, the nudge is contingent on elicited student thinking. In this case, students demonstrated a conceptual and strategic understanding of their work and could consider how to make it clear to others. Third, students must take up the nudge and co-construct its meaning. Here students ignore the nudge to explain but take up the idea of drawing a picture. Finally, the nudge maintains student sense-making and ownership over the work. While teachers suggest the avenue for productive work, students have the agency and authority to transform what is offered, in this case to decide what picture to create and how it will represent their thinking.

Fully two-thirds of the conferences (n=65 of 97) in this study were *approximate conferences*, in which teachers elicited student thinking, but attempts to push student thinking, when present, lacked the characteristics of the nudge. Over half of the transcribed approximate conferences (14 of 25) included an attempt by the teacher to nudge student thinking that students simply did not take up at all, reflecting the findings of others that responding to student thinking is the most challenging pedagogical move (Franke et al., 2009; Jacobs, Lamb, Philipp, & Schappelle, 2011).

Nudge Typology

Analysis of the 36 nudges in the data revealed that nudges come in five different types, distinguished by their purpose (see Table 1). A key finding is that each type focuses on advancing conceptual or strategic thinking or engagement in mathematical practices.

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Table 1: Typology of nudges

Nudge Type	Description
Conceptual Understanding	• Focuses attention on the underlying meaning of the task and the concepts it represents • May include attending to and reasoning about an error
Developing a Strategy	• Coaches students to develop a strategy for the current problem • Students maintain ownership over the strategy
Communication	• Prompts students to articulate thinking in a cohesive explanation • Often a rehearsal for writing or sharing orally with others
Representation	• Prompts students to develop a way of representing the task, current thinking, or the strategy used • May allow students to communicate more effectively, develop generalizable representations of a concept, identify a strategy, or uncover errors
Collaboration	• Orients students to each other's thinking (Chapin, O'Connor, & Anderson, 2013) and focuses efforts on joint work

Discussion

Identifying teacher-student discourse practices that deepen student thinking and engagement in mathematical practices is of critical importance for the daily work of math classrooms. But interactions cannot be planned in advance. Teachers would benefit from a framework for navigating these intimate moments of instruction. The explicit structure and foci of complete conferences found in this study provide such a framework.

This study demonstrates that conferring is a discursive practice that can be adapted for responsive instruction in the elementary mathematics classroom. The math conference centers on two critical elements, eliciting student thinking and nudging that thinking forward. Advancing mathematical thinking or practices hinges on co-constructing a nudge with students. This co-construction process depends on teachers eliciting enough student thinking to form an accurate interpretation and deciding on an avenue of response that students will take up and own.

Future research is needed to investigate how nudges affect student understanding beyond the interaction itself. In order to further implementation of conferring by practitioners, the field needs an understanding of how teachers might learn to confer and the professional development structures that could support growth of similarly responsive practices.

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SIGNIFICADO MOSTRADO AL RESOLVER ECUACIONES LINEALES

THE MEANING DISPLAYED WHEN SOLVING LINEAR EQUATIONS

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Esta investigación consistió en tres estudios de caso y fue realizada a alumnos de segundo año de preparatoria. Los datos se recabaron mediante entrevistas y cuestionarios. El estudio estuvo centrado en el análisis de los tratamientos y conversiones de las diferentes representaciones semióticas de sistemas de ecuaciones lineales exhibidas por los alumnos. El análisis se realizó desde el enfoque de la Teoría de los Registros de Representaciones Semióticas (TRSS) de Raymond Duval. Además, el estudio describió las experiencias y reflexiones relatadas por los estudiantes de cuando trabajaron con las ecuaciones lineales en el salón de clases.

Palabras clave: Afecto, Emoción, Creencias y Actitudes; Álgebra y Pensamiento Algebraico; Actividades Y Prácticas De Enseñanza; Educación Bachillerato

Introducción

Las ecuaciones lineales son un tema enseñado como parte de los programas de educación preparatoria en cada país del mundo y México no es la excepción. Schmidt et al. (2001) argumenta en su reporte del análisis del currículum que alrededor del mundo no hay un tema más ampliamente enseñado en preparatoria que el de ecuaciones. Por lo tanto resulta de gran interés el analizar el proceso de enseñanza-aprendizaje involucrado en la didáctica de las ecuaciones lineales desde la perspectiva de los estudiantes.

Cuando un estudiante se enfrenta a un objeto matemático en el salón de clases, empieza a generar hipótesis acerca del objeto matemático en sí, cómo funciona y para qué puede llegar a ser útil. Esto es debido a que intenta darle un significado a lo que está estudiando (Block y Papacostas, 2000). Es interesante este proceso, una vez que el alumno hace sus hipótesis, empieza a encontrar diferencias con lo que realmente es el objeto matemático presentado por el instructor. Una vez que encuentra estas diferencias y le da un nuevo significado al objeto podemos decir que el estudiante ha hecho un trabajo constructivista ya sea gracias a actividades del maestro o actividades del mismo estudiante pero que al final de cuentas producen un conocimiento a partir de la experiencia propia. Sin embargo, la didáctica de algunos maestros ni siquiera genera interés en los alumnos por aprender ciertos objetos matemáticos (Kline, 1989). Es por ello que resulta esencial el interés de los alumnos en el tema para que estén motivados a aprender.

D'Amore (2006) establece que: "en los años 80 y 90 se declaraba que, mientras el matemático puede no interrogarse sobre el sentido de los objetos matemáticos que usa o sobre el sentido que tiene el conocimiento matemático, la didáctica de la matemática no puede obviar dichas cuestiones" (p. 179). Es por ello que resulta importante reflexionar y analizar el proceso de enseñanza- aprendizaje dentro de un salón de clases.

Diversos estudios (e.g., Hiebert et al. 2003; Hollingsworth, Lokan, & McCrae, 2003) han revelado que en la mayoría de los países las actividades matemáticas relacionadas con las ecuaciones lineales en preparatoria tienen la estructura siguiente: revisión, introducción, ejemplo, ejercicio sentado en la banca y resumen de la actividad. Dicha estructura se denominará tradicional en este estudio. Aunque existen variaciones en la didáctica, la estructura básica es en esencia la misma. De hecho, estas investigaciones muestran que no importa el tema que se pretenda enseñar, la estructura en su mayoría es la misma.

Alrededor de la didáctica de las matemáticas, las teorías que se han desarrollado de este campo

son meramente constructivistas. Dichas teorías proponen actividades y situaciones donde los alumnos sean quienes generen el conocimiento a partir de reflexiones de experiencias propias y al final ellos mismos construyan un significado, gracias al razonamiento hecho previamente para solucionar un problema o actividad que sin el nuevo conocimiento adquirido hubiese sido imposible de resolver. Existen prácticas dentro del salón de clases que de acuerdo a la perspectiva de los maestros son "efectivas"; desafortunadamente, las percepciones de los alumnos suelen ser muy distintas a las de los maestros.

En general, durante la historia de la práctica de la enseñanza de las matemáticas el enfoque de la misma ha estado orientado solamente a la enseñanza sin darle el lugar que merece el proceso de aprendizaje. Dicha dualidad debe ser tomada en cuenta con igual importancia ya que sin una de ellas no existe la otra.

Debido a ello es que debe realizarse un estudio a fondo del proceso de enseñanza-aprendizaje llevado a cabo en el salón de clases enfocándose en lo que expresan los alumnos para tener una diferente perspectiva que enriquezca los conocimientos actuales acerca de la enseñanza de las ecuaciones lineales.

Preguntas de Investigación

Este estudio estuvo guiado principalmente por dos preguntas de investigación que son las siguientes:

1. ¿Cómo describen los estudiantes sus experiencias dentro del salón de clases acerca de la enseñanza-aprendizaje de los sistemas de dos ecuaciones lineales con dos incógnitas?
2. ¿Cuáles son los conocimientos mostrados por los alumnos en los tratamientos y conversiones de las diversas representaciones semióticas de los sistemas de dos ecuaciones lineales con dos incógnitas?

Marco Teórico

La Teoría de las Representaciones Semióticas fue desarrollada en los noventas por el francés Raymond Duval. Esta teoría hace énfasis en que los objetos matemáticos únicamente pueden ser manipulados por el hombre a través de sus múltiples representaciones semióticas, ya que los objetos matemáticos son intangibles para el ser humano a diferencia de los objetos físicos. Por lo tanto, es necesario manipularlos a través de sus representaciones. Esta manipulación del objeto matemático en sus diferentes representaciones es llamada tratamiento por Duval (2004).

Esta teoría, se enfoca básicamente en tres características de la semiótica: la representación, el tratamiento y la conversión (D'Amore, 2006). Un objeto tiene distintas representaciones como la verbal, la gráfica, la numérica, la textual, entre otras. El tratamiento no es posible en el objeto en sí, sino que el tratamiento se hace en las representaciones del objeto. La conversión se entiende como el cambio de una representación a otra cuidando no perder de vista cuál es el objeto matemático.

De acuerdo con Duval (1993) la única forma de conseguir una conceptualización auténtica de un objeto matemático es haciendo tratamiento en varias de las representaciones semióticas sin perder el significado del proceso al realizar alguna conversión. Es muy común ver que los estudiantes hacen tratamientos en buena manera dentro de alguna representación dada pero en la conversión pierden el sentido de lo que están realizando. Es por ello que entre más conexiones hagan entre las representaciones y no pierdan el sentido del objeto al hacer conversiones las probabilidades de adquirir una conceptualización del objeto matemático aumentan.

De acuerdo con Segura (2003) "la coordinación de varios registros de representación semiótica resulta fundamental para una asimilación conceptual de un objeto; además, es necesario que el objeto no se confunda con sus representaciones, pero debe ser reconocido en cada una de ellas" (p. 53).

Método

El estudio se desarrolló en Ciudad Juárez, una comunidad al norte del México que colinda con los Estados Unidos. El Colegio Bachilleres de Chihuahua Plante # 6 es la escuela preparatoria de donde se eligió la muestra. Dicha institución cuenta con alumnos de nivel medio socioeconómico. Los profesores en dicho colegio cuentan con al menos licenciatura para poder impartir clases. Los profesores que imparten clases de matemáticas son egresados de licenciaturas en diversas ingenierías. El colegio ofrece dos turnos para sus estudiantes: matutino de 7:00 am a 1:10 pm y vespertino de 1:10 pm a 7:50 pm. En el cuarto semestre los estudiantes pueden elegir de tres diferentes especializaciones: físico-matemático, humanidades y ciencias biomédica. Los estudiantes que forman la muestra provienen de diferentes especializaciones y turnos.

Debido a que el acceso a información de los estudiantes del Colegio como calificaciones requería un trámite especial y tardaban un tiempo considerable en responder a un requerimiento de esa índole, el investigador optó por utilizar el método de selección de muestra denominado 'bola de nieve'. El investigador tenía como conocidos a varios estudiantes dentro del Colegio. Sin embargo ellos no podrían ser sus participantes en el estudio debido a que habría conflicto de intereses, por lo que el investigador solamente les pidió que le identificaran estudiantes que cursaran el cuarto semestre y que se caracterizaran por expresarse de manera natural en las clases y que además vivieran cerca de la escuela y que tuvieran un promedio general mayor al 8.0. Además se solicitó que los participantes tuvieran facilidad de palabra y que tuvieran interés en el tema de investigación de este estudio. Fue así como tres estudiantes fueron sugeridos por otros estudiantes del colegio. Una vez que se determinó quiénes serían los participantes, el investigador se comunicó con ellos. Los tres participantes accedieron a participar en el estudio.

En esta investigación se emplearon dos instrumentos para la colección de información: la entrevista no estructurada y el cuestionario.

Resultados y Conclusiones

De acuerdo con la Teoría de Registros de Representaciones Semióticas, para lograr la conceptualización de un objeto matemático dado, en este caso un sistema de dos ecuaciones lineales con dos incógnitas, es necesario para el estudiante interactuar con diferentes registros de representación semiótica como pueden ser en este ejemplo el registro gráfico, el registro literario y el registro algebraico. Además, se resalta la importancia tanto el tratamiento como de la conversión de cada uno de los registros y así mismo, que no se pierda de vista en las conversiones, en los tratamientos y en sus representaciones el objeto matemático en sí.

En los resultados de la investigación, se pudo observar que la conversión del registro literario al algebraico no representó mayor dificultad para los alumnos participantes. Con respecto a la conversión del registro algebraico al gráfico solamente un participante no pudo realizarlo pero expresó que de haber tenido calculadora lo hubiera hecho. En ambos casos el registro algebraico estuvo presente y los alumnos mostraron habilidad para realizar los tratamientos y conversiones. En la sección 3 de nueva cuenta los participantes realizaron tratamientos al registro algebraico y no mostraron dificultad. Solamente un participante prefirió contestar que no estaba seguro. Al final del cuestionario se le cuestionó su respuesta y el participante mencionó que de haber tenido calculadora lo hubiera hecho sin dificultad. Debido a las respuestas a estas tres secciones pudimos concluir que donde está el registro algebraico presente los estudiantes no muestran dificultad para manipularlo ya sea mediante tratamientos o conversiones. Sin embargo, en la única sección donde no está presenta el registro algebraico, se pudo observar que la dificultad fue mayor. Ninguno de los estudiantes acertó las dos respuesta, solamente un estudiante acertó una de ellas. Con esto pudimos concluir que la representación gráfico representó un problema real para los estudiantes. Además, de acuerdo con lo que comentaron los estudiantes en sus entrevistas, se mencionó que el registro gráfico en el salón de clases no es tan frecuentemente visto como el algebraico.

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Es por ello que se pudo deducir con base en las entrevistas y los cuestionarios que las prácticas más comunes dentro del aula de clases es el registro algebraico y por lo tanto el tratamiento de este registro es dominado en su mayoría por los estudiantes. No obstante, el significado mostrado por los participantes careció en general de sentido. Los participantes sabían que se debían utilizar las ecuaciones en la vida diaria; sin embargo, no tuvieron claro ejemplos de donde aplicarlo. Por lo tanto, resultó necesario que las prácticas dentro del salón de clases deben dar igual importancia a los diferentes registros de representación así como a los tratamientos dentro de cada uno de ellos y brindarle vital importancia a las conversiones que representan comúnmente el mayor de los retos para los estudiantes.

Finalmente, de acuerdo con las categorías emergentes, los profesores utilizaron los exámenes como el medio más adecuado para evaluar a los alumnos. Sin embargo, podemos ver que los alumnos no tienen precisamente una idea muy clara de cuál es el uso en la vida diaria de las ecuaciones lineales y no dominaron todos los registros de representación semiótica. Además, las ecuaciones fueron mejor identificadas por los alumnos como método operativo más que una relación entre dos cantidades.

This research consisted of three case studies and it was conducted with high school students. The data was collected by interviews and questionnaires. The study focused on the analysis of the treatments and conversions of the different semiotic representations of linear equations displayed by the students. The analysis was based on the Theory of Register of Semiotic Representation (TRSR) developed by Raymond Duval. In addition, the study describes the experiences and reflections expressed by the students when they were working with linear equations in the classroom.

Keywords: Affect, Emotions, Beliefs, and Attitudes, Algebra and Algebraic Thinking, Instructional Activities and Practices, High School Education

Introduction

Linear equation is a topic taught as part of the high school education program in every country in the world and Mexico is not an exception. Schmidt et al. (2001) established in their curriculum analysis report that around the world there is no topic more broadly taught in high school than linear equations. Therefore, it is interesting to analyze the process involved in the teaching-learning of linear equations from students' perspective.

When a student faces a mathematical object in the classroom, they generate a hypothesis about the mathematical object itself. How does it work? How can it be useful? This happens because the student tries to give a meaning to what he is studying (Block & Papacostas, 2000). This process is interesting, as once the student makes his hypothesis, he begins to find differences with the real mathematical object presented by the instructor. Once the student finds the differences and gives a new meaning to the mathematical object we can say the student has constructed the object. It could be because of the instruction given by the teacher or activities generated by the student himself. However, in the end the student produces knowledge from his or her own experience. Unfortunately, teacher strategies sometimes do not create interest in students to learn certain mathematical objects (Kline, 1989). Thus, students' interest in the topic is crucial to support motivation to learn.

D'Amore (2006) stated: "in the 80's and 90's, while the mathematician might not be asked about the sense of the mathematical objects used or about the sense that the mathematical knowledge has, the mathematics didactics cannot obviate such questions" (p. 179). Therefore, it is essential to analyze and reflect about the teaching-learning process within the classroom.

Many studies (e.g., Hiebert et al. 2003; Hollingsworth, Lokan, & McCrae, 2003) have revealed that in most of the countries, mathematical activities related to linear equations in high school have

the following structure: review, introduction, example, exercises at students' desk and activity summary. Such structure will be called traditional in this research. Even though there are didactic variations, the basic structure is basically the same. In fact, these studies show that the topic being taught does not matter, the lesson structure typically remains the same.

Most of the theories developed in the field of mathematics education are constructivist theories. Such theories propose activities and situations where students should be the ones generating the knowledge from reflections on their own experiences and then they construct their own meaning. This meaning occurs because of students' prior reasoning to solve a problem or activity that without the new knowledge obtained would be impossible to solve. There are many practices in the classroom that according to teachers' perspective are "effective". Unfortunately, student's perceptions are often very different from teachers' perspectives.

In general, in the history of mathematics teaching practice, the focus has been oriented only towards teaching without giving adequate focus to the learning process. The duality of the teaching-learning process should be taken with equal importance because without one of them the other would not exist.

Hence, the importance of a deep research about the teaching-leaning process in the classroom focused on students' perspective. This research can help us to have a better understanding of the teaching-learning process involved in linear equations.

Research Questions

This study was guided mainly by the following two research questions:

1. How do the students describe their experiences in the classroom related to the teaching-learning process involved with systems of two linear equations with two unknowns?
2. What is the knowledge shown by the students in the treatments and conversions of the different semiotic representations of systems of two linear equations with two unknowns?

Theoretical Framework

The French scholar Raymond Duval developed the Theory of Register of Semiotic Representation (TRSR) in the 90's. This theory emphasizes that people can only manipulate mathematical objects through their multiple semiotic representations because mathematical objects are intangibles to human beings, in contrast to physical objects. Therefore, it is necessary to manipulate mathematical objects through their representations. This manipulation of mathematical objects in different representations is called treatment (Duval, 2004).

The TRSR focuses basically in three semiotic characteristics: the representation, the treatment, and the conversion (D'Amore, 2006). An object has multiple representations such as: verbal, graphic, numeric, text, among others. The treatment is not possible on the object itself. The treatment can be done only on the representations of the object. Conversion is the change from one representation to other representation, without changing the meaning of the mathematical object.

According to Duval (1993) the only way to get an authentic conceptualization of a mathematical object is by doing treatments on different semiotic representations without losing the meaning of the process when doing any conversion. It is common to see that students know how to make treatments within one of the representations but in the conversion they lose the meaning of what they are doing. Hence, if there are more connections between the representations and the students do not lose the object's meaning when doing the conversions, the chances of developing a conceptualization of a mathematical object increase.

Segura (2003) states that the coordination of different registers of semiotic representation is fundamental for the conceptual assimilation of an object; in addition, it is necessary that the object

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should not be confused with its representations. However, the object must be identified in each of its representations.

Methodology

The study was conducted in Ciudad Juarez, a community located in the north of Mexico in the border with the United States. The sample was selected from a high school named Colegio Bachilleres de Chihuahua Plantel # 6. Colegio Bachilleres de Chihuahua Plantel # 6 has students with middle socioeconomic status. The instructors need to have a bachelor's degree in order to teach classes. Most of the instructors that teach mathematics have a bachelor's degree in engineering. The high school has two shifts for students: one in the morning from 7:00 am to 1:00 pm, and one in the evening from 1:10 pm to 7:50 pm. In the fourth semester, the students can choose between three different specializations: physics-mathematics, humanities, or biomedical sciences. The students from the sample reflect different specializations and different shifts.

Since the access to students' data in this high school required a special application that would take some time, the researcher decided to use the sample selection method called "snow ball". The researcher knew some of the high school students. Unfortunately, these students could not be the participants in order to avoid conflicts of interest. However, these students helped the researcher to identify students that were studying in the fourth semester, lived near the school, had a GPA equal or greater to 3.0, and that expressed themselves in a typical manner in the classroom. Once the researcher had the contact information of some students, he contacted them and three of them agreed to participate in the study. An unstructured interview and questionnaire were the two instruments used to collect data from the participants.

Results and Conclusions

According to TRSR, in order to achieve a conceptualization of a mathematical object - in this case a system with two linear equations with two unknowns - is necessary for the student to interact with different registers of semiotic representation such as: the graphic register, the literary register, and the algebraic register. In addition, we highlight the importance of the treatment as well as the conversion of each of the registers. The meaning of the mathematical object should not be lost in the process.

In the results, the conversion from the literary register to algebraic register did not represent difficulties for the participants. In the conversion from the algebraic register to graphic register only one participant could not do it. However, the participant said that if a calculator had been allowed he would have been able to solve it. In both cases the algebraic register was present and the students demonstrated the ability to do the treatments and conversions. In the section about treatment to the algebraic register, the students did not show struggles. Just one student preferred to answer that she was not sure. At the end of the questionnaire the researcher asked her about her answer and the participant mentioned that if a calculator had been allowed she would have been able to solve it without difficulty. From the answers to these sections we concluded that students did not have problems working with the algebraic register, whether it was with treatments or conversions. However, in the only section where there was no algebraic register, the students struggled with treatments and conversions. None of the students answered both questions correctly, and only one student answered one of the questions correctly. This led us to conclude that the graphic register was a real problem for the students. Furthermore, students commented in the interviews that the instructors give lees time to explain the graphic register in the classroom than the algebraic register.

Therefore, from the interviews and questionnaire we determined that the most common practices in the classroom were related to the algebraic register. Thus, the treatment of this register is mastered by most of the students. Overall, the participants demonstrated lack of meaning of the algebraic register. The participants know there are real life applications for linear equations; nonetheless, they

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could not give examples of these applications. Hence, it is necessary that classroom practices give the same importance to the different registers of semiotic representation as well as the treatments within each of them. Most important would be a focus on the conversions between the different registers because that was the biggest challenge for students in this study.

Finally, according to the emergent categories, the instructors used tests as the better assessment for students. However, the students did not have a clear idea of the real life applications of linear equations, and they did not demonstrate competence with all of the registers of semiotic representation. Furthermore, equations were identified by the students as an operative method rather than a relation between two quantities.

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MECANISMOS DE DESARROLLO DE LA CULTURA DE RACIONALIDAD DEL SALÓN DE CLASES DE MATEMÁTICAS

MECHANISMS FOR DEVELOPING A CULTURE OF RATIONALITY IN THE MATHEMATICS CLASSROOM

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Siguiendo los principios metodológicos de la Teoría Fundamentada y los que Toulmin propone para sistematizar argumentos, y con base en un estudio de caso, en la investigación se identificaron dos mecanismos que inciden en el desarrollo de la Cultura de racionalidad de la clase de matemáticas analizada (de primer grado de secundaria): un mecanismo se refiere a los formatos de interacción presentes en el desarrollo de argumentos y otro, al tipo y funciones de los argumentos, destacando los productivos y los reproductivos. La metodología empleada permite sugerir que estos mecanismos tienen vigencia en aulas de matemáticas de distintas latitudes; por ello, el conocimiento que el profesor posea sobre dichos mecanismos puede serle de gran utilidad para planear la evolución de la Cultura de racionalidad de su clase de matemáticas.

Palabras clave: Razonamientos y Demostraciones, Conocimiento del Profesor

Objetivos del trabajo y Antecedentes

En escritos previos elaborados por los autores se aportaron evidencias de que en el salón de clases de matemáticas existe una Cultura de racionalidad (Rodríguez y Rigo, 2015). En la presente investigación se plantea identificar y explicar el proceso de construcción, evolución y consolidación de la racionalidad real que impera en el aula ordinaria de matemáticas de educación secundaria.

La noción de Cultura de racionalidad tiene filiación con las nociones de cultura y cultura de clase (Bertely, 2000), pero especialmente con la de normas sociomatemáticas (Yackel y Cobb, 1996). Estas normas regulan los argumentos matemáticos que ahí se dan y consisten, por ejemplo, en los acuerdos normativos acerca de las explicaciones matemáticamente aceptables. Sobre el mismo tema versa también el trabajo de Planas y Gorgorió (2001), quienes se centran en averiguar las "posibles interferencias en el aprendizaje que en el aula pueden derivarse de las diferentes interpretaciones de las normas matemáticas" (p. 135).

Marco Teórico

Para el análisis de los argumentos identificados en el estudio de caso se adoptó el Modelo de Toulmin (1974), conforme al cual, un argumento está integrado por una conclusión (C); por una evidencia (E), que aquí coinciden con los sustentos ofrecidos para obtener la conclusión; por una garantía (G), que aparecen descritas en la segunda columna del Cuadro 1, y por un respaldo (R), que ofrece el cimiento teórico, práctico o experimental del argumento. En este estudio se consideran dos clases de respaldo. El primero está relacionado con el tipo de sustento en el que se soporta el argumento. Para caracterizar estos sustentos se acude a los esquemas epistémicos, término que Rigo (2013) definió para hacer referencia a los mecanismos que una persona o una comunidad recurren habitualmente para sustentar los hechos de las matemáticas. La investigadora identifica esquemas epistémicos de tipo matemático y esquemas epistémicos extra-matemáticos. La segunda clase de respaldo alude al nivel de conocimiento sobre la teoría de proporcionalidad (operatoria, intuitiva o formal escolar) que el sujeto pone en juego al argumentar. Así que el respaldo permite distinguir de forma general la racionalidad en la que se inscribe el argumento. En la investigación cuyos resultados

parciales aquí se exponen, la Cultura de racionalidad de una clase de matemáticas está integrada, entre otras cosas por:

Normas de sustentación

Prácticas habituales y más aceptadas de sustentación o de elaboración de argumentos (integrados por una sucesión de esquemas epistémicos) sobre los hechos de las matemáticas, que los agentes de clase llevan a cabo en el aula (cf., normas sociomatemáticas de Yackel y Cobb, 1996).

Normas sociales sobre el reparto de responsabilidades y sobre las formas de interacción

Hace referencia al agente de clase que habitualmente le corresponde argumentar (i.e., dar las conclusiones y las evidencias, y eventualmente las garantías), y al que le toca sancionar los argumentos y las conclusiones dadas, y apunta también hacia las formas de interacción que para argumentar se dan cotidianamente en clase (cf., normas sociales de Planas y Gorgorió, 2001).

La Cultura de racionalidad es la suma de las racionalidades que están presentes en un salón de clase a lo largo de un período de tiempo. Un momento de racionalidad se da cuando se presenta un conjunto de argumentos que tienen en común los mismos respaldos.

Metodología de investigación y técnicas e instrumentos de recogida de datos

La investigación que aquí se reporta fue desarrollada aplicando los principios de la Teoría Fundamentada (Corbin y Strauss, 2015), cuyo objetivo central consiste en construir, a partir de apoyos empíricos, conceptos teóricos que expliquen fenómenos sociales. Las herramientas analíticas de la Teoría Fundamentada (e.g., la codificación axial y el análisis centrado en los procesos) permiten crear relaciones entre conceptos teóricos con base en las cuales es posible dar cuenta del cómo se genera, evoluciona y cambia el fenómeno bajo estudio pero también permite explicar y responder al por qué de algunos aspectos relacionados con él.

Los datos empíricos provienen de un estudio de caso. El caso lo conforma la profesora Noemí y su grupo integrado por 42 alumnos de primer grado de educación secundaria. Para el análisis que aquí se expone se examinó la secuencia didáctica que versa sobre el tema 'Reparto proporcional', impartido por la profesora en seis módulos de 50 minutos cada uno, los cuales fueron videograbados y transcritos. Para este reporte se analizaron 37 episodios y 74 argumentos.

Mecanismos de desarrollo de la Cultura de racionalidad en el salón de clases de matemáticas. Algunas evidencias empíricas

En el apartado se caracterizan e ilustran, con pasajes del caso de estudio, dos mecanismos de construcción y desarrollo de la Cultura de racionalidad de la clase analizada.

Mecanismo "Formatos de interacción para el desarrollo de argumentos"

Este mecanismo descansa en la idea de que los argumentos se construyen en un contexto de interacciones entre los agentes de clase (Krummheuer, 1995). Se identificaron tres patrones relacionados con este mecanismo, pero por cuestiones de espacio se describirá sólo uno.

Patrón sobre el nivel de los argumentos. Dado un primer argumento para una conclusión, los argumentos subsecuentes (para esa misma conclusión) son generalmente de un nivel de complejidad mayor. En el Cuadro 1 se han organizado ascendentemente los argumentos por nivel de complejidad: en la parte inferior se han incluido los de carácter extra-matemático y en la superior los que tienen una base matemática, comenzando por instanciaciones de reglas con bases intuitivas (teoremas en acción, Vergnaud, 1989) hasta llegar a los argumentos en los que se explicitan los procesos matemáticos o éstos se justifican. El patrón al que aquí se hace referencia se puede reconocer en ese Cuadro 1; para la conclusión cuatro por ejemplo, la primera evidencia, dada por un alumno, consistió en una instanciación intuitiva de una regla y en la segunda evidencia la Maestra justificó el proceso. En ese mismo Cuadro 1 es posible detectar estas regularidades; por ejemplo, de los 27 casos donde se

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identificaron primeras y segundas evidencias para una misma conclusión, en 23 de esos casos (el 85%) el nivel de profundidad de la segunda evidencia fue mayor.

Mecanismo "Tipos y funciones de los argumentos: los reproductivos y los productivos"

Evocando las funciones de la prueba (de Villiers, 1990; y Hanna, 1990), en este trabajo se han distinguido dos tipos de argumentos.

Los 'argumentos reproductivos' que comparten con otros argumentos análogos, evidencias y garantías similares pero respaldos iguales. Los agentes de clase exponen este tipo de argumentos a través de 'intervenciones de consolidación' que tienen como finalidad (explícita o implícita, consciente o no deliberada) fortalecer y arraigar una racionalidad y crear las condiciones sobre las cuales se puedan construir nuevos argumentos, lo que eventualmente dará lugar a una racionalidad modificada quizás más rica en formas de sustentación.

Los 'argumentos productivos' son aquellos cuyos respaldos ofrecen un soporte más enriquecido, en el sentido matemático, en relación a los argumentos precedentes, y comparten con éstos los mismos procedimientos, reglas o tareas por resolver. Los agentes de clase proponen este tipo de argumentos mediante 'intervenciones de cambio', modificando la racionalidad existente y dando eventualmente lugar a una nueva, en el caso en el que ésta se arraigue.

En el Cuadro 1 también se pueden ilustrar los tipos de argumentos y sus funciones. Intervenciones de consolidación de una racionalidad operatoria que involucra a la regla de tres son aquellas en las que se ofrecen los argumentos reproductivos 3, 5, 10, 11, 28, 29, 59, 60, 61 y 62, mismos que comparten igual respaldo; una intervención de cambio es en la que se expone el argumento productivo 63, el cual presenta ya un nuevo respaldo; la mayoría de estos argumentos fue proporcionado por la maestra quien muy posiblemente de forma deliberada ofreció el primer grupo de argumentos con el fin de consolidar una racionalidad, de tipo operatorio, y expuso el 63 con el propósito didáctico de profundizar matemáticamente en la regla involucrada, modificando así la racionalidad previa en torno a esa regla, e introduciendo una racionalidad basada en una teoría de la proporcionalidad formal escolar. Los argumentos productivos, cuando son compartidos y reiterados por el colectivo, pueden reconfigurar una nueva racionalidad, cuya consolidación crea las condiciones para que eventualmente se dé un nuevo ciclo. En la clase analizada, el 85% de los argumentos son reproductivos, y el 15% son productivos (v. Cuadro 1).

Las intervenciones de consolidación tienen como finalidad servir de nutrimento a las intervenciones de cambio. En este mismo sentido, Chevallard, Bosch y Gascón (1997) sostienen que en clase, la producción de técnicas nuevas "se apoya en el dominio robusto de las técnicas básicas" (p. 279) (argumentos reproductivos) y en los "momentos de la justificación de la técnica" (p. 263) (argumentos productivos).

Consideraciones finales

Con base en la metodología de la Teoría Fundamentada, adoptada en este trabajo, se puede suponer razonablemente que los mecanismos antes descritos están presentes en aulas de matemáticas de regiones diversas. Sería importante entonces que los profesores tomen en consideración la Cultura de racionalidad que prevalece en su clase y la dinámica de su desarrollo. En particular, que reflexionen sobre las consecuencias didácticas que puede acarrear el desequilibrio entre argumentos productivos y reproductivos –posiblemente la elaboración conceptual del grupo será bastante raquítica– o el desbalance a favor del profesor entre sus intervenciones y las de los alumnos, lo que les impedirá beneficiarse de los resultados cognitivos que propicia la acción y la interacción en clase, dando lugar otra vez a aprendizajes escasos.

Cuadro 1: Trayectorias de los argumentos y de participación

Número del argumento

Garantías	1–74 (argumentos)		AP / AR
Matemáticos — JE	P I M (7), P I M (9), V U (13), P I M / V U / F P (16–18), R3 (63)		AP
Matemáticos — EPIR	P I M (1), P I M (37), V U (43), V U (49)		
Matemáticos — RIR AVI	V U (14), P I M (21), V U (23), V U (24), P I M (26), P I M (29), P I M (31), P I M (38), P I M (40), P I M (47), P I M (57), P I M (59), R3 (73)		
Matemáticos — IR (I o FE)	P I M (1), P I M (6), P I M (8), V U (13), V U (14), P P I I M M (20–21), V U (23), V U (24), P I M (26), P I M (29), P P I I M M (30–31), P I M (33), P I M (36), P I M (39), P I M (41), V U (44), P P I I M M (47–48), P I M (50), V U (51), P P P P I I I I M M M M (52–55), V I U M (57), P I M (59), F F P P (62–63), R3 R3 R3 R3 (64–67), P I M (68), R3 R3 (69–70), R3 (74)		
Extra Mat — RIRO AVI	R3 (5), R3 (12), R3 (28)		AR
Extra Mat — IRO	R3 (2), R3 (11), R3 (28), R3 R3 R3 R3 (61–64)		
Extra Mat — RP	R3 (3)		

	EC	AE	C	AC
(valores por argumento)	1 2 3 4 5 1 2 3 4 5 6 1 2 1 2 1 1 1 1 2 1 2 1 2 3 4 1 2 3 4 1 2 1 2 1 2 1 2 1 2 1 2 1 2 3 1 1 2 1 1 1 1 1 1 2 3 1 2 1 1 1 1 1 2 3 4 1 2 1 1 2 1 2 3	A M A M M A M A A A M A M A M M M M A M A M A M A M A M A M A M A M A M A M M M A A M A A M A A A A A M A M M M M M A A M M M M A M A M M	1 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37	A A A A M M M A A A A A A A A A M A A A A A A A M M M A A A

AP: Argumentos productivos; AR: Argumentos reproductivos; EC: Evidencias que soportan la conclusión; AE: Agente que proporcionó la evidencia; C: Conclusión; AC: Agente que proporcionó la conclusión; JE: Justificación empírica de definiciones y reglas; EPIR: Explicitación de proceso de la instanciación de una regla; RIRAVI: Repetición de la instanciación de una regla para dar aval institucional; IR: Instanciación de una regla (soportada en consideraciones intuitivas "I" o escolares formales "EF"; RIROAVI: Repetición de la instanciación de la regla soportada en consideraciones operatorias para dar aval institucional; IRO: Instanciación de una regla soportada en consideraciones operatorias; RP: Razones prácticas. PIM: Propiedad isomórfica de la multiplicación; VU: Valor unitario; FP: Factor de proporcionalidad; R3: Regla de tres.

Following the methodological principles of Grounded Theory and those proposed by Toulmin for systematizing arguments and based on a case study, this research identified two mechanisms that have incidence in the development of a Culture of rationality in the mathematics class analyzed (first year of secondary school). One mechanism refers to the interaction formats that occur in development of arguments, and the other to the type and function of the arguments, among which productive and reproductive arguments are the most salient. The methodology employed allows for the suggestion that these mechanisms occur in mathematics classroom at different latitudes. Therefore, the teacher's knowledge on said mechanisms may be very useful in planning the evolution of the Culture of rationality in the mathematics classroom.

Keywords: Reasoning and Proof, Teacher Knowledge

Work Objectives and Background Information

Previous papers produced by the authors provide evidence that there is a Culture of rationality in the mathematics classroom (Rodriguez & Rigo, 2015). This research proposes to identify and explain the process of construction, evolution and consolidation of real rationality that reigns in the ordinary classroom of secondary school mathematics education.

The notion of a Culture of rationality is related to the notions of culture and classroom culture (Bertely, 2000), but specifically with the notion of sociomathematical norms (Yackel & Cobb, 1996). These norms regulate mathematics arguments that occur there and consist, for instance, of normative agreements concerning mathematically acceptable explanations. The work of Planas and Gorgorió

(2001) addresses the same topic, in that they focus on finding the "possible learning interferences that may be derived from different interpretations of mathematical norms in the classroom" (p. 135).

Theoretical Framework

The Toulmin Model (1974) was used to analyze the arguments in the case study, according to which an argument is made up of a conclusion (C); data (D), which in this case coincides with the grounds offered to obtain the conclusion; a warrant (W), which is described in the second column of Table 1; and a backing (B), which offers the theoretical, practical or experimental basis for the argument. This study considers two kinds of backing. The first relates to the type of grounds upon which the argument is backed. Epistemic systems are used to characterize these grounds, a term that Rigo (2013) defined to make reference to mechanisms that a person or community habitually resorts to in order to back mathematical facts. The researcher identifies epistemic schemes of a mathematical and extra-mathematical type. The second type of backing alludes to the level of knowledge concerning proportionality theory (operational, intuitive or formal scholarly) that the subject puts forth when arguing. Hence, the backing allows to distinguish, in general, the rationality on which the argument is built. In the research, of which partial results are presented here, the Culture of rationality of a mathematics classroom is integrated, *inter alia*, by:

Norms of sustentation

Habitual and accepted practices for grounds or development of grounds (integrated by a succession of epistemic schemes) regarding mathematics facts, which the agents of the class (students and teacher) carry out in the classroom (cf., sociomathematical norms by Yackel and Cobb, 1996).

Social norms related to the division of responsibilities and on forms of interaction

These norms make reference to the class agent that habitually argues (i.e., provide conclusions and data, and eventually warrants), and the agent that sanctions the arguments and the conclusions, and points to forms of interaction that routinely arise in class for purposes of stating arguments (cf., social norms of Planas & Gorgorió, 2001).

The Culture of rationality is the sum of rationalities that are present in a classroom throughout a period of time. A moment of rationality occurs when a set of arguments are presented that have the same backings in common.

Research Methodology and Data Collection Techniques and Instruments

The research reported in this paper was developed applying the principles of Grounded Theory (Corbin & Strauss, 2015), whose main purpose is to construct -from empirical support- theoretical concepts that explain social phenomena. The analytical tools of Grounded Theory (e.g., axial coding and process-centered analysis) allow for the creation of relationships amongst theoretical concepts on which basis it is possibly to appreciate how the phenomenon being studied is generated, how it evolves and changes, but it also allows for explanations and reasons as to the why of some aspects related to it.

The empirical evidence came from a case study. The case is composed of teacher Noemi and her group of 42 first year of secondary school students. The didactic sequence on the "Proportional division" topic given by the teacher in six fifty-minute modules was examined for the analysis presented here. This sequence was videotaped and transcribed. 37 episodes and 74 arguments were analyzed for this report.

Mechanisms for the Development of a Culture of Rationality in the Mathematics Classroom. Some Empirical Evidence

The section illustrates and characterizes, using passages from the case study, two mechanisms for

construction and development of the Culture of rationality in the analyzed class.

"Forms of Interaction for Development of Arguments" Mechanism

This mechanism is underpinned by the idea that arguments are constructed within a context of interactions among class agents (Krummheuer, 1995). Three patterns related to this mechanism were identified, but only one will be described due to space constraints.

Pattern on the level of the arguments. Once an argument has been given for a conclusion, subsequent arguments (for the same conclusion) are generally more complex. Table 1 shows arguments sorted in an increasingly complex arrangement: The lower portion includes extra-mathematical arguments and the upper portion includes those with mathematical basis, from instances of a rule with intuitive bases (theorems in action, Vergnaud, 1989) to arguments in which mathematical processes are made explicit or are justified. The pattern in reference can be recognized in Table 1; conclusion 4, for example, the first datum given by a student, is composed of an intuitive instance of a rule and in the second datum the Teacher justifies the process. In that same Table 1 it is possible to detect such regularities; for instance, of the 27 cases where first and second data for the same conclusion were identified, the second datum is deeper in 23 of those cases (85%).

"Type and Function of the Arguments: reproductive and productive" Mechanism

Bringing to mind the functions of the test (de Villiers, 1990; Hanna, 1990), in this work the authors have distinguished two types of arguments.

The 'productive arguments' share with analogous arguments, similar data and warrants but with equivalent backings. The agents of the class present this type of argument through 'consolidation interventions', which have the purpose (explicitly or implicitly, consciously or non deliberately) of strengthening and entrenching a rationality and creating conditions upon which new arguments may be constructed, which will eventually lead to a modified rationality that is perhaps richer in its forms of sustentation.

'Productive arguments' are those whose backings offer richer grounds, in a mathematical sense, in terms of the arguments that precede them, and they share the same procedures, rules or tasks to be solved. The agents of the class propose this type of argument through 'interventions of change,' modifying the existing rationality and eventually leading to a new one, whenever it takes root.

Table 1 also shows the types of arguments and their functions. Interventions aimed at consolidating an operational rationality that involves cross multiplication are those that use reproductive arguments 3, 5, 10, 11, 28, 29, 59, 60, 61, and 62, which also share a backing. An intervention of change is one in which productive argument 63 is expressed, and that argument already presents a new backing. Most of these arguments were provided by the teacher who, possibly deliberately, offered the first group of arguments in order to consolidate an operational-type rationality, and presented 63 with the didactic purpose of deepening mathematical insight concerning the rule involved, modifying thus the previous rationality regarding this rule and introducing a rationality based on a theory of formal scholarly proportionality. When productive arguments are shared and reiterated collectively, they can reconfigure a new rationality, which consolidation can create the conditions for a new cycle to eventually occur. In the class analyzed, 85% of the arguments are reproductive and 15% are productive (see Table 1).

Consolidation interventions serve to nourish interventions of change. In this sense, Chevallard, Bosch, and Gascón (1997) hold that in a class, production of new techniques" is supported by the robust mastery of basic techniques" (p. 279) (reproductive arguments) and in the "moments of the technique justification" (p. 263) (productive arguments).

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Final Considerations

Based on the Grounded Theory methodology that was adopted in this work, it can be reasonably assumed that the mechanisms described above take place in the mathematics classrooms of various regions. It would then be important for teachers to take the Culture of rationality that prevails in their class and the dynamics of its development into consideration. In particular, reflecting on the didactic consequences that can entail the imbalance between productive and reproductive arguments –it is possible that the group's conceptual formulation may be fairly weak- or an imbalanced teacher-student participation, favoring the former that prevents them from benefiting from the cognitive results driven by action and interaction in class, once again leading to scarce learning.

Table 1: Trajectories of Arguments and Participation

PA: Productive Arguments; RA: Reproductive Arguments; WC: Warrant supporting the conclusion, AW: Agent providing the warrant: C: Conclusion; AC; Agent providing the conclusion: JE: Empirical Justification of definitions and rules; EPIR: Explicit instance of a rule: RIRAVI: Repetition of explicit instance of a rule to provide institutional endorsement: IR: Instance of a rule (supported by intuitive considerations "I" or formal scholary "EF"; RIROAVI: Repetition of explicit instance of a rule supported by operational considerations to provide institutional endorsement: RP: Instance of a rule supported by operational considerations; RP: Practical Reasons. PIM: Isomorphic property of multiplication; VU: Unit Value; FP: Proportionality Factor; R3: Cross multiplication.

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EFFECTS OF TEACHER KNOWLEDGE AND QUALITY OF INSTRUCTION ON LINGUISTICALLY DIVERSE LEARNERS

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The main goal of this project is to empirically estimate whether and which classroom factors contribute to mathematics gains of English Language Learners in Texas schools. The emphasis is on mathematical knowledge for teaching (MKT), the mathematical quality of instruction (MQI), and language diverse quality of instruction in middle grade classrooms. First round of correlational analysis shows that there are strong associations between teachers' mathematical knowledge for teaching and some elements of MQI such as exposure to rich mathematics and attention to students as learners during instruction. Preliminary regression analysis shows that one of the significant factors that predict student learning is teachers' practices related to knowing their students not just as mathematics learners but affording their linguistic diversity.

Keywords: Mathematical Knowledge for Teaching, Instructional Activities and Practices, Equity and Diversity, Middle School Education

Motivation and Goal

Currently in the United States, significant achievement gaps in mathematics exist between linguistically diverse students and their majority counterparts. Nevertheless, recent mathematics achievement results in the state of Texas show that some districts outperform others despite having large percentages of socio-economically disadvantaged students. Therefore, the main goal of this study is to provide insights and explanations for this phenomenon from the teacher capacity perspective by investigating how teachers' mathematical knowledge, knowledge of students as linguistically diverse learners, and quality of instruction contribute to students' achievement gains. In particular, we explore the relationship between teachers' knowledge and their teaching practices in linguistically diverse middle school classrooms.

Theoretical Framework

Studies in US bilingual mathematics classrooms have at least three different categories of research foci. One deals with the relationship of learning mathematics with cultural issues such as students' households, parents, and communities (Civil, 2007). This perspective does not look at the practice of teaching but instead relates the world of the learners to the classroom. Another deals with the relation of mathematics and 'social activism' (Gutstein, Lipman, Hernandez, & de los Reyes, 1997) and issues of empowering minority students (Celedón-Pattichis, 2004). Gutstein and others looked at the teachers' and principals' beliefs and ideologies in relation to integrating the mathematics with the children's culture, while Celedón-Pattichis investigated the role of parents, educators, and administrators together with school policies and procedures that place English Language Learners (ELLs) in mathematics upon immigration to US schools. Finally, a third category, more pertinent to this study, deals with mathematics practices and language (Civil, 2007; Khisty & Chval, 2002; Moschkovich, 2007). Part of this body of research focuses on the structures of the lesson that provide opportunities for students to talk about mathematics. Civil (2007) explored the learning of mathematics in different settings like regular classrooms, after-school mathematics clubs, out-of-school activities, and with their parents. Moschkovich looks at the use of two languages (code switching) and gestures during mathematical conversations. She provides examples from classroom observations of how the use of two languages can provide resources for mathematical communication, contrary to the view that code switching indicates deficiency. Khisty and colleagues

focus on the mathematical discourse and analyze the role of the teacher. They argue that even students that are proficient in English have difficulty communicating in mathematics discussions because of the difference that exists between 'social language' and 'academic language'. Furthermore, they conclude that the teacher plays a significant role by using her/his own academic talk as a model and support for emerging mathematics discourse. Despite this extensive body of research about the teaching and learning of mathematics for ELLs, there is a lack of detailed understanding of *how* teacher knowledge and classroom instruction affects these students' achievement. Further, there is a lack of empirical evidence about how these variables relate to each other in this population of students. Hence, the two research questions addressed in this study are:

1. To what extent are teachers' knowledge measures and the mathematical quality of instruction in linguistically diverse classrooms associated?
2. What are the effects of teachers' mathematical knowledge, teacher knowledge of students as linguistically diverse learners, and mathematical quality of instruction on student achievement?

Methodology

Measures

The study utilizes established measures for teacher knowledge and mathematical quality of instruction with complementary measures to capture the special knowledge and instruction in linguistically diverse classrooms. The first measure is the Mathematical Knowledge for Teaching (MKT) instrument which includes domains conceived of as pedagogical content knowledge, as well as common content knowledge and specialized content knowledge. One of the main reasons for using this instrument for our study is because MKT has been linked to student achievement gains and to the mathematical quality of instruction (Hill, Ball, & Schilling, 2008). Finally, we capture the classroom practices with the 4-point version of the Mathematical Quality of Instruction (MQI) protocol. This instrument measures several dimensions that characterize the rigor and richness of the mathematics of the lesson, including the presence or absence of mathematical errors, mathematical explanations and justifications, mathematical representations, and related observables. The MQI was augmented by the ELL Mathematical Proficiency rubric developed by the authors based on previous research and validated with a focus group of teachers and a small set of video samples from linguistically diverse classrooms (Sorto, Mejía Colindres, & Wilson, 2014). The elements of this newly develop rubric are 1) Connections of mathematics with students' life experiences, 2) Connections of mathematics with language, 3) Meaning and multiple meanings of words (including gestures, use of two languages, cognates), 4) Use of visual support, 5) Record of written essential ideas and concepts on the board, and 6) Discussion of students' mathematical writing.

Sample and Data Collection

The participants of the study are 34 middle grade mathematic teachers representing all of the 11 middle schools in a large district in south Texas with even distribution among grades 6, 7, and 8. About two thirds of them self-identified as being either advance or high advance in Spanish language proficiency. The average teaching experience is 9.5 years and the majority of them (75%) teach in classrooms with at least half of students classified by the school district as ELLs. About 99% of the students in the district are from low-income homes (qualify for free/reduced lunch).

Teachers responded to the two surveys during a professional development session in summer 2013. The following school year, teachers were videotaped three times (beginning, middle, and end of the school year). A total of 98 lessons were videotaped. Thirty-one teachers were videotaped three times, two teachers were videotaped twice and one teacher was videotaped once.

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Data Analysis and Results

Scoring Quality of Instruction and MKT

Videotaped lessons were broken into seven and one half-minute segments which were then coded by two independent raters using the 30 codes designed to represent the elements of mathematical quality (outlined in Table 1) and ELL mathematical proficiency. The codes used a 4-point scale: *not present*, *low*, *medium*, and *high*. Finally, each lesson was assigned a holistic score that ranged from 1(*low*) to 5(*high*) which represents the overall MQI score. The two raters were trained using the MQI training program (http://isites.harvard.edu) prior to independently coding each lesson and reconciling conflicting codes afterwards. Video scores for each teacher in each element and overall lesson scores were computed by averaging all three lesson scores. The MKT pencil-and-paper responses of the 34 teachers were compared with a larger national sample using a two-parameter IRT model. Scale scores for each teacher were computed using parameters estimates for each item derived from the results of the national sample. The distribution of scores for our sample was very similar to the scores for the national sample of teachers.

Data Analysis and Results

To address the first research question, we explored the relationship between teacher knowledge and quality of instruction. Table 1 shows the correlation between the overall elements of mathematical quality and the MKT measure. All of the associations are positive, except for the errors and imprecisions, which indicate that teachers that tend to make fewer mistakes in class have higher levels of mathematical knowledge for teaching. Two elements, Richness of Mathematics and Working with Students and Mathematics have significant positive associations with MKT; the Overall ELL Math Proficiency is tending to significant ($p = 0.078$).

Table 1: Correlation between MQI and MKT Measures

MQI Elements	Correlation with MKT
Richness of the Mathematics	0.482**
Working with Students and Mathematics	0.359*
Errors and Imprecisions	-0.288
ELL Mathematics Proficiency	0.311^
Common Core Aligned Student Practices	0.256
Overall Mathematics Quality of Instruction	0.285

^ Significant at 0.078 level. * Significant at 0.05 level. ** Significant at 0.01 level.

To address the second research question, we performed hierarchical linear model (HLM) analysis with standardized learning gains as the dependent variable and controlling for social economic status and school effects. The factors that explain learning gains for all students are the level of MKT and Overall MQI, their effect is positive and significant. Positive effects on ELL students include teachers' implementation of tasks and activities that develop mathematics ($p = 0.045$) and ELL Mathematics Proficiency ($p = 0.064$).

Conclusions

The present correlational study yields preliminary results that demonstrate strong links between mathematical knowledge for teaching and the exposure of rich mathematics presented to students and the teachers' ability to attend to and remediate students during instruction. Links are moderate between teachers' knowledge and the ability to implement strategies that support the learning of mathematics for linguistically diverse students. Variables that explain or contribute to students' learning gains for English Language Learners are the teachers' practices related to knowing their

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students not just as mathematics learners but affording their linguistic diversity such as connecting mathematics with students' life experiences, emphasizing meaning and multiple meaning of words using synonyms, gestures, drawings, or cognates, visual support, and access to essential ideas, concepts, representations, and words.

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THE ROLE OF MATH TEACHERS IN FOSTERING STUDENT GROWTH MINDSET

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This paper presents on a year long mixed-methods study that investigated how math teachers might contribute to students' beliefs about their own math potential (mindset). The paper shares results from analysis of survey data that examined whether teachers' beliefs predict students' mindset. This paper also presents findings from analysis of classroom observations and artifacts that identified potential instructional practices that may explicitly or implicitly communicate mindset messages to students. This paper has implications for both researchers and practitioners.

Keywords: Affect, Emotion, Beliefs, and Attitudes, Instructional Activities and Practices

Introduction & Relevant Literature

The belief that math ability is innate – some people have it, others do not, and there is little one can do to change their basic math ability – is particularly prevalent in the United States (Stevenson, Chen, & Lee, 1993) and has been linked to maladaptive behavior such as avoiding challenging work and giving up when encountering difficulty (Dweck, 2000). Such beliefs may contribute to the persistent problem of relatively low performance and low interest in mathematics in the U.S. (Boaler, 2009), particularly among female (National Academy of Sciences, 2006), African American, and Latino (Flores, 2007) students. Improving U.S. students' math performance requires a better understanding of the sources of such beliefs. This mixed-methods study investigates the ways in which math teachers may contribute to students' beliefs and ideas about their own math potential.

People's beliefs about math ability tend to fall along a spectrum which ranges from a *fixed mindset*, the belief that one's math ability is innate and limited, to a *growth mindset*, the belief that math ability is something that is malleable and can be developed through hard work and perseverance (Dweck, 2006). Despite evidence suggesting the value of growth mindsets for student achievement (Aronson, Fried, & Good, 2002; Blackwell, Trzesniewski, & Dweck, 2007), little is known about how teachers might influence student mindsets, particularly in relation to mathematics. To further our understanding of math teachers' potential influence on student mindset beliefs, this paper addresses the following research questions: (1) What is the relationship between teacher and student beliefs related to mathematics and mindset? (2) How might teachers communicate mindset messages in the mathematics classrooms?

Methods & Analysis

The study consisted of two parts, a quantitative and a qualitative component. First, I designed a survey study to address the first research question, which sought to understand the relationship between teacher and student beliefs. I designed and tested survey instruments to measure four belief constructs: 1) student and teacher mindset beliefs related to math ability (e.g., You have a certain amount of math intelligence, and you can't really do much to change it; $\alpha= 0.688$), 2) teacher expectations (e.g., In my class(es) students who start the year low performing tend to stay relatively low performing; $\alpha= 0.726$), 3) teacher beliefs about which students should have access to more complex math (e.g., It is important for students to acquire basic skills before engaging in complex conceptual math; $\alpha= 0.836$), and 4) student and teacher beliefs related to the nature of mathematics (e.g., Mathematics involves mostly facts and procedures that have to be learned; $\alpha= 0.650$). All items were rated on a six-point agree-disagree Likert scale. The survey was administered to 40 middle school math teachers and approximately 3400 of their students at the beginning and the end of the

2013-2014 academic school year. All participants came from six middle schools with economically, linguistically, and racially diverse student populations in California. I examined various correlations between the different belief constructs. Using hierarchical linear modeling (HLM) (Raudenbush & Bryk, 2002), I also examined whether teachers' beliefs predicted students' mindsets and related beliefs at the end of the academic year.

To address the second research question, I examined teacher interactions with students to further our understanding of how teachers communicated mindset messages in the context of their math classes and how such interactions might influence students' beliefs. From the 40 surveyed teachers, I purposefully sampled (Marshall, 1996) eight teachers who represented a range of beliefs about math ability. Over the course of one year, I conducted parallel case studies (Yin, 2009) of these eight case teachers. Through the conduct of ethnographic methods (Eisenhart, 1988), such as teacher and student interviews, classroom observations with field notes and analysis of classroom talk and course materials (e.g., feedback on assessments, wall postings, syllabi, etc.), I examined differences in how these case math teachers communicated particular messages about math ability to their students. Building upon existing frameworks (Boaler, 2013), this study identified potential instructional practices that may explicitly or implicitly communicate mindset messages to students.

Findings

Results from this study found that one's perception about the nature of mathematics was correlated with having more of a growth or fixed mindset related to math ability. This was true for both teachers and students ($\beta = 0.313$, $p < 0.05$ and $\beta = 0.218$, $p < 0.01$, respectfully). Teachers and students with more procedural and rules-based (one-dimensional) views of math were more likely to have more of a fixed mindset. Results from the HLM analysis suggests that teacher mindset at the beginning of the year did not predict students' mindsets at the end of the year, while controlling for students' beginning of the year mindsets. Rather, teachers' beliefs about the nature of mathematics predicted student mindset at the end of the year (Coefficient = 0.212, $p < 0.01$, intercept $p < 0.001$). Thus, teachers who ascribed to more multi-dimensional views of mathematics at the beginning of the year, viewing math as more than procedures and rules, tended to have students with more of a growth mindset at the end of the year.

In the case study, I expanded upon the relationship between mindset and nature of math. Through examination of classroom instruction of the case teachers, I developed a framework to analyze math teaching using a mindset lens that I called the *math teaching for mindset framework*. Through triangulation of the qualitative data (Miles & Huberman, 1994), I identified particular teacher practices that had the potential to communicate mindset messages. These practices were initially identified using both a priori codes determined by past literature related to mindset and emergent codes. I then sorted and classified each practice into four dimensions of math teaching from Boaler's (2013) conceptualization of mindset and mathematics: 1) sorting, 2) norm setting, 3) using math tasks and 4) assessing and giving feedback. In total I identified 15 practices that were classified under these four dimensions. For each of the practices, I identified the nature of the mindset message communicated through each teaching practice, as being more fixed, more growth mindset, or more neutral mindset. I also examined how the practice was or was not connected to a multi-dimensional view of mathematics.

Through this process of sorting and classifying teacher practices, I created a continuum by identifying anchor instances for each of the various practices. These anchor instances represented the relative extreme of a particular practices as being more fixed or growth mindset oriented. Table 1 provides an example of an anchor instance for a sorting practice called "grouping strategies," which I elaborate upon in the proceeding sections[1].

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Table 1: Math Teaching for Mindset Framework

	FIXED MINDSET Math Teaching	GROWTH MINDSET Math Teaching
Dimension #1: Sorting		
Grouping Strategies	Groups based on **one-dimensional** of mathematical success and on **deficit-based** views of students.	Groups based on **multi-dimensional** perspective of mathematics and on **asset-based** views of students.

In classrooms where grouping strategies communicated more fixed mindset messages, teachers intentionally placed a "smart" student in each group who was identified by having a high test score. Such fixed labeling could communicate fixed ability (Dweck, 2006) because some students were viewed as smart and others were not. Additionally, this type of grouping was based on a deficit view (Valencia, 1997) of low achieving students. Students who were considered low performing were viewed as being in need of help and only the "smart" student or the "leader" was capable of helping the student. Under this grouping approach low achieving students felt that they had limited ability to contribute.

In contrast, grouping strategies that were based on a wide range of characteristics, and not solely on perceived math achievement or ability, communicated more growth mindset messages. Teachers who adopted this more growth mindset approach to grouping espoused a more multi-dimensional view of mathematics (Boaler, 1998, 2006) that accounted for multiple ways students could engage each other around mathematical ideas. In these classrooms multiple "smarts" (Cohen, Lotan, Scarloss, & Arellano, 1999) were valued, such as clearly communicating, asking questions, and drawing others out. This type of grouping strategy conveyed that there were multiple ways students could contribute and grow their math ability.

Across the math teaching for mindset framework, a key difference between more fixed and more growth mindset math instruction is the nature of the mathematics that was emphasized. For more growth mindset math instruction across sorting, norm setting, using math tasks and assessing, teachers make explicit connections to the multi-dimensional and open nature of mathematics (Boaler, 2006). Whereas, more fixed mindset math instruction tends to focus on closed (Boaler, 1998) and one-dimensional mathematics with a particular focus on rules and procedures (Boaler, 2002). Thus, as suggested by the survey study, the qualitative analyses also highlight how growth mindset is intricately linked to a multi-dimensional perspective of mathematics.

Ultimately, findings from the survey and case study analyses seem to confirm that having a multi-dimensional view about the nature of math may be an essential component of having a growth mindset towards math ability (Boaler, 2016).

Conclusion

Developing knowledge of the ways teachers might influence student mindset is crucial for leveraging the many benefits of growth mindset for students. Findings suggest that if teachers have closed (Boaler, 1998), highly procedural, or one-dimensional views of math and instruct in these one-dimensional ways, they are likely to communicate fixed mindset messages about math ability to students. In contrast if teachers have more multi-dimensional views of math valuing connection between ideas and multiple strategies and instruct in ways that align with this perspective, they are

more likely to communicate growth mindset messages. In sum, we cannot disentangle having a growth mindset from a multi-dimensional (Boaler, 2006; Cohen et al., 1999) perspective about the nature of mathematics. By empirically identifying opportunities for teachers to convey mindset messages in their math lessons, this study offers authentic examples of growth mindset math instruction. This study can inform best practices for how math teachers can help students develop more of a growth mindset.

Endnotes

[1] Other dimensions of the framework will be elaborated upon in the corresponding presentation.

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EXPLORING THE INSTRUCTIONAL CONSEQUENCES OF A SECONDARY TEACHER'S LEVEL OF ATTENTION TO QUANTITATIVE REASONING

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This paper reports findings from a study that explored the effect of a secondary mathematics teacher's level of attention to quantitative reasoning on the quality and coherence of his instruction of angle measure. I analyzed 37 videos of an experienced teacher's instruction of trigonometric functions to characterize the extent to which the teacher attended to supporting students in reasoning quantitatively, and to examine the consequences of this attention (or lack thereof) on the quality and coherence of the meanings the teacher's instruction supported. My analysis revealed that the incoherencies in the teacher's instruction were occasioned by his inattention to quantitative reasoning.

Keywords: Mathematical Knowledge for Teaching, Teacher Knowledge, Cognition

Introduction

A growing body of research (e.g., Ellis, 2007; Moore, 2012, 2014; Moore & Carlson, 2012; Oehrtman, Carlson, & Thompson, 2008; Thompson 1994) has identified *quantitative reasoning* (Smith & Thompson, 2007; Thompson, 1990, 2011) as a powerful way of thinking that supports students in constructing a meaningful understanding of a wide variety of mathematics concepts. Several researchers have noted that quantitative reasoning is especially foundational for supporting students' conceptual learning of angle measure and trigonometric functions (Moore, 2012, 2014; Tallman, 2015; Thompson, 2008). However, the instructional consequences of teachers' attention to quantitative reasoning (or lack thereof) are less frequently documented. For this reason, I explored the effect of a secondary teacher's level of attention to quantitative reasoning on the coherence of the meanings of angle measure his instruction supported.

Quantitative Reasoning

I leveraged Smith and Thompson's (2007) and Thompson's (1990, 2011) explicit formalizations of *quantitative reasoning* in the design of this study and in my analysis of its data. Quantitative reasoning is a characterization of the mental actions involved in conceptualizing situations in terms of *quantities* and *quantitative relationships*. A *quantity* is an attribute, or quality, of an object that admits a measurement process (Thompson, 1990). One has conceptualized a quantity when she has identified a particular quality of an object and has in mind a process by which she might assign a numerical value to this quality in an appropriate unit (Thompson, 1994). *Quantification* refers to the mental actions involved in conceptualizing an appropriate unit of measure as well as a measurement process, and results in an understanding of "what it means to measure a quantity, what one measures to do so, and what a measure means after getting one" (Thompson, 2011, p. 38).

The quantities one might construct upon analyzing a situation are not limited to those whose numerical values are attainable from direct measurements. Defining a process by which one might measure a quantity often involves an operation on two or more previously defined quantities. In such situations, we say that the new quantity results from a *quantitative operation*—its conception involved an operation on other quantities. Quantitative operations result in a conception of a single quantity while also defining the relationship between the quantity produced and the quantities operated upon to produce it (Thompson, 1990, p. 12). Arithmetic operations, in contrast, are used to calculate a quantity's value.

Methods

The sole participant for this study was an experienced secondary mathematics teacher, David, who taught Honors Algebra II at a large suburban high school in the Southwestern United States. I collected data throughout David's instruction of trigonometric functions. In this paper, I present only the results of my analysis of David's instruction of angle measure.

David taught two sections of Honors Algebra II every weekday during the spring semester of 2014. I video recorded both classroom sessions over a seven-and-a-half-week period of this semester, which resulted in 37 videos of David's teaching. In addition to the video recordings of David's instruction, I generated field notes during the class sessions that focused on characterizing the extent to which David supported students in reasoning quantitatively, and on documenting the mathematical meanings David's instruction promoted.

The procedures I used to analyze the video data are consistent with Strauss and Corbin's (1990) grounded theory approach. My analysis focused on documenting the degree to which David's instruction supported students in: (1) identifying quantities, (2) attending to units of measure, (3) constructing quantitative relationships, and (4) interpreting mathematical symbols and expressions as representing the values of quantities. I organized the coded segments of video into themes, examined the data within each theme, and characterized the extent to which the quality and coherence of the meanings David's instruction promoted was facilitated/impeded by his level of attention to supporting students in reasoning quantitatively.

Results

Meaningfully assigning numerical values to the "openness" of an angle requires that one has identified a quantity to measure and has specified a unit with which to measure it. David's instruction was often inconsistent with regard to the quantity one measures when assigning numerical values to the "openness" of an angle. On some occasions David supported students in conceptualizing angle measure as the length of an arc the angle subtends, while on other occasions he explained that measuring an angle involves determining the fraction of the circle's circumference subtended by the angle. These meanings are not the same. Understanding the fraction of a circle's circumference that an angle subtends as a measure of subtended arc length involves conceptualizing the circle's circumference as a unit of measure for the length of the subtended arc. Specifically, one must recognize that the resulting fraction represents a multiplicative comparison of an attribute being measured (subtended arc length) and the unit of measure (circumference). For example, to say an angle subtends 59/360[ths] of the circumference of a circle centered at its vertex is to say that the length of the subtended arc has a measure of 59/360 in units of one circumference. The two meanings of angle measure David conveyed were distinct since he did not support students in conceptualizing the circumference of the circle centered at the angle's vertex as a unit of measure for the length of the subtended arc. Due to space limitations, the following paragraphs illustrate only two occasions in which David's instruction supported inconsistent meanings of angle measure (from my perspective). I emphasize that the events discussed here are representative of several instances from David's teaching in which he promoted discrepant meanings.

David began his first lesson by asking a student to draw two angles on the whiteboard. The student drew one angle above the other. David then explained that the measure of the angle on top is larger than the measure of the angle on bottom because, if one were to construct two circles of equal radii respectively centered at the vertex of each angle, the angle on top would subtend an arc that is longer than the arc subtended by the angle on bottom. Immediately following this explanation, David asked the question in the following transcript.

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David: When we measure an angle what are we really measuring? I mean it's not like we're measuring a length, right? How would we describe the thing that I'm measuring when I just look at these two angles? …

Student: The openness of the angle.

David: Yeah. Which is weird. How do you measure openness? … I'm not measuring length. … We have to think about what we are actually measuring.

While David previously compared the openness of two angles by attending to the respective arc lengths these angles subtend, he claimed that quantifying the openness of an angle does *not* involve measuring a length. Following the dialogue above, David explained that two angles have the same measure if "the length of the [subtended] arc is the same, as long as I made the circle have the same radius and it was centered at the vertex." David therefore supported contradictory meanings of angle measure during the first lesson; he pronounced that measuring an angle is not a process of measuring a length and then proceeded to compare the openness of two angles, as well as define what it means for two angles to have the same measure, by attending to the arc lengths the two angles respectively subtend. In other words, when speaking of angle measure David did not consistently reference the same quantity being measured.

A few minutes after David's remarks in the above transcript, he projected the image displayed in Figure 1 on the whiteboard.
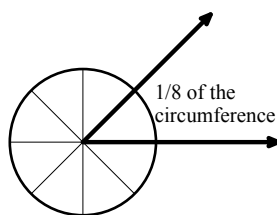
Figure 1. Angle measure as a fraction of the circle's circumference.

After acknowledging that the angle in Figure 1 "subtends 1/8th of the circumference of the circle," David exclaimed that the measure of the angle is a value without units. In particular, David explained that if one measured the subtended arc length and circumference in inches, the ratio of these quantities is unit-less because the inches "cancel" as a result of the division. Moreover, by claiming, "I'm not just measuring arc length. What am I measuring? I'm measuring arc length and comparing it to what? … Circumference!" David did not support his students in seeing the ratio of subtended arc length to circumference as the length of the subtended arc measured *in units of the circumference*. Generally speaking, David's instruction did not provide students with an opportunity to interpret the ratio of subtended arc length to circumference as a quantitative operation but rather as an arithmetic operation; that is, he did not communicate the division of these quantities as a measure of subtended arc length in units of circumference but simply as the ratio of two lengths. Such an emphasis is necessary if one is to support students in conceptualizing angle measure quantitatively (i.e., as a measure of some attribute in some unit). Therefore, while David's instruction during the first lesson overtly emphasized angle measure as a fraction of the circle's circumference subtended by the angle—and implicitly conveyed angle measure as the length of the subtended arc—he did not encourage students to see the former meaning as an application of the latter by failing to support them in conceptualizing the ratio of subtended arc length to circumference as a quantity that represents the length of the subtended arc measured in units of the circumference. In fact, David suggested that these meanings were incompatible by continually asserting that the process of measuring an angle is not one of measuring length.

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Discussion

The results of this study demonstrate that a secondary mathematics teacher's (David) inattention to quantitative reasoning contributed to his conveying incoherent meanings of angle measure and trigonometric functions. In particular, by not maintaining a consistent emphasis on supporting students in: (1) identifying quantities, (2) attending to units of measure, (3) constructing quantitative relationships, (4) interpreting mathematical symbols and expressions as representing the values of quantities, and (5) performing quantitative—rather than arithmetic—operations, the ways of understanding David's instruction promoted were often inconsistent. Moreover, the meanings David supported varied by context because they were not consistently governed by, or in the service of promoting, a particular way reasoning. Consequently, the results of this study suggest that teacher educators should design instructional experiences that allow pre- and in-service teachers to develop the disposition to support students' identification of quantities and quantitative relationships, particularly in the difficult context of trigonometry.

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ACTIVITIES IN CALCULUS I ASSOCIATED WITH STUDENT INTEREST

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Establishing and maintaining students' interest during class has been theorized as vital for learning (e.g., Dewey, 1913) and empirically connected to positive affect (e.g., Bye, Pushkar, & Conway, 2007), information retention (e.g., Siegle, Rubenstein, Pollard, & Romey, 2010), persistence (Hidi, 1990), and academic achievement (e.g., Bye et al., 2007; Koaler, Baumert, & Schnabel, 2001; Siegle et al., 2010). This study reports on a secondary analysis of data collected from the Characteristics of Successful Programs in College Calculus (CSPCC) project (Bressoud, Mesa, & Rasmussen, 2015), where student interest is analyzed using multiple level modeling. Results suggest that college students with more class time for explaining their reasoning, and working individually while taking the course with an experienced instructor tend to be more interested during class, even when controlling for their enjoyment of mathematics.

Keywords: Affect, Emotion, Belief, and Attitudes, Instructional Activities and Practices

Many regard the report by Steen (1988), former president of the Mathematical Association of America (MAA), as the beginning of Calculus reform efforts in the United States. However, following this work, no nationwide study to examine the conditions of college-level, introductory Calculus (calculus) was conducted for nearly 30 years. In 2009, the MAA set in motion a national study to investigate the conditions of college-level calculus and to identify aspects of programs that were consistently experiencing success in this area. The *Characteristics of Successful Programs in College Calculus* (CSPCC) project set out to achieve five goals:

1. To improve our understanding of the demographics of students who enroll in calculus.
2. To measure the impact of the various characteristics of calculus classes that are believed to influence student success.
3. To conduct explanatory case study analysis of exemplary programs in order to identify why and how these programs succeed.
4. To develop a theoretical framework that articulates the factors under which students are likely to succeed in calculus.
5. To use the results of these studies and the influence of the MAA to leverage improvements in calculus instruction across the United States. (Bressoud et al., 2015, p. vi)

To achieve these goals, CSPCC project investigators administered surveys to calculus students, instructors, and department chair persons to a nationally representative stratified random sample of colleges and universities. Additionally, in-depth case studies were conducted to further understand the aspects of successful programs.

Sonnert and colleagues (2014), used hierarchical linear modeling with the survey data to determine characteristics of calculus classrooms, reported by students, associated with students' attitudes towards mathematics. Results of their analysis indicated that, on average, college calculus has a substantial negative effect on students' mathematics attitudes; technology use had non-significant influences; *good teaching* was positively associated with students' attitudes towards mathematics, especially for students who entered college calculus with low attitudes; and *ambitious teaching* was negatively associated with students' attitudes on average, but tended to improve the attitudes of students who entered the course with high attitudes towards mathematics. Good teaching and ambitious teaching were terms coined by these investigators used to describe composites of

variables indicative of each set of pedagogical practices (e.g., listening to students' comments and questions [good] and frequent collaborative work [ambitious]).

The purpose of this paper is to delve deeper into the investigation of students' attitudes and consider their interest in calculus. Hence, this is a report of a secondary analysis of survey data from the CSPCC project. This work can provide further contribution towards the original project's second goal – identifying characteristics of calculus programs associated with student success. Psychologists suggest that interest is an important emotion influencing learning (Bergin, 1999; Dewey, 1913; Hidi, 1990).

Methods

Identifying factors of classroom environments that are positively associated with interest would allow for instructors and departments to make decisions about their practices that enhance the interestingness of their introductory calculus courses. This study conducts a secondary analysis of survey data collected for the CSPCC project in order to empirically describe factors of college calculus classrooms associated with high levels of student interest. Specifically, this analysis aims to address the following research questions: (1) what are the effects of opportunities for students to present, explain their thinking, work collaboratively with their peers, and work individually during class time on students' interest in college calculus when controlling for student enjoyment?, and (2) do these relationships depend on instructors' recent teaching experience with college calculus?

Description of the Data

The CSPCC project administered pre- and post-semester surveys to calculus students and their instructors and a single survey to department chairpersons or calculus coordinators to a stratified random sample of 212 institutions of higher education across the country. This undertaking resulted in survey data from nearly 14,000 students and roughly 700 instructors. However, due to incomplete responses, a subsample of 3,103 students and 308 instructors from 123 institutions are considered for this report. Sonnert and colleagues (2014) utilized the same subsample in their report on the relationship between students' attitude towards mathematics and instructor pedagogical practices. Their report completely describes the sample used here and compares it to the national population of college calculus students. This subsample represents the group of students (nested within instructors, nested within institutions) who have data from all five survey instruments. This dataset possesses a requisite nesting structure – students within instructors within institutions – for which multiple level modeling is appropriate (Raudenbush & Bryk, 2002).

Variables and Centering

One item on the student survey was, "My instructor makes class interesting." Students were instructed to rank their beliefs of this item on a 6 point scale, where 0 represents "Strongly disagree" and 5 corresponds to "Strongly agree." Here, student *interest* (or instructor *interestingness*) will serve as the dependent variable. Additionally, students were asked how often their instructors allowed them class time to collaborate with their peers, present solutions, explain their work, and work individually. Students were also asked to rank their enjoyment of mathematics. Each of these items was also ranked on a 6 point scale, where 0 represents "Not at all" and 5 represents "Very often." These will be used as student-level predictor variables.

Instructors were asked to indicate the number of terms they taught calculus I during the previous five years. This item was reported with a scale of ranged values (e.g., 3-5 times), so a linearized variable was created using the central value from each range. This is the instructor-level predictor variable used in this analysis. All predictor variables were grand-mean centered (Raudenbush & Bryk, 2002) to provide meaningful interpretations of results.

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Analysis

To begin, a three-level unconditional model was created to determine the amount of variation in interest located at each level. This model indicated only significant variation at the student and instructor level, so only two-level models were conducted for the remainder of the analysis. Thus a two-level unconditional model was conducted to examine the partitioning of variation.

Next, a completely unconstrained model was conducted (Model 2). This model estimates the extent to which students vary from the sample (average) slopes (or intercept) estimated for each level 1 predictor variable. Third, only the variability around the slopes of the collaboration and explanation slopes were left in the model (Model 3). Finally, a completely constrained model was conducted (Model 4). This model constrains all variability around the slopes of level 1 predictors to be zero. Models 2, 3, and 4 all contained cross-level interaction terms in order to respond to research question 2.

Results

The parameter estimates for all models were computed using PROC MIXED in SAS 9.4. Model 1 indicates that 26% of variation in student interest lies at the instructor-level, while 74% lies at the student-level. These values are significantly different from zero, which justifies the remaining models. Model 2 did not converge, and according to Singer's (1998) method Model 4 is better fitting than Model 3 ($\chi^2(5) = 74.7, p < .001$). Thus, only the estimates from Model 4 are interpreted.

On average, students agreed that their instructors made class time interesting ($\gamma_{00} = 3.17$, $t = 58.22, p < .001$). Frequency of class time to collaborate with peers ($\gamma_{10} = .04, t = 1.81, p = .07$) and to present in class ($\gamma_{20} = .02, t = 1.19, p = .23$) were not associated with student interest. However, students with more opportunities to explain their work ($\gamma_{30} = .36, t = 18.83, p < .001$) and to work individually ($\gamma_{40} = .07, t = 4.23, p < .001$) tended to report higher levels of interest even with enjoyment present in the model. Students who enjoyed mathematics tended to report higher interest ($\gamma_{50} = .21, t = 11.79, p < .001$). Instructors with more than the sample average experience were associated with higher interestingness ($\gamma_{01} = .04$, $t = 2.82, p = .005$). No cross-level interactions yielded statistically significant results. This suggests that the relationships between student interest and the student-level predictors do not depend on instructor experience ($\gamma_{11} = .004, t = .63, p = .53$; $\gamma_{21} = -.005, t = -.74, p = .46$; $\gamma_{31} = -.01, t = -1.94, p = .053$; $\gamma_{41} = -.001, t = -.12, p = .90$; $\gamma_{51} = -.004, t = -.75, p = .45$). Model 4 explains 28% of student-level variation in interest and 38% of instructor-level interestingness (Raudenbush & Bryk, 2002).

Discussion & Conclusion

Establishing and maintaining students' interest during class time has been theorized as paramount for learning (e.g., Dewey, 1913). In fact, research on the role of interest while learning has demonstrated that interest is positively associated with positive affect (e.g., Bye, Pushkar, & Conway, 2007), information retention (e.g., Siegle, Rubenstein, Pollard, & Romey, 2010), persistence (Hidi, 1990), and academic achievement (e.g., Bye et al., 2007; Koaler, Baumert, & Schnabel, 2001; Siegle et al., 2010). Recently, Pantziara & Philippou (2015), concluded that positive self-efficacy beliefs positively influenced students' interest in mathematics, so it is no surprise that student enjoyment of mathematics was positively associated with student interest.

This study provides empirical evidence for the importance of certain classroom activities and their positive associations with student interest. Additionally, this analysis suggests that department heads and decision makers should consider course assignments for instructors over more than one term. For this subsample of the CSPCC survey data, the 308 instructors had an average of 2 terms in the previous five academic years with teaching calculus, and those instructors with higher than average experience tended to be reported as more interesting. Bergin (1999) concluded that interest is reciprocally related to learning, in that, interest sparks a desire to learn, and learning more increases

interest. Thus, aspects of college calculus classrooms that are positively associated with student interest are synonymous with aspects of effective learning environments.

Acknowledgement

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"BIG SQUARE, ROUND CIRCLE": SPATIAL TALK BY EARLY CHILDHOOD EDUCATORS AND IF IT MATTERS

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The objectives of the present study were (a) to examine the types and frequency of spatial language that early childhood educators (ECEs) naturally engage in during circle times, and (b) to investigate whether spatial language input predicts children's mathematical knowledge. Twelve ECEs participated in the study. Seventy 3- to 5-year-old children's mathematical abilities were pre- and post-tested with The Test of Early Mathematics Ability (Ginsburg & Baroody, 2003). The circle times in six classrooms were video recorded over a six-week period and were transcribed and coded for the frequency and types of spatial talk in which the ECEs typically engaged. Results revealed that ECEs did not spend a substantial portion of time engaging in spatial input, and as such, the amount of spatial input by ECEs was minimally related to preschooler's mathematical competence.

Keywords: Early Childhood Education, Instructional Activities and Practices, Pre-School Education

Introduction

Children's mathematical knowledge prior to formal schooling has important implications for their future success. Mathematics comprehension in the preschool years remains highly stable until at least the second grade, and children with a high level of competence experience a faster rate of growth in their mathematical development than those that enter school with limited mathematical understanding (e.g., Aunola, Leskinen, Lerkkanen, & Nurmi, 2004).

One mathematical domain that is especially important is geometric and spatial ability. For instance, advanced spatial abilities in adolescence predict engagement and success in science, technology, engineering, and mathematics occupations in later adulthood (STEM) (Shea, Lubinski, & Benbow, 2001).

Adult input has an imperative impact on mathematical development. The amount of overall math talk that preschool teachers use predicts the growth of children's mathematical knowledge over the school year (Klibanoff, Levine, Huttenlocher, Vasilyeva, & Hedges, 2006).

However, no studies thus far have explored the spatial language input preschoolers receive at child care centres and how it is related to their overall mathematical competence. This is important because the spatial input children receive has implications for their overall mathematical ability (Cheng & Mix, 2014). Further, high quality interactions between early childhood educators (ECEs) can act as a buffer against poor cognitive outcomes if there is a lack of verbal input in the child's home (Vernon-Feagans & Bratsch-Hines, 2013). Thus, the objectives of this study were to explore (a) the nature of spatial language input in child care settings and (b) whether such input predicts children's mathematical knowledge.

Method

Data Source

The present study was part of a cross-sectional research project evaluating the effectiveness of an early numeracy program on children's mathematical development in a child care setting. The

program did not have a spatial focus, therefore it was not expected that the program would influence the results of the present study on spatial input.

Participants

Seventy (39 boys) children (M_{age} = 47.49 months, SD = 7.84; $Range$ = 34 months to 70 months) were recruited from six classrooms within four child care centres. Approximately 81% of the children were Caucasian. A total of twelve female early childhood educators from the six classrooms were also recruited for the study. All of the early child educators were Caucasian.

The mother's highest education level was used as a proxy for SES (Catts, Fey, Zhang, & Tomblin, 2001). The highest education level attained by mothers was as follows: 21% of mother's completed highschool, 13% had a college education, 26% had a university degree, and 40% had graduate/professional training.

Materials and Procedure

The Test of Early Mathematics Ability- Version 3 (TEMA-3). The TEMA-3 (Ginsburg & Baroody, 2003) is a standardized, one-on-one test that assesses three- to eight-year-old children's overall mathematical understanding. The TEMA-3 is not timed. Testing stops once a child answers five incorrect questions in a row. Children's TEMA-3 raw scores were converted into standard scores (M = 100, SD =15) for analysis. The TEMA-3 consists of two parallel forms: Form A and Form B to avoid practice effects for pre- and post-test purposes.

The current study was conducted over a ten-week period. The first two weeks of the study consisted of pre-testing children on TEMA-3. The first author conducted all of the testing. The circle times were videotaped over the next six weeks. Circle times were chosen for analysis because they are a period in which ECEs purposefully engage in activities and/or teaching with their preschoolers. Thus, these segments were used as a beginning step to evaluating such input. There was one ECE per circle time that was videotaped .The ECEs were asked to act as naturally as possible and to deliver their typical planned circle time activities, though they were aware that the study had a general math focus. No time constraint was imposed. Once the video tapings were completed, the children were post-tested on the TEMA-3 (form B).

Transcribing and Coding

The data collected consisted of a total of 32 videotapes and 37% (12 observations) were secondary coded for inter-coder reliability (Cohen's Kappa = 0.92, Rho = 0.99).

Spatial Talk. The video recordings were transcribed and coded to examine the total spatial talk per circle time in which the ECEs engaged, using the Noldus Observer XT software (The Observer XT 8.0, 2008). The coding scheme was adopted, with permission, from Cannon, Levine, and Huttenlocher (2007). This scheme categorizes spatial words into eight categories:

1. *Spatial Dimensions*: the size of physical objects, people, and spaces (i.e., big, little, small)
2. *Shapes:* any 2- or 3- dimensional object or space (i.e., circle, triangle, rectangle)
3. *Locations and Directions*: the whereabouts of objects, people, and spaces (i.e., left, under, above, between)
4. *Orientations and Transformations*: an object's or person's orientation or transformation (i.e., turn, flip, rotate)
5. *Continuous Amount*: the amount of continuous quantities such as objects, liquids, and spaces (i.e., half, whole, piece, more)
6. *Deictics*: words that are place deictics/ pro-forms and rely on the context of which they are used to determine whether they are used spatially (i.e., here, there, somewhere)

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7. *Spatial Features and Properties*: describe 2- and 3- dimensional objects, people, and spaces (i.e., straight, curvy, round)
8. *Pattern:* words used in a spatial pattern (i.e., next, first, repeat)

The present study used a "word-type level analysis" (Cannon et al., 2007). Spatial words from the coding scheme were identified in the transcripts. Then, each word was considered in the context it was used and only words used in a spatial context were coded. The total frequency of spatial words for each classroom was summed and divided by the total time of video recording gathered for that classroom to calculate a spatial talk per minute score for each classroom.

Results

The total time of all 32 observations was 606 minutes ($M = 18.93$ minutes, $SD = 6.14$, *Range* = 7 to 38 minutes). The ECEs produced a total of 3,675 spatial words across the 32 observations ($M = 114.84$, $SD = 51.01$, *Range* = 36 to 209 spatial words). On average, they produced 6.02 spatial words per minute ($SD = 1.80$, *Range* = 3.05 to 9.67 words per minute). The most frequent types of spatial talk were Location and Direction (58%), followed by Spatial Dimensions (15%), Continuous Amount (12%), and Deictics (10%). The least frequent types were Shapes (1%), Orientation and Transformation (1%), and Pattern (less than 1%).

The second objective was to determine whether the amount of spatial talk engaged in by ECEs predicted children's mathematical ability. Due to the children being nested within classrooms, the current analysis used a multilevel model analysis. Because the TEMA-3 post-test standard score was significantly correlated with SES ($r = 0.40$, $p = 0.001$), the TEMA-3 score at pre-test ($r = 0.84$, $p < .001$), and not correlated with gender ($r = -0.15$, $p = 0.22$), the children's pre-test scores and SES were the only covariates entered into the model.

The full model was: $Y_{ij} = \gamma_{00} + \gamma_{01}(\text{SPATIAL}_j) + \gamma_{10}(\text{SES}_{ij}) + \gamma_{20}(\text{Pretest}_{ij}) + \mu_{0j} + \varepsilon_{ij}$.

The MLM analysis revealed that spatial talk was not a significant predictor of the variability in TEMA-3 post-test standard score ($\gamma_{01} = 1.96$, $SE = 1.04$, $p = 0.06$), though this analysis was approaching significance. SES was also not a significant predictor of TEMA-3 post-test standard score. As expected, TEMA-3 pre-test scores significantly predicted TEMA-3 post-test scores ($\gamma_{20} = 0.73$, $SE = 0.07$, $p < .001$).

Discussion

The present study suggests that 3- to 5-year-old children's spatial environment, and in turn, their mathematical development, may not be adequately supported during circle time in child care centers.

Overall, spatial language input was limited. Many spatial words uttered by the ECEs were not explicitly uttered with the purpose of engaging in spatial teaching. One implication of our findings is that there was minimal talk about shapes, and spatial features and properties. Most children are aware of circle, triangle, square, and rectangle prototypes before preschool. However, by the age of five, children are already rigid in their thinking that a square is not a rectangle (Hannibal, 1999). ECEs should be exploring the characteristics of shapes so that children's concepts of shapes and their categories remain flexible (Clements, 2004, p. 285). The ECEs in the present study did not teach children about the characteristics of shapes, at least not during circle times – a period of purposeful teaching – and thus, children did not get opportunities to be exposed to atypical shapes or shape properties.

The limited input of ECEs' spatial talk may explain why it was not related to children's mathematical knowledge. This analysis, however, was approaching significance and may have been due to the small number of classrooms ($N = 6$) at the second level in the MLM model.

The main limitation is that we only analyzed spatial talk engaged in during circle times. However, this time of day was specifically chosen because of the purposeful teaching that occurs

during circle times. Further investigation throughout different activities during the child's day may provide additional insight to more and different types of spatial talk by ECEs.

The importance of providing a spatial environment is evident, given that spatial abilities provide the necessary tools required for success in STEM related occupations (Shea et al., 2001) and in mathematical achievement (e.g., Cheng & Mix, 2014). Further, studies that show that spatial abilities are malleable (Uttal et al., 2013) and that children's spatial abilities decline over the summer from kindergarten to first grade (Huttenlocher, Levine, & Vevea, 1998), point to the fact that the development of young children's spatial abilities strongly depends on adult input. With an appropriate enriched environment, early childhood educators can thus help foster children's success in mathematics and in their spatial abilities.

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INVESTIGATING THE QUALITY OF MATHEMATICS OFFERED TO STUDENTS IN POST-SECONDARY CONTEXTS

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Keywords: Post-Secondary Education, Instructional Activities and Practices

Several observational tools have been developed to study and describe elementary school mathematics instruction (see Boston, Bostic, Lesseig, & Sherman, 2015). Some of these tools have been linked to student outcomes (Blazar, 2015) and are being used for professional development purposes (Hill, Beisiegel, & Jacob, 2013). Researchers have begun to develop observational tools that would apply to teaching any discipline at the post-secondary level (Hora & Ferrare, 2014), while others have been designed specifically for post-secondary mathematics (Bergqvist, & Lithner, 2012), but few of these tools focus specifically on the mathematics that is taught to students. Since students rely on and are more likely to learn the mathematics they are taught, we argue that such an observational tool is critical for understanding the mathematics that is presented to students at the post-secondary level.

The specific goal of this study was to investigate and describe the mathematics offered by multiple instructors teaching a College Algebra course. The *Mathematical Quality of Instruction* (MQI) instrument (Hill, Blunk, Charalambous, Lewis, Phelps, Sleep, & Ball, 2008) was chosen as a first step in analyzing the mathematics in lessons because of its honed focus on mathematical content in the dimension the *Richness of the Mathematics*. In particular, this dimension includes the codes *Linking between Representations*, *Multiple Procedures and Solution Methods*, *Explanations*, *Patterns and Generalizations*, and *Mathematical Language*. Each of these codes has multiple score points that describe the varying levels of quality.

Over the course of two years, the lessons of three instructors teaching the same College Algebra course were video recorded. Three researchers watched and scored these lessons with the *Richness* dimension, while also paying attention to meaningful mathematical contributions that were not captured by the MQI. Our findings include: (1) the MQI provided a valuable starting point and we were able to single out, score, and describe some of the mathematics that was taught, and (2) in order to capture more, if not all, of the mathematics taught by the instructors, new dimensions need to be created and tested. These findings as well as exemplars from the lessons will be presented.

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HOW CALCULUS STUDENTS AT SUCCESSFUL PROGRAMS TALK ABOUT THEIR INSTRUCTORS

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Keywords: Affect, Emotion, Beliefs, and Attitudes, Post-Secondary Education

According to the PCAST report (2012) increasing the retention rate of the students who enter college intending to major in a STEM field has the potential to significantly decrease the gap between the number of STEM degrees produced and the projected number of STEM degrees needed to sustain the United States position in the global market. While there are many reasons students leave STEM fields, there is a growing body of research that suggests that intending STEM students are switching out of STEM fields due to experiences in their introductory mathematics courses (PCAST, 2012; Rasmussen & Ellis, 2013), including experiencing poor instruction (Bressoud, Mesa, & Rasmussen, 2015; Seymour & Hewitt, 1997). To this end, we seek to better understand student experiences in successful Calculus courses by answering the question, *how do students in successful Calculus programs talk about their instructors?*

The data reported here comes from student focus group interviews undertaken as part of the Characteristics of Successful Programs in College Calculus (CSPCC) project - a five year study focused on Calculus I instruction at colleges and universities across the United States with overarching goals of identifying institutional and departmental factors that contribute to student success. Qualitative analysis was conducted on a subset of questions from the student focus group interviews; this subset of questions helped to identify overarching ways in which students at these institutions talk about their instructors. Our findings include four distinct perceptions students have of calculus instructors and their roles/behaviors in the classroom: (1) Students report instructors concern for their *conceptual understanding* of foundational concepts in calculus; (2) students perceive their instructors as accessible, approachable, and *available* both in the class and out; (3) the instructors promoted an *encouraging atmosphere* in the classroom where students can interact with mathematics; and, (4) students report their instructor as *mathematically diverse*, offering multiple approaches to problems.

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SHIRKERS, WORKERS, AND SHOWBOATERS: SHIFTING SOCIAL POSITIONING TO INCREASE EFFECTIVENESS OF MATHEMATICAL DISCOURSE

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Keywords: Affect, Emotion, Beliefs and Attitudes

Shirkers, Workers, and Showboaters: Breaking Borders

Student self-selected social positioning is a border- constructed to protect beliefs about our capacity to be a mathematician. Communication in the mathematics classroom can suffer from behaviors oriented towards protecting self-identity, rather than actively engaging in learning (Barron, 2003). In order to break through this border, we asked the students to identify the behaviors they use when acting as a novice (Shirkers) or an expert (Showboaters). This was contrasted with a co-regulation stance, showing a willingness to be influenced by others (Workers). Students knew exactly what it looked like to shirk or showboat (see Table 1). Students were able to describe underlying beliefs when shirking- "afraid that people will find out they can't do it", and showboating- "wanting attention, think it is a way to get people to like them". Additionally, these negative behaviors increased when the mathematics became more challenging.

A Common Language for Change

By distinguishing behaviors from mathematical ability, we were able to increase attentiveness and co-regulation among the class. During the study, Jessica suddenly put her head down on her desk and began crying; she was angry and would not respond to peer or teacher help. Later, she identified her behavior as "shirking", but underneath she was "frustrated". We used the Worker list to identify what she can do when frustrated, she chose to ask questions. Jessica became a regular participant in our mathematical discourse and improved her achievement on assignments. This type of intervention was repeatedly used during the teaching experiment. We found that students were able to participate in a discourse about the mathematics at hand, rather engaging in social positioning.

Table 1: Student Generated List of Behaviors

Shirkers	*Workers*	*Showboaters*
Escape to bathroom, nurse	Get started right away	Brag
Play with things	Work quietly, ask questions	Say, "I told you so"
Get someone else to do their work	Say, "I can do it!"	Won't listen to others' ideas
Whine about the work	Help others	Try to finish first
Take the easy way out	Try to understand other people's ideas	Interrupt the teacher and other kids

Adjusting Sociomathematical Mindset Norms

During the teaching experiment, the research team tracked behaviors of the class which impeded learning. The teacher reported that she usually "ignored the behaviors" during prior math lessons. Calling attention to shirker and showboater behavior adjusted negative sociomathematical mindset norms. The teacher asked students to set positive mindset goals before math lessons. The class was praised for choosing the worker moves they generated on their list. This approach broke through self-protective borders, which allowed for a strong mathematical community to blossom.

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TEACHER LEARNING AS A TASK OF TEACHING

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In this poster, I present findings from a qualitative case study that brings boundaries between teaching and learning into question. The literature has many examples of teachers as learners. Teachers are positioned as students, needing knowledge related to content and practice (e.g. Ball, Thames, & Phelps, 2008), or professionals, drawing situated knowledge to select and bring classroom artifacts to a community of practice for continued learning (e.g. Franke & Kazemi, 2001). I posit another conception of "teacher as learner" is needed, wherein learning is framed as a fundamental task of teaching; necessary for carrying out day-to-day work in classrooms.

I analyzed the practice of two teachers who participated in a NSF-funded PD program aimed at helping K-3 teachers develop mathematical knowledge for teaching. A central component of the program was cultivating teacher characteristics thought to be important when teaching in ways that are responsive to students' mathematical ideas. Data analyzed includes observations, transcripts, teacher interviews, and lesson plans.

When responding productively to students' mathematical ideas, teachers set up and took advantage of opportunities to learn while teaching. In other words, teachers who used students' mathematical ideas productively transformed their own learning into a task of teaching. Findings suggest tasks of teaching and tasks of learning are cyclical and mutually reinforce one another. Findings also suggest it may be useful to conceptualize lists like Ball, et al.'s "mathematical tasks of teaching" (p. 400) as comprising both tasks of teaching and tasks of learning (Figure 1), and to consider how teachers might develop their capacity for learning while teaching. Implications for teacher education and professional development are explored.
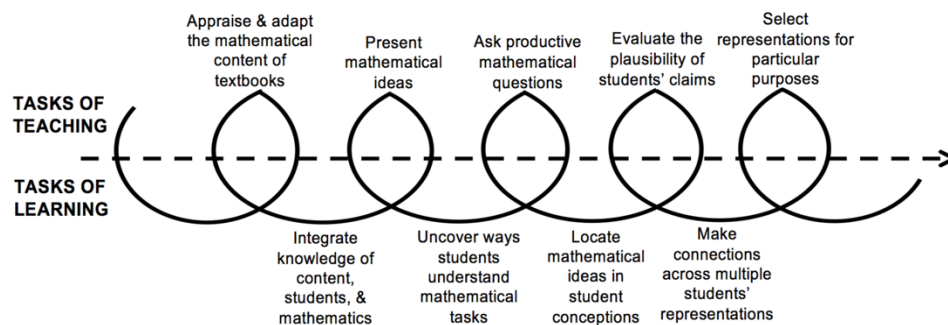
Figure 1. Tasks of teaching and tasks of learning.

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CREENCIAS DE DOCENTES Y ESTUDIANTES DE NIVEL MEDIO SUPERIOR ACERCA DE LA RESOLUCIÓN DE PROBLEMAS

HIGH SCHOOL TEACHERS' AND STUDENTS' BELIEFS ABOUT MATHEMATICS PROBLEMS SOLVING

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Palabras clave: Educación Media Superior, Resolución de Problemas, Creencias de los Maestros

Importancia de las Creencias de Docentes y Estudiantes

Las creencias permiten entender lo que los docentes y alumnos hacen en clase y por qué lo hacen (Lebrija, Flores, & Trejos, 2010). En un estudio realizado en México, De la Peña (2002) encontró que las creencias de los docentes de bachillerato acerca de las matemáticas propician frecuentemente el rechazo de los estudiantes hacia las matemáticas.

Leder, Pehkonen y Töner (2002) plantean que las creencias tienen una gran influencia en cómo aprenden y utilizan las matemáticas los alumnos, que las creencias del profesor regulan sus decisiones, la planificación, el desarrollo y la evaluación de los procesos de enseñanza aprendizaje de las matemáticas, además, que las creencias y las prácticas en la clase forman un círculo difícil de romper. Vila y Callejo (2005) consideran que para destruir este círculo primero es necesario diagnosticar aquellas creencias que no son adecuadas y, después, diseñar actividades dirigidas a los estudiantes que propicien experiencias que puedan desestabilizar esas creencias. Con base en lo anterior el objetivo de este trabajo es identificar las creencias de un grupo de profesores y estudiantes de bachillerato acerca de la enseñanza aprendizaje en la resolución de problemas matemáticos en México. Las preguntas de investigación planteadas son: ¿Cuáles son las creencias predominantes de docentes de bachillerato acerca de la enseñanza de las matemáticas? ¿Cuáles son las características de las creencias de estudiantes de bachillerato acerca del papel del profesor de matemáticas en la enseñanza?

Métodos y Análisis

Participaron en total 730 estudiantes de todos los grados del nivel de bachillerato de escuelas públicas y privadas, y 24 maestros de matemáticas. Las escuelas pertenecen al Estado de Puebla en México. Se aplicó el *Mathematics-Related Beliefs Questionnaire*, a los alumnos, y *Enseñar matemáticas*, a los profesores. Ambos instrumentos son de la autoría de Gómez-Chacón, Op't Eynde, y De Corte (2006).

Las creencias identificadas, tanto en los profesores como en los alumnos, se pueden clasificar en las de etapa ontológica (referentes a qué es la matemática, qué es aprender matemáticas, qué es la resolución de problemas), las de etapa gnoseológica (referidas al aspecto didáctico de las matemáticas) y las de etapa de validativa (que tratan con cuestiones de quién y cómo valida el aprendizaje matemático), tanto en alumnos como en profesores.

Conclusiones

Se destaca que si bien existe una importante dificultad para identificar las creencias y concepciones, entre ellas las referentes a las matemáticas, el uso de instrumentos como los empleados en este trabajo facilita aproximarse a su estudio. Se reflexiona acerca del influjo del tipo de creencias identificadas entre los profesores en aspectos como la búsqueda de estrategias didácticas que les permitan superar las dificultades y las exigencias de enseñanza.

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Keywords: High School Education, Problem Solving, Teacher Beliefs

Importance of Beliefs of Teachers and Students

Beliefs allow us to understand what teachers and students do in class and why they do (Lebrija, Flores, & Trejos, 2010). De la Peña (2002) in a study in Mexico found that beliefs of high school teachers about math, often promote the rejection of students towards mathematics.

Leder, Pehkonen, and Töner (2002) argue that beliefs have a strong influence on how students learn and use mathematics, that teachers' beliefs govern their decisions, planning, development and evaluation of the processes of teaching and learning mathematics and that beliefs and classroom practices form a circle that is difficult to break. Vila and Callejo (2005) consider that to change and improve this circle, first it is necessary to diagnose those beliefs that are not appropriate, and then design activities that foster experiences for students that could destabilize those beliefs. The purpose of this work is to identify the beliefs of a group of high school teachers and students about the teaching and learning process toward solving mathematical problems in Mexico. The research questions are: What are the prevailing beliefs of high school teachers about teaching mathematics? What are the characteristics of high school students' beliefs about the mathematics teachers' role in teaching?

Methods and Analysis

Twenty-four math teachers and 730 High school students from public and private schools participated in this study. Schools are located in the state of Puebla in central Mexico. The instruments used to collect the data were *Mathematics-Related Beliefs Questionnaire* to students, and *Teach mathematics* to teachers. Both instruments authored by Gómez-Chacón, Op't Eynde, and De Corte (2006).

The identified beliefs, both in the teachers and in the pupils, can be classified in the ontological stage (concerning what mathematics is, what learning math is, what problem solving is), those of epistemological stage (referring to the math teaching aspect) and the validative stage (dealing with questions of who and how mathematical learning is validated), for students and teachers alike.

Conclusions

Findings suggest that although there is a major difficulty in identifying the beliefs and conceptions, including those relating to mathematics, the use of instruments as used in this paper facilitates approaching to its study. We reflect upon the influence that the type of beliefs identified among teachers' areas such as the search for teaching strategies that allow them to overcome the difficulties and demands of teaching.

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IMPLEMENTATION OF AN INTERDISCIPLINARY CO-PLANNING TEAM MODEL AMONG SECONDARY MATHEMATICS AND SCIENCE TEACHERS

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Academic subjects in school are traditionally isolated, but drawing connections, especially between mathematics and science, may increase student engagement, performance, and critical thinking skills (Becker & Park, 2011). However, obstacles limit widespread implementation of integrated curricula in schools (Vars, 2001). The Interdisciplinary Co-planning Team (ICT) model was created to overcome these obstacles to offer a sustainable interdisciplinary experience for teachers and students. In the model, teachers of different subjects participate in co-planning sessions on a regular basis to discuss connections that can be drawn between their subjects. They then implement their plans in their own classrooms. The ICT model is iterative, so after implementation, the teachers meet to reflect on their lessons and to co-plan future lessons.
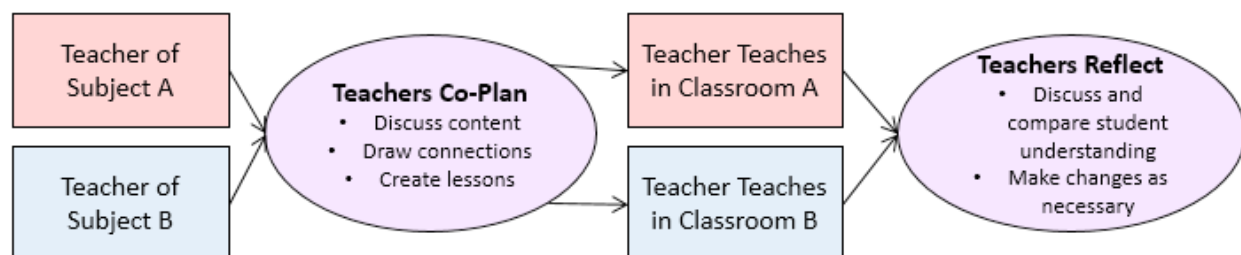
Figure 1. The interdisciplinary co-planning team model.

This study uses a qualitative multiple-case intervention design to examine the nature of teachers' planning processes, plans, and beliefs during an 8 week implementation of the ICT model. The conceptual framework, informed by literature regarding teachers' beliefs and planning practices, expresses the reciprocal relationship among the co-planning process, teacher beliefs, and teaching practices. Data includes observations of co-planning sessions, artifacts of planning (i.e., lesson plans, student worksheets, reflective journals), and interviews.

Preliminary findings include that routine is quickly established, teacher experience seems to affect the nature of co-planning discussions, and that pairs are most productive when they discuss specific concepts, processes, or activities. Regarding teacher plans, science teachers tend to use extended labs whereas math teachers tend to use shorter examples to draw connections. Regarding teacher beliefs and dispositions, co-planning seems to result in courage for teachers to leave their comfort zones, and teachers who engaged in co-planning seemed to have an overall positive experience and plan to continue to collaborate with their partners.

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EXPLANATION AND ARGUMENTATION: A CASE STUDY

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Keywords: Teacher Beliefs, Classroom Discourse, Instructional Activities and Practices

One important aspect of mathematical discourse in a classroom environment is collective argumentation. Yackel (2002) suggested that the teacher plays a particularly important role in facilitating collective argumentation. Pedagogical choices about when to press for justification and what norms are negotiated in the classroom have consequences for how and when students engage in mathematical arguments and even what counts as appropriate explanation and justification (Yackel & Cobb, 1996). In our framework for teachers' support for argumentation, we describe specific moves teachers may use to facilitate this type of discourse (Conner, Singletary, Smith, Wagner, & Francisco, 2014). This research uses our framework to describe one teacher's support for argumentation and link that support to her beliefs about mathematics, teaching, and proof to answer the question: What aspects of Kylee's beliefs explain her choices in supporting argumentation in her classroom practice?

Using qualitative methods that align with a situative perspective on learning to teach mathematics (Peressini, Borko, Romagnano, Knuth, & Willis, 2004), we described her beliefs about mathematics, teaching, and proof as well as her actions in support of collective argumentation. Kylee contributed many of the warrants in her class, and her students' warrants were surrounded by her supportive actions (such as repeating or rephrasing her students' contributions). In Toulmin's (1958/2003) model of argumentation, the warrant is crucial in linking the data to the claim, explaining relationships within the argument. Kylee often expressed a belief that explaining was an important part of teaching: "Knowing that your explanation helped somebody to get it really is exciting to me…I love math anyway, and I was excited about explaining things to people." Her beliefs about mathematics and proof were also related to explanation. Kylee believed that explanation was important, and her patterns of support illustrate a belief that she was responsible for the explanations within her classroom.

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THIS IS NOT MATHEMATICS

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René Magritte's famous painting *The Treachery of Images* depicts a smoker's pipe with writing under it that says "Ceci n'est pas une pipe," which means "This is not a pipe." We apply this play on language to mathematics classroom communication to raise questions about the teaching and learning of mathematics.

When we were mathematics teachers, we generally thought students should be doing mathematics, and we considered them "off task" if they talked about anything else. Perhaps not surprisingly then, our own and many other research analyses of mathematics classroom discourse often distinguish mathematical discourse from nonmathematical discourse (e.g., Setati, 2005). However, drawing on others' work (e.g., Chazan & Ball, 1999) we now explore the storylines intersecting during communication acts expressed by groups of Grade 10 French immersion students, and reflect on the potential for "off task" mathematics for expanding students' language and mathematical repertoires. In one exchange, for example, in a group comprising two males and one female, storylines came together to influence group dynamics and mathematics. The two male students took charge of the mathematics and the communication. Very little "off task" communication occurred but at one point students brought storylines to the table that were seemingly unrelated to the task at hand (discussing their parents). We question how these storylines affected students' communication choices (e.g., choosing to not participate in the case of the female student, a male student choosing to disclose his knowledge of the female student's personal details) and thus the mathematics at hand. Significantly, we wonder if one reason for these bilingual students staying "on task" when doing mathematics is that they are linguistically less equipped to say things "off task" in French.

The classic interpretation of Magritte's painting is that it is not a pipe. It is an image of a pipe. We suggest that being "on task" with mathematics is like an image of mathematics, different from real mathematics used as a tool in complex situations. We call for work that develops ways for engaging students in more complicated situations and explores differences in the way students connect various storylines to their mathematics in schoolwork, especially in additional-language contexts.

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"THEY NEVER TAUGHT US THAT": ACCESS TO MATHEMATICS DISCOURSE PRACTICES AMONG PRE-ENGINEERING COLLEGE STUDENTS

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A sociocultural perspective on the specialized ways of thinking, being and using language in mathematics is valuable in order to understand ways in which underrepresented minority groups (URMs), including Latinos and English learners (ELs) develop specialized discourses. URMs acquisition of these discourses may be compromised by a limited access to these discourses. In this paper, we report on an analysis of the discourse practices of pre-engineering students. Part of a larger study in which a college course for freshman pre-engineering students at a Hispanic Serving Institution (HSI) was re-designed, in this study we focus on a teaching and learning activity. Student teams presented a lesson to their fellow students on an assigned precalculus topic. All students in the course were randomly placed into teams of four, resulting in teams that were composed of students with varying levels of math achievement, (as determined by test placement scores). We video recorded all presentations. Data for this study were selected based on feedback from interviews. Three out of five students noted that the presentations on trigonometry (including functions and unit circle) were the most helpful. We asked: How do college students at varying levels of mathematics achievement differ in their mathematical discourse practices? Using discourse analysis (Gee, 1999), we analyzed the use of multimodal resources including oral language, writing and gestures. We identified several mathematical Discourse practices, suggesting differences among students as to their identity.

Results

The first practice, giving definitions, is reading or recalling memorized phrases. Students with the lowest placement scores were limited to the use of this practice. These students spoke the least during the presentation with an average of 85 words for the entire lesson delivery.

The second practice, giving instructions, is giving test-taking tips, memorization strategies and mnemonic devices. Students might engage in this practice by positioning themselves as "a traditional teacher" in the sense that they seemed to assume that mathematics is about answering test items correctly and doing well on exams.

The third practice was constructing meaning. This practice consisted of making connections to audiences by drawing on various meaning-making resources including the use of writing and visual representations on the whiteboard. Students who had been placed in Calculus and above happened to use this practice the most, though they also used other prior practices as well. These students explained and justified their thinking drawing on multiple representations.

In this paper we have shown differences in mathematical discourse practices to argue that differential access to these discourse practices indexes achievement in college mathematics.

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CHARACTERISTICS OF EFFECTIVE QUESTIONING FOR MATHEMATICAL MODELING TASKS

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Complex modeling tasks on their own may not be sufficient to achieve the desired outcomes of mathematical modeling in the classroom. The literature identifies at least two reasons: the challenges for both the student and the teacher to engage in modeling tasks (Blum & Borromeo Ferri, 2009) and the need for teachers to shift their focus onto students' current knowledge and understanding (Doerr, 2006). Thus there is a need to consider the interactions between student and teacher to harness the potential for mathematical modeling as a learning tool. The strong link between teacher moves (e.g. questioning, scaffolding, or leading discussions) and student outcomes in cognitively demanding mathematical tasks (Drageset, 2014; Henningsen & Stein, 1997) suggests that exploring the ways in which questioning directs students' attention toward the connection between the real world and mathematical aspects of the task would be beneficial. We present a preliminary analysis of the impact of questions posed to 12 high school and middle grades students in task-based, think-aloud interviews. The tasks were mathematical modeling and applications problems. We first analyzed interview transcripts according to a modeling cycle (Czocher, 2016). We then used frameworks for exploring classroom interactions (Drageset, 2014; Henningtsen & Stein, 1997) and qualitative coding techniques to document the impact of interviewers' questions on the students' progress and success. Analysis suggests that any one interviewer question or interaction may not be universally helpful. However, there are ways to scaffold students' mathematical reasoning about real world situations without trivializing the connection between real world assumptions and mathematical constraints. We found that when the interviewer used questioning to initiate parts of the modeling cycle for students it created a collaborative modeling process that led to the student's continued progress within the modeling cycle. For example, the interviewer's questions often became part of the student's validation of her own modeling choices. As Stender and Kaiser (2015) suggested, we also found that asking a student to explain her work gave the interviewer insight into the student's metacognitive process and allowed the student to refocus her thoughts on the task. An emphasis on validating resulted in greater understanding of student thinking, more focused follow-up questions, and sustained student interaction with the modeling cycle.

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TRANSLATING RADICAL CONSTRUCTIVIST RESEARCH INTO PRACTICE: FIRST STEPS

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The Nation's Report Card (National Center for Educational Statistics, n.d.) makes clear that mathematical proficiency for U.S. students, as a group, is not only shockingly low, but is declining. Lack of proficiency is not evenly distributed: Gaps continue between groups based on gender, ethnicity and socio-economic status, creating borders that limit access for some and generate opposing trajectories for others. The intransigence of these problems in mathematics education may be due to the fact that we continue to organize the mathematics curriculum from the adult's frame of reference (Steffe, 2010), so that the mathematics of children is not capitalized on, but rather suppressed, in favor of more conventional ways and means generated from the adult perspective. Radical constructivist researchers call for teachers to organize their instruction around the mathematics of their students, using the models of children's mathematics that can be built by researchers and teachers. A critical need exists for a translating group that knows the current research on the mathematics of children and the realities of the classroom well enough to bridge the divide.

Inspired by the line of research initiated by Leslie Steffe, we are using research, video, teaching experiments and the extant epistemic subjects developed by researchers to assist teachers in building second-order models of student thinking that may explain and predict the student's mathematics (Steffe, 1991). Having knowledge of epistemic subjects influences the teacher/researchers to understand potential differences among student's understandings and to employ a prescriptive course of action that works within a student's zone of proximal construction and might eventually engender a reorganization in the students' thinking (Ulrich, Tillema, Hackenberg, & Norton, 2014).

By focusing on knowledge translation we are able to weave together research and action in a way that is beneficial to the classroom teacher and different from the pressures felt by either the researcher or the teacher (Bennett & Jessani, 2011). The essential aspects of the mathematics of students must be understood deeply by the teacher so that s/he may seamlessly weave between work with the individual and the class as a whole. Therefore, the translator(s) must maintain the integrity of the research, but make decisions about the language and level of detail to use in order to be accessible to the teacher. The translation of research to classroom practice may well be the missing link to true equity in access.

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CHALLENGES OF MAINTAINING COGNITIVE DEMAND DURING LIMIT LESSONS

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In this study, we examined 10 limit lessons to understand students' opportunities to learn cognitively challenging tasks and maintain cognitive demand during limit lessons. Since our results are based directly on day-to-day instructional practices, we hope our results will have more widespread impact in the actual teaching of calculus. The results show that over 90% of tasks were at a low level and both mathematicians were not able to maintain cognitive demand for most tasks. With our findings, we can suggest the following for Dr. A and Dr. B. For Dr. A, more mathematical questions that require explanations and reasoning are needed because over 80% of her questions only required simple and short answers. These questions often sought only correct answers. In addition, instead of taking over the discussion, Dr. A needs to give her students opportunities to reflect and monitor their work. For Dr. B, it is critical to ask more questions, give them time to think about tasks, and provide students opportunities to explain and monitor their thinking. Dr. B tended to lecture and explain everything and, as a result, his students worked on only a few limit tasks and were asked only a handful of mathematical questions. We also recommended professional development for calculus instructors. Professional development at secondary and elementary levels is common, but not often seen often at the undergraduate level. Reflecting on one's own teaching and changing the concept of one's teaching are keys to successful professional development (Hannah, Stewart & Thomas, 2011; Paterson et al., 2011). Such an approach is "bottom-up" rather than "top-down," meaning that instead of developing a policy to be imposed on faculty members, each faculty member reflects on his/her teaching practices. With self-reflection and collaboration, we should be able to identify and provide class-based individualized feedback for each calculus instructor and take a step toward reforming instructional practices in calculus.

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INTEGRAL STUDENTS' EXPERIENCES: MEASURING INSTRUCTIONAL QUALITY IN CALCULUS 1 LESSONS

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In this study, we examined 10 integral lessons to observe the quality of instruction in these lessons. The results of this study confirmed the well-known fact that lecture is still the main mode of teaching in Calculus 1. With our findings, the real question is, "How can we use these results to improve calculus teaching?" One way is to collaborate with mathematicians to share our findings, so they can reflect on their teaching. With our findings, we can suggest the following for Dr. A and Dr. B. Both mathematicians should carefully select and implement mathematical tasks with high level cognitive demand and instead of taking over class discussions, ask more novel questions to elicit responses from students. Also, even if many tasks required only low-level cognitive demand, both Dr. A and Dr. B should attempt to access students' prior knowledge to assess understanding, so that students have opportunities to make connections between procedures and concepts. We also suggest professional development for calculus instructors, similar to what previous studies on secondary and elementary mathematics teachers (Boston, 2009; Henningsen & Stein, 1997). As part of professional development and collaboration, calculus instructors can be informed of the Mathematical Tasks Framework, the maintenance of cognitive demand, and more importantly, be provided with opportunities to reflect on their teaching. These suggestions may sound obvious, but mathematicians are often not trained in education, so it is beneficial to collaborate with them and provide professional development opportunities. Reflecting on one's own teaching and changing the perception of one's teaching are keys to successful professional development (Hannah, Stewart & Thomas, 2011; Paterson et al., 2011). With self-reflection and collaboration, we should be able to take a step toward reforming instructional practices in calculus.

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TEACHERS' BELIEFS: HOW THEY ARE ENACTED IN THE CLASSROOM

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One of the essential elements needed to have effective mathematics instruction for all students is a commitment to access and equity (NCTM, 2014). Unfortunately, this commitment has not been fully realized in many U.S. mathematics classrooms. Many African American students are consistently demonstrating low mathematical achievement, and many educators have refused to acknowledge the racial inequities faced by African Americans. Too often, teachers hold beliefs that do not facilitate the academic participation and learning of African American students (Martin, 2007). The purpose of this study was to examine two middle level mathematics teachers' beliefs about African American students and how they learn mathematics. Specifically, we investigated how these beliefs were enacted during mathematics instruction.

In this qualitative study, we analyzed data from three different contexts: (1) teacher interviews, (2) videotaped classroom observations, and (3) classroom artifacts including lesson plans, teacher handouts, and student work. The data were analyzed using a data reduction approach (Miles & Huberman, 1994) and a constant comparative approach (Charmaz, 2006). Coding was used to generalize patterns and find conceptual coherence in the data (Grbich, 2007).

Both teachers articulated that it was important they relate mathematics to their students' lives and provide opportunities for students, especially their African American students, to talk with their peers about the mathematics. One teacher noted that African American students seem "to be more outspoken…they get really excited and kind of get louder" (Initial Interview, 2015). Although the teachers expressed students, particularly African American students, learn mathematics through sharing and discussing the mathematics with their peers, about 65% of the observed instructional time the students were working individually. When the students were given the opportunity to work in groups or with a partner, the teachers spent a considerable amount of time quieting the students, even when the noise level in the classroom was minimal.

The commitment to access and equity centers on teachers having productive beliefs and high expectations of students who have been traditionally marginalized in the mathematics classroom. Most importantly, the commitment must be enacted in teacher's daily practices and classroom instruction in order to ensure all students, including African American students, have access and opportunities to learn and engage in challenging and rigorous mathematics.

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STUDENT TEACHERS' AND TEACHER LEADERS' TAKE-UP OF EXPLORATORY ("ROUGH DRAFT") TALK

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During classroom discussions, some students prefer to wait to share their thinking until they are certain that their ideas or solutions are correct. These students may not recognize that it is helpful to engage in exploratory or "rough draft" talk (Mercer & Hodgkinson, 2008). Rough draft talk is talking about ideas while they are still in progress by using hesitant, imprecise, and bumpy language. Talking is used as a tool to develop thinking. Instead, students may default to presentational or "final draft" talk. Final draft talk aligns with *performing* rather than *learning*.

Promoting rough draft talk in the mathematics classroom has benefits for both students and teachers. Mathematics students may not be aware that they can *continue* to learn during classroom discussions by refining their thinking through rough draft talk. Mathematics teachers benefit from eliciting students' thinking in progress so that they can access and build upon students' thinking during instruction (Leatham, Peterson, Stockero, & Van Zoest, 2015).

How do middle school mathematics student teachers and secondary mathematics teacher leaders take-up exploratory ("rough draft") talk? Data from this study were drawn from two professional development contexts: a pedagogy course during a student teaching semester and a statewide professional development experience for teacher leaders. Participants in both contexts engaged in a semester-long online study group about exploratory talk. They reflected on readings (chapters from Mercer and Hodgkinson (2008)) and discussed self-selected video clips from their classrooms that exemplified exploratory talk. This work parallels others who have studied how teachers take up a construct introduced in professional development (c.f., Herbel-Eisenmann, Drake, & Cirillo, 2009; Turner, Christensen, Kackar-Cam, Tracano, & Fullmer, 2014).

All participants reported that exploratory talk supported students with making connections collaboratively during discourse and reduced risks during whole-class discussions. When reflecting upon ways to create occasions for exploratory talk, student teachers focused more upon eliciting talk generally and teacher leaders focused more upon opportunities for *exploratory* talk in particular. Teacher leaders additionally reported that they explicitly talked with students about how people learn mathematics, especially the role of talk in learning. Teacher leaders also reported using rough draft talk to position more learners as mathematically competent. Although it is unsurprising that these groups of teachers differed, contrasting the ways that teacher leaders' thinking differed from student teachers' thinking suggested a trajectory for teachers' learning.

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RIGHTS OF THE LEARNER: A FRAMEWORK FOR PROMOTING EQUITY THROUGH FORMATIVE ASSESSMENT IN MATHEMATICS EDUCATION

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Despite calls to address the inequitable opportunities for all children to be successful in school (Sleeter & McLaren, 1995), not all students have such an opportunity, specifically in mathematics (U.S. Department of Education & National Center for Education Statistics, 2015). In order to help more students learn mathematics, there are existing frameworks that help teachers leverage students' mathematical, cultural, and linguistic resources while teaching mathematics (Gonzalez, Andrade, Civil, & Moll, 2001; Turner, Drake, McDuffie, Aguirre, Bartell, & Foote, 2012). If as Bob Moses and the Algebra Project argue (Moses & Cobb, 2002) that mathematics education is seen as a civil right, then teachers should provide specific opportunities to validate the mathematical thinking of each student; and as a result, teachers can help more students connect their in-school mathematical learning with how they use mathematics outside of the classroom. This poster describes another framework aimed at helping teachers to promote equity in the mathematics classroom: The Rights of the Learner.

Framed by sociocultural learning theory (Werstch, 1990), the Rights of the Learner (RotL) was first conceived by an elementary mathematics teacher, Olga Torres, and is comprised of four elements: 1) the right to be confused, 2) to make mistakes, 3) to say what makes sense to you, and 4) to write what makes sense to you. The RotL assumes that all students are afforded an equitable opportunity to take explicit ownership of their learning, both in writing and oral communication. Furthermore, teachers (including teacher educators) can use the RotL to help students embrace productive struggle as a part of the learning process and to recognize mathematical misconceptions and mistakes as being an crucial insights into students' mathematical thinking. Finally, the RotL frames formative assessment as a tangible tool that teachers can use when supporting *all* students to learn mathematics while honoring students' diverse mathematical resources (e.g., native language, out-of-school knowledge and experiences). Implications for future research and teacher education will be suggested.

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REACHING BEYOND STRUGGLE TO PROMOTE MATHEMATICAL PROFICIENCY IN ALL LEARNERS

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Providing high quality instruction for all students has been a unifying theme in mathematics education for years (NCTM, 2014). However, additional research on instructional supports used to develop mathematical proficiency (Kilpatrick, Swafford, & Findell, 2001) in struggling learners is still needed. While it is expected that students will engage in struggle at some point during instruction, the ways in which teachers and students reach resolutions impact students' opportunities to learn substantive mathematical concepts and practices. Consequently, a systematic investigation of the way teachers support struggling learners in mathematics may offer insight on maintaining rigorous instruction when students grapple with complex ideas.

In this exploratory study, I examined how a first grade teacher assisted students who struggled with differentiated tasks while working in math centers. In this research, I defined struggle as moments when "students expend effort to make sense of mathematics, to figure something out that is not immediately apparent" (Hiebert & Grouws, 2007, p. 387). In particular, instances of struggle were identified when the teacher engaged students in tasks designed to remediate students' lack of understanding demonstrated on an assessment or when a student could not make independent progress toward a solution. I identified instances of struggle across 12 lessons related to number sense and measurement and consulted with a second researcher as I developed codes from the data. Three students who consistently struggled to grasp mathematical ideas were the central focus of the data analysis.

Preliminary findings indicate that the teacher used student struggle as an opportunity to engage students in mathematical practices such as providing justifications, evaluating arguments, and refining solutions to problems. She used probing questions that encouraged students to connect their work to other mathematical representations and provide "proof" for claims they made about content. Using these supports usually allowed students to arrive at a resolution independently, but the findings also reveal circumstances in which teacher-directed resolutions tended to occur. Finally, my findings will illustrate similarities and differences between the instructional supports used in lessons related to number sense and measurement.

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PATHWAYS TO LEARNING OPPORTUNITIES FOR CONCEPTUAL DEVELOPMENT: CASE OF KOREAN HIGH SCHOOL CLASSROOMS

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Keywords: Classroom Discourse

This study presents a classroom discourse interactional model that enhances development of students' conceptual understanding, in particular of derivations of the formulas for the finite series, $\sum_{k=1}^{n} k^2$ and $\sum_{k=1}^{n} k^3$. Previous research has pointed out that secondary mathematics classrooms in the United States often lack the intellectual engagement of important mathematical ideas or explicit attentions to connections among ideas, procedures and various representations (Hiebert & Grouws, 2007; Hiebert et al., 2003). This study illustrates a classroom model using detailed examples from a Korean high school mathematics classroom where classroom activities ultimately focused on making connections to the core concepts. This model provides an image of mathematically productive learning opportunities for development of conceptual understanding, particularly in secondary mathematics classroom contexts.

The analysis with the mathematical discourse framework (Sfard, 2007, 2012) focused on how the teacher (1) provided mathematical commognitive conflicts for students to explore the patterns of the finite series $\sum_{k=1}^{n} k^2$ and $\sum_{k=1}^{n} k^3$, (2) guided them to make connections between the manipulatives (e.g. unicube blocks) and visual representations, and (3) led them to make explicit connections to the symbolic expressions. The results show that students did not immediately take into account the representations in a traditional mathematical sense but their mathematical understandings emerged and developed in social discourse activity where the participants—teacher and students—supported development of conceptual understanding through discourse. In the poster, the specific features of the teacher's use of words and visual mediators will be provided with detailed examples from the classroom observation data.

This model sheds light on our understanding on the process of teaching-learning interaction as a social activity in a mathematics classroom. This study also methodologically contributes to the potential of the detailed analysis on the development of conceptual understanding in the process of teaching-learning interaction in classroom research. Implications on improving practice such as professional development for concept-focused instruction or curriculum design will be discussed. This study particularly addresses the conference theme, *questioning borders within mathematics education across two countries,* by serving as a tool for understanding on productive learning opportunities for development conceptual understanding using international classroom data.

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STUDENT PRESENTATIONS AS A LOCUS FOR DEVELOPING INTELLECTUAL AUTHORITY AND MATHEMATICAL PRACTICES

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Keywords: Instructional Activities and Practices, Standards, Middle School Education, Classroom Discourse

Mathematics reform movements, most recently, the Common Core State Standards for Mathematics (CCSSM, 2011) focuses on the mathematical practices intertwined with mathematical content across grade levels as critical aspects of mathematical learning. In an effort to understand how classrooms can become environments that support students' development of the mathematical practices, we focused our research inquiry toward a particular structure, student presentations. During student presentation time, students are more genuinely responsible for developing their mathematical arguments, having opportunities to present their thinking, learning to analyze other peoples' arguments and reaching agreement on shared understanding (Engle & Conant, 2002; Lampert, 2001). These presentations are both public and place the spot light directly on one or two individual students. Hence, student presentations may have substantial potential to promote students' intellectual authority and to develop their mathematical practices.

In this study, we are investigating pursuing the following questions: (1) In what ways do students engage in the mathematical practices during presentation to their classmates? And (2) In what ways do the student presentations promote the intellectual authority of the student presenters?

The study uses video data of three lessons in two middle school mathematics classrooms, drawn from a larger research project, examining classroom use of lessons specifically developed to support the mathematical practices and formative assessment. Video recordings of the student presentation portions of each of the lessons are transcribed and coded for opportunities for students to engage in reasoning quantitatively and abstractly and development and critiquing mathematical arguments. The interactions between the student presenters and the teacher in preparation for the presentation, during and immediately following the presentation are analyzed based on Engle's (2010) notion of four kinds of intellectual authority: *intellectual agency, authorship, contributorship,* and *local authority*.

Preliminary results indicate substantial differences between the two classes in student engagement with mathematical practices during mathematical presentations. These differences correspond to differences in the intellectual authority afforded to these students. In addition, the analyses reveal strengths in how each teacher prepares for and interacts with the student presentations, as well as suggesting how student presentation can be better leverages to support learning. This study particularly addresses the conference theme, *questioning borders within mathematics education,* by serving as a tool for understanding on development of student authority and of mathematical practices across grade levels.

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ADDRESSING DILEMMAS OF SOCIAL JUSTICE MATHEMATICS

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What constitutes student success?

Social Justice Mathematics (SJM) educators explicitly aim to develop students' sociopolitical consciousness in addition to teaching mathematics content (Gutiérrez, 2013; Gutstein, 2006). These multiple goals, create a question of "What constitutes student success?" If a student uses mathematics to save their homes from being demolished through advocacy work with city officials, but the student does not pass the required math exit exam, would this student be considered successful, or vice versa? Critical race scholar William Tate (1995) poses the question, "Is it possible to develop high-level mathematical competence for African American students within a Eurocentric paradigm?" Rochelle Gutiérrez (2013) suggests "playing the game while changing it" to improve students' facility with both *critical* and *dominant* mathematics.

What is the curriculum for SJM instruction?

Second, is the dilemma of the actual SJM curriculum, or the projects and activities to be developed for one's students. Students themselves should propose the social issue(s) they wish to investigate and use mathematics to analyze and take action to solve such problems. This empowers students and fosters a co-constructed classroom space, rather than the teacher choosing and designing a mathematics project around a social issue s/he thinks is relevant to his/her students. Opportunities for collaboration to develop SJM lessons and projects are helpful, with teams of teachers, student advisors, and/or with outside guests from local universities.

How can teachers gain sociopolitical consciousness?

When developing SJM lessons and projects, teachers must have an awareness of students' lives. However, teachers may or may not be from nor live in students' neighborhoods and therefore may or may not possess the sociopolitical consciousness needed to create meaningful SJM projects. Students may be able to help teachers gain sociopolitical consciousness through respectful dialogue between students and teachers and community members and families. This is a great way to cultivate a co-constructed classroom space because students take the lead as experts. These dilemmas of SJM may be addressed through collaboration of students, educators, families, and even university professors and graduate students to empower students to succeed in both critical and dominant mathematics.

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THE INFLUENCE OF TEACHER AFFECT ON RETENTION OF URBAN MATH TEACHERS OF COLOR

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Objectives

Urban schools often struggle to find certified math teachers who remain in the classroom. Teachers of color are more likely to work in diverse urban schools (e.g. Ingersoll & May, 2011), making their retention a potential means of reducing math teacher shortages. Studies also indicate that teachers of color produce better academic results for students of color than white teachers (e.g. Achinstein et al., 2010). The present study investigates reasons for longevity of eight math teachers, six teachers of color and two white teachers, in one urban Title I public school with high teacher retention, Beachside High School, or BHS (pseudonym). BHS serves 98.5% students of color with 100% considered economically disadvantaged. I focus on these research questions.

1. What reasons do participants give for job satisfaction (and how might this differ by race)?
2. What conditions, factors, and/or experiences influence participants' longevity in the classroom (and how might this differ by race)?

Methods

All eight math teachers with more than five years' experience at Beachside High School, or BHS, sat for a 60-minute interview. All interviews were audio recorded with participants' consent and transcribed through a transcription service. Interview data was coded and re-coded with Atlas.ti through a grounded theory approach to uncover themes to answer the research questions.

Results

Student Interactions

All teacher participants, both of color and white, enjoy interactions with students. Participants enjoy students' "aha" moments, or when the "lightbulb" goes off. They also enjoy learning from their students. Teachers expressed that they are "learning from students every day," and are "inspired" and "empowered" by students. Third, not surprisingly, participants enjoy gratitude and thank-yous from students.

Meaningful Work in an Urban School

Participants explained that they felt more useful, or more needed, in an urban under-resourced school like BHS and did not express fear of the community. When asked if participants chose to teach at BHS intentionally or by coincidence they affirmed their commitment to teaching in an urban school. Teachers described teaching at BHS because it is "where the most need is," and "where the struggle is." Math teacher longevity in urban schools may be improved by cultivating positive relationships between teachers and students.

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DOES OBSERVATIONAL INSTRUMENT CHOICE MATTER IN TEACHER EVALUATION?

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Along with other accountability measures, high-stakes teacher evaluation is on the rise in the United States. Although teacher evaluation systems vary district by district and state by state, all fifty states require observation to be part of their evaluation systems (Hull, 2013). This study focuses on the role of subject matter content in various observational instruments used for both instructional improvement and decisions about teacher retention. To study whether and how instrument choice affects outcomes on observational assessments of mathematics teachers, ten videotaped mathematics lessons were coded using six different instruments. The video analysis followed Erickson's "manifest content approach" (2006). Four of the instruments used were mathematics-specific (RTOP; M-SCAN; TRU; MQI); two were not (Danielson; Marzano). In order to compare ratings, nine major constructs present in all six instruments were developed.

For nine of the lessons viewed, the choice of observational instrument made little difference on ratings outcomes. For one lesson, however, the choice of instrument produced different results. Although this only represents 10% of our sample, in a high-stakes environment, this is cause for concern. These difference results can be attributed to the following: a) Instruments vary in density about teaching and learning of mathematics. b) Mathematical content substance varies across mathematics-oriented instruments. c) Some instruments highlight differently the standards of mathematical practices, and other math processes. d) Instruments are sensitive to mathematical content differentially, and lesson types vary.

These findings suggest that observational instruments might be productively employed for establishing shared language for instruction and improving instruction, and instrument choice can be directed towards targeted areas of need. But since a lesson could be rated differentially on different instruments, caution is called for in the use of observation assessments to inform high-stakes teacher retention decisions.

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APREHENSION DE CONCEPTOS POR MEDIO DE LA INVESTIGACIÓN: ESPIRAL DE CONOCIMIENTO MATEMÁTICO

APPREHENSION OF CONCEPTS THROUGH RESEARCH: MATHEMATICAL KNOWLEDGE SPIRAL

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Palabras clave: Actividades y Prácticas De Enseñanza

Antecedentes

El desarrollo de competencias matemáticas en el estudiante implica una serie de retos en los procesos de enseñanza. Una alternativa es el uso de recursos tecnológicos, sin embargo estos deben ir articulados con una estrategia didáctica para alcanzar los propósitos de aprendizaje. Este trabajo aborda el ¿cómo los estudiantes desarrollan competencias matemáticas al aplicar la estrategia de investigación con el apoyo de sensores? Se sustenta en Hitt (2003), para apoyar el uso de los recursos tecnológicos para la visualización así como el manejo de diferentes representaciones semióticas de los conceptos, (Duval, 1993). Los resultados muestran que los estudiantes desarrollan diferentes competencias matemáticas: A partir de lo anterior, se detalla una de éstas experiencias. El salón de matemáticas es un laboratorio permanente para explorar estrategias didácticas. Es así como desde el 2010, en la clase de análisis numérico del sexto semestre de la licenciatura en matemáticas aplicadas, se explora la estrategia didáctica de investigación. La tarea: los estudiantes deberían realizar un trabajo de investigación para aplicar y profundizar los contenidos del semestre. Es así como, al inicio del semestre se les presenta los lineamientos de ese trabajo de investigación: delimitación del tema clara y pertinente, justificación, definición de objetivos, sustento teórico matemático relacionado con los objetivos, ejecución práctica del tema investigado, contrastación entre la teoría y la práctica.

Objetivo, Metodología, Desarrollo y Resultados

El objetivo fue documentar las competencias matemáticas que desarrollan los estudiantes al implantar la estrategia didáctica basada en investigación y con el uso de sensores. La investigación fue cualitativa y la metodología ACODESA Hitt (2007) y Hitt y Cortés (2009), cuyas fases son: 1. El trabajo individual, comprender la tarea, 2. El trabajo en equipo con los procesos de discusión y validación, 3. Debate, 4. Auto reflexión y 5. Institucionalización del conocimiento. La población: un equipo conformado por dos estudiantes de licenciatura. Resultados por fases: 1. Los estudiantes delimitaron su trabajo: comparar los valores del Ph del suelo en diferentes zonas de la ciudad de Durango (Dgo). 2. Identifican y precisan los conceptos matemáticos: interpolación lineal y pruebas de hipótesis. 3. Se genera el debate, en forma de espiral permeando todas las fases.4. Los procesos de auto reflexión se constatan en cada una de las asesorías. 5. La institucionalización de conocimientos se comprueba en tres momentos. El primero, al término de su trabajo semestral, ellas articularon las representaciones gráficas, tabulares y los modelos algebraicos de regresión lineal. El segundo, ellas retoman el problema de la medición del Ph con diseño de experimentos. Tercero, ellas desarrollan su tesis en las zonas agrícolas en la ciudad de Dgo. Las competencias desarrolladas fueron: pensamiento reflexivo, analógico, toma de datos y procesamiento de información (López-Betancourt, 2013). La aprehensión de conceptos matemáticos es una evolución de los procesos mentales de los estudiantes. Al enfrentar la solución de un problema en contexto detona una activación de procesos mentales para acceder a los conceptos matemáticos y darles significado.

Keywords: Instructional Activities and Practices

Background

The development of mathematical skills in a student involves a number of challenges in the teaching processes. An alternative is the use of technological resources; however, these must be articulated with a teaching strategy to achieve the purposes of learning. This paper addresses: How do students develop math skills by applying a research strategy supported by sensors? Based on Hitt (2003), to support the use of technological resources for visualization and handling different semiotic representations of concepts (Duval, 1993). The results show that students develop different mathematical skills: from the above, one of these experiences is detailed. The math classroom is a perpetual laboratory to explore teaching strategies. Thus, since 2010, in the class of numerical analysis of the sixth semester of a degree in applied mathematics, teaching strategy research is explored. The task: students should conduct a research to apply and deepen the contents of the semester. Thus, at the beginning of the semester, they are presented the guidelines of this research: delineation of clear and relevant topic, justification, definition of objectives, mathematical theoretical basis related to the objectives, practical implementation of the research topic, contrasting between theory and practice.

Objective, Methodology, Development and Results

The aim was to document the math skills that students develop to implement a research-based teaching strategy and the use of sensors. The research was qualitative and used the ACODESA methodology Hitt (2007) and Hitt and Cortés (2009), whose phases are: 1. The individual work, understand the task, 2. Teamwork with the processes of discussion and validation, 3. Debate, 4. Self-reflection, and 5. Institutionalization of knowledge. Population: a team made up of two undergraduate students. Results by phases: 1. Students delineated their work: compare soil pH values in different areas in Durango city (Dgo). 2. Identify and acquire mathematical concepts: linear interpolation and hypothesis testing. 3. The debate generated a spiral permeating all phases. 4. Self-reflection processes are detected in each of the devices. 5. The institutionalization of knowledge is checked three times. The first, at the end of its six-month work, they articulated the graphics, tabular and algebraic linear regression models. The second, they retake the problem of measuring pH with design of experiments. Third, they develop their thesis in agricultural areas in Durango city. The skills developed were: reflective thinking, analogic, data and information processing (López-Betancourt, 2013). The apprehension of mathematical concepts is an evolution of the mental processes of students. Facing the solution of a problem in context triggers mental processes activation to access mathematical concepts and give them meaning.

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MATHEMATICS AND DIGITAL CITIZENSHIP: BRIDGING CONCEPTS WITH TECHNOLOGY

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Keywords: Technology, Instructional Activities and Practices, Teacher Knowledge

Middle school is a tremendous period of change for students. Among notable changes, student interest and self-perception in mathematics declines most in middle school (Holdren, 2010). Additionally, Gonzalez, Andrade, Civil, and Moll (2001) finds individuals are more likely to think of math as a "school" subject with complex algorithms instead of recognizing the mathematical thinking underlying their daily lives. This poster presents how one teacher used the Analyze Design Develop Implementation Evaluate (ADDIE) model to design a technology-based lesson for bridging mathematical reasoning and digital citizenship.

"Technology is an "indispensable feature of [the mathematics] classroom… [T]he platform of the technology is less important than the functionality that it provides" (NCTM, 2014). Koh (2015) argues in order to design a lesson utilizing technology, teachers should incorporate design thinking (ADDIE model). The interactive lesson used for this study embedded mathematical questions in a presentation about digital citizenship and allowed each student the opportunity to answer questions using Nearpod. The use of technology allowed the teacher to monitor and get instant feedback on posed questions.

Student responses highlight which questions sparked the most effective connections between mathematical reasoning and digital citizenship. Question 4, "In 2013 Bruno Mars had his songs illegally download 5,783,556 times. If each time Bruno sells a song he makes a quarter, how much money was stolen from Bruno because of illegal down loads?" and Question 5 "If people were stealing a million dollars from you how would you feel, and what would you do if you were Bruno Mars?" work together to blend mathematics and digital citizenship. Students did not have to accurately complete mathematical computations (question 4) to apply reasoning and problem solving skills (question 5) when evaluating the ideals of good digital citizenship. After evaluating student learning, the teacher returned to the cyclical ADDIE model reanalyzing the lesson goals to redesign the content focus. Redeveloping the lesson using ADDIE created an effective and efficient bridge between digital citizenship and mathematics.

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CONNECTING MATHEMATICS WITH LITERACY: COMPREHENSION STRATEGIES AND MATHEMATICS PROBLEM SOLVING

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Keywords: Elementary School Education, Problem Solving, Instructional Activities and Practices

Research Purpose

Many research studies have demonstrated a strong correlation between reading and mathematics achievement (e.g. Fuchs, Fuchs, Eaton, Hamlett, & Karns, 2000). Understanding, text and mathematical concepts, are the goal for both literacy and mathematics teachers. Using comprehension skills with mathematics has not kept pace with other disciplines. There is a need for more research that links specific comprehension strategies with specific mathematics content (e.g. algebra, number and number sense, geometry). If such connections exist, mathematics instruction could include specific comprehension strategies. In particular, our research examined released standardized mathematics tests to explore the possibility of this connection.

Methods

Our initial findings come from a larger content analysis (Krippendorf, 2013) study. We used three third grade released standardized tests and our unit of analysis was each test problem. We conducted a qualitative analysis to identify the comprehension strategy or strategies using the constant comparison method (Glaser, 1965) on the first two tests and axial coding on the second set. We conducted inter-rater (two literacy experts) reliability measurement on the third test.

Results/Conclusions

Our inter-reliability measure was .91, outstanding (Landis & Koch, 1977). First, we identified 14 comprehension strategies that we used as codes. Reduced to eight, we created operationalized definitions for each: author and me, clarifying, cognitive flexibility, compare and contrast, fluency, graphics, prior knowledge, think and search. We found that number and number sense may best be supported by the comprehension strategy graphics; computation and estimation by fluency, author and me, and graphics; measurement and geometry by graphics and prior knowledge; probability and statistics by graphics and clarifying; and algebra by clarifying, think and search, and fluency. Currently, we are analyzing three fifth grade state released mathematics tests. Next, in-service teachers will use our codes to identify the comprehension strategy or strategies for each problem using the same tests to validate our findings.

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STUDENT MATHEMATICAL CONNECTIONS IN AN INTRODUCTORY LINEAR ALGEBRA COURSE EMPLOYING INQUIRY-ORIENTED TEACHING

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Keywords: Instructional Activities and Practices

Linear algebra is a subject that many undergraduate students find difficult for a number of reasons; one such reason is the overwhelming number of mathematical connections students are expected to learn. Many of these connections can be illustrated through theorems like the Invertible Matrix Theorem (IMT), which presents a list of statements logically equivalent to the statement "a square matrix A is invertible" (Lay, 2011). While the IMT is arguably the most famous theorem in introductory linear algebra, it is only applicable to square matrices; however, similar theorems of logical equivalence can be formed for matrices of any size. Viewing logical equivalence as a bidirectional logical implication, many of the connections in linear algebra at an introductory level could be viewed as *connections as logical implications*.

There is currently a growing body of research on inquiry-oriented learning in linear algebra. While such research is an important contribution in providing resources for instructors to implement inquiry-oriented materials in a linear algebra class, factors such as time constraints and class size can make it difficult to foster inquiry in an undergraduate class. Rather than abandoning all hope of implementing inquiry in the face of these challenges, I aim to find areas in linear algebra where inquiry could be implemented in a practical, realistic sense. To that end, I have followed Rasmussen and Kwon's (2007) suggestion of viewing inquiry in a mathematics classroom through Richard's (1991) definition of *mathematical inquiry* as the mathematics of mathematically literate adults; this includes participating in mathematical discussion, solving new mathematical problems, and forming mathematical conjectures. I then define *inquiry-oriented teaching* as a practice of creating opportunities for students to participate in mathematical inquiry. With this definition, I have investigated the following research questions:

- How can a teacher implement inquiry-oriented teaching in an introductory linear algebra course?
- How do students connect linear algebra concepts through logical implications and logical equivalencies when provided with opportunities to engage in mathematical inquiry?

In an attempt to answer these research questions, I have conducted an action research study in which I taught an introductory linear algebra course that implemented both inquiry-oriented teaching and traditional lecture. Preliminary results suggest that while there are both advantages and disadvantages to this hybrid approach, there appears to be important differences in how inquiry-oriented teaching and lecture each provide opportunities for students to form logical implications.

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ANALYSIS OF SINGULAR-PLURAL DIALOGUE OF MATHEMATICS TEACHERS

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Introduction

This presentation reports on the continuing analysis of an empirical comparison of singular-plural usage during lessons by specially chosen mathematics teachers (whose classrooms demonstrated healthy student dialogue) with the usage seen in "typical" American mathematics lessons (Ricks, 2015). One indicator of functional mathematical learning communities is healthy student dialogue (when students talk *to each other* about mathematical ideas); for students to talk to each other about their mathematical ideas—especially in the public space of whole-class discussions—a classroom cultural climate must be carefully cultivated. I hypothesized that in classrooms with such healthy student dialogue, teacher dialogue would manifest lower singular (*me*, *I*, and *my*) usage compared to plural (*us*, *we*, and *our*) usage.

Methods and Results

The teachers demonstrating classrooms with healthy student dialogue participated in *The Mathematics Class as a Complex System* (MCCS) study, described in greater detail elsewhere (Ricks, 2007). For comparison, I chose the four American lessons with publicly available transcripts of classroom dialogue that were highlighted in the 1999 Trends in International Mathematics and Science Study (TIMSS) video study as "typical" classes (NCES, 2003). As reported earlier (Ricks, 2015), previous analysis of classroom dialogue (one lesson per MCCS teacher) demonstrated that the MCCS teachers had similar singular-plural usage with each other (average: .42), which differed significantly from the TIMSS mathematics teachers (average: 1.26). Thus, all the MCCS teachers had low singular-plural usage (using singulars less than half as much)—as hypothesized, while the "typical" American teachers generally had a high singular-plural usage (using singulars more than plurals, some higher than twice as much). Further analysis also revealed similar growth patterns of the ratio of singular-plural usage during lessons for the MCCS teachers. The comparable consistent rise of cumulative usage is a surprising regularity across the MCCS teachers, especially when compared to the irregularities of the ratios for the TIMSS teachers. This presentation focuses on this similar growth pattern over time.

Discussion and Conclusions

These additional findings suggest that an underlying dynamic may be regulating the MCCS teachers' singular-plural usage because their language growth patterns over the course of their lessons were so similar. More research is needed to understand the relationship of singular-plural usage as an indicator of potential healthy student dialogue, and what might encourage its growth.

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HOW DO SECONDARY MATHEMATICS TEACHERS MANAGE (APPARENTLY) INCORRECT STUDENT RESPONSES?

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Though there are numerous teaching practices that impact student learning (e.g. facilitating whole class discussions, giving explanations, etc.), one ubiquitous practice embedded within many of these is the mathematics teaching practice of managing students' (apparently) incorrect responses. Enacting this practice skillfully necessitates being "responsive" to student thinking and ideas (Pierson, 2008) which is no small feat. Managing students' responses involves complex cognitive processes and resources utilized by mathematics teachers in-the-moment to hear, interpret (hence, the qualification that responses are *apparently* incorrect), and respond to what a student has said (Jacobs, Lamb, & Philipp, 2010). These needed cognitive resources include both specialized skills and specialized content knowledge, such as mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008), that a teacher must use while simultaneously navigating her obligations to the mathematical discipline, individual students, the class as a whole, and the institution in which she works (Herbst & Chazan, 2012). It is no wonder, given these obligations and this complexity, that this practice can be incredibly stressful and difficult to enact, even for veteran teachers.

Though several lines of research are germane to the practice of managing apparently incorrect student responses, the practice itself is neither well-understood nor described in mathematics teaching. This poster presents preliminary findings from a study that explores this practice by using records from classroom observations of four secondary mathematics teachers across four to five consecutive lessons in each of their classrooms. Pre and post lesson interviews also provided information about what teachers anticipated students might struggle with and how teachers thought about particular instructional moments when students provided an apparently incorrect response. In general, teachers struggle with managing their obligations to help students demonstrate an understanding of the mathematics and creating an environment it is safe for students to publicly make mistakes. There were noticeable patterns in how teachers responded to errors that had anticipated versus those that they had not anticipated. Information about teachers' general anxieties and beliefs about teaching and learning mathematics (Stipek, Givvin, Salmon& MacGyvers, 2001) were also explored to contextualize the classroom findings.

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MEASURING ACTIVE LEARNING WITH AN OBSERVATION PROTOCOL IN COLLEGE CALCULUS CLASSROOMS

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Keywords: Instructional Activities and Practices, Classroom Discourse, Post-Secondary Education

Active learning, defined by Freeman et al. (2014) "engages students in the process of learning" and "emphasizes higher-order thinking." This type of student-centered instruction has been shown to increase learning (e.g. Prince, 2014). The purpose of our research was to determine the extent to which the Mathematics Classroom Observation Protocol for Practices (MCOP^2) (Gleason, 2015) measures active learning in classrooms. The MCOP^2 consists of 16 items scored on a scale of 0 to 3, with 8 items addressing student engagement/active learning and 8 items addressing teacher facilitation (Gleason and Cofer, 2014).

Three college calculus instructors (I1, I2, and I3) were videotaped as they taught the same topic (the Mean Value Theorem), and the MCOP^2 was used to score the lessons. I1 and I2 lectured for the entirety of the lesson with questions frequently posed to the class; in contrast, I3 lectured for about half of the lesson, using a large portion of time for students to work in groups and have a whole class discussion about what the groups found. We observed that I1 and I2 had identical active learning totals of 8/24, while I3 received an active learning sub score of 22/24. Due to the similarities between the active learning sub scores for I1 and I2, we looked for any large, noticeable differences in the class sessions. After analyzing the videos, it became clear that the types of questions asked by I1 and I2 were very different. The researchers used a condensed version of Bloom's Taxonomy (Piggott, 2011) to categorize instructors' questions. This categorization indicated that I2 asked questions that required a higher-level of student thinking.

Although the scores on the MCOP^2 did correlate with the amount of active learning in the classroom, the scores did not accurately reflect the level of difficulty of questions posed, which is an important part of active learning. While the MCOP^2 did capture the levels of questioning in one item, we believe that the differences in scores on this one item did not fully illustrate the vast differences in the quality of the lessons. In summary, we found the MCOP^2 to be a useful tool to quantitatively compare active learning in classrooms, but was not a good tool for comparing the quality of multiple lectures.

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CO-TEACHING AND MATH DISCOURSE TO SUPPORT STUDENTS AND TEACHERS IN LINGUISTICALLY DIVERSE ELEMENTARY CLASSROOMS

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There are cognitive advantages to speaking more than one language; yet, linguistic diversity can impact teaching and learning in complex ways (Moschkovich, 2005). For math, all students are expected to develop rigorous understanding of content and math practices (CCSSO & NGA Center, 2010). However, instruction for English learners (ELs) may focus on procedures and vocabulary rather than cognitively demanding activities (Moschkovich, 2005); thus, ELs may not be held to the same high expectations as other students. ELs need opportunities to participate in rich math activities and discussion that take into account their competencies and also provide necessary support. This need may require *innovative educational ideas*. Co-teaching, where both teachers assume the responsibility of planning and implementing instruction (Bacharach, Heck & Dahlberg, 2010), may provide innovation for increasing teacher confidence and flexibility in responding to student needs – thus "questioning borders" (conference theme) of traditional solutions for supporting linguistically diverse mathematics classrooms.

This ongoing, small-scale study investigates how supported, co-teaching practices may enhance engagement with math discourse in linguistically diverse elementary school classrooms. Four co-teaching teams each include one experienced teacher and one master's intern (post-student teaching). The teams participate in professional development (PD) and ongoing collaborative support related to co-teaching and math discourse. Data include: field notes and video and audio recordings (PD and math lessons), co-teaching team reflections, and classroom artifacts. Data collection and qualitative analysis (Creswell, 1998) are ongoing.

Preliminary results suggest that co-teaching models, when accompanied by focused PD and co-planning opportunities, have the potential to increase student-teacher interaction, instructional flexibility, and attention to individual needs – and, thus, may promote more meaningful mathematical discourse and, in turn, positive student learning outcomes. Further, findings suggest benefits and challenges of different co-teaching approaches (e.g., parallel teaching; station teaching; one teach, one assist; and team teaching) (Bacharach, Heck & Dahlberg, 2010) and their relationships to supporting math discourse. The poster will provide greater detail of the findings. This work is significant because it has the potential to identify best practices, strategies, and tools to support teacher education and PD with specific emphasis on the intersection of co-teaching and math discourse to support linguistically diverse classrooms.

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EXAMINING WHOLE GROUP MATHEMATICAL DISCOURSE: WHAT IT MEANS TO BE STUDENT-CENTERED

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Keywords: Classroom Discourse, Elementary School Education

Introduction

Engaging in whole group mathematical discourse affords early elementary students the opportunity to consider and respond to mathematics in discussions, as well as engage in collaborative problem solving (Perry, McConney, Flevares, Mingle & Hamm, 2011). One aspect central to mathematical discourse involves the degree to which interactions are student-centered (Deboer, 2002), where students' thinking drives the discussion and they take agency for their learning (Hufferd-Ackles, Fuson & Sherin, 2004). What does this mean for first grade students for whom mathematics-focused discussions are a new experience? This poster presents an example of a student-centered interaction within a first grade mathematics classroom.

Methods

Utilizing a case study design, classroom observations were video-recorded twice a week for six-week intervals, during November, February, and May. Researcher field notes were taken during observations, with semi-structured teacher interviews audio-recorded once a week. Recordings were transcribed and segments of interactions coded and categorized into themes.

Findings & Discussion

Open-ended problems were regularly posed to students in a way that allowed them to make sense of the tasks and resolve any questions they might have. After working through the problem in small groups, a whole group discussion ensued in which up to four students presented their solution methods to the class. Through regularly engaging in this type of activity, the norm of explaining one's mathematical thinking was established, and afforded opportunities for the rest of the students to examine each other's thinking.

One interaction highlights the student-centered nature of mathematical discussions that might develop over time. Two first graders were provided with the opportunity, tools and facilitation to explain their solution method to the class, thereby making their mathematical thinking explicit. A classmate then asked several key questions which pushed the discussion forward, bringing forth an opportunity for the examination of a numerical pattern by the entire group. With the teacher's facilitation, students' thinking became central to the discussion. They explained, questioned each other, found relationships, and determined the direction of the discussion, all aspects that constitute what it means to engage in student-centered mathematical discourse.

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Chapter 12

Technology

TECNOLOGÍA DIGITAL Y FORMULACIÓN DE PROBLEMAS DURANTE EL PROCESO DE RESOLUCIÓN DE PROBLEMAS

DIGITAL TECHNOLOGY AND FORMULATION OF PROBLEMS DURING THE PROCESS OF SOLVING PROBLEMS

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Se reportan y analizan diferentes tipos de preguntas y problemas planteados por futuros profesores de educación media superior durante el proceso de reconstruir configuraciones dinámicas de una figura dada en el enunciado de un problema geométrico. ¿En qué medida el uso de un Sistema de Geometría Dinámica (SGD) ayuda a los participantes para implementar actividades de planteamiento de problemas durante todas las fases de la resolución de problemas que incluyen la construcción de modelos dinámicos, la búsqueda de distintas soluciones y la extensión de la tarea inicial? Los participantes exhibieron estrategias que involucran el arrastre, deslizadores, medición y lugares geométricos para construir configuraciones dinámicas que los llevó a formular y explorar diversos problemas.

Palabras clave: Resolución de Problemas, Tecnología, Geometría y Pensamiento Geométrico y Especial, Maestros en Formación

Introducción

En los últimos treinta años, la resolución de problemas ha sido un dominio o área de investigación en educación matemática que relaciona el quehacer de la disciplina con el aprendizaje de los estudiantes (Santos-Trigo, 2014). Un principio fundamental en la resolución de problemas es la importancia de formular preguntas relevantes como medio para de entender, representar, explorar y resolver problemas (Polya, 1965). Así, estas preguntas son el medio para explorar conceptos y para desarrollar habilidades de resolución de problemas. Misfeldt y Johansen (2015) resaltan que la actividad de formular preguntas y problemas es un aspecto crucial y no trivial de la práctica profesional de los matemáticos. También, se reconoce que el avance científico y tecnológico generalmente se origina y relaciona con del planteamiento y seguimiento de preguntas o problemas novedosos. Einstein e Infield (1938) argumentan:

> La formulación de un problema es generalmente más importante que su solución, la cual puede ser simplemente una cuestión de habilidades matemáticas o experimentales. Para generar nuevas preguntas y nuevas posibilidades, para revisar viejas preguntas desde nuevos ángulos y para lograr avance real en la ciencia se requiere de imaginación creativa. (citado en Ellerton & Clarkson, 1996, p.1010).

En este contexto, un aspecto central en los ambientes de aprendizaje es que los estudiantes tengan la oportunidad de participar en actividades de planteamiento de problemas para profundizar en conceptos matemáticos durante todo el proceso de la resolución de problemas (Cai, Hwang, Jiang & Silber, 2015). Para este fin, resulta importante investigar en qué medida el uso de distintas herramientas, en particular el uso de SGD, permite a los participantes involucrarse en actividades de planteamiento de problemas a partir de interrogantes que surgen en ambientes de resolución de problemas. En este camino, la pregunta de investigación que guió el desarrollo de este estudio es: ¿Cómo y en que medida el uso de un SGD (GeoGebra) ofrece a los participantes estrategias para formular y dar seguimiento a preguntas o problemas durante las fases de resolución de problemas que

incluyen la construcción de modelos dinámicos, búsqueda de distintas soluciones, extender las tareas iniciales y comunicar resultados?

Marco Conceptual

Planteamiento de problemas y el uso de la tecnología

El planteamiento de problemas se puede caracterizar a partir de tres actividades: (1) generación de un problema original a partir de una situación dada, (2) reformulación de un problema que se está resolviendo o (3) formulación de un problema nuevo modificando los objetivos o condiciones de un problema que ya ha sido resuelto (Rosli et al., 2015). Kılıç (2013) menciona que el planteamiento de problemas "involucra la generación de nuevos problemas y preguntas con el objetivo de explorar una situación dada así como a la reformulación de un problema durante el proceso de resolverlo" (p. 145). En esta perspectiva, el proceso de planteamiento de problemas puede ocurrir en cualquier fase de la resolución de problemas.

Actualmente se reconoce que el uso de tecnologías como GeoGebra genera oportunidades para desarrollar conocimiento matemático, pero también permite transformar escenarios de enseñanza alrededor de la resolución de problemas (Santos-Trigo, Reyes-Martínez & Aguilar-Magallón, 2015). En esta dirección, resulta necesario investigar cómo el uso de distintas herramientas digitales ofrece a los estudiantes caminos y oportunidades para representar, explorar, entender, resolver, generalizar y extender problemas. Algunos estudios reportan que las herramientas tecnológicas liberan al individuo de trabajo técnico relacionado con cálculos o gráficas y le permiten concentrarse en el análisis y el sentido de las relaciones o conceptos matemáticos (Lavy, 2015). Abramovich y Cho (2015) afirman que "El uso apropiado de herramientas tecnológicas digitales comúnmente disponibles pueden motivar y soportar actividades de planteamiento de problemas" (p. 71).

Lavy (2015) argumenta que el uso de un SGD puede jugar un papel importante en el planteamiento de problemas, pues permite de forma sencilla manipular, mover y deformar objetos geométricos, pero además facilita conjeturar y explorar relaciones matemáticas a partir de distintas representaciones. Leikin (2015) menciona que una forma de plantear problemas con ayuda de un SGD es mediante tareas de investigación; donde se formulan interrogantes y problemas a partir de la exploración de configuraciones dinámicas que surgen al resolver problemas geométricos de demostración. Una idea fundamental en el trabajo de Leikin (2015) son las tareas o actividades para favorecer o promover episodios de resolución y planteamiento de problemas. Imaoka et al. (2015) consideran algunos elementos fundamentales de este tipo de tareas donde se usa un SGD. Por un lado, las tareas deben tener características que el estudiante pueda usar para plantear problemas; por ejemplo, los problemas deben motivar múltiples representaciones o deben involucrar la medición de atributos, como áreas, perímetros o longitudes. Por otro lado, las tareas deben estar pensadas para trabajarse dentro del SGD, en otras palabras, que la solución a las tareas pueda encontrarse por medio de la formulación y exploración de conjeturas más que por la aplicación de algoritmos conocidos.

Metodología

En este estudio participaron nueve estudiantes de una Maestría en Educación Matemática a lo largo de catorce sesiones de trabajo semanal con duración de tres horas cada una. Las sesiones se desarrollaron en un laboratorio de cómputo donde cada participante tuvo acceso a una computadora. En cuanto a la forma de trabajo, se intentó crear una comunidad de aprendizaje donde los participantes tuvieran la oportunidad de enfrentarse a tareas de resolución y formulación de problemas con el apoyo de diversas herramientas digitales (GeoGebra). Esencialmente se diseñaron e implementaron tres tipos de problemas: problemas de investigación, problemas de demostración y problemas de construcción. Siguiendo las ideas de Leikin (2015), en los problemas de investigación

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los participantes analizaron y exploraron configuraciones geométricas dinámicas prediseñadas dentro del SGD con el objetivo de formular problemas. Otro tipo de tarea fue la resolución de problemas geométricos de demostración. Por último, en los problemas de construcción el objetivo fue construir figuras o elementos geométricos básicos (rombos, tangentes, triángulos, rectángulos, cónicas, entre otros) a partir de condiciones iniciales dadas.

El trabajo durante las sesiones estuvo dividido en dos etapas. En una primera etapa los participantes trabajaron de manera individual. En la segunda etapa se generaron discusiones plenarias donde los participantes presentaban sus ideas y había una retroalimentación directa del grupo y de los coordinadores del grupo.

Resultados

En esta sección se discutirán algunos episodios que muestran cómo los participantes se involucraron en actividades de planteamiento de problemas durante el proceso de resolver un problema tradicional de geometría. En particular, el análisis se centra en cómo el uso de GeoGebra ayudó a los participantes para plantear problemas novedosos como resultado de reconstruir figuras presentes en el enunciado del problema.

Planteamiento de problemas antes de la resolución de un problema

Existe una gran variedad de problemas geométricos en los cuales la información o condiciones son presentados por medio de figuras o configuraciones geométricas. Por ejemplo, la Figura 1 es parte del enunciado de un problema donde se solicita probar cierta relación entre los objetos matemáticos involucrados. Comúnmente, cuando se resuelven problemas de forma tradicional (con papel y lápiz) el estudiante no se plantea la tarea de reconstruir las figuras que aparecen en el enunciado del problema. Cuando se trabaja el problema dentro de un SGD, resulta esencial pensar en las formas de obtener una representación dinámica que generalmente involucra el enunciado del problema. Por ejemplo, durante una de las sesiones se pidió a los participantes que trabajaran con el siguiente problema:

Problema 1. La Figura 1 muestra dos círculos, un ángulo y un triángulo equilátero. Demuestre que la suma de los radios de las circunferencias es igual a la altura del triángulo. (adaptado de Leikin, 2007).
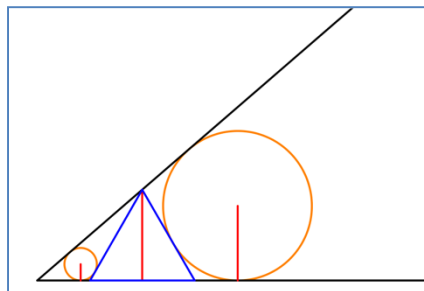
Figura 1. Probar que la suma de los radios de las circunferencias es la altura del triángulo.

¿Cómo reconstruir una representación dinámica del problema? Los participantes observaron que la figura del problema incluía cuatro elementos (un ángulo, dos circunferencias y un triángulo) y formularon la siguiente pregunta: ¿con qué elemento comenzar? Esta pregunta los condujo a plantear problemas asociados con diferentes caminos para reconstruir la configuración geométrica. La Tabla 1 muestra algunas rutas sugeridas por los participantes y los problemas formulados. Los participantes identificaron seis formas distintas de obtener la configuración dinámica del problema (en total existen 18 formas de construirla). En algunos casos, obtener la representación del problema inicial se redujo a formular y resolver problemas similares o idénticos. Por ejemplo para la primera y tercera

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opción, al final se requirió construir las circunferencias tangentes a los rayos del ángulo y al triángulo equilátero.

Tabla 1: Algunas rutas identificadas por los participantes para obtener la configuración geométrica del problema

Opción	Orden de la construcción	Problemas formulados
1	Ángulo Triángulo equilátero Circunferencias	1.1 Trazar el triángulo equilátero, a partir del ángulo formado por dos rectas. 1.2 Trazar las circunferencias tangentes, a partir del ángulo y el triángulo.
2	Circunferencias Ángulo Triángulo equilátero	1.3 Trazar el ángulo, partiendo de las dos circunferencias. 1.4 Trazar el triángulo equilátero, a partir del ángulo y las circunferencias.
3	Circunferencia Ángulo Triángulo equilátero Circunferencia	1.5 Trazar el ángulo, teniendo una circunferencia. 1.6 Construir el triángulo equilátero, teniendo el ángulo y una circunferencia. 1.7 Trazar la circunferencia tangente a los rayos del ángulo y al lado del triángulo.
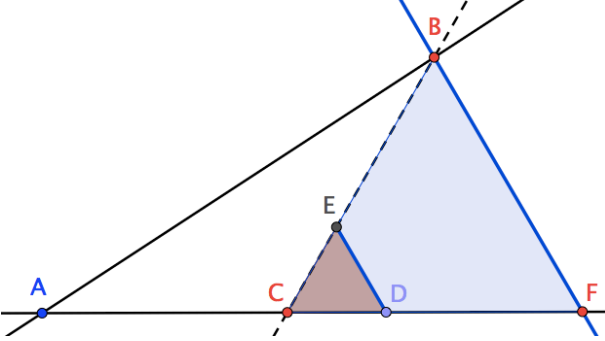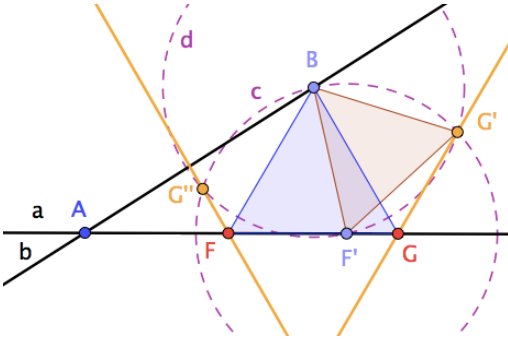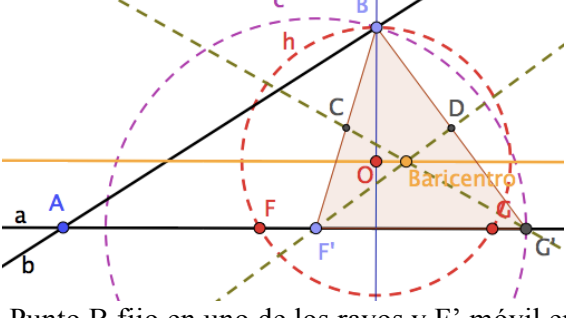
Después de identificar distintos caminos para construir la configuración geométrica del problema 1 los participantes resolvieron algunos de los problemas formulados. En la siguiente sección se describen algunas soluciones obtenidas para el problema 1.1.

Planteamiento de problemas durante la resolución del problema

Durante la resolución del problema 1 se exhibieron actividades de planteamiento de problemas. Por ejemplo, los participantes formularon y exploraron diferentes formas de reconstruir la figura involucrada en el enunciado del problema. Para este fin, fueron útiles algunas heurísticas de resolución de problemas vinculadas directamente con estrategias de reformulación; por ejemplo relajar las condiciones del problema, resolver un problema similar o más simple, descomponer el problema en problemas subsidiarios, analizar casos particulares o extremos, entre otras (Cai et al., 2013).

Considerar trazos auxiliares y lugares geométricos como estrategia de resolución y planteamiento de problemas. En una sesión posterior se discutieron, de forma plenaria, distintas rutas para obtener la configuración dinámica del problema inicial. En esta discusión, se resaltaron estrategias de reformulación basadas en la heurística de relajar las condiciones del problema para simplificarlo. Por ejemplo para resolver el problema 1.1 se discutieron tres soluciones distintas. En la Tabla 2 se muestra la descripción de las construcciones, el proceso de reformulación y las estrategias de solución del problema 1.1.

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Tabla 2: Procesos de reformulación y estrategias de solución del problema 1.1

Construcción	Proceso de reformulación	Estrategia de solución
1. Punto C fijo sobre un rayo del ángulo. Triángulo CDE equilátero. Punto de intersección B de la recta CE con el rayo del ángulo. Recta BF paralela a ED.	Problema: construir un triángulo equilátero CDE que tenga su base CD en un rayo del ángulo. Condición relajada: el vértice E no está en el otro rayo del ángulo.	Una vez obtenido el triángulo CDE equilátero, trazar un triángulo CFB semejante (equilátero) por medio de una recta paralela.
2. Punto B fijo en uno de los rayos y F' móvil en el otro. Triángulo BF'G' equilátero.	Problema: Construir un triángulo equilátero BF'G' que tenga sus vértices B y F' sobre los rayos del ángulo. Condición relajada: la base del triángulo equilátero no está sobre uno de los rayos del ángulo.	Visualización del lugar geométrico que genera el vértice G' del triángulo equilátero BF'G' cuando se mueve el vértice F'. Solución cuando el vértice G' está sobre el rayo del ángulo.
3. Punto B fijo en uno de los rayos y F' móvil en el otro. Triángulo isósceles BF'G' tal que BF'=F'G'. Baricentro del triángulo BF'G'. Altura de triángulo BF'G' asociada al vértice B.	Problema: construir un triángulo isósceles BF'G' que tenga un lado sobre un rayo del ángulo y un vértice sobre el otro. Condición relajada: el triángulo no es equilátero.	Visualización del lugar geométrico descrito por el baricentro del triángulo BF'G' cuando se mueve el punto F'. Solución cuando el baricentro está sobre la altura (en un triángulo equilátero baricentro, ortocentro, circuncentro e incentro coinciden; punto O).

Durante la construcción del triángulo equilátero, los participantes formularon preguntas acerca del comportamiento o propiedades de algunos objetos usados para construir dicho triángulo. Por ejemplo, en el acercamiento que involucra la construcción del baricentro del triángulo plantearon: ¿qué propiedades tiene el lugar geométrico descrito por el baricentro del triángulo? ¿es una línea

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paralela a la base? ¿en qué proporción divide el punto O a la altura del triángulo BF'G' (asociada con el vértice B)? ¿la proporción es 2/3?

Planteamiento de problemas después de la resolución del problema

Resolver el problema 1.1 condujo a resolver el problema 1.2, es decir, trazar las circunferencias tangentes a las rectas y al triángulo equilátero. En la discusión plenaria, los participantes resolvieron el problema 1.2 por medio del trazado de las bisectrices de los ángulos externos del triángulo y la bisectriz del ángulo formado por las dos rectas (Figura 2). Finalmente, el problema inicial (Problema 1) se resolvió de diversas maneras; en las Figuras 2 y 3 se muestran dos de ellas. La primera solución involucra el Teorema de Viviani que menciona que para cualquier punto sobre un triángulo equilátero se cumple que la suma de las distancias de dicho punto a los lados del triángulo es igual a su altura. La segunda solución implica construir un rectángulo ELHI con EH=IL igual a uno de los radios, para después demostrar que el cuadrilátero FGIC (donde FG es igual al otro radio) es un paralelogramo, como se observa en la Figura 3.
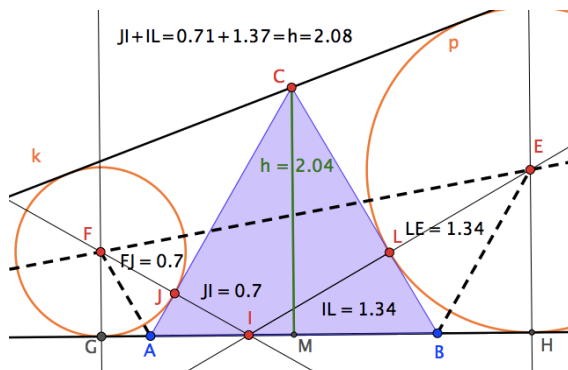
Figura 2. Solución probando que FJ=JI y LE=IL para aplicar teorema de Viviani.
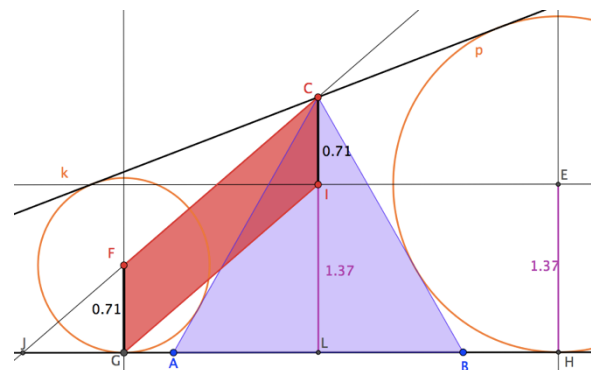
Figura 3. Solución probando que el cuadrilátero FGIC es un paralelogramo.

¿Qué preguntas o problemas formularon los participantes después de resolver el problema inicial? En general, los participantes se centraron en formas de generalizar y extender los métodos usados para resolver el problema como plataforma para plantear nuevos problemas. Algunos problemas planteados por los participantes fueron los siguientes:

- Problema 1.8. Teniendo dos rectas que se intersecan y dos circunferencias tangentes a ellas ¿siempre es posible construir un triángulo equilátero entre las dos circunferencias como en la Figura 1? ¿Qué relación debe existir entre la posición de las circunferencias para que sea posible construir el triángulo equilátero?
- Problema 1.9. Teniendo dos rectas que se intersecan y dos circunferencias tangentes a ellas ¿es posible construir un triángulo isósceles entre las dos circunferencias?
- Problema 1.10. ¿Si el triángulo es isósceles se cumple que la suma de los radios es igual a la altura del triángulo?

Los participantes examinaron el modelo dinámico construido para explorar el problema 1.8 y concluyeron que no siempre es posible construir un triángulo equilátero dados el ángulo y las circunferencias tangentes. Más aún, descubrieron que dados el ángulo y las circunferencias siempre es posible trazar un triángulo isósceles entre las dos circunferencias y tangente a ellas. Un vértice del triángulo es la intersección de un rayo del ángulo con la mediatriz del segmento que une los centros

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de las circunferencias. La configuración dinámica mostrada en la Figura 4 se usó para mostrar que la suma de los radios de los círculos es igual a la altura del triángulo isósceles ABC.
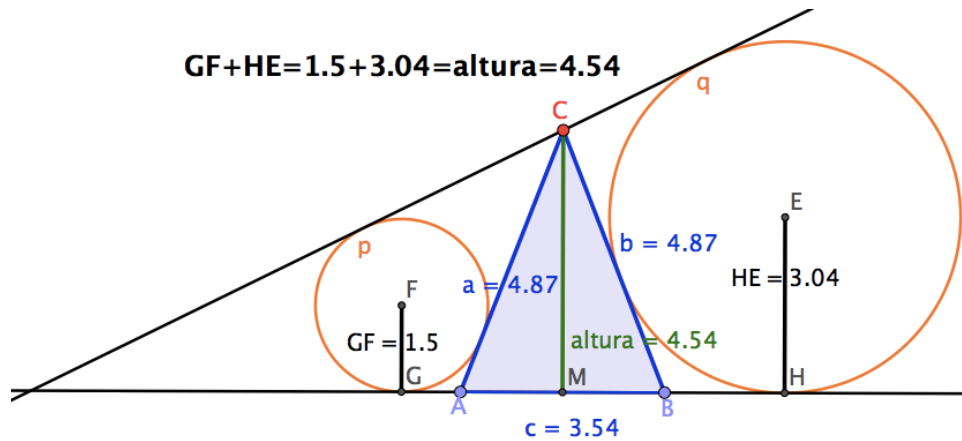
Figura 4. La proposición del problema 1 se cumple cuando el triángulo es isósceles**.**

Discusión de resultados

Normalmente, cuando se resuelven problemas la atención se enfoca principalmente en obtener un resultado. El uso de GeoGebra permitió a los participantes poner atención en otros aspectos de la resolución de problemas. Uno de estos aspectos fue considerar distintas maneras de representar una situación problemática lo cual motivó un episodio de planteamiento de problemas. El reflexionar sobre los posibles caminos para obtener una representación dinámica del problema 1 permitió a los participantes problematizar y conectar diversas propiedades de figuras básicas como triángulos, circunferencias y rectas tangentes. Además, los participantes identificaron un conjunto de relaciones que surgieron a partir de la exploración del modelo dinámico. Por ejemplo, encontraron que para obtener la configuración dinámica del problema inicial el ángulo definido por las dos rectas debe ser menor de 60º.

Gracias a las herramientas que ofrece el SGD para dar movimiento a objetos matemáticos, los participantes lograron implementar procesos de reformulación de problemas. En este camino, el uso de la heurística "relajar las condiciones del problema" fue fundamental. Construir trazos auxiliares y visualizar lugares geométricos fueron estrategias usadas por los participantes no solo para resolver problemas de formas novedosas, sino también como plataforma para plantear problemas. Después de visualizar lugares geométricos, una actividad recurrente de formulación de problemas, fue explorarlo y describirlo en términos de sus propiedades.

Conclusiones

¿Qué tipo de preguntas o problemas plantean futuros profesores cuando se apoyan de herramientas digitales para representar y explorar problemas matemáticos? En este estudio se documentó que futuros profesores pueden involucrarse en actividades de planteamiento de problemas durante todas las fases presentes al resolver un problema. Para este fin, el objetivo de reconstruir una figura dada (que involucra dos circunferencias, un ángulo y un triángulo equilátero) en un problema tradicional, fue el punto de partida para formular problemas relacionados con el orden para obtener dicha configuración. Después, identificaron e implementaron diferentes formas de construir objetos particulares, como un triángulo equilátero, a partir de ciertas condiciones. De forma similar, la exploración del modelo dinámico del problema inicial no solo les ayudó a formular nuevas preguntas, sino también a buscar argumentos para soportar resultados matemáticos. Al final, los participantes se enfocaron en las condiciones necesarias para construir un modelo consistente y robusto del problema; lo que los condujo a encontrar que la medida del ángulo tiene que ser menor

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que 60º y también les permitió concluir que siempre es posible construir un triángulo isósceles en medio de los círculos tangentes. Varias estrategias de resolución de problemas les ayudaron a examinar casos particulares (relajando las condiciones iniciales) para explorar el comportamiento de objetos e identificar relaciones matemáticas. En particular, algunas permisibilidades de la herramienta (GeoGebra) como el arrastre de objetos, lugares geométricos, medición de atributos y el uso de deslizadores fueron importantes para encontrar y soportar conjeturas. Inicialmente, estas conjeturas fueron soportadas por medio de argumentos empíricos; pero después los participantes presentaron argumentos geométricos y algebraicos.

We analyze and discuss ways in which prospective high school teachers pose and pursue questions or problems during the process of reconstructing dynamic configurations of figures given in problem statements. To what extent does the systematic use of a Dynamic Geometry System (DGS) help the participants engage in problem posing activities throughout problem solving phases that include the construction of dynamic models, the search for different solutions methods and the process of extending initial statements? The participants relied on technology affordances such as dragging objects, using sliders, quantifying relations, and drawing loci to construct dynamic configurations that led them formulate and pursue different problems.

Keywords: Problem Solving, Technology, Geometry and Geometrical and Spatial Thinking, Teacher Education-Preservice

Introduction

In the last thirty years, problem solving has been a research domain in mathematics education, which relates the practicing or development of mathematics and the ways in which students learn the discipline (Santos-Trigo, 2014). A fundamental principle in a problem solving approach is that learners engage in an inquiring process as a means to understand, represent, explore, and solve mathematical tasks (Polya, 1965). Thus, questions are the vehicle for learners to explore concepts and to formulate and develop problem solving competencies. Misfeldt and Johansen (2015) pointed out that the activity of formulating problems is a crucial aspect that distinguishes the work and professional practice of mathematicians. Indeed, it is also recognized that progress and developments in scientific and technological fields involve both the formulation of questions and the process to follow up those questions. Einstein and Infield (1938) argue that:

> The formulation of a problem is often more essential than its solution, which may be merely a matter of mathematical or experimental skills. To raise new questions, new possibilities, to regard old questions from new angle, require creative imagination and marks real advance in science. (Cited in Ellerton & Clarkson, 1996, p.1010).

In this context, a central aspect in learning environments is that students have an opportunity to engage in problem posing activities in order to delve into mathematical concepts and during the entire process of solving problems. (Cai, Hwang, Jiang & Silber, 2015). To this end, it becomes important to investigate the extent to which the students' use of different digital technologies and in particular the use of a DGS provides opportunities to engage them in problem posing activities and ways to follow up related questions in problem solving environments. Thus, the research question that guides the development of the study is: How and to what extent does the systematic use of a Dynamic Geometry System (GeoGebra) provide the participants the affordances to formulate and pursue questions or problems during problem solving phases that include construction or reconstruction of dynamic models, thinking of and implementing various solution methods, extending mathematical tasks and communication of results or solutions?

Conceptual Framework

Problem posing and the use of digital technology

Learners can engage in problem posing activities in contexts and situations that involve: (1) the generation of an original problem from a given information or data (2) the reformulation of a problem such as solving a simpler or related problem as a means to delve and solve an initial problem or (3) the formulation of a new problem by modifying the objectives or conditions of a given problem (Rosli et al., 2015). Kılıç (2013) mentions that the problem posing "is defined as the creation of new problems or the reformulation of a given problem" (p. 145). In this perspective, the process of problem posing might occur during or at any stage of the solution process.

It is now recognized that the use of digital technologies such as GeoGebra provides learners an opportunity to extend ways of reasoning about problems and learning environments need to encourage students to rely and value the use of several digital technologies in their problem solving experiences (Santos-Trigo, Reyes-Martínez, & Aguilar-Magallón, 2015). In this direction, it is necessary to investigate how the use of different digital tools offers students ways and opportunities to represent, explore, understand, solve, generalize and extend problems. Some studies report that the use of technological tools releases the individual technical work that involves calculations or graphs and helps students to concentrate on the analysis and making sense of relationships or mathematical concepts (Lavy, 2015). Abramovich and Cho (2015) state that "the appropriate use of commonly available digital technology tools can motivate and support problem posing activities" (p. 71).

Lavy (2015) argues that the use of a DGS can play an important role in the problem posing, which allows a simple way to manipulate, move and deform geometric objects, but also facilitates the formulation of conjectures and the exploration of mathematical relationships from different representations.

Leikin (2015) mentions that one way to pose problems with the help of a DGS is through investigation tasks; where questions and problems are formulated from exploring dynamic configurations that arise when geometric problems are solved. A fundamental idea in the work of Leikin is to ask learners to work on tasks or activities to facilitate or promote both the formulation and solution of problems. Imaoka, Shimomura, and Kanno (2015) consider some fundamental elements of these tasks where a DGS is used. Tasks should have features that students can use to pose problems; for example, problems should encourage multiple representations or should involve the measurement of attributes, such as areas, perimeters and lengths. Also, tasks must be designed to be worked within the DGS, in other words, the solution to the tasks should be found through the development and exploration of conjectures rather than the application of known algorithms.

Methodology

In this study, nine students participated in a Problem Solving course as part of a Master program in Mathematics Education over fourteen weekly working sessions lasting three hours each. The sessions were developed in a computer lab where each participant had access to a computer. As for the way they work, we tried to create a learning community where participants had the opportunity to face posing and solving problems with the support of digital tools (GeoGebra). Essentially three types of task were designed and implemented: investigation problems, demonstration problems and construction problems. Following the ideas of Leikin (2015), in investigation problems, the participants discussed and explored dynamic geometric configurations through the use of GeoGebra with the aim of formulating problems. Another type of task was to solve geometric demonstration problems. Finally, in the construction problems the objective was to draw or represent simple mathematical objects (rhombus, tangents, triangles, rectangles, conics, etc.) dynamically in order to find, formulate, and support mathematical relations.

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The dynamic of the session involved two main activities: In the first stage the participants worked individually and later discussed his/her ideas with another participant (pair work). In the second stage there was collective or plenary discussion where they presented their ideas and received direct feedback from the group and the group course coordinators.

Results

In this section, we present and discuss some episodes that show how the participants engaged in problem posing activities during the process of solving a traditional geometry problem. In particular, we focus on analyzing how the use of GeoGebra opened up novel ways for the participants to formulate problems as a result of reconstructing figures associated with problem statements.

Problem posing before solving a problem

There is a variety of geometric problems for which the information or conditions are given through figures or geometric configurations. For instance, Figure 1 is part of a statement in which learners are asked to prove a certain relation of involved objects. Usually, when problems are solved in a traditional way (with paper and pencil) students do not ask themselves how to reconstruct the figures shown in the problem statement. However, with the use of GeoGebra it is natural to ask about ways to reconstruct the figure(s) involved in problem solving statements. For example, during one of the sessions, the participants were asked to work on the following task:

Problem 1. The Figure 1 shows two circumferences, an angle and an equilateral triangle. Prove that the sum of the circles' radii is equal to the height of the triangle (adapted from Leikin, 2007).
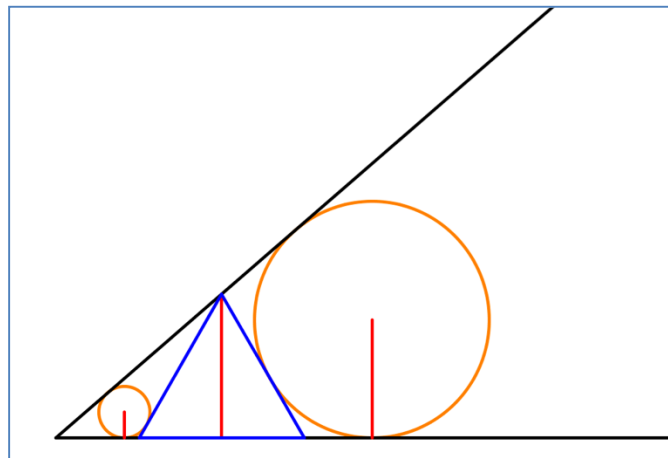
Figure 1. Prove that the sum of the circles radius is equal to the height of the triangle.

How to draw a dynamic representation of the problem? Participants noted that the figure of the problem included four elements (an angle, two circles and a triangle) and formulated the following question: with which elements do we begin? This question led the participants to select and pursue different ways to reconstruct the geometry configuration. Table 1 shows some approaches suggested by the participants and the problems they formulated to construct the figure. Participants identified six different ways to get the dynamic configuration of the problem (in total there are 18 ways to get it). In some cases, representing the initial problem consisted of formulating and solving similar or identical problems. For example, for the first and third option, in the end it was required to construct the circumferences that are tangent to the ray and to the equilateral triangle.

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Table 1: Approaches identified by the participants to draw the geometric configuration of the problem

Option	Order of the Construction	Formulated problems
1	Angle Equilateral triangle Circle	1.1 Draw an angle formed by two straight lines and construct the equilateral triangle. 1.2 Draw tangent circles to an angle and the triangle.
2	Angle, tangent circles Equilateral triangle	1.3 Get the angle. Draw two tangent circles to the angle. 1.4 Construct the equilateral triangle tangent to the angle and the circles.
3	Circle, Angle Equilateral triangle Circle	1.5 Draw a circle and construct the angle whit two tangent lines. 1.6 Construct the equilateral triangle, having the circle and the angle. 1.7 Construct tangent circle to the sides of the angle and the side of the triangle.
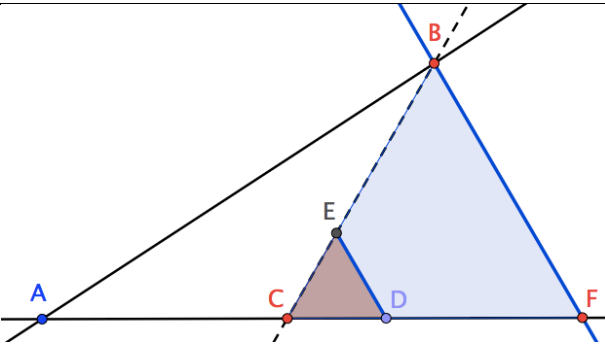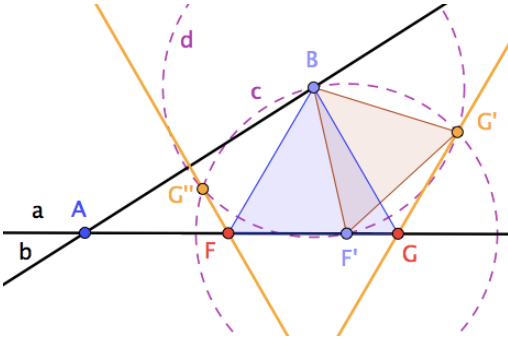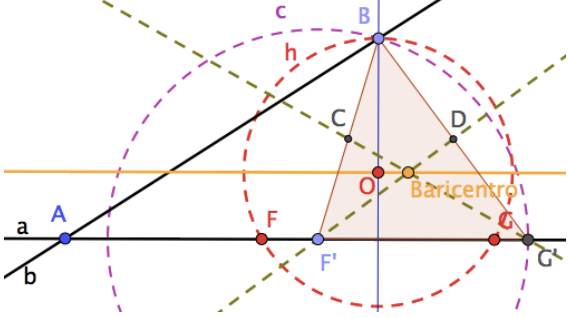
After identifying various ways to construct the geometrical configuration of the problem 1 the participants solved some of the formulated problems. The following section provides some solutions obtained for the problem 1.1.

Problem posing during the resolution of the problem

Problem posing activities were shown during the process of solving problem 1. For instance, the participants formulated and explored different ways to draw the figure in the problem. To this end, some problem solving heuristics helped them pursue some ways to draw a dynamic model of the figure. These heuristic involve: relaxing the task conditions of the problem, solving a similar or simpler problem, decomposing the problem in simpler problems, analyzing particular or extreme cases, etc. (Cai et al., 2013).

Considering auxiliary objects and geometric loci as a strategy for posing and solving problems. In a later session the participants discussed as a whole group ways to draw a dynamic model of figure 1. The heuristic that involves relaxing the initial conditions to simplify the problem was implemented. Thus, the participants showed three different routes to construct an equilateral triangle (problem 1.1). Table 2 shows the main ideas associated with the constructions, the process of reformulation, and solution strategies.

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Table 2: Reformulation process and solution strategies of problem 1.1

Construction	Reformulation process	Solution strategy
1. Fix point C on a ray of the angle and draw an equilateral triangle CDE. Identify the intersection point B of line EC and ray AB (side of angle) and draw line BF parallel to ED.	Problem: Draw an equilateral triangle CDE having its base CD on a ray of the angle. Relaxed condition: the vertex E does not belong to the other ray of the angle.	Once the equilateral triangle CDE is obtained, draw a similar triangle CFB (equilateral) through a parallel line.
2. Let point B be a fixed point on one of the rays of the angle and F a mobile point on the other ray. BF'G' equilateral triangle.	Problem: Draw an equilateral triangle BF'G' whose vertices B and F' are on the rays of the angle respectively. Relaxed condition: the base of the equilateral triangle is not necessary on the ray of the angle.	Draw the locus generated by the vertex G' of the equilateral triangle BF'G' when the point F' is moved along one ray of the angle. The solution is achieved when vertex G' is on the ray of the angle.
3. Let point B be a fixed point on one of the rays and F' a mobile point on the other ray. Draw an isosceles triangle BF'G' such that BF' = F'G'; centroid of triangle BF'G'. Draw the height of triangle BF'G' that passes through vertex B.	Problem: draw an isosceles triangle BF'G' having a side on one ray of the angle and a vertex on the other side. Relaxed condition: the triangle is not equilateral.	Draw the locus generated by the centroid of the isosceles triangle BF'G' when the point F' moves along the side of the angle. Solution is achieved when the centroid is on the height (in an equilateral triangle centroid, orthocenter, circumcenter and incenter all coincide; point O).

During the construction of an equilateral triangle, the participants formulated questions about properties or behaviors of objects that appeared or were used to draw the triangle. For example, in the

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approach that involved the construction of the triangle centroid, they posed: What are the properties of the locus described by the centroid of the triangle? Is it a line parallel to the base? In which proportion does point O divide the height of the triangle BF'G' (associated with the vertex B)? Is it a ratio of 2/3?

Problem posing after the resolution of a problem

Solving problem 1.1 led the participant to solve problem 1.2, i.e. they traced the tangent circles to the angle and equilateral triangle. In the whole group discussion, participants solved the problem 1.2 by drawing the bisectors of the exterior angles of the triangle and bisecting the angle formed by the two rays of the angle (Figure 2). Finally, the initial problem (Problem 1) was solved in various ways; Figures 2 and 3 show two of the ways. The first solution involves Viviani's Theorem which states that for any interior point on an equilateral triangle it is true that the sum of the distances from that point to the sides of the triangle is equal to its height. The second solution involves building a rectangle EHLI (EH = IL equal to one of the radii) to demonstrate that the quadrilateral FGIC (where FG is equal to the other radius) is a parallelogram, as shown in Figure 3.
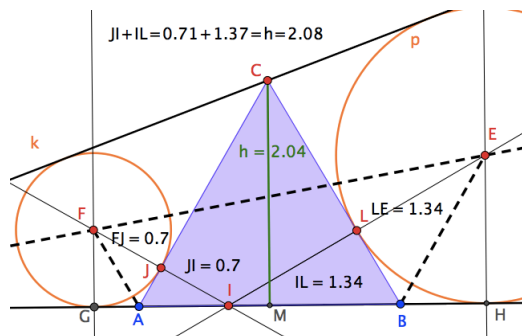
Figure 2. Solution proving FJ = JI and LE = IL to apply Viviani's theorem.
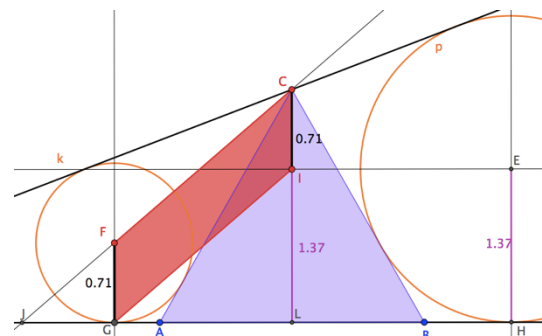
Figure 3. Solution proving that the quadrilateral FGIC is a parallelogram.

What questions or problems did the participants formulate after solving the initial task? In general, the participants focused on ways to generalize and extend the methods they used to solve the task as a means to propose new questions. Some examples of problem that the participants formulated included:

- Problem 1.8: Given two intersecting lines and two circles that are tangent to them is it always possible to construct an equilateral triangle between the two circumferences as shown in Figure 1? What relationship should exist between the positions of the circumferences to make it possible to construct the equilateral triangle?
- Problem 1.9: Given two intersecting lines and two circles that are tangent to them is it possible to construct an isosceles triangle between the two circles?
- Problem 1.10: If the given triangle is isosceles, instead of equilateral, does it hold that the sum of the radius is equal to the height of the triangle?

The participants examined the dynamic model that they had constructed of the task (problem 1.8) and concluded that it was not always possible to draw and equilateral triangle from a given angle and tangent circles. Indeed, they found that given an angle and the tangent circles it is always possible to draw an isosceles triangle between the circles that is tangent to them. A vertex of this triangle is the intersection of the perpendicular bisector of the segment that joins the centers of the circle and one ray of the angle. Figure 4 was used to show that the sum of the radii of the tangent circles is equal to the height of the isosceles triangle ABC.

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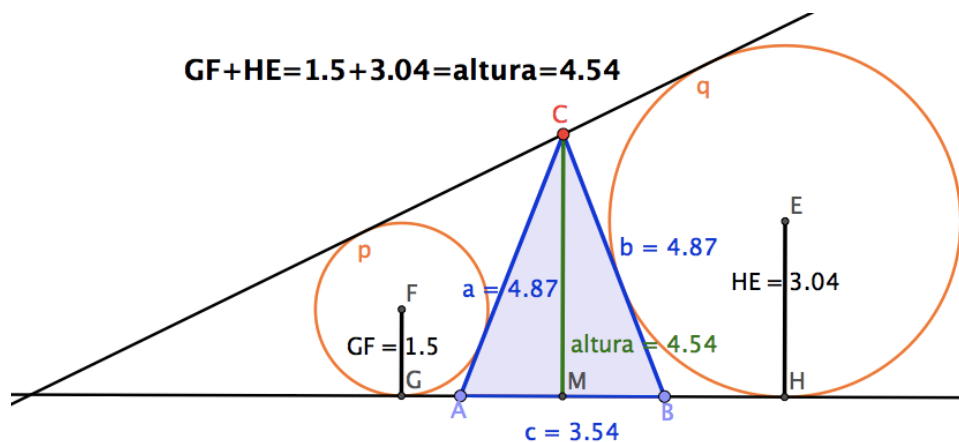
Figure 4. The sum of the radii of the tangent circles is equal to the height of the isosceles triangle.

Discussion of Results

In general, when learners are asked to solve problems they mainly focus on getting a solution to the problem. The use of GeoGebra allowed participants to pay attention to other aspects of problem solving. One of these aspects was to consider different ways to represent the given situation, which led them to formulate and pursue new problems. Focusing on possible ways to construct a dynamic representation of the problem 1 enabled the participants to problematize and connect various properties, concepts and results associated with triangles, circles and tangents. In addition, the participants identified a set of relations that emerged during the dynamic exploration of the model. For instance, they found that in order to construct the model of the problem, the angle defined by the two lines must be less than 60 degrees.

The affordances provided by the DGS allowed the participants to move objects within the model and to constantly pose questions about the objects' behaviors. "Relaxing the initial conditions of the problem" became an important heuristic. Similarly, the participants used strategies such as introducing auxiliary objects and visualizing geometric loci not only to solve problems in new ways, but also to pose new problems. After visualizing geometric loci, a recurrent activity in the formulation of problems, exploring and describing in terms of its properties also happened.

Conclusions

What types of questions or problems do prospective high teachers pose when they rely on digital technology affordances to represent and explore mathematical problems? In this study, we document that prospective teachers can engage in problem solving activities throughout all problem solving stages. To this end, the goal of reconstructing a given figure (that involved an angle, two tangent circles and an equilateral triangle) in a traditional problem became a point of departure to formulate problems related to the order to draw the elements involved in the figure. Later, focusing on ways to draw a particular object, an equilateral triangle, led them to identify and implement several ways to achieve this task. Similarly, the exploration of the dynamic model of the task not only helped them formulate new questions; but also to look for arguments to support mathematical results. At the end, the participants asked about conditions to construct a consistent model of the task and this led them to find out that the measure of the angle must be less than 60 degrees and also that it was always possible to draw an isosceles triangle between the tangent circles. In addition, several problem solving strategies helped them to examine particular cases (relaxing initial conditions) to explore objects' behaviors and to identify mathematical relations. In particular, GeoGebra software affordances that include dragging objects, finding loci, quantifying attributes and using sliders

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became important to find and support conjectures. These conjectures were initially supported through empirical arguments; but later the participants gave geometric and algebraic arguments.

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COMMUNICATING PROFESSIONAL NOTICING THROUGH ANIMATIONS AS A TRANSFORMATIONAL APPROXIMATION OF PRACTICE

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This paper explores the use of animations as an approximation of practice to provide a transformational technology experience for elementary mathematics preservice teachers. Preservice teachers in mathematics methods courses at six universities (n=126) engaged in a practice of decomposing and approximating components of a fraction lesson. Data analysis focused on the extent to which preservice teachers were specific or general with respect to mathematics in written and animated accounts of noticing. Findings illuminate preservice teachers' degrees of specificity, with most preservice teachers being more specific about mathematics in their animations, showing promise for animation as a tool for communicating what is noticed. Further, preservice teachers perceived the use of animations a transformational experience, meaning the technological medium provided learning and access beyond what could have been accomplished without the technological support.

Keywords: Teacher Education-Preservice, Technology, Instructional Activities and Practices

As technology changes, new advances permeate teacher education programs and traditional methods of teacher preparation. These recent developments have "made feasible the rich presentation of classroom processes in a way that can capture much of the complexity inherent in education" (Miller & Zhou, 2007, p. 332). For example, showing video of classroom scenarios provides an opportunity for preservice teachers (PSTs) to view a setting they otherwise may not have observed. These technologies afford PSTs growth in learning from the technology and in learning how to use the technology. As teacher educators consider how they will teach their methods courses, they must recognize the role of technology in the process and maintain cognizance about the affordances and constraints of such innovations. The purpose of our study was to analyze the role technology played to support the specificity of PST noticing. We analyzed PST noticing to gain an understanding of affordances and constraints for the incorporation of technology within a mathematics methods course and answered the following questions: (1) How does the technological medium of animation as an approximation of practice enable specific communication of PST noticing?, and (2) Based on these findings, does the animation technology replace, amplify, transform or hinder written processes? We use the term *methods courses* to refer to university or college courses that prepare future teachers on the pedagogical approaches for teaching a specific content area. In this paper, we specifically focus on elementary mathematics methods courses, meaning those courses designed to prepare future elementary teachers with research-based methods for teaching mathematics.

Theoretical Framework

As teacher educators prepare courses and consider the incorporation of technology, it is important they are aware of the function of technology in their courses. Cognizance of the intent of technology integration is integral for positive outcomes for PSTs (Amador, Kimmons, Miller, Desjardins, & Hall, 2015; Hughes, Thomas, & Scharber, 2006). Teacher educators should consider how technology infused into the course functions, meaning the extent to which it is replacing (R),

amplifying (A), or transforming (T) current methodological practices (Hughes et al., 2006). If a technology takes the place of a process currently performed without technology, it is considered replacement (R). When a technology adds to a current practice in a methods course, amplification (A) occurs. Finally, when a technology provides opportunities that could otherwise not be accomplished without the given technology, transformation (T) takes place. Coupled with this framework of replace, amplify, and transform (RAT), some researchers have proposed hindrance (H) as a fourth technology function, claiming that not all technology incorporation is positive (Amador et al., 2015). Consequently, we frame our work theoretically with the Hughes et al. (2006) RAT (replace, amplify, transform) model with the addition of H (hindrance) as a lens for understanding the role of technology in supporting PSTs' development of research-based practices.

Related Literature

A variety of techniques, ranging from traditional drawings to GIF environments, can be used to produce an animation that depicts the phenomenon of movement. For this research, animation refers to the use of a digital medium in which a sprite (Amador & Soule, 2015) is manipulated and speaks audible utterances and the sequence of interactions is playable. Animation can be used to generate fictional interactions, such as classroom episodes, or can be produced by the teacher educator and provide straightforward depictions of a concept or practice, which can be productive when helping novices learn (Chazan & Herbst, 2012). For example, Chen (2012) had PSTs create comic-based lesson depictions to describe the nature of teaching. Similarly, Herbst, Chazan, Chen, Chieu, and Weiss (2011) argued for the use of Lesson*Sketch*, a comic-based representation, as a way to support PSTs in making decisions about teaching. As Hoban and Nielsen (2013) indicated, "animations could provide a motivation for engaging with content if learners became the designers and creators rather than consumers of information as in expert-generated animations" (p.121). Although our focus was not to explore PST motivation and engagement, we were interested in exploring how the use of PST-generated animations could provide a window into PST noticing to support communication about what they notice.

Professional noticing is fundamental to the work of teaching (Sherin, Jacobs & Philipp, 2011), and many instances simultaneously demand teachers' attention and impact teachers' decisions (Star, Lynch, & Perova, 2011). Research indicates that differences exist between novice and expert teachers' noticing skills, where veterans are more adept at interpreting a situation and novices tend to be descriptive and miss key elements (Huang & Li, 2012). Research also indicates that it is possible to improve PST noticing with support (Schack et al. 2013). As a result of studying the development of noticing among PSTs, Star et al. (2011) argued that teacher education programs should explicitly teach PSTs to notice. We were interested in studying what the use of PST-generated animations could enable in regards to their noticing, so that we as teacher educators could better support PST development of this complex skill.

Within practice-based teacher education, the primary goal of a methods course is to develop PSTs' professional practices in a specified content area, meaning the focus is on PSTs learning to do the work of teaching (Grossman, Hammerness, & McDonald, 2009). Grossman and colleagues (2009) developed a framework for pedagogies of professional practice to conceptualize processes that support teachers' development of professional skills. This framework includes three elements: representations, decompositions, and approximations of practice. Approximations of practice are opportunities for PSTs to enact one or some practices of teaching themselves with scaffolding. Turning again to technology, various technological mediums have been used as a way to approximate practices with the aim of enhancing PSTs' development of professional practices. One example of this is the use of simulations, including SimSchool, which is an on-line teaching application (Gibson, 2007) as well as the TeachLive program (Dieker, Rodríguez, Lignugaris, Hynes, & Hughes, 2014). These technological features, including animation, provide an advantageous way for PSTs to engage

in an approximation of practice and simultaneously provide us, as teacher educators, insight into their noticing.

Method

PSTs (n=126) from mathematics methods courses at six universities in the United States participated in a task involving the use of animation. Data for this study come from a larger study, and were analyzed to measure PST noticing to further understand the specificity of noticing that existed across the different mediums employed, namely written records and animations. To design and implement the task, we built on previous literature of professional noticing in the context of methods courses (e.g., Star et al., 2011) by using video, and added the use of PST-generated animations. The project design involved three phases.

Phase One: Task Design

In Phase One, we created a task to provide insight into PSTs' noticing that could be administered in each of the six methods courses. We showed the PSTs a video clip from a mathematics classroom that featured a student-centered classroom and included segments of elementary students sharing their mathematical thinking. We used the publicly available video, *Cookies to Share*, (http://www.learner.org/resources/series32.html). The PSTs were required to watch the clip (beginning to 13:45) in class, identify a pivotal moment of student(s)' mathematical thinking specific to mathematical learning or teaching, and then record this moment in writing. After writing about what they noticed, preservice teaches created their own scenes of the most pivotal moment they noticed using the website, goanimate.com. Goanimate is a cloud-based platform for generating and disseminating animated videos. Figure 1 is a screenshot of an animation. The intent of this design was to provide opportunities for PSTs to communicate what they noticed from the video through written and animated mediums.

Figure 1. Goanimate screenshot example.

Phase Two: Task Implementation

For Phase Two, the task was administered in one section of an elementary mathematics methods course at each of six different universities during the fall 2013 semester. The exact timing of project implementation within the semester schedule was left to the discretion of the individual methods course instructors. The methods instructor of the respective methods course administered the project him or herself. The data set included all of the PSTs' written responses, animations, and transcriptions of the animations: 126 written documents and 106 animated videos in all. It should be noted that due to limited access to technology PSTs in one teacher educator's class did not animate the written medium, thus accounting for the difference between the written and animation data.

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

Phase Three: Data Analysis

In Phase Three data were analyzed to measure PST noticing and further understand the specificity of noticing that existed across the different sections of the elementary mathematics courses and the different mediums employed. First, we analyzed each written account and animation with a focus of whom and what PSTs noticed, using the coding framework in Figure 2, adapted from (DeAraujo et al., 2015). Each data item was coded as one of the following: Teacher General Noticing, Teacher Specific Noticing, Student General Noticing, or Student Specific Noticing.

WHO	Teacher		Student	
	Attend to the Teacher		Attend to Whole Class Attend to a Group of Students or Attend to a Student	
WHAT	**GENERAL** Focus on General Teaching Strategies, Pedagogy, Content of a Lesson, Context of a Problem	**SPECIFIC** Focus on a connection between the teacher and another person(s) or aspect (e.g. pedagogical strategies interactions	**GENERAL** Focus on general actions, aspects, ideas, topics, lesson structure or contextual features (e.g. physical environment, home environment, dispositions, etc.)	**SPECIFIC** Focus on a connection between a student and another person(s) or aspect (e.g. his/her mathematical thinking, his/her interactions with others, or specific teaching strategies)

Figure 2. Framework for Level One coding, general or specific.

Following the Level One coding, we were interested in knowing more about the content of the data items coded as specific to understand what the PSTs were identifying as pivotal. We coded these data using a Level Two Coding Framework that included the following codes: 1) Addition, 2) Slope, 3) Fair Sharing, 4) Justification, 5) Procedural Explanation, 6) Written Notation, 7) Clarification, and 8) Meaning of Equal. All written and animated data deemed as *specific* in Level One coding were then coded again based on these Level Two themes.

To analyze data, we built on the aforementioned data analysis with a focus on the specificity of PSTs as they noticed. We took all codes that were general or specific (Level One coding) and the codes of content (Level Two coding) and analyzed the data by examining changes from the written to animated mediums. We analyzed for one of the following four scenarios: General in Writing to General in Animation, General in Writing to Specific in Animation, Specific in Writing to Specific in Animation, or Specific in Writing to General in Animation. We then analyzed for the extent to which the change occurred, meaning the quantity of specific and general comments, such as changing from one specific focus in the written account to two specific foci in the animation (see Table 1). Following this analysis, four of the researchers discussed major themes across the data set. This information resulted in discussion about: (a) the type of change that occurred (i.e. specific to general), and (b) the content of the change each ultimately informing the role animation played in replacing, adding, transforming or hindering PST noticing.

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Table 1: Codes for Level of Specificity in Communication

Code Number	Change Description	Animation Content Compared with Written Content. Animation has…
Code 1	Specific to Specific	Same Content
Code 2	General to General	Same Content
Code 3	Specific to Specific	Same Content with Additional Specificity
Code 4	Specific to Specific	Specific Content is Different
Code 5	General to Specific	One Specific Focus
Code 6	General to Specific	More than One Specific Focus
Code 7	General to General	Different General Focus
Code 8	Specific to Specific	Different Specific Codes and Written Account had more than one Specific Focus
Code 9	Specific to General	General Stemming from one Specific Focus in Animation
Code 10	General to Specific	Specific stemming from more than one General Code
Code 11	General to Specific	More than one General Code shifted to more than one specific code

Findings

Based on analysis of the written versus animated accounts of PST noticing, four main categories of data were present when comparing PSTs' levels of specificity: general to general, general to specific, specific to specific, and specific to general. We describe the change with respect to specificity from the written account to the animated account, both intended to capture the salient moment of PST noticing. Figure 3 shows the distribution of the categories of change for the PSTs.

N = 106

5	53	46	2
General → General	General → Specific	Specific → Specific	Specific → General
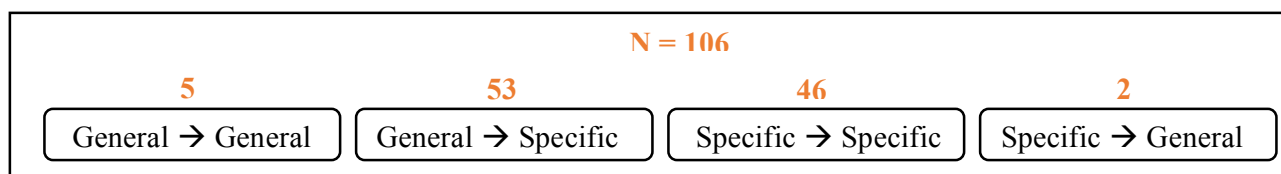
Figure 3. Distribution of change from written to animated noticing based on degree of specificity.

Chi-Square for type of change was 80.94 (df=3, $p < .001$). Therefore, changes from the written to animated noticings towards general occurred less than expected by chance, and moves toward specific occurred more than expected by chance. The two main categories that occurred more than expected by chance and represented the greatest number of PSTs were general to specific (n=53) and specific to specific (n=46). Consequently, we focus our results on these two categories because they represent the greatest number of PSTs and are statistically distinguished from the other two categories.

Following analysis of the overall categories, we analyzed additional details about the changes that were occurring across mediums to further recognize how PSTs were expressing their noticing both through the written and animated mediums. For example, Code 10 states, "Two general to one specific" this indicates that in the written record of noticing, the data from the PST in this category had two general topics that were present and when the PST animated, the animation had one specific topic included (see Table 1). Similarly, as another example, in data coded with Code 4, the PSTs' written data were each coded with one specific code and when they animated, they focused on a different specific code.

After recognizing the various paths taken by the PSTs, we sought understanding for the changes that occurred from the written to animated mediums that would provide insight to mathematics teacher educators about how PSTs' thinking was communicated differently across the two mediums.

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The most common scenarios for PSTs who transitioned from written general accounts to specifics in their animations were to transition from one general aspect in the written noticing to one specific focus in the animated noticing (code 5, n=29) or to transition from one general aspect to more than one specific focus when animating (code 6, n=20). Analysis of those who were specific in their written accounts and maintained specificity in their animations showed that it was common for PSTs to write about one specific topic and then animate that same aspect (code 1, n=18). Additionally, many PSTs began with one specific focus and then animated that topic along with at least one more specific topic (code 3, n=11). There were also PSTs who wrote about one specific topic and then animated a different specific topic (code 4, n=16). These five cases were further distinguished in the data set because they represented the majority of the PSTs and posed interesting situations about how the PSTs were maintaining or adjusting their noticing with the use of different mediums.

General to Specific Noticing Across Mediums

The PSTs who had general descriptions of what was noticed in their written analysis and then included a specific connection between the teacher or student and another person or aspect in the animation accounted for 49 of the 106 cases, being the most common change noted across the data set. Within these cases, 29 PSTs shifted their noticing to one specific aspect while 20 focused on two or more aspects in their animation.

One general to one specific. There were 29 PSTs who focused on one general aspect in their written account and one specific aspect in their animation. As the shift from general to specific occurred, two main features were identified. First, most PSTs introduced additional people into the animation who were not mentioned in the written account of noticing. For example, one PST focused on students when describing her pivotal moment in writing, "When the student expressed that everyone (all 8 people) will have the same amount of cookies and none left over." When the same PST animated the pivotal moment (see animation at https://youtu.be/ZUGf1x8FOkQ), she included the teacher and had the teacher prompt a student (named Grinch) to explain an equal sharing problem and justify his thinking. As seen in this example, a shift from a general account in the written medium typically involved the addition of other people in the animated medium. Second, as the PSTs shifted from a general focus in their written accounts to specific connections in their animations they increased the frequency comments related to mathematics. In the animations, the PSTs most notably included fair sharing, the explanation of a method, or the meaning of equal or equal parts.

One general to more than one specific. There were 20 PSTs who focused on multiple aspects within their animations. Nineteen of the 20 PSTs focused on the use of justification or explanation as one of their paired aspects which was often not noted in the written noticing across these cases. For example, one student wrote, "I chose the moment when the teacher asked the students if ½ a cookie would work." In the animation of this moment, the PST connected this action to the notion of student explanation or justification. The second theme among these 20 PSTs was the specific use of justification or explanation to support student mathematical understanding of fair share within the animated video. This occurred in 80% of the cases. The final theme across these cases was the use of *anchor* questions within animations to support the PST noticing. A majority of the students had the teacher asking questions such as, "How do you know you are right? What did you come up with? What did you guys do?" These questions served as an entry point and focus of conversation within the animated video, but were rarely mentioned in the written reflection.

Analysis across all 49 cases of PSTs who wrote about general aspects and animated specific aspects indicates: (a) an increased quantity of people present, (b) an increased mathematical focus, especially with fair sharing, (c) the inclusion of explanation and justification, and (d) the inclusion of anchor questions embedded in dialogue in their animations. These findings indicate that when the PSTs animated their noticings, they were more specific and included elements of effective mathematics teaching and learning that were not present in their written accounts.

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Discussion and Implications

We argue that the practice of implementing the animation platform as a mechanism for PSTs to communicate their noticing was transformative (Hughes et al., 2006) because the medium afforded opportunities for PSTs to convey their thoughts about classroom interactions in a way that was not afforded through the written medium alone. We support this claim with the increased specificity that was apparent in the animations as compared to the written records. Further, we consider this practice of animating, as a transformational experience, to serve as an approximation of practice for teaching; the transformational features of the animation enabled PSTs to engage in approximating actual teaching practice (Grossman et al., 2009).

According to Grossman et al. (2009) approximations of practice engage participants in practices that are proximal to the practices of a profession, meaning closely related to teaching in this case. As previously mentioned, current research has examined approximations of practice in multiple forms, including through technological mediums and other avenues (e.g. Herbst et al., 2014). In fact, Herbst et al. (2014) focused on media rich, web-authoring tools for PSTs to create scenarios as an approximation of practice. We argue that our use of animation provides similar opportunities for approximating teaching as does their platform, *LessonSketch*. The ability to manipulate audio and visual components of figures (i.e. student(s) and teacher) in a scene in the present study further increases the opportunities for proximal practice through transformational means. More specifically, when considered in combination with the literature on professional noticing, this medium afforded opportunities for PSTs to approximate their practice as they communicated what they professionally noticed.

Further, in the present study, animation was used in the context of a mathematics methods course. We recognize the affordances of this medium for providing PSTs with opportunities to communicate their noticing about a mathematics scenario. However, future research should examine how the platform could be used in other disciplines and contexts to provide proximal practices that closely mirror actual teaching (Grossman et al., 2009). These opportunities are essential for teacher development because they provide opportunities for deliberate practice, allow for elaborations of practice, and highlight PST considerations about the profession—all key components for learning to teach.

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THE USE OF STUDENT-CREATED DYNAMIC MODELS TO EXPLORE CALCULUS CONCEPTS

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Multiple representations, multiple modalities, and technology can be helpful in the understanding of mathematical concepts when used in an appropriate manner (Shah & Freedman, 2003; Goldman, 2003), but this alone does not account for the student benefits of creating and using dynamic models over teacher generated graphs to construct connections between representations. By uncovering the dynamic nature of mathematics, calculus becomes more transparent as relationships and patterns emerge. The struggle to understand becomes worthwhile and rewarding for students as they create and observe the action of a dynamic mathematical object. This study shows an improvement in attitude, and academic achievement when students develop dynamic mathematical object to understand calculus and poses new questions to explore.

Keywords: Technology, Curriculum, High School Education, Learning Theory

Introduction

Mathematics students across all ages and subtopics are encouraged to explore multiple representations of mathematical relationships (e.g., graphical, symbolic, tabular) using a variety of appropriate tools (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010; National Council of Teachers of Mathematics [NCTM], 2000). The use of tools ranging from pencil, paper, and rulers to calculators, computer algebra systems, and dynamic geometry software are encouraged to aid in the solution and exploration of mathematical problems and concepts. Since the invention of the computer in the mid-twentieth century, computers have decreased considerably in size, increased tremendously in processing power, and have become inexpensive enough to permeate private and public sectors, including education. The increasing availability of technology in education has led to digital textbooks, advanced handheld calculators, and one-to-one technology initiatives. New technologies such as these lead to new questions and studies regarding the efficacy of technology in the classroom, and the results have varied greatly (Bebell & Kay, 2010; Donovan, Hartley, & Strudler, 2007; Maninger & Holden, 2009). The opportunity to explore the affordances of effective practices is expanding as the technologies evolve, and more studies must be done in order to identify these affordances to help maximize learning in a mathematics classroom.

Although computers grant access to the Internet, powerful dynamic software, and the ability to collaborate in new ways, it is necessary that teachers thoroughly explore the multitude of options and determine what methods are effective and beneficial to teaching and learning. Studies of effectiveness should parallel the teachers' and students' explorations to confirm whether particular methods are beneficial, or possibly detrimental, to learning objectives and also strive to identify the particular affordances of the experience. This process will refine educators' understanding of the usefulness of technology and may help to evolve current methods of instruction by identifying the affordances of useful procedures. In this paper, these matters of efficacy and affordances will be explored by examining the use of software to create dynamic representations of functions, referred to as dynamic models, in a secondary calculus classroom. Specifically, this paper will address the question: does student creation of, and interaction with, dynamic models using Wolfram *Mathematica* increase performance on calculus assessments composed of a variety of subtopics (i.e., derivatives, tangent lines, relative extrema)? And, does this experience have an effect on student

attitudes toward mathematics? First, the Literature Review section will provide context for the various tools that were utilized during the experiment. Then, the Design and Methods section will describe the quasi-experimental design and provide details about the intervention that took place during the experiment. Finally, sections will explore certain affordances of the learning activity and make suggestions for further clarifying research.

Literature Review

Dynamic mathematic learning environments allow students and teachers to explore and uncover relationships by manipulating aspects of a particular concept. These dynamic learning environments can be applied in a variety of ways to a diverse assortment of topics. Students can use dynamic geometry software (e.g., Geometer's Sketchpad, GeoGebra) to construct geometric figures and explore properties of the figures by clicking and dragging components to investigate patterns, make conjectures, and verify relationships. Bu and Haciomeroglu (2010) explore the specific use of sliders (see figure 1) in dynamic learning environments. A slider acts as a single mathematical object that has two representations. Algebraically, a slider acts as a variable within a defined interval that can be conceptualized as a constant in certain settings. "Graphically, a slider appears as a segment which allows the user to adjust the value of the corresponding variable through dragging" (Bu & Haciomeroglu, 2010, p. 214). The presentation of both graphical and algebraic representations gives learners the ability to make abstractions more visible and the opportunity to make connections between representations (Martinovic & Karadag, 2012). When sliders are used to represent a constant or a constant variable that acts as a parameter of a function, learners can explore multiple cases of a function without having to change the function definition (Bu & Haciomeroglu, 2010). By comparison, graphing multiple of cases of a function using pencil and paper, or even a traditional graphing calculator (e.g., the TI-84+), would invariably take more time. Among the various studies that examine the benefits of dynamic and interactive mathematics environments, there still exists a need to explore the importance of having learners generate dynamic representations as opposed to having them generated for them (Goldman & Petrosino, 1999; Schwartz & Bransford, 1998).

Design and Methods

During the course of a two-year period, assessments were collected from two introductory level calculus classes consisting of high school seniors at a midsized, rural high school in Texas. As seniors, students in this course had taken an advanced algebra course and a precalculus course prior to entering the introductory level calculus course. At the beginning of the experiment, the teacher had about four years of teaching experience and had taught the course multiple times before. During the 2013-2014 school year, the course content was taught to a class of 29 high school seniors in a traditional manner. The first semester of the course was designed to review algebra concepts while the second semester focused on connecting precalculus concepts to topics of differential calculus. Multiple representations were explored using pencil and paper. Graphing calculators (e.g., TI-84+) were used on occasion to verify relationships and check the reasonableness of solutions. The teacher often used graphical representations during instruction. Although graphical representations were an emphasis of the course, symbolic manipulation and interpretation were largely the primary foci, and the graphs that were explored were not dynamic. During the 2014-2015 school year, the same teacher taught course content to 25 high school seniors. Periodically throughout the second half of the year, the teacher guided students to create dynamic models that utilized sliders using Wolfram *Mathematica,* after which the teacher would formatively assess the students' understandings of the concept under investigation through the lens of the dynamic applet. All other instruction, assessments, and assignments were unchanged.

An Overview of the Tasks

The students were first introduced to *Mathematica* as a replacement of the graphing calculator in order to gain familiarity with the software. By making use of *Mathematica's* freeform input, which essentially allows the user to input commands using English phrases and basic math notations, students could easily plot functions and note how *Mathematica* reformats the input into formal code. With this feature and varying amounts of support and guidance from the teacher and the Wolfram website, the students programmed an applet that accounted for the effect of one parameter of a sinusoidal function (e.g., amplitude). Then, the students were challenged to explore and manipulate the families of trigonometric functions by creating an applet in which all parameters of the functions could be changed (see figure 1). Upon completion, the teacher checked for understanding by asking questions about the effects of changing each parameter and by challenging the students to create sinusoidal functions that had certain attributes (e.g., a period of π, an amplitude of 4).
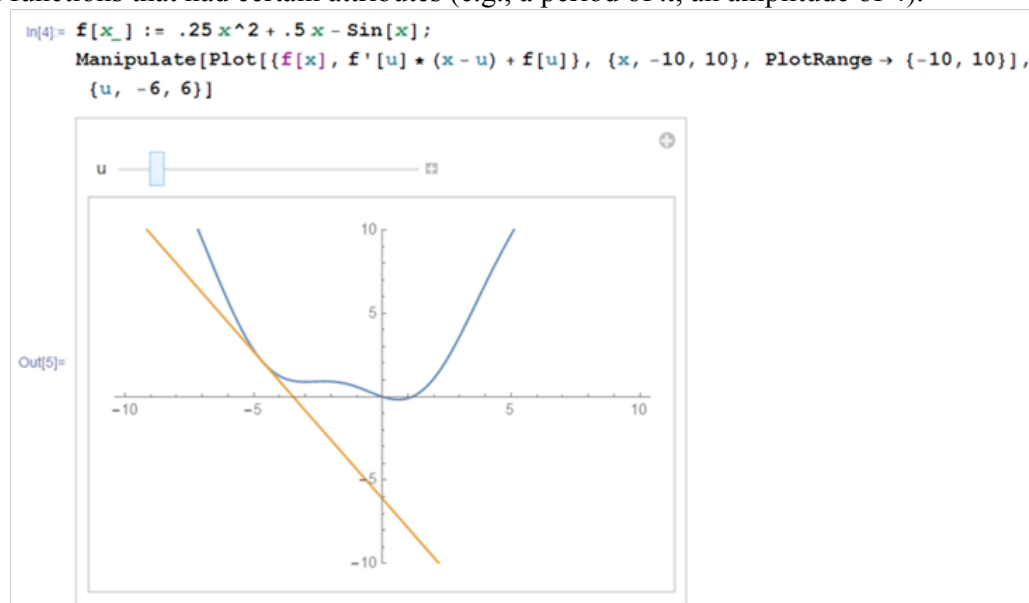
Figure 1. Screenshot of student created exploration of tangent lines.

Data, Analysis, and Results

All assessments were collected from the two senior classes during the 2013-2014 (Group1) and 2014-2015 (Group 2) school years. The formal assessment that occurred after the final use of *Mathematica* was chosen as the dependent measure to be compared in the quasi-experimental design. The assessment covered topics that aligned with the goals of the final applet-creation activity including the understanding of tangent lines and relative extrema. This assessment will be referred to as the post-test. The comprehensive semester exam (covering a review and extension of algebra topics) was chosen as a covariate to act as a control for the variability in math skills and experience between the two groups. The semester exam was graded on a traditional 100-point scale. The post test included an opportunity to score five bonus points for a maximum score of 105. As seen in table 1, the means and ranges of the scores on the semester exam vary between the groups, and it appears that the means of the post-test scores vary considerably. The Attitudes Toward Mathematics Inventory (ATMI) was given to each group, at the same time each year, after the post-test was given (see Tapia & Marsh, 2004). The ATMI is a brief survey consisting of 40 questions that is designed to measure high school and college students' attitudes toward mathematics.

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Table 1: Descriptive Statistics

		Minimum	Maximum	Mean	Std. Deviation
Group 1 N = 29	Semester Exam	49.00	101.00	75.93	12.23
	Post Test	46.00	99.00	69.48	16.40
Group 2 N = 25	Semester Exam	41.00	95.00	70.08	15.99
	Post Test	58.00	103.00	80.56	11.90

Results

A One-way ANCOVA revealed a statistically significant difference between the post-test scores of each group controlling for prior mathematical skill level and inconsistencies using the semester exam scores, $F(1, 51) = 14.075$, $p < 0.001$. It is important to note that the results were also significant when other prior assessments were used as a covariate. A partial eta-squared value of about 0.185 indicates that approximately 18.5% of the variance in the post test scores is attributable to the independent group variable (i.e., group 1 as the control group, group 2 as the post-intervention group).

A chi-square test of independence revealed a significant difference between the reported attitudes of group 1 and group 2, $\chi_2(2, N = 50) = 680.835$, $p < .001$. Students from group 2 reported a higher proportion of neutral and positive responses and reported fewer negative responses. Figure 3 shows the frequencies of responses for each group and level of response. The responses utilized a Likert scale from which the students chose the extent to which they agreed or disagreed with a statement involving mathematics. A response of rating one (1) corresponds to a choice that reflects the most negative attitude toward mathematics (e.g., Strongly Agree to the statement: "Mathematics makes me uncomfortable."; Strongly Disagree to the statement: "I think studying advanced mathematics is useful."). A response of rating three (3) corresponds to a selection of "neutral," and a response of rating five (5) corresponds to a choice that reflects the most positive attitude toward mathematics. Ratings of four (4) and two (2) correspond to intermediate responses, such as "Agree" or "Disagree."

Discussion and Conclusion

The significant difference in post test scores and reported attitudes suggests that the experience of creating and interacting with dynamic spaces did indeed aid the students in their quest to understand the nuances of calculus. This section will aim to explore some of the constructive aspects of this experience that may have implications for creating effective technological learning experiences in mathematics instruction. First and most obvious, the use of sliders allows for more time for productive discourse (Bu & Haciomeroglu, 2010). Once students learn how to operate the more advanced technology (i.e., *Mathematica*), students and teachers are enabled to spend more class time talking about the concept under investigation. As employed by the teacher in this study, discourse can evolve and move beyond addressing procedural features of a task (e.g., graphing multiple functions to identify a pattern) to attend to the qualities of the concept in a shorter amount of time while also granting more opportunities for feedback, revision, and reflection—a vital aspect of technology use in education (National Research Council [NRC], 2000). For example, during the task, questions (e.g., how can you use this model to find the where the lowest point on the function occurs?) used the dynamic aspect of the model to quickly make connections to concepts (e.g., relative extrema, concavity) while providing immediate feedback to each student. The efficiency of the dynamic models allows students and teachers to address broader conceptual qualities with less of an opportunity for students to lose interest or get distracted from the goal of the lesson.

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The act of modeling and programming dynamic spaces can parallel the construction of underlying mathematics (Tall,1991). On the surface, the students are mirroring the common task of constructing tangent lines when they define a function and situate the correct values in position (e.g., $f''(u)$ as the slope, $f(u)$ as the y-intercept) (see figure 1). In this particular activity, several students initially programmed incorrectly by positioning functions (e.g., $f'' x$ as the slope) rather than functions evaluated at a point (e.g., $f'' u$ where u represents a constant). This mistake was quickly realized when the student's output returned graphs that clearly did not represent a tangent line which, in turn, led to conversations between the students and the teacher about the difference between a variable and a variable that represents a constant—an example of an opportunity for revision and reflection. While other software (e.g., GeoGebra, Geometer's Sketchpad) can construct similar dynamic spaces, programming using *Mathematica* introduce a practical application of a function. Specifically, by using the "Plot" and "Manipulate" functions of Mathematica, students gain experience with functions of multiple arguments (see figure **2**). At a deeper level, when students embed a function into another function (e.g., Manipulate [Plot[{f(x),…), they are reintroduced to the concept of function composition, a crucial topic in the understanding of the chain rule.
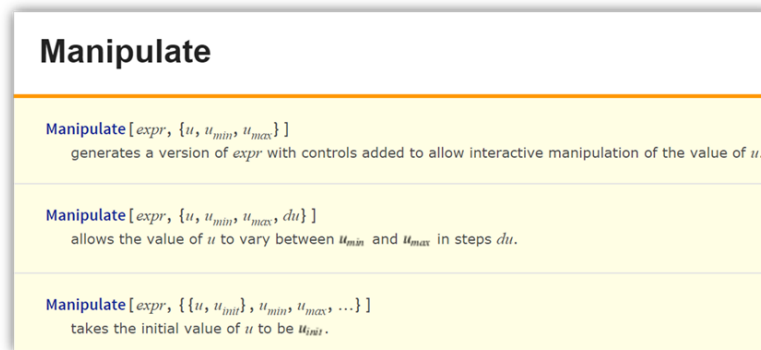
Figure 2. Screenshot showing multiple arguments of the Manipulate function.

The positive results are also consistent with NCTM Standards (2000) suggesting that the exploration of multiple representations is beneficial to the understanding of mathematical concepts. Since it is believed that "student difficulties in understanding calculus concepts result from an inadequate understanding of graphic and algebraic aspects of these concepts" (Haciomeroglu & Andreasen, 2013, p. 7), it is plausible that the experience with the dynamic graphical representations, the algebraic nature of coding *Mathematica,* and the use of a slider that acts as a medium between representations contributed to the easing of such difficulties and increased understanding evident in the results. For example, when using derivatives to determine relative extrema (i.e., a maximum or minimum value of a function) the analytic procedures employed align with a graphical interpretation. Specifically, to determine possible relative extrema, or critical points, of a single variable function f analytically, one may compute the derivative of the function f' and solve for the values at which the derivative evaluates to zero (i.e., solve for c such that $f' c = 0$). This process aligns with the graphical interpretation of finding the points along the function at which the line tangent to the point has a slope of zero (figure 3).

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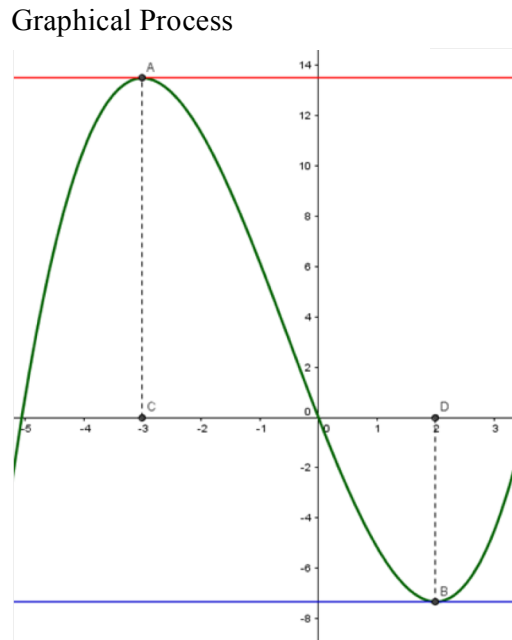
Graphical Process

Analytical Process

Given   –   –
$f(x) = 1/2\ x^2 + 1/4\ x^4 - 6x,$
Compute $f'(x)$,
$f'(x) = x4 + x - 6$
$= (x - 2)(x + 3)$
Let $f'(x) = 0$, solve for x,
$x4 + x - 6 = 0$
$(x - 2)(x + 3) = 0$
$x = 2, -3$

Conclude that f may have relative extrema at
$x = 2$ and/or $x = -3$.

Figure 3. Relationship between graphical interpretation and analytic.

The visualization of the graphical interpretation can act as a reference when selecting the appropriate analytic method to employ. In this case, by visualizing an example function, one can deduce that because the extrema occur where the tangent line has a slope of zero and the derivative of the function at a point can be interpreted as the slope of the tangent line, then the analytic process to find the extrema begins by setting the derivative equal to zero because that is the point where the slope is horizontal. When students fail to make this connection between the graphical and analytic procedures through visualization, many students haphazardly resort to analytic procedures without fully understanding what to use or why they are using them (Haciomeroglu & Andreasen, 2013).

Multiple representations, multiple modalities, and technology can be helpful in the understanding of mathematical concepts when used in an appropriate manner (Shah & Freedman, 2003; Goldman, 2003), but this alone does not account for the benefits of creating and using dynamic models over teacher generated graphs to construct connections between representations. One benefit of the dynamic feature of the models used is that they provide an external representation of tangent lines at various points on a function. Consequently, students do not have to maintain or mentally transform a mental representation of the model which may reduce the cognitive load required to comprehend the new concept being presented (Shah & Freedman, 2003). As Tall (1991) suggests,

> A computer can also give much-needed meaning to mathematical concepts that students may feel are not of the physical world but in the mind, or in some ideal world. It is generally agreed that ideas are easier to understand when they are made more "concrete" and less "abstract". When an abstract idea is implemented or represented in a computer, then it is concrete in the mind, at least in the sense that it exists (electro-magnetically, if not physically). Not only can the computer construct be used to perform processes represented by the abstract idea, but it can itself be manipulated, things can be done to it. (Tall, 1991, p. 235)

Tall's (1991) explanation may help shed light on how the dynamic model involving tangent lines aids in a concrete understanding of the abstract idea that a derivative of a function is, in many cases, a function itself. That is, the derivative of a function is more than just a slope at a single point, but

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rather a sort of formula that contains enough information for the practicing mathematician to find a slope, or rate of change, at any point along a curve (granted the derivative is defined at each point on the curve). Also, Tall (1991) posits that it is often true that "whenever a person constructs something on a computer, a corresponding construction is made in the person's mind" (p. 235). Although this is a very bold remark, it is suggestive of constructivist learning theory in the sense that what is being learned is inextricably tied to how it is learned. Thus, by using animated (i.e., dynamic) visuals to decrease the cognitive load of the learner at the time of conceptualization (Shah & Freedman, 2003), students are enabled to make connections between graphical and analytic procedures through previously unrealized dynamic visualization and avoid haphazardly resorting to analytic procedures without a deep understanding of the reasoning behind them.

Finally, Shah and Freedman (2003) suggest that students' attention is drawn to electronic visual displays, that students are more apt to study electronically delivered content for longer periods of time, and that visualizations in electronic learning environments can be attractive and motivating. These ideas help to explain the increase in positive responses on the mathematical attitude survey.

The implications of this research align with the current trend and push to integrate technology in the mathematics classroom. Although it may be detrimental to assume that the use of technology automatically implies learning, it is evident that effective use of technology in the classroom by both teachers and students can have substantially beneficial impacts. The affordances involved with the effective use of technology should be identified, and this technology should continue enhancing learning opportunities in mathematics classrooms by taking "advantage of what technology can do efficiently and well - graphing, visualizing, and computing" (NCTM, 2000, p. 26).

Limitations

This study highlights the beneficial aspects of incorporating technology into the calculus classroom but has many limitations and leads to more questions about maximizing learning through the use of student-created explorative spaces. The study was primarily limited by the small sample sizes involved. The sample sizes could be expanded to include a more diverse population by including other schools and classes. This could also help eliminate the possibility that the positive results were due to teacher engagement or an excitement about using "new" technology, rather than the actual exploration of concepts and representations.

Opportunities for Future Research

Although the large effect size implies that the applet-creation and exploration was substantially valuable, from this study alone, it is unclear to what extent each aspect of the creative and investigative processes was beneficial to student learning. For this reason, among others, this study raises many more questions to be explored. Would the results have been different if the dynamic models were to be created and supplied by the teacher rather than created by the students? Would the results differ if another program were to be used? To distinguish between the benefits of investigating student-created dynamic spaces versus teacher supplied dynamic spaces, multiple groups could be included in a study. Particularly, the control group could be taught traditionally, a second group could explore concepts using teacher supplied dynamic spaces, while a third group could explore concepts using student-created dynamic spaces. This design need not be limited to concepts of calculus. Rather, it could be expanded to explore other statistical models and spaces used for real-world problem solving tasks. A similar three-group design could be used to distinguish between benefits of one program over another, given a particular topic, which might aid in determining if there is a strong benefit to the programming language employed by *Mathematica* over a more elementary software program. Although current popular programs used to explore mathematical concepts and create dynamic spaces (e.g., *Desmos, GeoGebra, Geometer's Sketchpad, Mathematic*a, *Matlab*) share some overlapping features of their functionality (e.g., the ability to

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create sliders) and user interfaces (e.g., programming language, inputting geometric figures by clicking and dragging), certain programs may be more appropriate for certain levels of development and age groups and may also vary among particular concepts and explorations. As Tall (1991) posits, it is important to note "the principal aim of the programming system of *Mathematica* is predominantly for doing mathematics, rather than learning mathematics" (p. 242). Since the technology that is used as an instructional tool develops over time due to updates in software, it could prove beneficial to identify the affordances of common features that aid in the development of mathematical skills in order to maximize the advantages of using technological tools in education. The future of technology in the classroom is promising; there are many questions left to explore.

Acknowledgments

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EXAMINING PRESERVICE TEACHER THINKING ABOUT TECHNOLOGY-BASED TRIGONOMETRIC EXPLORATIONS THROUGH A REPLACING, AMPLIFYING, AND TRANSFORMING FRAMEWORK

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Researchers promoting the inclusion of technology for teaching and learning have recently called for the integration of mathematical technologies into the preparation of future teachers. This report analyzes the dynamic geometry sketches produced by preservice secondary mathematics teachers when investigating trigonometric relationships. We analyzed preservice teachers' approaches to a particular task based on the extent to which the technology replaced, amplified, or transformed learning opportunities about tangent. We highlight the prominent aspects of the diagrams produced and discuss technological affordances these diagrams present for supporting preservice teachers' development of content knowledge and pedagogical approaches related to trigonometry.

Keywords: Technology, Teacher Education – Preservice, Geometry and Geometrical and Spatial Thinking

There have been recent pushes to incorporate technology into the teaching and learning of K-12 mathematics (National Council of Teachers of Mathematics, 2000; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). As Lee and Hollenbrands (2008) point out, "Whether technology will enhance or hinder students' learning depends on teachers' decisions when using technology tools, decisions that are often based on knowledge gained during a teacher preparation program" (p. 326). Subsequently pre-service teachers (PSTs) need opportunities to incorporate technology to learn and teach mathematics (Association of Mathematics Teacher Educators, 2006; International Society for Technology in Education, 2008). Incorporating technological tools into mathematics content courses allows conversations about new ways in which mathematical topics can be explored and new mathematical investigations that were not previously possible with more traditional tools (Hughes, Thomas, & Scharber, 2006).

One mathematical topic PSTs particularly struggle with is trigonometry (Moore, 2009), which spans both geometric and algebraic perspectives where an algebraic approach uses the unit circle as the object of study and a geometric approach uses right triangles as the object of study. Although researchers advocate for students to make connections between these two approaches, teachers often have difficulty coherently understanding trigonometric relationships (Moore, Paoletti, & Musgrave, 2014; Weber, 2005). This paper examines a set of activities used to develop 20 secondary preservice mathematics teachers' thinking about trigonometric relationships. These tasks required teachers to create dynamic geometry sketches in Geogebra or Geometer's Sketchpad to create multiple representations of trigonometric relationships and make connections between these representations. Specifically, the activity asked the PSTs to examine the slopes of lines, the slope-triangles created on these lines, the ratios of side lengths within these triangles, and the angles within these slope triangles. This research examines the following question: How does the production of dynamic geometry sketches by preservice teachers support their understandings of tangent, and what is the role of technology (i.e. *replace*, *amplify*, *transform*) in supporting their content knowledge about the tangent relationship?

Frameworks

The Technological Pedagogical Content Knowledge (TPACK) framework, described by Koehler and Mishra (2009) unpacks the unique kinds of knowledge that go into the effective teaching of

content (in this case mathematics) with technology. The framework identifies ways in which pedagogical, technological, and content knowledge interact with each other in a given context. The emergent forms of knowledge are "the basis of effective teaching with technology, requiring an understanding of the representation of concepts using technologies" (p. 66). This framework is also a useful lens for examining the technological artifacts that teachers create and use to explore mathematical content. Through these artifacts, mathematical, pedagogical, and technological issues are highlighted and minimized when they are used as tool to illustrate and investigate mathematics. By investigating the technological tools created by PSTs, we gain insight into their understanding of the complex interactions between technology, pedagogy, and content.

As new technologies are developed and utilized in mathematics classrooms their effects commonly *replace*, *amplify*, or *transform* non-technological means of teaching the same content (Hughes et al., 2006). New technological features (e.g. dynamic presentations, simultaneous representations, communication, etc.) offer the potential to change the mathematical explorations and discussions that occur in classrooms. However, this is only possible if the technology is used in ways that make use of this potential. The Replacing, Amplifying, and Transforming (RAT) framework provides a lens for analysis of the mode of technology integration into an activity (Hughes et al., 2006). This framework draws attention to whether the technology *replaces* a similar presentation without the technology, *amplifies* the learning process that was present in the non-technology version, or *transforms* the learning experiences to provide possibilities that were otherwise not possible without the technology. This framework has been a useful tool for analyzing the integration of technology into teacher preparation courses as a means for understanding how teacher educators can improve instruction (Glassmeyer, Brakoniecki, & Amador, 2016a, 2016b). The RAT framework allows for the analysis of technology and the range of ways it is used in tasks to advance learning outcomes.

The tasks used with these PSTs were focused on exploring mathematical content, namely the tangent relationship (as these tasks were given in a mathematics content course for beginning teachers). In our study, the TPACK framework is a theoretical lens for understanding how PSTs conceptualized the tangent relationship while using technology tools and considering their future careers as secondary mathematics teachers. The RAT framework functions as an analytic lens to describe how the PSTs used sketches when exploring this relationship and how the technology provided for learning opportunities. Together, these frameworks for technology integration provided understanding about how PST-generated dynamic geometry sketches supported their understandings of tangent as technology was integrated into a given task.

Method

This study took place at a large Southern university within a content course for prospective secondary teachers. In the course, PSTs regularly engaged in learner-centered instruction incorporating collaborative learning and technology such as graphing calculators, Geometer's Sketch Pad, Geogebra, and Desmos. Approximately half of the course meetings were devoted to having PSTs explore trigonometric relationships to support their quantitative reasoning and ultimately their conceptual understanding of mathematical topics, with secondary attention going to enhancing their pedagogical content knowledge by considering how they would teach this material to their own students.

The study focuses on a three-day lesson where the 20 PSTs investigated the tangent relationship. The task was modified from the CPM Core Connections Geometry textbook (Kysh, Dietiker, Sallee, Hamada, & Hoey, 2013) by requiring teachers to create dynamic geometry sketches to answer parts of the lesson. The lesson has teachers explore connections between the slope-ratio triangle (a right triangle produced by a line and it's vertical-to-horizontal change) and the base angle of that slope triangle angle. (See figure 1 for an example of a slope ratio triangle in a pre-constructed applet).
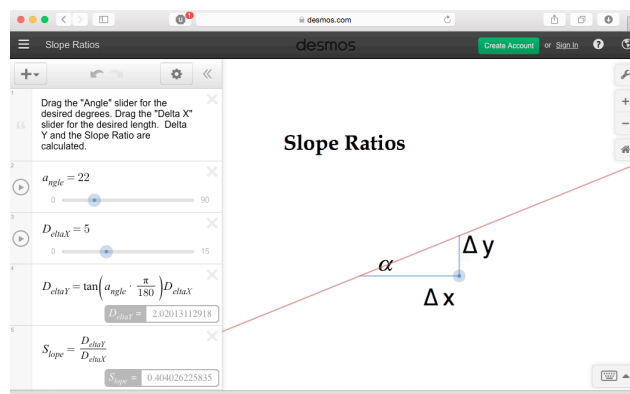
Figure 1. Screenshot of an app that can be used to explore relationships of slope and angle.

Near the beginning of this exploration, the PSTs determined (with a paper and pencil task) that for a slope-ratio triangle with an 11º base angle, the slope ratio is approximately 1/5. Additionally, for a triangle with a 22º base angle, the slope ratio is approximately 2/5. These beginning teachers were asked to construct their own dynamic geometry sketches to explore this supposed fact from their pencil and paper sketches.

Data for this study include PST homework and subsequent class discussions in which they used the dynamic geometry software to explore the tangent relationship. The preservice teachers were explicitly asked to create dynamic geometry sketches where they could "click and drag a point to get different slope triangles." The PST then met in class and presented their sketches to their peers. These whole class discussions were audio recorded as well as conversation of three small groups of 3-4 PSTs as they worked through the task. The series of tasks lasted three days, resulting in a total of nine transcripts, three small group recordings a day for three days. As a part of a larger project, the transcripts were initially read by all three researchers of the project, one being the instructor of record for the course in which the data were collected. The data were originally open coded using Strauss and Corbin (Corbin & Strauss, 2007) constant comparative methods by each of the researchers. The researchers then met and identified technology as an emergent theme in the data as a way to support the PSTs' understanding. Following this, cognizant of the theoretical framing of TPACK (Koehler & Mishra, 2009) and the analytic framework for RAT (Hughes et al., 2006) the three researchers each independently recoded the entire data set with a focus on the role of technology in supporting PSTs' understandings of the tangent relationship. The identified chunks in the data that related to technology and assigned a code of *replace*, *amplify*, or *transform*. The three researchers then met and compared codes of the data. Themes related to technology use were then derived and agreed upon. The researchers then collectively generated themes around the *replace*, *amplify*, and *transform* uses of the technology as related to learning tangent to describe the PSTs' use of the dynamic geometry software for exploring the tangent relationship. An emphasis was on understanding variations of technology use to support understanding. Finally, the dynamic geometry sketches were analyzed to corroborate findings to further understand how PSTs were using technology to understand the tangent relationship.

Findings

This section describes several of the variations among the multiple sketches produced by the PSTs. Each variation is discussed including its mathematical, pedagogical and technological implications, with specific emphasis on the role of technology to *replace*, *amplify*, or *transform* learning processes.

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Static vs. Dynamic

One of the features of dynamic geometric software is the ability to drag objects in a sketch and observe what happens to the relationships among lengths, angles, and the connected shapes. This provides an *amplification* over static sketches where multiple sketches must usually be created in order to observe a change in various relationships. In this study, some of the sketches produced by the PSTs contained no dynamically moving parts. Either this was because all pieces of the sketch were "locked" together (any attempt to move a single aspect of the sketch actually moved the entire sketch), or there was no dynamic and relational aspects included in a picture (e.g. lengths of sides were written in, not linked to actual lengths, there were no preserved aspects of the diagram evident such as lines remaining horizontal or vertical)—in other words, these uses *replaced* traditional methods for learning. Here, PSTs treated their sketch the same as a regular pencil and paper sketch, only done on a computer, choosing not to utilize one of the key advantages of dynamic geometry software. This choice may have come from a lack of technological knowledge around how to display these measures or construct these relationships within their sketches. Additionally, their understanding of the activity may have centered on producing a sketch that displayed the relationship as opposed to exploring the relationship.
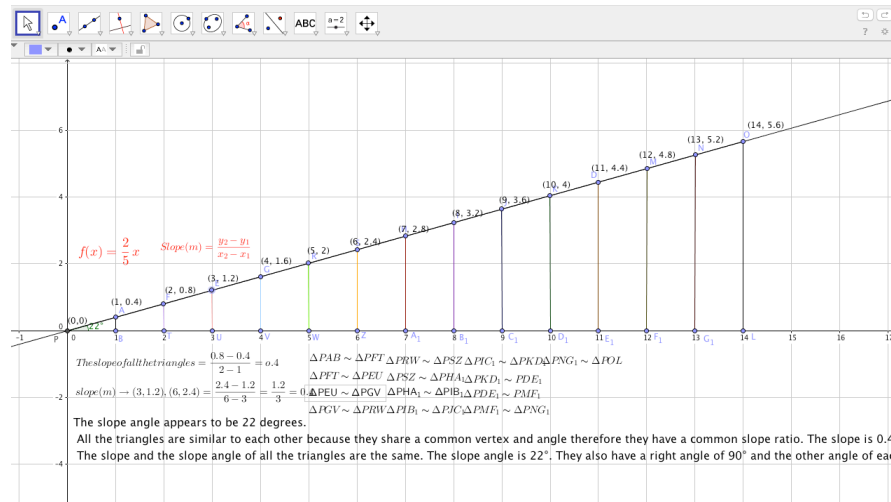
Figure 2. Screenshot of PST-created sketch with multiple slope triangles.

Multiple Overlapping Triangles

When using dynamic geometry software, the dragging of points and lines often allows the user to, in essence, see multiple different arrangements of figures within quick succession of each other. While at any given time, only one version of the figure is visible, the moving of one aspect allows users to visualize patterns and relationships in the figure and explore what may vary or remain invariant. With paper and pencil static sketches, often multiple versions of the figure will be included in a single diagram, with appropriate aspects labeled, in an attempt to "illustrate" these relationships. Some of the PSTs in the study created dynamic sketches and also included multiple slope triangles in their diagrams (see figure 2). While it was possible to dynamically manipulate their sketches to see relationships, the existence of multiple triangles made this interaction unnecessary as the relationships for consideration were already displayed upon opening the sketch. Here, again, the sketches seemed to be digital representations of static diagrams *replacing* a sketch of a similar representation, and the pedagogical advantage of exploration with dynamic geometry software was no longer a key feature of these sketches.

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Right Angle

When creating dynamic sketches, the objects are often created in a parent-child relationship. For example, when creating a line segment, two points are first placed (the parents) then the line segment (the child) connects the two points. Whenever a parent is moved, objects that are dependent on the parent are adjusted accordingly. With many digital sketches, there are often aspects of a diagram that users would like to vary, and other aspects of a diagram that creators may wish to remain invariant. For example, when investigating parallelograms, a learner may wish to have the side lengths and angles be modifiable, but always have the opposite sides of this quadrilateral be parallel and congruent in length. For the activity in our task, one of the crucial elements of a slope triangle is comparing the "rise" to the "run" of the triangle, which must be at right angles to each other. In many of the PST-generated sketches, the sketch of the slope triangle initially showed a right triangle. However, when vertices of the triangle were moved with the technology, their triangles no longer remained right triangles (see figure 3 for a sketch before and after manipulation). When these PSTs created their triangles in the software, the right angle was not an invariant part of the diagram. When the right angle was set to be invariant, the sketch *amplified* what might have been presented in other static representations, making it clearer for those looking at the sketch the relationships that exist in the diagram. It is possible that these PSTs did not know how to arrange the parent-child relationships among the aspects of their, diagram which would ensure that the angle of the slope-ratio triangle always remained at 90º. Additionally, they might not have understood the importance of this feature of their diagram. Consequentially, they may have felt that by closely approximating a right angle in their diagram through the manipulations of multiple points, you could still see the relationship between side lengths, and the base angle of the slope-ratio triangle. However, this introduced a potential source of error in measurements and the conclusions based off of those measurements, in addition to being a less efficient way of moving vertices and preserving the right angle.
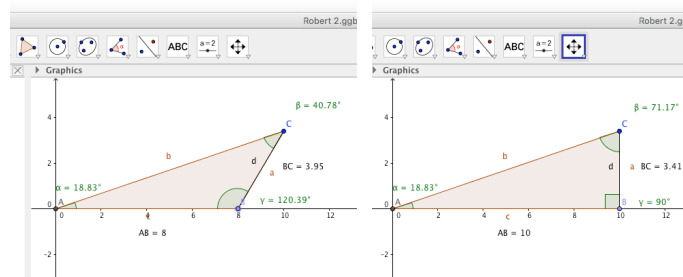
Figure 3. Screenshot of PST-created sketch with a non-preserved right angle.

Fixed Angle vs. Fixed Slope

In this activity, PSTs were asked to explore the relationship between the degree of the base angle in the slope-ratio triangle and the slope-ratio of that triangle. To create these slope-ratio triangles, there were two approaches PSTs used. One approach was to use a horizontal base for the triangle, and construct a line off of that base at a given angle for the hypotenuse of the triangle. In these sketches, the exploration allowed PSTs to see, given a particular degree measure, approximately what slope-ratio results exists in that triangle? A second approach in constructing these triangles was to again, begin with a horizontal base length. To create the hypotenuse, a line defined by a function with a given slope was created.

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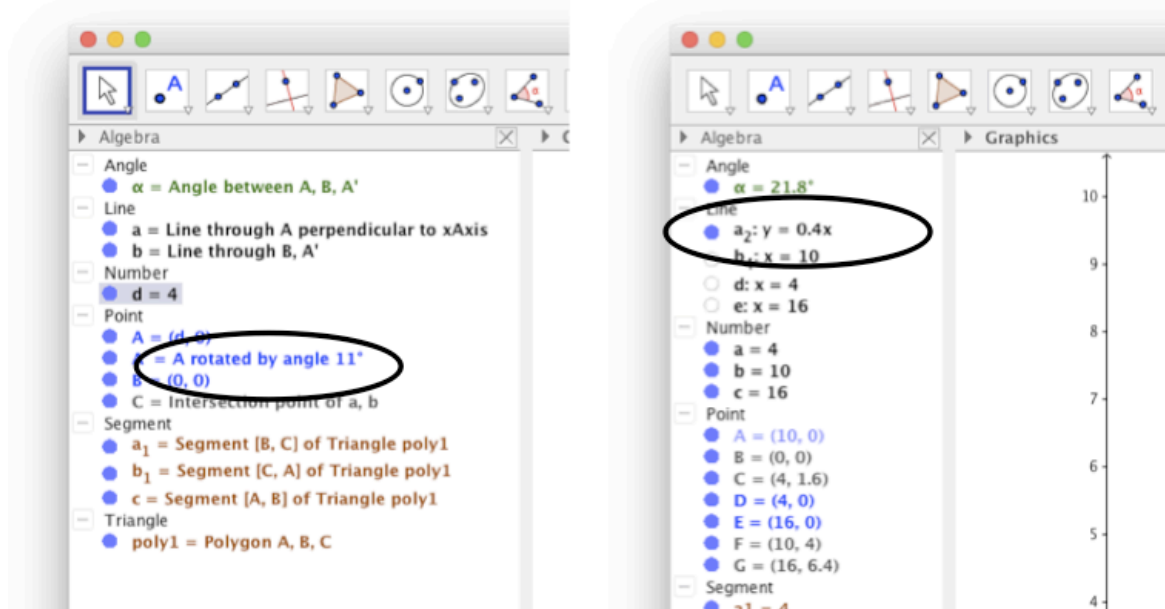
Figure 4. Screenshot of PST-created sketches with triangle created by angle, and by linear function with slope.

With these diagrams, the exploration allowed PSTs to see, given a triangle with a particular slope ratio, what is the base angle of that slope ratio triangle? Figure 4 provides an example of each approach used by the same PST in different sketches. The circled aspect of each diagram highlights how they defined their slope-ratio triangle. One of the key features to note with this different approach to constructing the triangles is that it actually enabled different mathematical explorations, in essence *transforming* the learning process in mathematics by allowing users to retrace exactly how a sketch was created. Furthermore, these different approaches emerged from the same activity around exploring the relationship. It's unclear whether the PSTs were aware of how their constructions impact the mathematics that they were investigating, or how these two approaches are complementary and can work together to support the investigation of the same relationship.

Unit Circle

This overall activity focuses on investigating right triangle trigonometry (relationships between the angles and side lengths in a right triangle). In addition to this perspective, trigonometry can also be investigated with the unit circle (a circle of radius 1 unit). In this approach, often times triangles are drawn inside of these unit circles, with the hypotenuse of each triangle extending from the origin to the circle, and the triangle is drawn connecting the point on the hypotenuse/unit circle to the x-axis so it intersects perpendicularly. In these unit circle triangles, the vertical length of this triangle is the sine of the angle, and the horizontal length of this triangle is the cosine of the angle. In some of the sketches produced by the PSTs, there appeared to be some efforts made to merge these perspectives of trigonometry. Figure 5 provides one instance of this approach by a PST. In these sketches, the slope-ratio triangle was embedded in circles centered at the origin (although not always unit circles). This attempt at merging was important for several reasons. First, it was an attempt to bridge mathematics content, usually presented differently in two domains (algebra and geometry), in essence *transforming* the learning process in mathematics. While there are mathematical similarities in these perspectives, the differences shape the range of relationship explorations.

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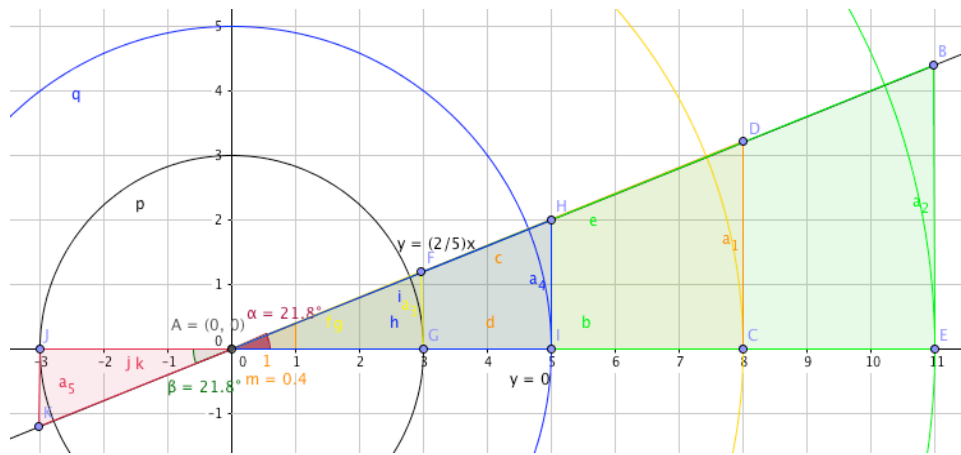
Figure 5. Screenshot of PST-created sketches with slope triangles and circles.

Right triangle trigonometry is limited to exploring angles between 0º and 90º while unit circle trigonometry is presented only with triangles with a hypotenuse of 1 unit. Additionally, the sketches that merge the right triangle and unit circle approaches have the potential for helping PSTs understand the strengths and limitations of each perspective and also offer ways that PSTs can simultaneously draw upon both perspectives when reasoning through a problem.

Discussion

Through this task of exploring the slope ratio triangle and its angles using dynamic geometry software, the PSTs were exposed to different approaches for thinking about trigonometric content and the tangent function and had opportunities to use technology in ways that would *replace*, *amplify*, or *transform* their previous experiences (Hughes et al., 2006). The activity was designed to engage these beginning teachers in mathematical explorations, but to also encourage them to think about the use of this task with their own students, and use technology to explore and explain some of the conjectures they were making. In analyzing the diagrams produced by the PSTs, we uncovered the variety of approaches used in their constructions, the role of the diagram with the activity, and the mathematical, pedagogical, and technological choices made by these PSTs. Specifically, they were able to recognize the role of technology in affording opportunities to *transform* their previous experiences through technological manipulation that otherwise would not have occurred with a pencil and paper drawing.

This study raises points about our work with, and study around, the role of technology and dynamic geometry programs with PSTs. As teacher educators, we need to be more explicit with PSTs about the strategic advantages that individual technologies offer (e.g. dynamic movement, simultaneous changes in representation, etc.) and how they may transform learning for their students in ways analogous to the transformations that occurred within the context of this secondary content course. One aim is to ensure that PSTs doing work to incorporate technology in their teaching are aware of the ways that the technology is best suited to enhance their instruction, albeit through replacement, amplification, or transformation of learning processes (Hughes et al., 2006). This may involve explicitly focusing different forms of TPACK knowledge (Koehler & Mishra, 2005) PSTs might be drawing upon when they create and use these sketches or other technologies. As we look toward supporting future teachers' uses of technology for their own teaching and learning, we recognize the complex interactions of mathematics content, the incorporation of technology into practice, and the attention to the ways in which learners will interact with that technology. By understanding how PSTs are currently making sense of these interactions, we hope to better support their future efforts.

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ASSESSING MATHEMATICAL ARGUMENTATION THROUGH AUTOMATED CONVERSATION

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Mathematical Argumentation skills have historically been overlooked in assessment, but the inclusion of Mathematical Argumentation in the Common Core State Standards (CCSS) as one of the Standards of Mathematical Practice challenges assessment developers to assess this mathematical practice. Explanation and justification of one's own thinking to a specific audience is considered a fundamental part of this mathematical practice. Based on student demonstration of argumentation skills when engaging with peers, we have developed automated conversations with virtual teachers and peers to investigate how alternative conversational patterns influence types of student responses. This technology allows assessment developers an innovative avenue for exploring new task designs that adapt to individual users and produce additional data not found in traditional measures. Preliminary findings from this investigation are presented.

Keywords: Assessment and Evaluation, Classroom Discourse, Elementary School Education, Technology

Defining Mathematical Argumentation

Mathematical Argumentation has been defined most generally as "understanding relationships to make connections to new ideas" (Mueller & Maher, 2009). Hunter (2007) defines argumentation in the classroom as collaborative argumentation in which students work together through mathematics discourse "to critically explore and resolve issues which they all expect to reach agreement on ultimately." (Hunter, 2007, p. 3-18). More specifically, argumentation includes making a conjecture, proving a proposition, justifying an inference, or explaining a point. Students have been found to demonstrate argumentation skills when arguing with and asking questions of peers. In addition, explanation and justification of one's own thinking such that it can be understood by a specific audience is considered a fundamental part of this mathematical practice.

Conversation-Based Assessment

To address the challenge of assessing mathematical argumentation, we turned to the prospects of using Conversation-Based Assessment (CBA). Such automated conversations with virtual agents have been widely used to support student learning in intelligent tutoring systems (ITS) (e.g., Graesser et al., 2004; Halpern, Millis, Graesser, Butler, Forsyth, & Cai, 2012; Millis, Forsyth, Butler, Wallace, Graesser, & Halpern, 2011). Students' interactions with these agents can be used to gather evidence about their knowledge and skills, and provide them with appropriate help (e.g., feedback, scaffolding). The use of CBA, and more specifically, "trialogues" (three-party conversations among a human student and two virtual agents) for assessment is more recent (see Yang & Zapata-Rivera, 2010), but this area of application is a natural fit for assessment purposes due to the underlying requirement for ITSs to assess relevant skills that will enable intelligent and adaptive responses. Leveraging this requirement allows assessment developers an innovative avenue for exploring new task designs that adapt to individual users and include additional data not found in traditional measures (i.e., conversational responses related to specific scaffolding).

Trialogues are one way to create learning environments that can be used to simulate particular learning strategies or social interactions (Butler, Forsyth, Halpern, Graesser and Millis, 2011). This makes this type of environment an ideal one for the assessment of argumentation skills since we are

able to recreate not only the mathematical content learned in the classroom, but also the interactions that accompany them.

In the development of the task that serves as the basis for our automated conversation, we looked at what design principles could be used to structure tasks so that students' collaboration, and their discourse in particular, will be "thought-revealing" (Kelly & Lesh, 2000). Hoover, Hole, Kelly and Post (2000) proposed a set of principles for developing thought-revealing activities: 1. The model construction principle, 2. The reality principle, 3. The construct documentation principle, 4. The construct shareability and reusability principle, and 5. The effective prototype principle.

These principles suggest that a thought revealing task should require the development of "an explicit construction, description, explanation or justified prediction;" (p. 609) involve a situation that requires students to engage in meaningful mathematics; result in the creation of a product that itself provides information about student understanding; require students to produce explanations of process and not just a final product/answer; and result in the creation of an idea that can be referred back to in another context. We developed not only our underlying task, but also structured our conversations around these principles.

Figure 5: Screenshot of CBA prototype.

We developed a CBA that involves students engaging in an automated trialogue with a virtual teacher and virtual peer agents. The trialogue occurs in a simple chat-like interface as the student is led through solving a problem that involves both linear algebra and mathematical argumentation (Figure 1 shows a screenshot of the prototype, including the problem the students were asked to solve). We found that students are able to showcase their skills with mathematical argumentation through explanation, refutation, evidence, and position-taking just as would be the case in a classroom setting. Further, by utilizing automated scoring engines already integrated into the design of the CBA, we were able to come up with scores for mathematical argumentation that are more objective than the subjective scores of classroom observation or teacher rating.

Researchers have explored how best to support students' skills to support deep conversations and question-asking (see Graesser, Ozuru & Sullins, 2010). We aimed to build on this body of literature by continuing onto the next step in the development process, the framing of the questions and prompts to provide the continued support of thought-revealing responses and therefore student

argumentation contained within. For instance, do students provide the most information when countering misconceptions or when responding to direct questioning? Do students respond differently when the question comes from a virtual teacher as opposed to a virtual student? These are the design questions we aimed to better understand through this study.

The Problem Statement

In drafting the structure of automated conversations, just as in a real-life classroom situation, we must make multiple decisions as to how to query a student to elicit certain information. Sometimes, a very small change in wording of a query may elicit more, less, or different information. However, unlike a real-life situation, we do not have the opportunity to listen to nuances in the student response and ask the question again in a different way. This study looks at small changes to automated questions at three important points in the mathematical argumentation conversation to determine what impact these changes have on student responses.

Main Research Question: How do alternative conversational prompts influence the types of math responses gathered from students in the math prototype?

Sub-questions:

1. Do students respond differently in a situation where they are first asked to explain in their own words or when they are responding to another student's ideas? [Manipulation 1]
2. Do students respond differently to a question asked by a virtual student versus a virtual teacher? [Manipulation 2]
3. Do students provide a more complete argument when prompted to respond to an unlikely answer or a likely answer? [Manipulation 3]

Study Design and Procedure

Sample

The study was conducted with students in 8th-grade algebra at four schools in different regions of the U.S.A. We investigated the three research sub-questions simultaneously, using the same sample of students. Students were randomly assigned to one of the eight possible conditions as shown in Figure 2 (numbers shown were planned). We had aimed for a total of 120 students but due to technical difficulties at one school that caused them to end before all students were complete, we ended with 123 records for Manipulation 1, 107 for Manipulation 2, and 74 for Manipulation 3.
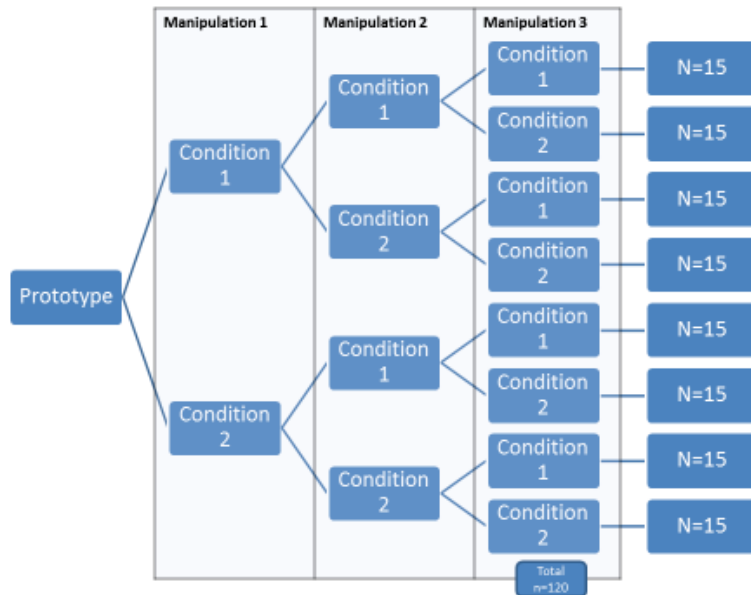
Figure 6: Study design.

Instruments

Students were administered a short pretest focusing on linear algebra skills as a baseline measure. They then engaged in the automated conversation on a computer, with random assignment to one of eight conditions, as described above. They then answered a short post survey about their perceptions of the activity.

Data Analysis

Scoring and Analysis. The pretest items were all automatically scored. Most of the analyses of the conversation data were also automatically scored with the exception of one longer argumentation item (Manipulation 3), which had to be scored by human raters.

Analysis. Each of the three manipulations has two discrete conditions that are being compared in their outcomes. Each of those manipulations was analyzed using Chi Squared Tests of Independence.

Results

Manipulation 1

Manipulation 1 varied by whether the initial response by the student was in reaction to a misunderstanding by the virtual student, Pat (Condition 1), or whether it was in reaction to a direct question by the virtual teacher, Ms. Turner (Condition 2). In both conditions, we were looking for the student to answer that y is the dependent variable and represents the total cost. In Condition 1, Pat offers an incorrect answer where y is the independent variable and is the cost per shirt. Figure 3 shows the conversation diagram for the first cycle of Condition 1. Condition 2 differs in the opening such that Pat does not offer an [incorrect] answer and instead Ms. Turner directs her question directly to the student.

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Opening			Cycle1	
Ms. Turner	X (Student)	Pat	Ms. Turner	X (Student)
Let's think about graphing the information about all three companies. To begin with think about the axes and their scale. Pat, how should you label the y-axis?		I think that y-axis is the independent variable, and here it is the cost for each shirt.	Let me restate what you just said. Y is the independent variable. Y represents the cost per shirt. _X_, do you agree with Pat? Why or why not?	A. Correct answer with full correct explanation of dependent variable: y=dependent y=total cost
				B. Correct Answer with incorrect explanation of dependent variable: y=dependent y=total cost
				C. Correct answer, no additional information: y=dependent y=total cost
				D. Misunderstanding: y=dependent; y=cost per shirt (agrees with Pat)
				E. Misunderstanding: y=independent (agrees with Pat); y=total cost
				F. Misunderstanding: y=dependent y=amount of shirts (confuses x and y axes)
				G. Incorrect answer: y=independent y=cost per shirt; or anything else
				H. Irrelevant response
				I. No response
				J. I don't know; I have no idea; I don't understand what you mean.

Figure 7: Conversation Diagram for Manipulation 1, Condition 1.

First, we looked directly at how the CBA system evaluated the student responses to the manipulated question. The results are shown in Table 1, where A is a completely correct response that leads directly to the end of the conversation, and C-F are variations of which components were correct or partially correct (B is missing as no one went down that path). G is the completely incorrect, but relevant response where the student states the same thing as Pat, that y is independent and the cost per shirt (or other relevant but incorrect labels for *y*). Other represents a path that the system computed as irrelevant, blank (i.e., no response provided by the student), or metacognitive (e.g., "I don't know") (H, I, or J).

Table 1: Manipulation 1 Cycle 1 Response

	Student Response Evaluation								Pretest Score
	A	C	D	E	F	G	Other	*Total*	% correct
Condition 1 (misunderstanding)	1	2	5	5	2	29	16	*60*	66%
Condition 2 (direct question)	1	0	8	0	6	5	43	*63*	69%
Total	2	2	13	5	8	34	59	*123*	68%

Students in Condition 1 were most likely to have both parts of the response incorrect (*G*) while students in Condition 2 were most likely to have a response that was interpreted as irrelevant or blank (*Other*; χ^2=38.9, *p*<.001). This result was interesting as the misunderstanding by Pat was intentionally built into the script to make it clear which pieces of information were relevant to the

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question while allowing the student to correct the information with their own response. This approach may have backfired as it indicates that the students given the response by Pat initially were likely to agree with him. On the other hand, more than half of the students that were directly questioned by the teacher with "how should you label the *y*-axis" did not produce any relevant response, indicating that both groups may have been lacking this basic knowledge of how to contextualize linear functions. Students in Condition 1, however, had Pat's answer to copy or restate while the other group did not have any information to use.

To explore this further, we looked at student responses to follow-up questions posed after this initial response in Cycle 1 to see which students eventually arrived at the correct answer through further questioning (Table 2).

Table 2: Manipulation 1 Final Answer

	Correct	Incorrect	Other	*Total*
Condition 1 (misunderstanding)	23	21	16	*60*
Condition 2 (direct question)	9	11	43	*63*

None of the students who began down the "Other" paths were able to eventually reach the correct answer. Of the remaining students, approximately half in each conditional eventually reached the correct answer (23/44 students in Condition 1 and 9/20 students in Condition 2), demonstrating that it was only the direct response to the manipulated prompt that caused student differences, there was no further chain reaction to this manipulation.

Manipulation 2

For Manipulation 2, the original version of the manipulated question (Condition 1) has Ms. Turner explicitly telling the students to use $y=mx+b$ and asks what m and b represent. In Condition 2, Pat says "I know we're supposed to use $y=mx+b$ for the equations. But I'm not really sure what m and b stand for. [Student], can you help me? What do you think m and b stand for?" This is similar to Manipulation 1 in that the student is responding to either Pat or Ms. Turner, but it also differs from that manipulation in that the question in this case is near-identical. There is no misconception introduced, and in both conditions the student is explicitly asked to define m and b in $y=mx+b$. In this particular instance, the flow chart is nonlinear, that is, the students are expected to say that m is the slope (or cost-per-shirt) and b is the y-intercept (or set-up fee) but there is no prescribed order to those two events. As shown in Table 3, there were more respondents in Condition 2 who met both expectations (defined both m and b), but the difference between the groups was not statistically significant ($\chi^2=3.13$, $p=0.37$).

Table 1: Manipulation 2 Results

	Only *m* is defined	Only *b* is defined	*m* and *b* are defined	Neither	*Total*	Pretest
Condition 1 (Ms. Turner)	11	5	17	17	*50*	65%
Condition 2 (Pat)	10	4	29	14	*57*	70%
Total	*21*	*9*	*46*	*31*	*107*	68%

After the manipulated question in Manipulation 2, all students were asked to write the equation for one of the companies, which was posed in an identical manner for all students. We investigated whether students' initial responses to the manipulated question led to response differences on this new, non-manipulated question. The results are shown in Table 4. No statistically significant difference in performance was found between the groups ($\chi^2=0.81$, $p=0.27$).

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Table 2: Manipulation 2 and Later Prompts

	Full Equation Written Correctly	Incorrect	*Total*
Condition 1 (Ms. Turner)	30	20	*50*
Condition 2 (Pat)	40	17	*57*
Total	*70*	*37*	*107*

Manipulation 3

The focus of Manipulation 3 was on the final mathematical argument. This manipulated question asks students to develop an argument for which company should be used for the school fundraiser. They do this by responding to an email from the student council stating they will go with either EZ Tees (Condition 1) or Perfect Printing (Condition 2). We chose these two conditions based on evidence from preliminary data of human trialogue interactions (teacher and two students). In the human trialogues, most triads arrived at the conclusion that Perfect Printing was the best choice of the three companies because it is the cheapest for the greatest range of shirts ordered. Thus, we intended to compare an argument for/against an unlikely choice with an argument for/against a likely choice. However, as can be seen in Table 5, most students in both conditions chose Shirts for Less (SfL), the third company, as the best choice. Therefore, the two conditions were each prompting students to respond to a choice that most thought unideal and, we did not have a condition with the most common choice.

Table 3: Manipulation 3 Final Argument

	EZ Tees	SfL	Perfect Printing	Other	*Total*	*Pretest*
Condition 1 (unlikely choice)	9	16	10	2	*37*	*67%*
Condition 2 (likely choice)	4	23	9	1	*37*	*68%*

The data seem to indicate that students in Condition 2 (Perfect Printing prompt) were more likely to choose Shirts for Less than those in Condition 1 (EZ Tees prompt), but the difference was not statistically significant ($\chi^2=3.22$, $p=0.19$).

We then scored student arguments along a rubric that was designed to align with an Argumentation Learning Progression (Cayton-Hodges et. al., 2014). The argument was scored 1-5, with 5 being the most complete and convincing argument. Results are shown in Table 6.

Table 4: Manipulation 3 Argumentation Score

Argument Score	**1**	**2**	**3**	**4**	**5**	**Total**
Condition 1 (unlikely choice)	10	14	8	3	2	*37*
Condition 2 (likely choice)	13	8	4	7	4	*36*

The results indicate that students in Condition 1 may have been writing slightly more proficient arguments than those in Condition 2. However, we achieved only 61% reliability (exact score matches) over multiple scorers in the rubric, so we did not perform inferential statistics on this data. It is clear that, as a whole, the sample did not perform well on the final argument. We see this as indicating a weak performing population, which was also shown in Manipulation 1, which could also be one reason for the discrepancy with the human trialogues, as that population of students was overall quite strong. We plan to investigate this question further using a sample of 9th and 10th grade students to see how the findings compare.

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Conclusion

This study aimed at better understanding the design choices made when developing CBA questions, which could also translate to choices made when encouraging argumentation in the classroom. We found that introducing misconceptions, a common approach to encourage argument in CBA, could actually lead students to repeat the misconceptions later in the assessment as opposed to argue against them. Meanwhile, other changes such as a direct question by a virtual teacher versus a virtual student had little, if any, effect responses from students in our sample.

Finally, we were unable to test the premise of responding to a likely versus unlikely answer since a majority of students in both cases chose a different answer than intended by the problem. This was overwhelmingly true in Condition 2, which was supposed to be the "likely" answer.

Further research on this prototype is ongoing, including increasing sample sizes and assessing students in later grades who should have more command of the material, to see if the results change with a stronger population.

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THE RHYTHM OF TOUCHCOUNTS: COUNTING ON TECHNOLOGY

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This research reports on a class of grade one students engaging with number concepts using the iPad application TouchCounts. It is theorized that students develop understandings of number based on a materially engagement with both social and material resources. Rhythm is the fundamental unit of analysis used to attend to student engagement with TouchCounts. A class activity as well as three students working together will be analyzed in terms of emerging rhythms. It is concluded that rhythm is an emergence of form, modeling and aligning with the mathematical notions of number.

Keywords: Technology, Elementary Education, Number Concepts and Operations

Early learning of number is significant for future success in mathematics (Libertus, Feigenson & Halberda, 2011). Concepts, ranging from counting and subitising to skip counting, are important in early development of number sense as a foundation for higher development. Students, however, find number concepts difficult (Lockwood, 2012) partly because they often do not have many ways of engaging with number. For example, when children learn to count, they often learn a song in the form of the audible sounds, 'one', 'two', 'three', and so on. While this method can be a good starting point for learning counting, it is linear and only supports an audible approach to counting. New technologies provide new opportunities for students to become more closely engaged with mathematical concepts.

New technologies such as the iPad, including the numerous applications that can be used on this platform, provide new and unique ways of engagement for students. Multi-modal approaches that include embodied actions such as gestures can help support the learning of number by allowing different ways to interact. In this study, the application *TouchCounts* (Sinclair & Jackiw, 2011) is used because it provides an open-ended experiential approach to interacting with number. Pappert (1980) drew on this notion of learning with expressive technologies in his development of Logo. In Logo, a turtle, which was controlled by student's programmed instructions, drew shapes on a computer screen. Pappert argued that when students created these shapes they 'were' the turtle and 'felt' the movements and turns in creating shapes. *TouchCounts* is similar in that it provides a new form of engagement where number is understood in multiple visual, audible, and gestural modalities among others.

Research with new technologies is still a new field as research is striving to keep up with new technology. There are frameworks that have been implemented but the overall number of studies is still very small. Since technology is changing so fast and new applications are being created everyday, different approaches are needed for new insights in learning mathematics, particularly with digital technology, since it is so new. In this study, the construct of rhythm is used as a unit of analysis when students engage with number in a grade one classroom, rhythm being a way to attend to what occurs during interaction rather than framing the human mastering the tool. Rhythm itself is a way to embrace the fusing of tool and user because it is a clearly material and easily identifiable result of an activity. While many frameworks identify the user of technology as the central figure in an interaction, this study adopts a new materialist approach that sees the user and the technology as being fused together, thus expanding the boundaries and consequently the potential of user and technology as one. Rhythm has been implemented in mathematics education (Roth, 2011; Radford et al., 2007). Moreover, rhythm can be seen as an aspect of learning number in how children learn to count. There is a temporal component in how children are introduced to the 'one', 'two', 'three'

song. As an expression of interaction we are familiar with in music, rhythm shifts attention to the material. It is evident that rhythm aligns new forms of engagement with multi modal technology. This research report explores what rhythmic variations occur in different approaches to learning number, particularly with the use of the iPad app *TouchCounts*? [For information on TouchCounts see: http://www.touchcounts.ca/]

Theoretical Framework

New materialist scholars (de Freitas & Sinclair, 2014; Barad, 2007) are challenging the duality and binary components of Cartesianism which separates mind and body. Consequently, one can approach the activity of a student working with technology as a material engagement and not rely on black boxed phenomenon such as thinking or mental construction. Critics of the immateriality of mathematics challenge the idea of what mathematics is and whether ideas can exist independent of physical materiality such as the body, the social collective, or even spoken words. de Freitas and Sinclair have developed a monist framework they term inclusive materialism that does not distinguish between digital tool and user thereby expanding the boundary of thinking subject as both technology and user as one material. This is important in this study because using a digital technology does not depend on mental contemplation and intention since that centralizes the 'thinking' in the student. de Freitas and Sinclair criticize representationalism because it assumes a mediation of knowledge that is based on the immaterial. According to Roth (2011), thinking is spontaneous and is not interrupted by constant mediation of interpretation and intent. The mediating aspect of representation is aligned with a computer model approach in which input is stored and operated on. Real life engagements do not seem to follow such a constantly interrupted flow of living forces. Roth argues that thinking is improvised in-the-now, particularly when engaged in a new activity.

New approaches in theory demand new considerations of methods. In this study, rhythm is taken to be the unit of analysis because it is its own unique phenomenon and yet something we are quite familiar with. Rhythm can be identified as a repeated action within a temporal consistency. It is not mechanical nor metronomic (Ingold, 2011) but has variance making it both organic and structured. It is fully material since what is seen and heard and felt contributes to our adoption of the construct. While rhythm is discounted as something extraneous or epiphenomenal, Ingold asserts that it is part of the development of knowledge and helps us understand learning. It is a significant part of the in-the-moment. However, it is also develops familiarity with patterns. Thus, it can be evaluated at the present time and it can also be used to predict what is to come.

It is the form of rhythm and how it aligns with the form of mathematics that makes it so significant. Rhythm provides evidence of the structuring of a human engagement with a mathematical idea. For example, ordinal numbers can be counted, rhythmically as 1, 2, 3. In that counting there is form, a structure of consistent beat and of always being able to add 1. These forms fuse together in practice, the rhythm emerging from their fusing so that the rhythm becomes a leader or guide to the structure. It takes over, and one becomes a part of its flow indicating that the social adoption of rhythm is where meaning begins and becomes.

Radford et al. (2007) describe rhythm as an indicator of generalizing. They position rhythm as a semiotic resource that is mobilized by students in their apprehending of a pattern and its subsequent generalization to convey a sense of generality. However, rhythm can be considered less as emanating from within students or as a constructed resource but more as an emerging engagement between student and tool. Rhythm need not necessarily be viewed as a choice or a personal expression but as a expression of material interaction. Radford et al. also draw on a notion of form in expressing rhythm. They describe two levels of rhythm when students express 'one, one, plus three', 'two, two, plus three' and 'three, three, plus three'. They describe the first rhythm as the verbal expression of the four beats of one, one, plus, three, where single syllable words one, one, plus, three, repeated in the

next two expressions, indicate a level of rhythm in each single set of words. They refer as well to a rhythm of expression. That is, one, one plus three could be seen as beat one, two, two plus three as beat 2 and this beat 1, 2, 3 indicates a rhythm that can go on. While the first rhythm with words draws attention to repetition and to slight changes in that repetition, the second rhythm is about continuation and how each expression in its length, cadence and expression is the same, continuing with no end.

Roth (2011) has also explored rhythm, basing his critical approach on the concept of material body and metaphysical mind being distinct. He asserts that both are modalities of the flesh. His analysis is based not upon using new tools but upon focusing on people speaking and listening to each other. He argues that periodic features such as rhythm are produced and reproduced in interactions of verbal dialogues and that rhythm of speech is integral to the expression and recognition of knowing (p. 160). He identifies pauses in speech as important because they are part of speaking, the pauses giving temporal structure to the speech event. Roth further proposes that rhythmic phenomenon is not merely evident with individuals speaking or listening but that is also manifested collectively.

As a unit of analysis, rhythm, initiates a research practice that moves away from representationalism and pays attention to material force as a way to notice how and when it is expressed, how to account for changes, and how to highlight forms of knowing. In this study, rhythm is advocated as an emergence of form and structure when students engage in varying modalities of interacting. From this, further questions arise: What do we notice about the change in rhythms in an activity? How do rhythms begin, how do they end, and how do they connect to the meaning making practices in mathematics learning?

Methods

The research took place in a multi cultural grade 1 classroom with 26 students in a French speaking school in Western Canada. Grade 1 was selected for observation because the curriculum centered on number sense, particularly on one to one correspondence, skip counting, numbers in depth as well as addition & subtraction to 20. It was on this foundation and curricular outline *TouchCounts* was introduced. The teacher had been teaching for 7 years and had not used *TouchCounts* before. *TouchCounts* was subsequently used in addition to some other techniques that the teacher was familiar with such as using hundred charts, Gattengo charts as well as using her hands.

Two researchers, including myself, visited the class approximately once a week for three months. We videotaped the classes which were taught both by the teacher and by one of the researchers. The videotaping focused at time on the whole class and at other times when students worked together in pairs or triplets. Notes were taken during the research by one of the researchers, noting relevant events. Videos were analyzed later, focusing upon rhythms of sound and movement.

Data

Two episodes present different activities in which students are engaged. Not only is the organization of the activity different but the content is also different. The first sequence covering skip counting draws from a collective class activity using *TouchCounts*. The second is drawn from a group of three girls sharing an iPad using *TouchCounts* to create herds of five. What these two episodes contrast is the variations of rhythm that emerge in each scenario. While the rhythms need not be contrasted they do highlight how rhythms are unique in each context and how each rhythm has its own unique connection to mathematics.

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Episode 1: Skip counting

With *TouchCounts* set up on the front projector and students sitting on the floor, one of the researchers was teaching the class and introduced skip counting (Figure 1). *TouchCounts* supports skip counting in a very rhythmical way because one successively touches above the shelf, below the shelf, above the shelf, below the shelf and so on. In this way, rhythm is established not only in a spatial and temporal way both above and below but also hints toward continuation. The researcher modeled how to create even numbers on the shelf by saying 'one down here,' touching below the shelf, 'one up here,' touching about the shelf, and so on until 2, 4, 6, 8, 10 were on the shelf. She then asked the students to verbally repeat the numbers and the students said 'two,' 'four,' 'six,' 'eight,' 'ten' in a very consistent beat. While the students were uttering the numbers, the researcher, turning slightly to point at the screen, was moving her hand toward each number when it was said and then moving her hand back as if she were pulling away from *TouchCounts* (Figure 2). The researcher could have just moved her hand horizontally, pointing at each number as her hand moved, but instead seemed to let the movement on the iPad affect her movements in the air as she pointed. After resetting *TouchCounts*, the researcher called up a volunteer from the class to put 2, 4, 6, 8, 10 on the shelf. As a student was going up to the iPad, a girl beside the researcher side lifted one hand to mimic the iPad, used her other hand to point towards the tops of her fingers and said, "one on the top" and pointed lower down towards her hand, saying, "one on the bottom.". It did not seem like a question but a confirmation that she would be able to do it.

Figure 1. Children sitting on floor.

Figure 2. Pointing at numbers.

The student who came up to the iPad put the numbers on the shelf in the exact consistent beat that the researcher had put the numbers on. The researcher asked the students to state the numbers again and the same rhythm and pointing of the researcher was enacted. Two more students came up

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to put the even numbers on the shelf and followed steady, consistent, and in flow, in the exact rhythm. Each time the researcher asked the students to put one extra even number on the shelf. Each time the students read the numbers.

At one point a student was asked to point to the numbers as they were repeated. The pointing was as if she were using *TouchCounts*. She pointed and pulled away, pointed and pulled away, just like the researcher had done in the same timing as before.

When the fourth student was asked to put the even numbers on the shelf, the student put the number 2 in middle of the shelf. He matched the rhythm for his first two touches but then sped up for 3, 4, 5, 6, 7, 8 and then stopped because he was at the edge of the screen of the iPad. The class was quiet. After about 10 seconds, he fit 10 on by touching 9 below and 10 above in the same constant rhythm that had been established earlier. The researcher at this point indicated that he could put discs on the other side of the shelf. When moving to the left of the screen, he touched above the shelf and 11 landed on the shelf, the researcher asking if this was even. The student shook his arm in the air, back and forth, rhythmically, while the rest of the class, said no.

Episode 2: Making 5s in *TouchCounts*

After a class activity took place in which students sat on the floor and the teacher performed 'friends of five', which entailed holding up a number of fingers. Raising two fingers, she then asked how many fingers on her other hand should be held up to make five. After this activity, students moved to work with *TouchCounts*. A group of three girls could be seen making 'friends of five' in this episode. The first girl touched the screen with two fingers, a herd of two being instantly created. She looked at her fingers, making three, and touched the screen creating a herd of three. Combining the two herds, a herd of 5 had been created. A second girl then moved the herd to the right and then moved her hands back towards her body. After adjusting her hair, she touched the screen with one finger, creating a herd of one and then touched with four fingers creating a herd of four, she then combined them and moved the 5 to the right of the screen under the first 5 (Figure 3).
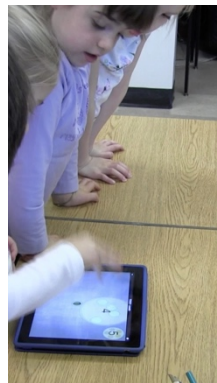
Figure 3. Second girl making herds.

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Figure 4. Third girl making herds.

These two girls established a rhythm. The timing of the movement was identical: touch once, look at fingers, touch again, combine, move to the side. The actions of the girls occurred in exact temporal alignment. The movement of their arms, the turning of the fingers, and the timing were all aligned thereby setting up a improvised but rhythmically negotiated approach to the activity. The third girl, however, did not match the rhythm of the previous two girls because she was interrupted. She touched the screen with two fingers and, as soon as she did that, the first girl moved her body toward the iPad, saying, 'You can't do the same as I did,' causing the third girl to not let go of the 'two' and just waited. An observing researcher attempting to prompt her with a question, asked, 'What can you add to two to make five?' She then touches three, combines the two numbers very slowly somewhat unsure of what may happen (Figure 4). The slowness is a new rhythm, possibly brought about by the interruption. As soon as a third girl created a five, the second girl from the previous turn-taking moved the five to the side and continued to repeat the same rhythm as before with the same timing and the same pattern of looking at her fingers.

Discussion

In the first episode, the mathematics was based upon skip counting. The rhythms that emerged were consistent among the researcher and the students. The consistent gestures that appeared, both in the touching of *TouchCounts* and the pointing at the numbers on the projector screen, remained within a very specific rhythmic pattern. Almost all the touching and pointing approximated the beat of the second hand on a clock. This rhythm addresses the notion of the social collective and how both the technology and social engagement with the technology had remained similar throughout the whole activity. It could be assumed that each student who participated in touching *TouchCounts* was mimicking what the researcher had done but this would imply that a mental intention is present as opposed to the improvisation alluded to by Roth (2011). Furthermore, if we see the student using technology to accomplish a goal, we lose sight of the central thinker in the interaction. While it might seem that the student is placing the numbers on the shelf in a very controlled way, this puts all important activity in the mind of the student. However, by seeing rhythm as a result of social becoming and as evidence of material engagement of a student-digital tool, we begin to see the importance of how rhythm carries students along. Rhythm seems to convey that a certain collectivity precedes knowledge.

As a researcher, I differentiate two different rhythms in this activity like Radford et al. (2007). The constant timing and movement of the above, below, above, below touching was a rhythm of repetition. As previously noted, this is important because it identifies a connection between social rhythm and personal rhythms. There is, however, also the unit of repetition that repeats which exists when the researcher, then one student after another, all express the same repetitive rhythm which has become a rhythm of continuity. This higher level rhythm gives a structure to the idea that the skip

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counting can continue onward. Each student who was observed was requested to put one more even number on the shelf. Consequently, there were slight differences, not found either in the constant rhythm of touching the screen nor in their movements in how they touched the screen. The last boy who did not fulfill the task did not follow the rhythm of the previous students. He sped up, he stopped, and then moved to the left side of the screen of the iPad. This is evidence that there is room for variation of engagement leading to different expression of rhythm, thus highlighting the significant of the same rhythm set by the researcher and three students who followed her.

In the pointing at the numbers on the shelf, both the researcher and the student pointed in a unique way, one that had been established in using *TouchCounts*. Here we see how rhythm establishes itself in the activity. For the fourth student, spacing became important when we saw rhythm develop or breakdown. The student's rhythm had been interrupted so that a new negotiation with the technology was needed. However, the previous rhythm did not seem to want to provide any more time and so he touched with 11 and was incorrect.

In the second episode, the rhythm established by the first two girls indicated an alignment of meaning making. The fingers, making the 2, then the 3, then combining the 2 and the 3 and then moving 5 to the side, created a form of engagement that is mathematical. Although moving 5 to the side could be argued to be extraneous, the movement of the 5 as an object was an action that was not present in whole class activities. However, this movement in the rhythm, as a repeated step and within a temporal framing, was afforded by *TouchCounts* so that in moving the five as a step in the activity, the girls legitimized the herd of 5 as a mathematical object, demanding its own gesture and it own space. There is an important aspect here of how rhythm, improvised as it was, creates an opportunity of understanding the mathematical symbol as being more than something to look at or say its name. The rhythm creates an opportunity for the object 5 to be touched and moved, just as any material object could be. While this may happen without rhythm, one still wants to know how this could occur. Without being told by a teacher and without a reason to touch 5 and move it, one wonders if an activity could be thought up to support this new form of engagement. However, in the social negotiation of creating 5s, there was a need for space, and there is a strong argument in the rhythm of 2, 3, combine, move, and 1, 4, combine, move, that the repetition and the exact temporality plays a part in the creation of number beyond sight and sound.

The result for all three girls was the same. Each created a herd of five. However, the third girl had a unique rhythm after having been interrupted. After the interruption, each movement was unique. She touched the screen quickly but kept her hand on the screen, later combining 2 and 3 very slowly. Her rhythm did not match the previous rhythms but became an attempt to renew a form of engagement. The important point is that each student ended up with a herd of five but the differences in how the first two were achieved, compared with the third creation, draws attention to process as opposed to product. If rhythm is the process of engagement, the third girl had a different 'understanding' of combining 2 and 3 than the first two girls.

Conclusion

Mathematics can be seen as a set of propositions that can be acquired, stored and applied to different situations. This supports the idea of representation which raises the question as to whether this approach to learning and knowing is too dependent on interpretative mediation. Life and movement, according to critics of representationalism, do not reflect such stop-and-reflect kinds of practices. Inclusive materialism, a new materialism established in mathematics education, privileges material interactions as knowing. With new digital technologies, methods that align with theoretical materialist approaches are needed to understand mathematical classroom practices. In the present study, rhythm was formulated as a correlative between new technologies and the social and the personal.

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In early education, counting is very much rooted in a rhythmic expression. However, new technologies support new rhythms. More specifically, the use of TouchCounts, both in a whole class activity and in a smaller group, enables the creation of a rhythm which provides an opportunity to experience mathematics within the context of an improvised and unique structure. In conformity with inclusive materialism, rhythm as an unfolding emergence is an expression of living forces and materials. In this study, we have seen TouchCounts provide new ways of engaging with skip counting. A gestural approach attends students both to the skip counting pattern and to the numbers that are skipped as well as to the temporal aspect of developing a sense of sequence. In adding 2 and 3, a new engagement with the number 5 emerges, not solely because of the affordances of the technology but because of the emergent improvised rhythms that create a need for such an interaction. Rhythm can help anticipate what is to come. Teachers may use emergent rhythms as a way of understanding collectivity or individuality, noticing as well how the in-the-now rhythm structures affect the rhythm that is to come.

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USING ROBOTICS AND GAME DESIGN TO PROMOTE PATHWAYS TO STEM

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This research report presents the results of a STEM summer program on robotics and game design. The program was part of a three-year study funded by the National Science Foundation. Children in grades four through six participated in a two-week summer camp in 2015 to learn STEM by engaging in LEGO® EV3 robotics and computer-based games using Scalable Game Design. Twenty-eight students participated in the study that took place in a small urban community in the Rocky Mountain West. This paper reports on the results of this part of the study, specifically, how children's computational thinking skills developed and how their self-efficacy in technology, attitude toward engineering and technology, and 21st century skills changed as a result of their participation.

Keywords: Affect, Emotion, Beliefs, and Attitudes, Informal Education

According to the Bureau of Labor Statistics, U.S. Department of Labor (2016), the need for mathematicians, computer systems analysts, and biomedical engineers is projected to grow from 21-23% by 2024. Preparing students to succeed in mathematics and technology is crucial to ensuring access to these and other STEM occupations of the future. Moreover, the number of engineering students is not increasing and in some instances is declining while the demand for engineers is expected to continue to grow (Hirsch, Carpinelli, Kimmel, Rockland, & Bloom, 2007). One reason that students are not choosing to study engineering is lack of information about the field, what it entails, and what engineers do (Hirsch et al., 2007). Exposing underrepresented students to pre-engineering skills through robotics and game design has the potential to increase their interest and to provide them with the skills needed to broaden participation to create a diverse STEM workforce (National Research Council (NRC), 2011).

To address this need in STEM education, a student- and teacher-focused project funded by the National Science Foundation was implemented in Wyoming. The goals of the three-year quasi-experimental study were to examine how spatial reasoning, computational thinking, children's self-efficacy in technology, and attitudes toward STEM and STEM careers changed as a result of participation and how well teachers implemented the STEM program. Specifically, we provided students with access to LEGO® EV3 robots, its accompanying software MINDSTORMS®, and Scalable Game Design software and protocols (Repenning, Webb, & Ioannidou, 2010; Webb, Repenning, & Koh, 2012). Teachers received training to deliver the instruction through an online professional development course.

The purpose of the two-week summer program was to field test the double effect of teaching both gaming and robotics to improve students' spatial visualization and computational thinking skills. Robotics and game design have not only been extolled for their role in learning but have also been identified as pathways to broaden participation in STEM and STEM-related careers (Caron, 2010; Sheridan, Clark, & Williams, 2013). We implemented the summer program to inform Year 3 deliverables. In Year 1, we piloted the instruments and the iterative project model. In Year 2, teachers taught robotics in the fall and game design in the spring to measure the effect of a single treatment. The research questions that guided this part of the study were as follows:

1. How did students' self-efficacy in technology, attitudes toward engineering and technology, and 21st century skills change as a result of engagement in robotics and game design?
2. How did children's computational thinking (CT) compare and contrast on Maze Craze, Frogger, and Pac Man games?
3. What learning preferences and STEM interests did students report during focus group interviews?

Theoretical Framework

The framework that undergirds this study is Learning-for-Use (LfU) (Edelson, 2001). LfU is a technology design framework that is based on four principles: (a) knowledge construction is incremental in nature, (b) learning is goal directed, (c) knowledge is situated, and (d) procedural knowledge needs to support knowledge construction (Edelson, 2001). These principles inform robotics applications and game design and lend themselves to the interventions implemented in this study.

The first and fourth principles of the LfU model are the incremental development of new knowledge and procedures. The goal behind the progression of two intervention components—robotics and game design— is to engage students in an incremental process. This idea allows students to incrementally add new concepts to memory, while subdividing existing concepts or making new connections between concepts. Procedural strategies for supporting and reinforcing incremental learning include observation, discussion, reflection, and application. New knowledge informs and empowers students to become proactive in their own learning. In its second and third principles, LfU recognizes that acquisition of knowledge is goal directed and situated. The realization of gaps in one's knowledge, perhaps as the result of an elicited curiosity or external demand, can be used as a motivational goal for acquiring new knowledge. The intervention was designed to encourage goal-directed tasks as students created computer games to learn and apply spatial reasoning and computational thinking skills, which are needed for computer science and information and communications technology (ICT) careers.

Literature Review

The bodies of literature that support this study are robotics and digital gaming. The literature that support this study is presented below.

Robotics

Robotics programs have resulted in an increase in students' comfort level with applications of STEM, development of 21st century skills, and increased interest in pursuing STEM-related programs beyond high school (Brand, Collver, & Kasarda, 2008). LEGO® robotics, specifically, is widely used in K-8 settings as an authentic and kinesthetic way to improve children's problem-solving skills, reinforce science applications and concepts, and build upon informal learning activities often done at home (Karp & Maloney, 2013). Informal STEM experiences in robotics extend students' experiences into alternative spaces where there is greater potential for learning to be "...self-motivated, voluntary, and guided by the learner's needs and interests" (Dierking, Falk, Rennie, Anderson, & Ellenbogen, 2003, p. 109). In this study, robotics provided students with opportunities to engage in authentic learning, inquiry, and scientific processes such as observing and recording data, evaluating and providing feedback, and using evidence to revise thinking, planning, and actions.

Digital Gaming

Digital game playing has also been used successfully to teach mathematics problem solving (Chang, Wu, Weng, & Sung, 2012) and can be used as a social practice to support the development of "strategic thinking, planning, communication, application of numbers, negotiating skills, group

decision-making, and data-handling" (Li, 2010, p. 429). Studies revealed that children showed considerable improvement in regard to developing positive attitudes toward learning mathematics through gaming (Kebritchi, Hirumi, & Bai, 2010). Using digital games in mathematics classrooms has led to favorable attitudes toward learning mathematics and to increases in mathematics achievement and student success. Several studies explore the use of games to improve student learning and computational thinking (Li, 2010; Repenning et al., 2010; Webb et al., 2012). For example, Scalable Game Design (SGD) project consists of instructional units to support game computational thinking through the use of game designs such as AgentSheets and AgentCubes (Repenning et al., 2010).

Methodology

Participants and Setting

Twenty-eight students (23 males; 5 females; 4 underrepresented minorities; and 4 students with disabilities) were recruited into the program. The two-week summer camp was held in August 2015 at the Starbase facility (Starbase is a Department of Defense program that seeks to motivate fifth-grade students to explore STEM through inquiry-based curriculum) in Cheyenne from 9 a.m. – 3 p.m., Monday – Friday with morning and afternoon breaks and a 1-hour lunch and recess. We ran two concurrent classes—one on robotics and the other on game design—with 14 students each. Students switched classes in the afternoon to receive four hours of instruction on robotics and game design each day.

The main instructor for the robotics class was an engineer at Starbase. The principal investigator was the main instructor for the game design class. Lesson plans in the robotics course focused on using MINDSTORMS® to engage in basic programming to incorporate ultrasonic sensors, color sensors, and touch sensors. Students used LEGO® EV3 robotic kits while working in pairs to make the basic car, two- and three-wheeled rovers, and a sumo bot that were programmed to traverse a color-coded map, race on a simulated track, or bot fight in a ring, respectively. Lesson plans in the game design class focused on guiding students to create Maze Craze, Frogger, and Pac Man games using Scalable Game Design.

Data Analyses and Data Sources

Mixed methods were used to analyze quantitative and qualitative data in this phase of the study. The Self-Efficacy in Technology and Science instrument (SETS) developed by Ketelhut (2010) was used in this study. Self-efficacy as defined by Bandura (1977) is the belief that one can successfully perform specific tasks. We administered three of the SETS subscales to measure students' self-efficacy in technology: videogaming (8 items), computer gaming (5 items), and using the computer to solve problems (5 items). Cronbach alpha reliability ratings ranged from 0.79 to 0.93, which were in the acceptable range. The Student Attitudes toward STEM survey developed by the Friday Institute (2012) was modified to include two subscales: engineering and technology (9 items) and 21st century skills (11 items). Cronbach alpha coefficients were in the acceptable range ($\alpha > 0.83$). A 5-point Likert scale that ranged from 1 (strongly disagree) to 5 (strongly agree) was used to rate items on the SETS and the Student Attitudes toward STEM surveys. The T-statistic was used to analyze pre-post scores on these surveys. The confidence interval was .95, and statistical significance was established at $\alpha = .05$.

Computational thinking (CT) was analyzed using a rubric developed by the principal investigator to rate students' games. A three-point rubric was established with an interrater reliability of 86%. Scores on the rubric ranged from 1 for emerging, 2 for moderate, and 3 for substantive evidence of CT. Focus group data were collected the final week of the summer camp during an interview with several randomly selected students. These data were analyzed using the constant comparative method

(Strauss & Corbin, 1990) to identify emergent themes and patterns related to the participants' learning and interest in STEM. Finally, two members of the research team collected field notes and conducted classroom observations.

Results

Self-Efficacy, Engineering/Technology Attitude, and 21st Century Skills

Twenty-one students completed the pre-post surveys. The data (see Table 1) reveal no significant differences on the SETS for computer gaming and computer use. However, there was a significant decline on the videogaming subscale: ($t = 2.126$; $p = 0.046$; Cohen's $d = 0.477$). Cohen's d shows a moderate effect size for this decline. While pre-post scores increased on Attitudes toward Engineering and Technology (M = pre: 3.87 (Std. Dev.: 0.80); post: 3.93 (Std. Dev.: 0.72)) and 21st Century Skills (M = pre: 4.12 (Std. Dev.: 0.72); post: 4.24 (Std. Dev.: 0.65)), results of a paired t-test show no statistically significant differences on either subscale.

Table 1: Results of Self-Efficacy in Technology Scale by Cohort

Construct (n=21)	Pre-Survey Mean	Standard Deviation	Post-Survey	Standard Deviation
Videogaming	4.32	0.59	4.20*	0.65
Computer gaming	4.08	0.76	4.04	0.68
Using the Computer	4.05	0.99	4.04	0.80

* $p < 0.05$

Computational Thinking

Descriptive statistics were also used to rate students' games for computational thinking using a 3-point rubric. Students were taught how to design three different types of games: Maze Craze, Frogger, and Pac Man. To analyze students' games, we evaluated the students' code, worksheets, and game functionality using the six International Society for Technology in Education (ISTE) definition for CT (i.e., problem formulation; abstraction; logical thinking; algorithms; analysis and implementation; and generalization and transfer). The majority of students completed the Pac Man game, which was the most difficult of the three games.

The results of analyzing the Pac Man game show that four students exhibited emerging CT strategies (threshold from 1 – 1.5), nine students exhibited moderate CT strategies (threshold from 1.6 to 2.5), and five exhibited substantive CT strategies (threshold from 2.6 to 3.0). Thus, the majority of students exhibited moderate or substantial CT strategies on one of the most difficult games. To further understand CT on all of the games, we completed additional analysis on four focal students (all names pseudonyms) using the same rubric described above. They were randomly selected from among eight students who completed all three of the games. The descriptive data are shown in Figure 1.

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.
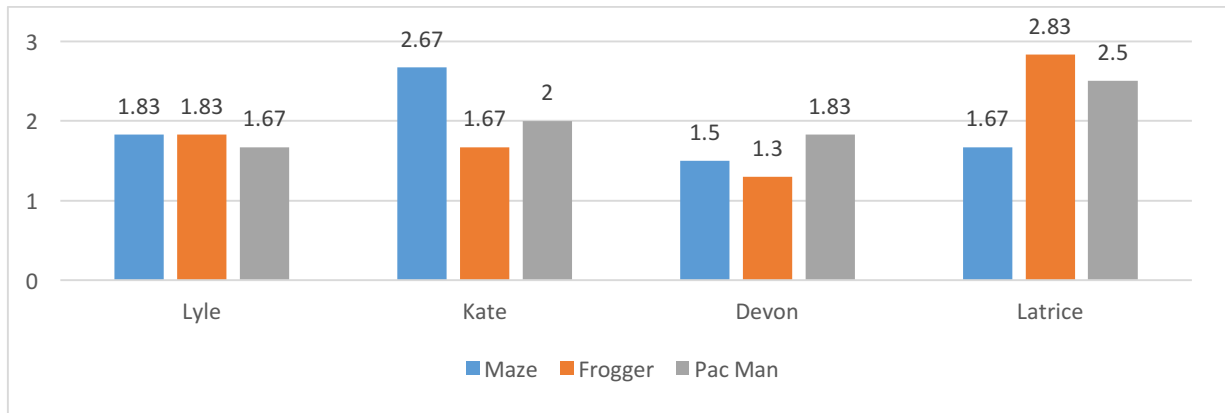
Figure 1. Focal Students' Computational Thinking Scores by Game Type**.**

Interestingly, Kate and Latrice, who were underrepresented females in a class that was predominantly male, showed greater levels of computational thinking than their male counterparts on two of the games. Latrice, who was also a student with disabilities, had the highest average CT score. When these data are analyzed by game type, the Pac Man game had the highest average CT rating (M = 2.00) compared to Maze Craze (M=1.92) and Frogger (M=1.92). Overall, mean scores support the finding that the CT scores of students in the summer cohort fell into the moderate range. Screenshots of each type of game are shown in Figures 2-4.
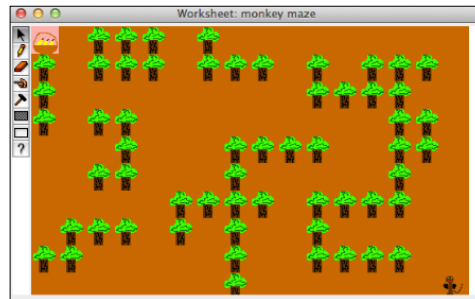
Figure 2. Maze Craze Screenshot (Kate).
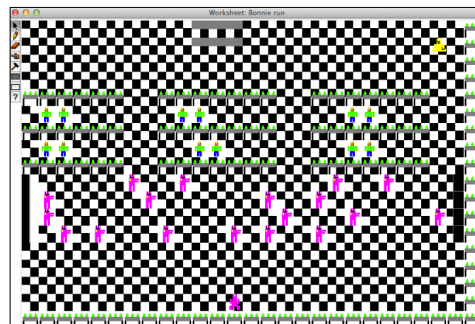
Figure 3. Frogger Screenshot (Latrice).

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.
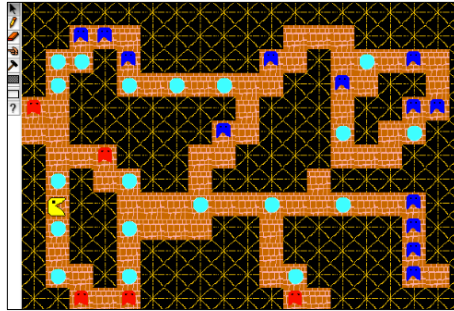
Figure 4. Pac Man Screenshot (Devon).

Qualitative data included excerpts from a questionnaire during focus group interviews with seven randomly selected student participants. The focus group interview with two girls and five boys was transcribed verbatim. Portions of the transcribed interview are presented below:

Interviewer: What one or two things did you think were cool about participating in robotics?

Student 1: What I thought was cool about…robotics was that we got to build our own robots and try out different things with our sensors, and we got to activate our robots….

Student 2: I liked building the robots, and I also liked programming the robots and finding out how far you could make the thing go.

Student 3: I liked it so much I have been to Starbase three times…two camps and once at school.

Student 4: I like building robots and putting on the sensors.

Interviewer: What one or two things did you think were not so cool about robotics?

Student 1: What I did not think was so cool about the robots was that we had to do certain steps, like we struggled through, and it was hard. I think they should make it easier.

Student 4: They should try to bring in people that have jobs with robots to help us out. That would be helpful.

Interviewer: How many of you are doing robotics for the first time? Oh, one. How many for the second time? Three. And how many are pros? What! Oh, all of the rest of you.

Interviewer: What one or two things did you think was cool about gaming?

Student 3: What I liked about the gaming was that we learned how to make our own avatar thing out the little pixels, and programming was fun, too.

Student 4: I get to create my own game. Just creativity, and you can actually play the game.

Student 6: I like what we are doing right now with Pac Man. My favorite game was my first game. I made snakes, and it was really fun. I think snakes are cool.

Student 7: I like how you can make it impossible for people to beat your game and that the teacher told me she never tried [a game like mine] before.

Student 2: I don't really like doing gaming. I like putting things together. It was really hard for me. The computer did not work very well. This is the first time [doing gaming].

Interviewer: How many of you are doing gaming for the first time. (Hands raised.) Oh, all of you are doing this for the first time.

Student 7: Gaming was not as easy as robotics. It's hard to follow the instructions. You can easily get confused and can do something wrong….

Themes and patterns were found in the qualitative data. The first theme—*building* robots—emerged among three of the seven students. Two students mentioned enjoying the sensors. A second theme that emerged from the transcribed data was *creating* or *programming* robots or games. Three students mentioned *creating* or *programming* explicitly, while two others were implicit in their references to creativity: *"I made snakes, and it was really fun; I like how you can make it impossible to beat your game."* The third emergent theme was the level of difficulty involved in programming

either robots or designing games. Three students stated "*it was hard*" and that specific steps had to be followed to make the robot move or to create games. The final theme that emerged was *prior knowledge* and *experience* with robots. However, none of the students had prior experience with game design. The novelty and high learning curve for developing computer games influenced students' attitudes toward videogaming and computer gaming, even though most of the students mentioned they enjoyed playing several off-shelf computer games (i.e., Mario, Minecraft, etc.) before participating in the study. While only one student mentioned the importance of having *STEM professionals* as guest speakers, this was a good suggestion in terms of helping students understand different aspects of engineering and technology and could improve their aspirations to pursue a STEM career.

Discussion

The results of this study reveal a great deal about student efficacy, computational thinking, and student interest in robotics and game design. The data suggest these children self-selected the summer camp because they had a high interest in robotics, gaming, or both. Their pre-scores on the surveys were high, indicating that results may have been impacted by a ceiling effect. While non-significant, STEM attitude scores in engineering and technology increased slightly along with 21st century skills. The highest score on the SETS survey was videogaming, but the scores declined significantly from pretest to posttest. Focus group data suggest some reasons for the decline, including a preference for robotics over game design, the difficulty in following directions to make the game, and making games that were impossible to win. One of the survey questions on the videogaming subscale specifically addressed students' efficacy as it related to winning a game. Most of the students' scores declined on this item after participating in the program. While the sample is small, we noticed that some girls were active gamers, and one female student with Asperger's Syndrome enjoyed game design and benefited from the step-by-step process. Moreover, an examination of her code revealed a departure from the SGD protocol to include her own nuances. Her game (see Figure 3 above) was judged as one of the most creative by her peers during a showcase on the final day of the camp.

In terms of developing computational thinking, focus group data suggest CT scores may be correlated with interest, enjoyment, and exposure to game design. None of the students had participated in game design before the summer program while almost all of them had prior experience in robotics. Field notes and classroom observations revealed most of the students in the program enjoyed working on robotics and playing each other's games. Engagement in robotics and gaming provides opportunities for students to engage in STEM content and creates pathways to STEM careers (Caron, 2010; Sheridan et al., 2013).

Significance

The findings presented in this research report informed the project team about the importance of combining the treatments of robotics and game design to increase students' spatial reasoning and computational thinking skills prior to implementing the Year 3 study. Instruction in both robotics and game design courses should be made more explicit (to be addressed during professional development). While minorities, females, and students with disabilities had high engagement, they were underrepresented in the program. This is an important finding that needs to be explored further. Recruiting females and other underrepresented students has been difficult in informal STEM programs (Leonard, J. Buss, A., Gamboa, R., Mitchell, M., Fashola, O. S., Hubert, T., et al., in press; Repenning et al., 2010), particularly in rural settings. This study is no exception. Perhaps using intact classrooms, as suggested by Webb et al. (2012) is one important mechanism for broadening participation. Our work is ongoing and will culminate in 2017.

Acknowledgments

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ESCENARIOS DE APRENDIZAJE Y EL USO SISTEMÁTICO DE TECNOLOGÍAS DIGITALES EN AMBIENTES DE RESOLUCIÓN DE PROBLEMAS

LEARNING SCENARIOS AND THE SYSTEMATIC USE OF DIGITAL TECHNOLOGIES IN PROBLEM SOLVING ENVIRONMENTS

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Existing technologies pose challenges for the education system related to what contents, strategies and skills students should learn and what types of teaching scenarios should be considered in learning. In this study, prospective high school teachers were encouraged to systematic use various digital technologies in a problem solving course during one semester. Results indicate that the participants developed and implemented a set of strategies and ways of thinking that complement and extend the approaches based on the use of pencil and paper. In addition, the use of technology allowed the participants extend mathematical discussion beyond formal settings.

Palabras clave: Resolución de Problemas, Tecnología, Maestros en Formación

Introducción

En los últimos diez años, el rápido desarrollo y disponibilidad de tecnologías digitales han cambiado sustancialmente la manera de interactuar de los individuos, las formas de comunicarse, obtener información y de realizar varias actividades cotidianas. Por ejemplo, con un dispositivo móvil (tableta o teléfono inteligente) es posible consultar información en línea, leer un periódico o, por medio de una aplicación, contratar los servicios de un taxi, reservar una habitación de hotel o conocer la ruta más eficiente para viajar de un lugar a otro. Las tecnologías existentes plantean retos para el sistema educativo relacionados con los contenidos, estrategias y habilidades que los estudiantes deben aprender y sobre qué tipos de escenarios de enseñanza se deben considerar en el aprendizaje. En este estudio interesa analizar y documentar los elementos de un escenario de aprendizaje donde los participantes utilizan diversas tecnologías digitales en un ambiente de resolución de problemas que promueve la interacción entre pares, el trabajo colaborativo y la extensión y solución de los problemas matemáticos planteados.

Para guiar este estudio se planteó la siguiente pregunta de investigación: ¿Qué formas de razonamiento construyen y exhiben un grupo de profesores de Bachillerato en formación cuando utilizan diversas herramientas (GeoGebra, Google Classroom, Padlet) en un ambiente de resolución de problemas? En particular, existe interés por documentar en qué medida el uso de herramientas para compartir información en línea (Padlet) puede ayudar a los estudiantes a modificar sus acercamientos iniciales de solución y desarrollar nuevas formas de explorar y extender los problemas a partir de trabajarlos con el Sistema de Geometría Dinámica (SGD) GeoGebra.

Marco conceptual

El estudio se sustenta y estructura a partir de tres áreas o dominios relacionados: (i) la idea de problematizar el aprendizaje de las matemáticas como principio básico en la resolución de problemas, (ii) el uso sistemático y coordinado de diversas tecnologías digitales en los procesos de representación y exploración de los conceptos matemáticos, y (iii) el diseño de las tareas como vehículos para organizar y promover el desarrollo del conocimiento matemático de los estudiantes.

La resolución de problemas y la actividad de interrogar o cuestionar

Un principio básico en una perspectiva de resolución de problemas es que los estudiantes desarrollen un enfoque inquisitivo para desarrollar y utilizar los conocimientos matemáticos. En este proceso, el aprendizaje de las matemáticas se concibe como una actividad de búsqueda de sentido en la que los estudiantes constantemente formulan y responden preguntas como una manera de representar, explorar, comprender y resolver situaciones matemáticas. Este principio es consistente con lo que Hiebert, et al. (1996) refiere como permitir que los estudiantes problematicen los temas o contenidos como una manera de profundizar y reflexionar sobre los conceptos matemáticos, esto significa "permitir que los estudiantes se pregunten porqué las cosas suceden, para indagar, para buscar soluciones, y para resolver las incongruencias" (p. 12), los autores utilizan el término problematizar en el sentido de que a los estudiantes se les debe permitir y alentar a problematizar lo que estudian, para definir los problemas que provocan sus curiosidades y habilidades para crear sentido. Por su parte Postman y Weingartner (1969), argumentan:

El conocimiento se produce en respuesta a preguntas. [...] Una vez que ha aprendido cómo preguntar -preguntas relevantes, apropiadas y sustanciosas- el estudiante ha aprendido cómo aprender y ya nadie lo puede detener en el camino de seguir aprendiendo lo que necesite y quiera conocer (p. 23).

El uso coordinado de herramientas digitales en la resolución de problemas matemáticos

Santos (2007) reconoce la importancia de que los estudiantes problematicen el estudio de la disciplina. Es decir, resulta esencial que los estudiantes formulen preguntas cuando se encuentran resolviendo problemas o intentan comprender ideas matemáticas, que utilicen distintas representaciones, busquen conjeturas y relaciones y donde el uso de distintas herramientas digitales les ayude a explorar y sustentar sus argumentos con acercamientos que pueden ir desde aspectos empíricos hasta acercamientos formales. El uso coordinado de varias herramientas digitales ofrece a los estudiantes diversas maneras de identificar, formular, representar, explorar y resolver problemas desde varias perspectivas (Santos-Trigo y Reyes-Martínez, 2016). El amplio desarrollo de tecnologías digitales implica decidir cuáles y cómo utilizarlas en los ambientes de aprendizaje con el objetivo de que los estudiantes comprendan conceptos matemáticos y resuelvan problemas. Santos (2007) sostiene que el uso sistemático de herramientas digitales en la construcción del conocimiento matemático de los estudiantes no sólo influye en la manera de presentar y explorar con las ideas matemáticas sino también en las formas de razonar, sustentar y presentar relaciones o propiedades matemáticas. También, Santos-Trigo y Ortega-Moreno (2013) documentan que las tecnologías digitales no solo extienden las representaciones y estrategias que aparecen en acercamientos basados en el uso de lápiz y el papel sino que generan nuevas formas de razonamiento para desarrollar conocimiento matemático.

Así, el uso de varias herramientas digitales ofrece diferentes oportunidades para que los estudiantes formulen y discutan preguntas que puedan guiarlos hacia la representación y exploración de las tareas matemáticas desde perspectivas distintas y complementarias pero también ofrecen a los profesores nuevas formas de organizar el trabajo el clase, para gestionar las tareas y materiales con mayor eficiencia, para mantenerse en constante comunicación con sus estudiantes y para crear ambientes que permitan extender la discusión de los problemas más allá del aula. En esta perspectiva, los problemas y el ambiente que se genere al resolverlos son aspectos cruciales para guiar y promover el desarrollo del conocimiento matemático y el aprendizaje de los estudiantes.

La importancia de las tareas matemáticas

En el marco de la resolución de problemas, las tareas o problemas representan un punto de partida y una oportunidad para que los alumnos busquen diferentes representaciones y formas de

resolverlas y ampliarlas, son un ingrediente fundamental para que los profesores guíen, fomenten y analicen los procesos que realizan los estudiantes en la construcción o desarrollo de un pensamiento matemático. Hiebert y Wearne (1993) afirman que "lo que los estudiantes aprenden se define en gran medida por las tareas que se les dan" (p. 395). Para Mason y Johnston-Wilder (2006), el objetivo de las tareas es hacer que los estudiantes participen de manera activa en la creación de sentido y en el desarrollo de sus habilidades matemáticas. Por lo tanto, las tareas son el vehículo para que los estudiantes comprendan los conceptos y desarrollen formas de pensamiento consistentes con el quehacer de la disciplina.

Se argumenta que el uso coordinado de diversas herramientas digitales ofrece a los estudiantes la oportunidad de ampliar las formas de razonar y trabajar los problemas y brindan la posibilidad de organizar, identificar y acceder a cierta información relevante en línea en todas las etapas de la resolución de problemas.
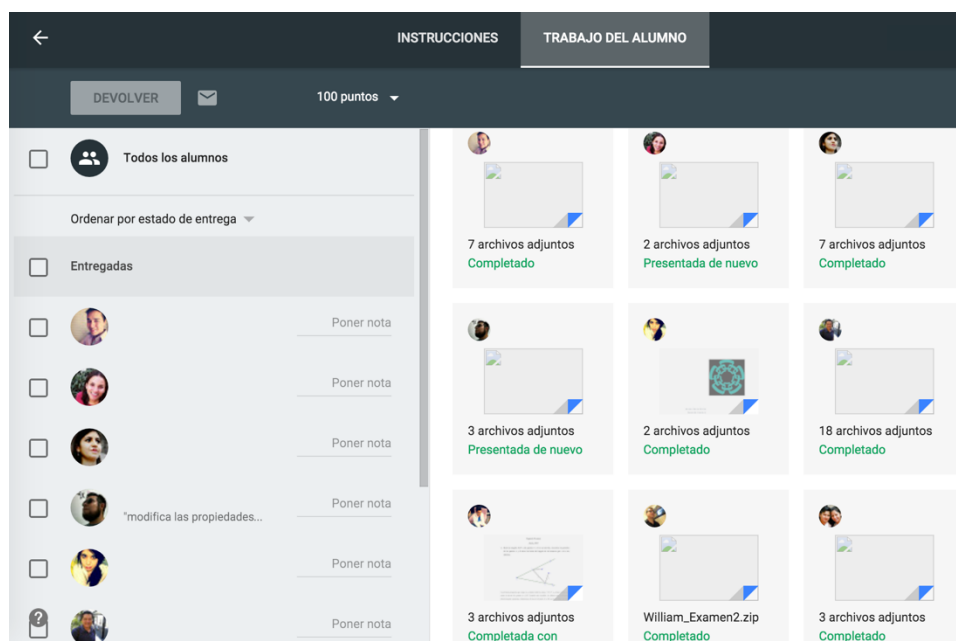
Figura 1. La plataforma se utilizó para enviar notificaciones a los estudiantes, compartir materiales y retroalimentar las tareas.

Participantes, problemas y procedimientos

En este trabajo se describen los elementos de un entorno de aprendizaje que busca extender las interacciones que ocurren en una aula tradicional de matemáticas. En este nuevo escenario se busca que los participantes usen de forma sistemática y coordinada un conjunto de tecnologías digitales en la resolución de problemas matemáticos. Tecnologías como: un SGD (GeoGebra) para explorar y representar los problemas de forma dinámica, una plataforma para administrar los contenidos y gestionar las tareas (Google Classroom, Figura 1), un muro digital (Padlet) para compartir ideas y nuevos acercamientos de solución y hacerlos accesibles durante el proceso de resolver problemas, servicios de Internet (Wikipedia, WolframAlpha, YouTube, KhanAcademy) para acceder, seleccionar y discriminar información relevante durante todas las etapas de resolución de problemas, una tableta (iPad) como principal medio de comunicación y acceso a información y contenidos interactivos (iBooks) y un servicio de mensajería instantánea (Google Hangouts) para comunicar y compartir información de forma eficiente con sus compañeros así como participar en una comunidad de aprendizaje en donde se discutan y compartan ideas y estrategias que les permitan resolver las tareas matemáticas.

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Para esta investigación participaron nueve profesores en formación inscritos en un curso de un semestre de resolución de problemas que formaba parte de un programa de maestría en educación matemática. Todos los participantes tenían un grado universitario en el área de matemáticas y se encontraban cursando el segundo semestre del programa. El curso incluía dos sesiones de tres horas cada una durante un semestre. En el desarrollo de las actividades se intentó que los participantes desarrollaran una actitud inquisitiva, la cual consiste, fundamentalmente, en el hábito de formular y discutir constantemente preguntas acerca de una situación y que siempre se buscaran diferentes formas de explorar y resolver los problemas con el uso sistemático de un conjunto de herramientas digitales.

En este contexto los problemas representaron un punto de partida para que los estudiantes se involucraran en una reflexión matemática que se extendiera más allá del aula y que les permitiera, inicialmente, resolver el problema y posteriormente buscar otras relaciones o resultados. En este sentido, el uso de herramientas digitales ofrece a los estudiantes oportunidades de participar en situaciones que los involucren en la formulación de conjeturas, la exploración de relaciones matemáticas, la creación de nuevos problemas, la extensión y generalización de las condiciones iniciales (Autores, 2015). Para este reporte, nos enfocamos en la discusión de un problema implementado a la mitad del semestre.

El problema. Dada una circunferencia *c* y un segmento *a*, construir un rombo de lado *a* de tal manera que la circunferencia *c* quede inscrita en ese rombo.

Este problema se planteó a los profesores como parte de una evaluación que consistió en tres etapas. En la primera etapa los estudiantes trabajaron de forma individual durante tres horas, durante este periodo utilizaron diferentes herramientas digitales para explorar el problema. Todos los participantes utilizaron GeoGebra para construir sus modelos dinámicos, esto resultó crucial para representar y explorar los objetos matemáticos involucrados en la construcción del rombo. Otra de las herramientas que fue utilizada por todos los participantes fue Wikipedia, mediante esta enciclopedia en línea pudieron consultar información (como definiciones, teoremas, propiedades y formulas: https://es.wikipedia.org/wiki/Rombo) que contribuyó a que los participantes comprendieran y le dieran sentido a la tarea. En la segunda etapa de la actividad se pidió a los estudiantes que compartieran en Padlet las ideas que utilizaron en sus exploraciones individuales, esto con el objetivo de crear un repositorio de ideas que pudieran ser utilizadas posteriormente por los demás participantes para explorar el problema. Por último, se pidió a los estudiantes que exploraran el problema a partir de las ideas compartidas por sus compañeros en el muro, con esa información los participantes se volvieron a enganchar en la exploración del problema, esta actividad la realizaron durante una semana.

Primera etapa. Trabajo individual. Al término de la sesión de trabajo individual, se reportaron 14 soluciones diferentes, que incluyen acercamientos empíricos, algebraicos, trigonométricos y geométricos. En esta sección se muestra el acercamiento más representativo que surgió durante esta sesión.

Cinco participantes construyeron el mismo modelo dinámico trabajando de forma independiente, para ello utilizaron la heurística de *relajar las condiciones iniciales del problema*, de esta manera construyeron un rombo cuyos lados tuvieran longitud *a* y luego hicieron que los lados fueran tangentes a la circunferencia *c*. Ubicaron el punto E sobre el eje *y* y trazaron una circunferencia de radio *a* que interseca al eje *x* en los puntos F y G, después reflejaron E respecto a A para encontrar el punto E', de esta manera se consigue una representación parcial de a solución (Figura), el rombo de vértices EFE'G tiene los 4 lados congruentes (de longitud igual al segmento *a*), al mover el punto E es posible visualizar una posición en la que la circunferencia *c* esté inscrita al rombo EFE'G, para que eso ocurra los lados del rombo deben ser tangentes a la circunferencia *c*. ¿Cómo se logra esta construcción?

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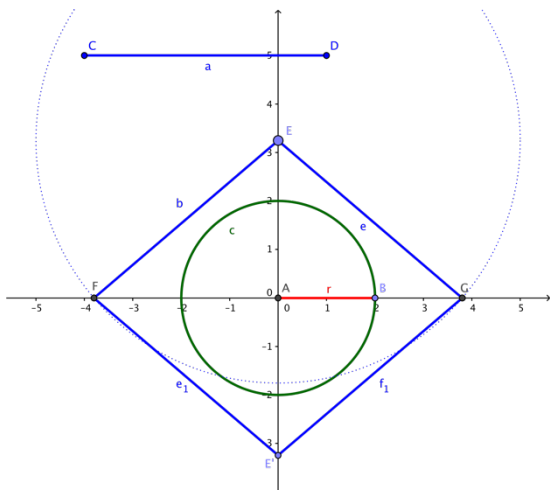
Figura 2. El rombo EGE'F es una representación parcial.
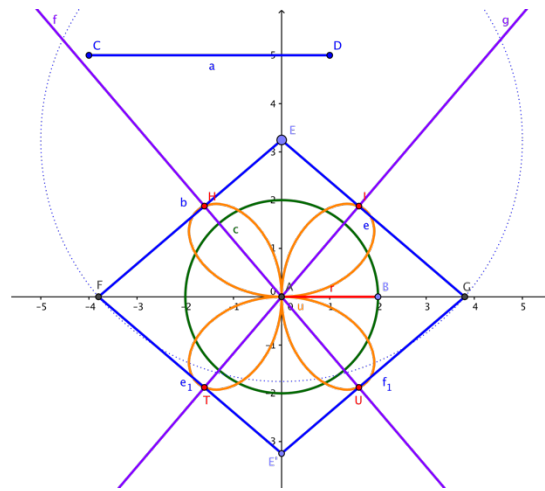
Figura 3. En naranja se muestra el lugar geométrico de los puntos H, T, I y U.

Al trazar rectas perpendiculares a los segmentos *e* y *b* por el punto A, éstas se intersecan con los lados del rombo se obtienen los puntos H, T, I y U (Figura 3). El lugar geométrico que describen los cuatro puntos cuando se mueve el punto E sobre el eje *y* se muestra en color naranja. Las intersecciones del lugar geométrico y la circunferencia *c* marcan las posiciones en las que los segmentos del rombo serán tangentes a la circunferencia *c*. Debido a la simetría del rombo y la circunferencia, este problema se puede simplificar y explorar solo en uno de los cuatro cuadrantes. Si centramos la atención en el primer cuadrante, se puede observar que el lugar geométrico que describe I interseca a la circunferencia *c* en dos posiciones, marcando así las dos soluciones del problema (Figura 4 y Figura 5).
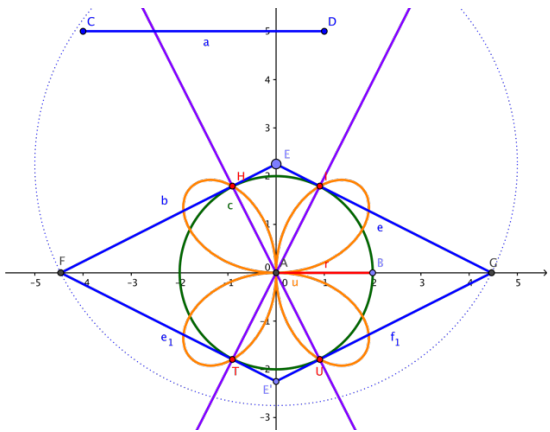
Figura 4. Primera solución**.**
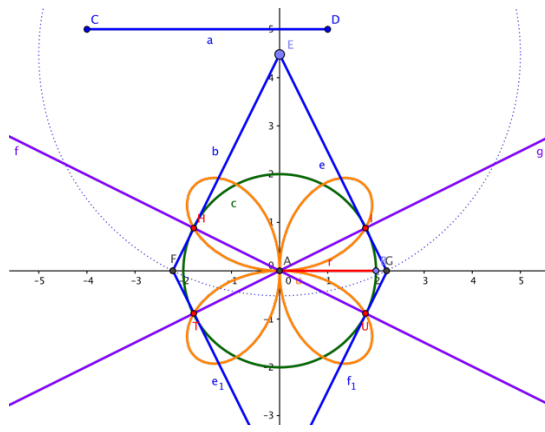
Figura 5. Segunda solución**.**

Segunda etapa. Ideas compartidas en Padlet. Después de trabajar de forma individual en la construcción del rombo, se pidió a los participantes que compartieran en Padlet las ideas medulares de sus exploraciones individuales. Se hizo hincapié en que no compartieran las soluciones sino las ideas que los llevaron a construir los modelos dinámicos. En la Figura 6 se muestra una parte del muro digital en el que se compartieron las ideas que se utilizaron para resolver el problema del rombo.

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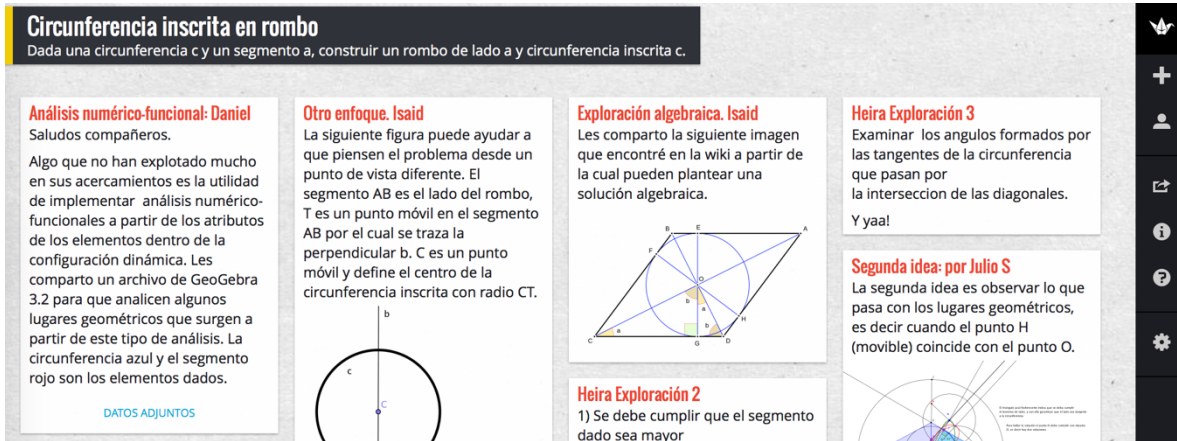
Figura 6. Muro que contiene todas las aportaciones hechas para el problema del rombo.

En total se compartieron 25 ideas en el muro (http://es.padlet.com/isaid/rombo). Se pidió a los participantes que seleccionaran algunas de las ideas de sus compañeros y que volvieran a explorar el problema a partir de esas ideas.

Tercera etapa. Exploraciones a partir de las ideas compartidas en el muro. Tres de los 9 participantes no lograron construir un modelo dinámico, fue hasta que tuvieron acceso a las ideas de sus compañeros que lograron resolver el problema. A continuación se muestra una solución que parte de una idea que se compartió en el muro. La Figura 7 hace referencia a una propiedad importante de los rombos: *La altura del rombo es igual al diámetro de la circunferencia inscrita*. Esta idea fue recuperada por una de las participantes que no había conseguido resolver el problema durante la primera etapa. En la Figura 8 se obtienen de forma algebraica los vértices y el centro del rombo a partir de construir un triángulo rectángulo de atura *2r*, base *x* e hipotenusa *a*. La participante utilizó deslizadores para variar los valores de *r* y *a*, esto le permitió definir el valor mínimo de *a* como *2r*, de esta manera siempre es posible construir el rombo.

Figura 7. Idea compartida por Karen.
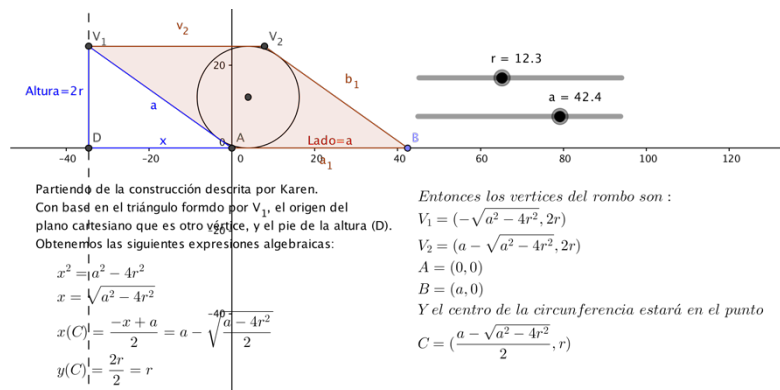
Figura 8. Solución algebraica a partir de considerar la idea compartida por Karen.

Todos los participantes mostraron una marcada participación respecto al número de soluciones que reportaron después de la exploración de las ideas del muro, en especial los que no habían podido resolver el problema por si solos. Resalta el caso de una participante que durante la primera etapa (trabajo individual) presentó 4 exploraciones en las que no pudo obtener una solución, tras explorar el problema a partir de las ideas del muro presenta quince exploraciones que incluyen dos soluciones que no surgieron durante la primera etapa.

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Comentarios. Durante esta actividad se resalta el caso de una participante que durante la sesión de trabajo individual no logró resolver el problema, después de realizar 4 intentos, pero despés de tener acceso al muro consigue presentar 15 exploraciones exitosas que incluyen dos soluciones que no se presentaron durante la sesión de trabajo individual. En este sentido, el muro digital le permitió a la participante recuperarse de intentos no exitosos y retomar ideas del muro que la llevaron a presentar varias soluciones al problema.

Conclusiones

Durante este reporte se ha argumentado que el uso de un SGD (GeoGebra) por parte de los profesores en formación puede depender de las *affordances* de la herramienta para construir y explorar los modelos dinámicos de las tareas matemáticas. Esta exploración los llevó a observar invariantes, patrones y relaciones entre los elementos de los modelos. Del mismo modo, el uso de las tecnologías de comunicación permiten que los participantes extiendan y continúen con las discusiones matemáticas más allá del salón de clases. De esta manera las discusiones pueden ocurrir prácticamente en cualquier lugar, sin limitaciones de tiempo y espacio.

Con el uso de las tecnologías digitales es posible restructurar y extender los ambientes tradicionales de los salones de clase, promoviendo que los estudiantes continúen trabajando, explorando, interactuando y reflexionando de forma continua virtualmente desde cualquier sitio sin limitaciones de tiempo y espacio. Para ello es importante que los estudiantes desarrollen formas de pensar y trabajar en esos ambientes que les permite el desarrollo de habilidades para utilizar de manera coordinada un conjunto de herramientas digitales.

De esta manera, el uso coordinado de diversas herramientas digitales fue esencial para que los participantes lograran identificar y apoyar las relaciones matemáticas y para crear un ambiente de aprendizaje que favoreciera y promoviera las interacciones de los participantes dentro y fuera del aula. En este contexto, el uso de Google Classroom permitió a los participantes acceder a los materiales del curso desde cualquier lugar y recibir retroalimentación de otros participantes. El uso de GeoGebra fue importante para la construcción de los modelos dinámicos que les ayudaron a identificar y examinar relaciones matemáticas. El uso de Internet les dio la oportunidad de obtener información en línea que les permitió aclarar o ampliar información sobre los conceptos involucrados en las tareas. Del mismo modo. El uso de Mathematica les ayudó a realizar cálculo matemáticos complejos y de esta forma concentrarse en el significado de los resultados. Compartir sus ideas en Padlet y utilizar Hangouts favoreció la interacción con los demás, ya que fue importante para buscar diferentes maneras de cómo resolver las tareas matemáticas. El uso de Padlet resultó transcendental durante esta actividad ya que impactó en todas las fases de la resolución de un problema (Polya, 1973) al permitirles:

1. Tener claridad acerca de lo que trataba el problema (Fase de comprensión del problema).
2. Considerar varias formas de resolver el problema y seleccionar un método particular a partir de una evaluación en relación con su utilidad (Fase de diseño de un plan).
3. Monitorear el proceso de solución y decidir cuando abandonar algún camino que no estuviera produciendo resultados (Fase de implementación).
4. Revisar el proceso de solución y evaluar la respuesta obtenida (visión retrospectiva).

Existing technologies pose challenges for the educational systems related to what contents, strategies and skills students should learn and what types of teaching scenarios should be considered in learning. In this study, prospective high school teachers were encouraged to systematic use various digital technologies in a problem solving course during one semester. Results indicate that the

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participants developed and implemented a set of strategies and ways of thinking that complement and extend the approaches based on the use of pencil and paper. In addition, the use of technology allowed the participants extend mathematical discussion beyond formal settings.

Keywords: Problem Solving, Technology, Teacher Education-Preservice

Introduction

In the last ten years, the rapid developments and availability of digital technologies are substantially changing ways in which individuals interact, communicate, access and get information to perform daily activities. For example, with a mobile device (tablet or smartphone) you can look for online information, read a newspaper or, through an application, request the services of a taxi, book a hotel room or make the most efficient route for going from one place to another. This study aims at analyzing and documenting the extent to which the use of digital technologies helps prospective high school teachers represent and explore mathematical tasks and to foster pairs' interaction, collaborative work and to extend mathematical discussions beyond formal settings.

To guide this study, the following research question was posed: What forms of reasoning do prospective high school teachers construct as a result of using different digital tools (GeoGebra, Google Classroom, Padlet) in a problem-solving environment? In particular, there is interest in documenting the extent to which the use of tools in sharing information online (Padlet) can help students to change their initial problem solving approaches and to develop new ways to explore and extend initial tasks by working on them with the Dynamic Geometry System (DGS) GeoGebra.

Conceptual framework

The study is based and structured around three areas or related domains: (i) the idea of problematize the learning of mathematics as a basic principle to learn mathematics and develop problem solving competencies (ii) the systematic and coordinated use of various digital technologies in the processes of representing and exploring of mathematical concepts, and (iii) the design of the tasks as vehicles to organize and promote the development of students' mathematical knowledge.

Problem solving and the problematizing activity

A basic principle in a problem-solving perspective is that students develop an inquisitive approach to develop and use mathematical knowledge. In this process, learning mathematics is conceived as a search for meaning activity in which students constantly ask and answer questions as a way to represent, explore, understand and solve mathematical situations (Santos-Trigo, 2014). This principle is consistent with what Hiebert et al. (1996) refers to as allowing students to problematize topics or content as a way to deepen and reflect on mathematical concepts, this means "allow students to ask why things happen, to investigate, to find solutions, and to resolve the incongruities" (p. 12), the authors use the term problematize in the sense that students should be allowed and encouraged to problematize what they study to define the problems that cause their curiosity and skills to create sense. Postman and Weingartner (1969), argue:

> Knowledge is produced in response to questions. [...] Once you've learned how to ask- relevant questions, appropriate and with substance- the student has learned how to learn and nobody can stop them to continue learning what they need and wants to know (p. 23).

The coordinated use of digital tools in solving mathematical problems

Santos (2007) recognizes the importance for students to problematize the study of discipline. That is, it is essential for students to ask questions when they are solving problems or try to understand mathematical ideas using different representations, to look for conjectures and relationships and where the use of different digital tools help them explore and support their

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arguments with approaches that can go from empirical reasoning to formal approaches. The coordinated use of various digital tools offers students different ways to identify, formulate, represent, explore and solve problems from various perspectives (Santos-Trigo & Reyes-Martínez, 2016). The extensive development of digital technologies involves deciding what and how to use them in learning environments with the objective that students understand mathematical concepts and solve problems. Santos (2007) argues that the systematic use of digital tools in the construction of students' mathematical knowledge not only influences the way to present and explore mathematical ideas but also the ways of reasoning, supporting and presenting relationships or mathematical properties. Also, Santos-Trigo and Ortega-Moreno (2013) pointed out that digital technologies not only extend the representations and strategies that appear in approaches based in the use pencil and paper but also generate new ways of thinking to develop mathematical knowledge.

Thus, the use of various digital tools offers different opportunities for students to develop and discuss questions that can lead them to represent and explore mathematical tasks from different and complementary perspectives, and also offer teachers new ways of organizing and managing the class work and class materials more efficiently. Similarly, teachers can keep constant communication with their students and create environments to extend the discussion of problems beyond the classroom. In this perspective, the problems and the teaching atmosphere are crucial aspects to guide and promote the development of mathematical knowledge and student learning.

The importance of mathematics tasks

In the context of problem solving, tasks or problems represent a starting point and an opportunity for students to find different representations and ways to solve and extend them. Mathematical tasks are essential tools for teachers to guide, promote and analyze their students' processes to construct, develop, and use mathematical knowledge. Hiebert and Wearne (1993) state that "what students learn is defined largely by the tasks they are given" (p. 395). For Mason and Johnston-Wilder (2006), the objective of the tasks is to make students participate actively in the creation of meaning and develop their mathematics skills. Therefore, the tasks are the vehicle for students to understand concepts and develop ways of thinking that are consistent with mathematical practices.

We argue that the coordinated use of various digital tools offers students an opportunity to expand the ways of thinking involved in solving problems and to organize, identify and access some relevant online information at all problem solving phases
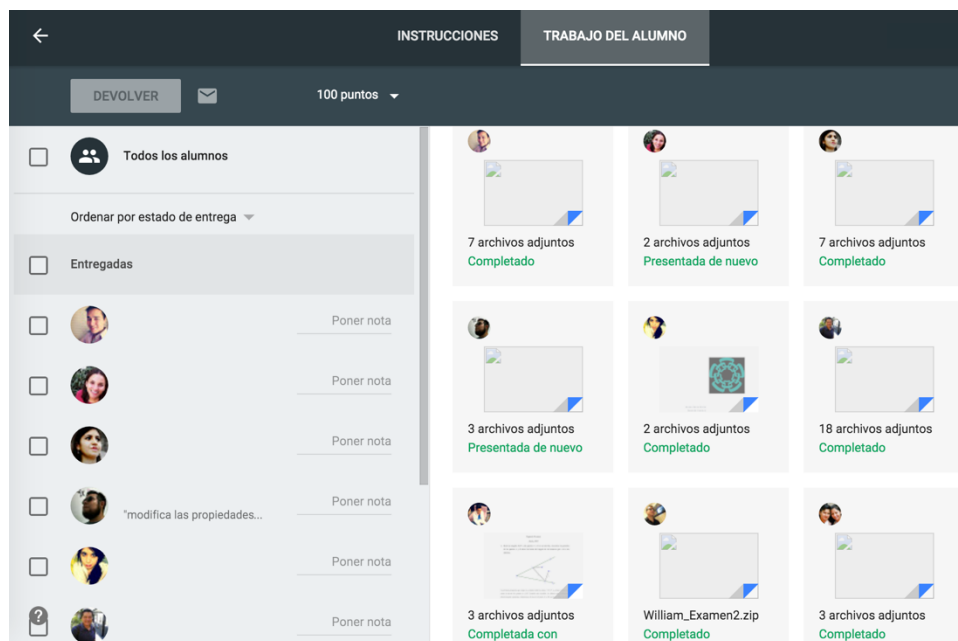
Figure 1. The platform that was used to send notifications to students, share materials and to provide feedback to students**.**

Participants, problems and procedures

In this paper we describe aspects of a learning environment that seeks to expand the interactions that take place in a traditional mathematics classroom. In this new environment the goal is that participants use in a systematic way a collection of digital technologies when solving mathematical problems. These digital technologies included: A DGS (GeoGebra) to explore and represent problems dynamically, a platform for managing content and manage tasks (Google Classroom, Figure 1), a digital wall (Padlet) to share ideas and new approaches to solve the tasks that were available during the problem solving process, Internet services (Wikipedia, Wolfram Alpha, YouTube, Khan Academy) to access, select and discriminate relevant information during all stages of problem solving, a tablet (iPad) as a primary means of communicating and accessing information and interactive content (iBooks) and instant messaging (Google Hangouts) to communicate and share information efficiently with their peers and participate in a learning community where they discuss and share ideas and strategies to solve mathematical tasks.

This study involved nine prospective high school teachers who were taking a semester long problem solving course that was part of a master's program in mathematics education. All participants had a bachelor's degree with a mathematics major and were enrolled in the second semester of the program. The course included two sessions of three hours each. During the development of sessions, all participants were encouraged to engage and develop an enquiring approach, which consists mainly in the habit of formulating and constantly discuss questions about a situation and different ways to explore and solve problems with the systematic use of a set of digital tools.

In general terms, all tasks or problems that were given to the participants represented a starting point for them to engage in mathematical thinking that could extend beyond the classroom and that would allow them to first solve the problem and then look for other connections and solutions. In this sense, the use of digital tools offers the participants an opportunity to share their ideas or conjectures, the exploration of mathematical relationships, the creation of new problems, the extension and generalization of initial conditions (Santos-Trigo, Reyes-Martínez, & Ortega-Moreno, 2015). For this report, we focus on the discussion of a problem implemented in the middle of the semester.

The problem. Given a circle *c* and a segment *a*, construct a rhombus with side *a* such that the circle *c* remains inscribed in the rhombus.

This problem was posed as part of the participants' assessment and consisted of three stages. In the first stage students worked individually for three hours, using different digital tools to explore the problem. All participants used GeoGebra to construct a dynamic model of the task. This model became a key source for the participants to identify, represent, and explore mathematical objects involved in the construction of the rhombus. Another tool that was used by all participants was the encyclopedia Wikipedia in which they looked for information (such as definitions, theorems, properties and formulas: https://es.wikipedia.org/wiki/Rombo) that helped them understand and give meaning to the task. In the second stage, the students were asked to share in the Padlet board the ideas they used in their individual explorations. That is, they created a repository of ideas which later could be used by other participants to explore the problem. Finally, the participants were asked to explore the problem using their colleagues' ideas; with that information, participants were again engaged in the exploration of the problem; this activity was conducted for a week.

First stage. Individual work. At the end of the individual work session, 14 different solutions, including empirical, algebraic, trigonometric and geometric approaches were reported. In this section the most representative approaches that emerged during this session are presented.

Five participants proposed the same dynamic model working independently, to this end they relied on the heuristic to *relax the initial conditions of the problem,* so they drew a rhombus whose sides have length *a* and then they made the sides tangent to the circumference *c*. They placed the point E on the *y* axis and drew a circle of radius *a* that intersects the *x* axis at points F and G, then reflected E with respect to A to find the point E ', so a partial representation of the solution is achieved (Figure 2); the rhombus with vertices EFE'G has 4 congruent sides (of length equal to segment *a*); if you move the point E it is possible to find a position where the circumference *c* is inside rhombus EFE'G, for this to occur the sides of the rhombus should be tangent to the circle *c*. How is this construction achieved?
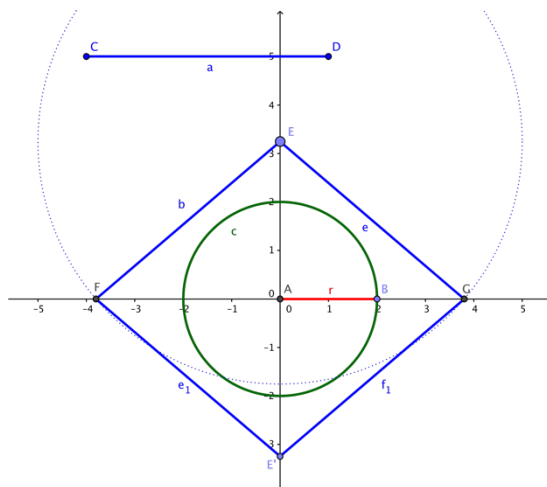
Figure 2. The rhombus EGE'F is a partial representation.
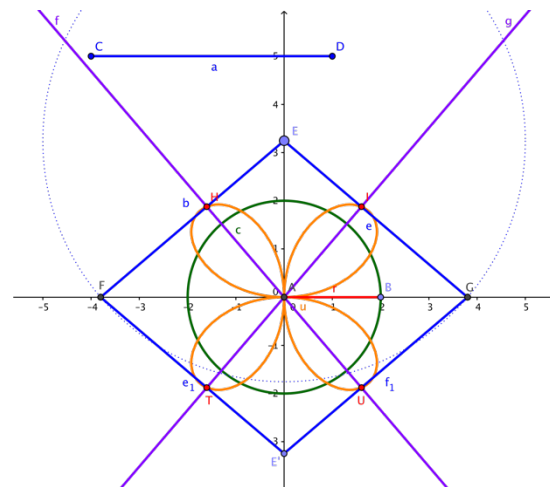
Figure 3. The geometric locus of the points H, T, I and U is shown in orange.

When drawing perpendicular lines to the segments *e* and *b* through point *A*, these perpendicular lines intersect the sides of the rhombus at the points H, T, I and U (Figure 3). The locus of the four points when point E moves on the *y* axis is displayed in orange. The intersections of the loci and the circumference *c* identify the positions in which the segments of the rhombus will be tangent to the

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circumference *c*. Due to the symmetry of the rhombus and the circumference, this problem can be simplified and explored by only looking at one of the four quadrants. If we focus on the first quadrant, we can see that the locus described by I intersects the circumference *c* in two positions, showing the two solutions of the problem (Figure 4 and Figure 5).
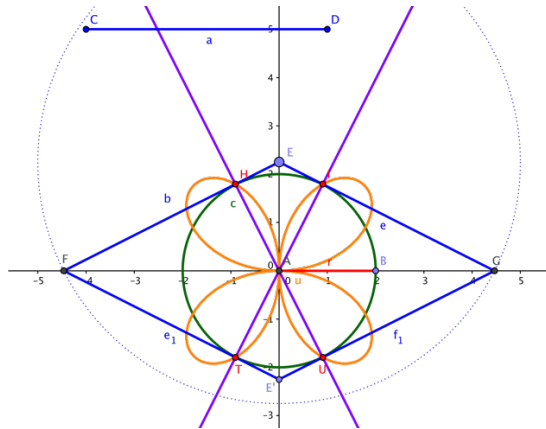
Figure 4. First solution.
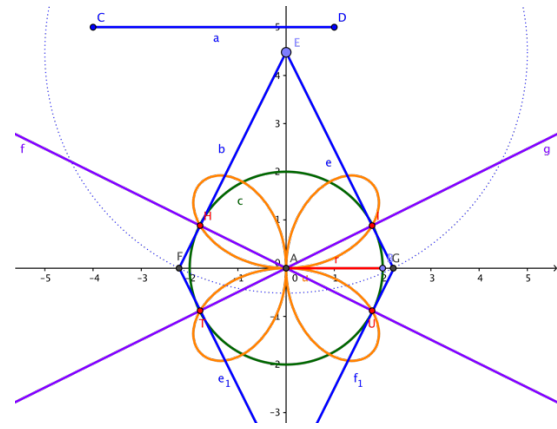
Figure 5. Second solution.

Second stage. Padlet shared ideas. After working individually in the construction of the rhombus, participants were asked to share via Padlet the core ideas of their individual approaches. It was emphasized that they should not share the solutions, but the ideas that led them to build the dynamic models. Figure 6 shows part of the digital wall where the participants shared the ideas that later were used to solve the problem.

Figure 6. Digital wall that contains all participants' contributions made to the rhombus problem.

In total 25 contributions were shared on the wall (http://es.padlet.com/isaid/rombo). The participants were asked to select some of their peers' ideas and to pursue them to explore the problem again.

Third stage. Explorations that consider shared ideas from the wall. Three of the 9 participants failed to initially build a dynamic model, but they were able to construct such a model later on based on the ideas that were shared at the wall. In what follows we show a solution that emerged from one idea that was shared in the wall. Figure 7 refers to an important property of the rhombus: *The height of the rhombus is equal to* the *diameter of the inscribed circle.* This idea was

recovered by one of the participants who had failed to solve the problem during the first stage. Figure 8 shows an algebraic approach that involves the vertices and the center of the rhombus and a right triangle with height *2r*, base *x* and hypotenuse *a*. The participant used sliders to change the values of *r* and *a,* this allowed her to set the minimum value of *a* as *2r,* so it is always possible to build the rhombus.
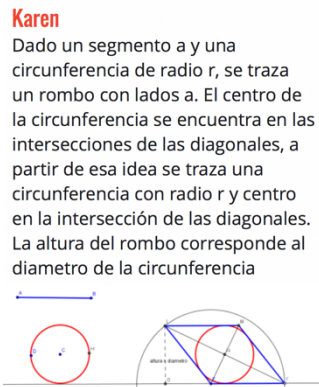
Figure 7. Idea shared by Karen.
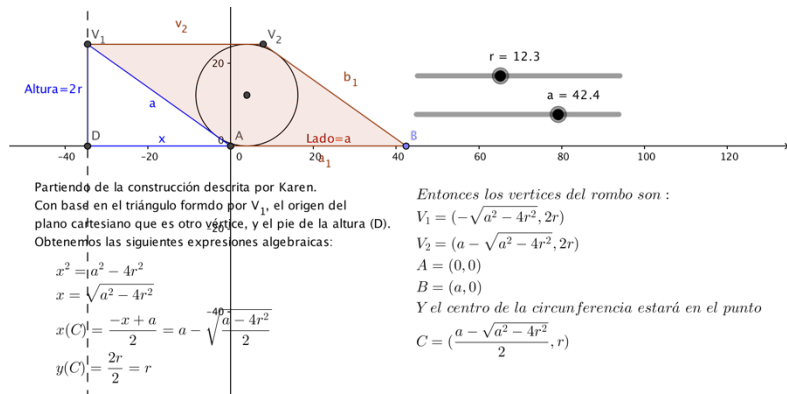
Figure 8. Algebraic solution from considering the idea shared by Karen.

All participants showed a marked participation regarding the number of solutions that were reported after exploring the ideas that were shared through the wall, especially those who had not been able to solve the problem by themselves. It is highlighted the case of a participant that during the first stage (individual work) presented four explorations in which she could not obtain a solution, after exploring the problem from the ideas of the wall, she presented fifteen explorations including two solutions that did not emerge during the first stage.

 Comments. One participant experienced difficulties during the individual work time and could not solve the problem after 4 attempts; however, after having access to the wall, she was able to present 15 successful explorations that included two solutions that were not presented during the individual work session. In this sense, the digital wall allowed the participant to recover from unsuccessful attempts and take into account ideas that led her to present several solutions to the problem.

Conclusions

Throughout this report we have argued that the use of a DGS (GeoGebra) by the prospective high school teachers can depend on the tool affordances to construct and explore dynamic models of mathematical tasks. This exploration led them to observe invariances, patterns and relationships among elements of the model. Similarly, the use of communication technologies allows the participants to extend and continue mathematical discussions beyond formal settings. This task discussion can happen virtually anywhere without limitations of time and space.

With the use of digital technologies, it is possible to restructure and extend the traditional classroom environments, promoting that students keep working, exploring, interacting, and reflecting from any place without limitations of time or space. To this end, it is important that students develop forms of thinking and working in these environments so that they can develop the ability to use in a coordinated way the digital tools.

Thus the coordinated use of various digital tools was essential for the participant to identify and support mathematical relations and to create a learning environment that favored and fostered interactions among participants inside and outside the classroom. In this context, the use of Google Classroom allowed the participants to access course materials from anywhere and receive feedback

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from others participants. The use of GeoGebra was important to construct dynamic models that helped them identify and examine mathematical relationships. The use of Internet gave them the opportunity to obtain online information that helped them clarify or extend what they recall from concepts involved in the tasks. Similarly, using Mathematica helped them to perform complex calculations and focus on the meaning of the results. Sharing their ideas via Padlet and using Hangouts favored the interaction with others, and accessing others' ideas was important to look for distinct task paths solution. The use of Padlet was essential for the participants during the process and problem solving phases during this activity (Polya, 1973) by allowing:

1. To fully understand the problem (Problem understanding phase).
2. To consider several ways to solve the problem and selecting a particular method from an assessment regarding its usefulness (Design of a plan phase).
3. To monitor the solution process and decide when to abandon a path that was not producing results (Implementation phase).
4. To review the solution process and evaluate the results (Review/extend phase).

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TRADEOFFS OF SITUATEDNESS: ICONICITY CONSTRAINS
THE DEVELOPMENT OF CONTENT-ORIENTED SENSORIMOTOR SCHEMES

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Mathematics education practitioners and researchers have long debated best pedagogical practices for introducing new concepts. Our design-based research project evaluated a heuristic framework, whereby students first develop acontextual sensorimotor schemes and only then extend these schemes to incorporate both concrete narratives (grounding) and formal mathematical rules (generalizing). We compared student performance under conditions of working with stark (acontextual) vs. iconic (situated) manipulatives. We summarize findings from analyzing 20 individually administered task-based semi-structured clinical interviews with Grade 4 – 6 participant students. We found tradeoffs of situatedness: Whereas iconic objects elicit richer narratives than stark objects, these narratives may detrimentally constrain the scope of potential sensorimotor schemes students develop in attempt to solve manipulation problems.

Keywords: Cognition, Learning Theory, Number Concepts and Operations, Technology

Introduction: Forging an Embodiment Middle Ground
Between Formalisms-First and Progressive Formalization

Scholars of mathematics education tend to hold two diametrically opposed positions on best pedagogical practices for introducing new mathematical concepts (Nathan, 2012). The formalisms-first approach (e.g., Baird, 2004; Kaminski et al., 2008; Sloutsky et al, 2005; Stokes, 1997; Uttal, Scudder, & DeLoache, 1997) posits that students should first learn symbolical representations of a new concept and only then apply their formal strategies to situated contexts. The progressive-formalization approach (e.g., Goldstone et al., 2005, 2008; Harnad, 1990) posits that students should begin from concrete situations and then progressively formalize their understandings of the situations towards normative abstract representations.
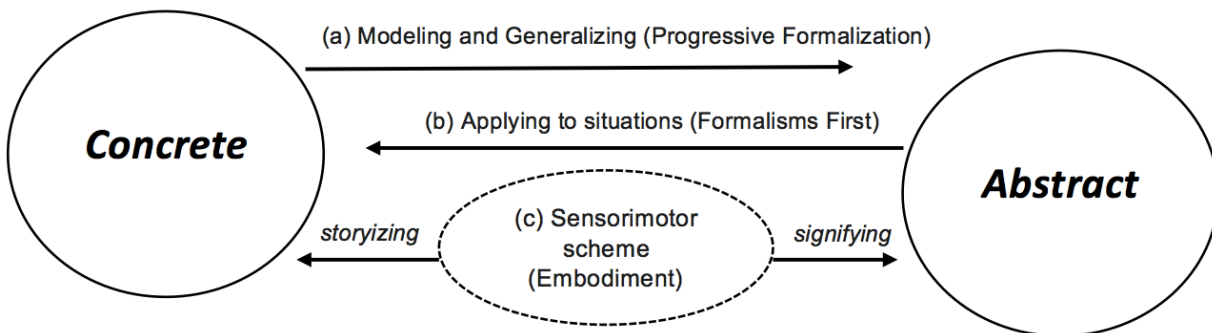
Figure 8. Positioning the (c) Embodiment approach with respect to the
(a) Progressive-Formalization approach and the (b) Formalisms-First approach.

And yet in between these two oppositional stances there may be a third position (see Figure 1). Inspired by the embodiment approach (Campbell, 2003; Chemero, 2009; Clark, 2013; Nemirovsky, 2003; Varela, Thompson, &, Rosch, 1991), this position implicates sensorimotor schemes as the epistemological core of mathematical learning and knowing. We conjectured that students could encounter new mathematical concepts by first developing sensorimotor schemes and then both grounding these schemes in concrete situations (storyizing) and signifying the whereas we embrace

the proposal to ground mathematical meaning in "our direct physical and perceptual experiences" (Nathan, 2012, p. 139), we decompose this idea by foregrounding and differentiating its two inherent phenomenological dimensions, sensorimotor schemes and situatedness (contextuality). We argue that these two dimensions have been conflated in historical debates (e.g., Barab et al., 2007; Bruner, 1986; Burton, 1999). That is, we maintain that learning activities can be created such that sensorimotor schemes are fostered either in contextual or acontextual situations. We hypothesized that different levels of contextuality would have different effects on learning, and we assumed that sensorimotor schemes would mediate this effect. We believed that students would develop different sensorimotor schemes in low- vs. high-context activities, and that the low-context condition would prove advantageous.

To evaluate this hypothesis, we designed and implemented a learning activity complete with materials, tasks, and facilitation techniques based on the embodied-design framework (Abrahamson, 2006, 2009, 2014). In the empirical study reported herein, we varied the contextuality of a manipulation problem by either incorporating or not incorporating iconic information that would cue narrative framings of the situation, and we measured for effects of this experimental variation on qualities of students' behaviors as they engaged in solving the problem. Our study thus aimed to empirically evaluate this in-between embodiment position with respect to the ongoing debate of formalisms-first vs. progressive formalization.

Theoretical Background

Nathan (2012) has characterized two opposing approaches to mathematics education as follows:

- *progressive formalization* proposes that students should encounter new concepts in the context of meaningful concrete situations and then abstract toward formal models of these situations by progressively adopting mathematical forms and nomenclature (Goldstone, Landy, & Son, 2008; Harnad, 1990); and
- *formalism first* proposes that students should encounter concepts through abstract procedures and then map formalisms to concrete situations via application problems (Baird, 2004; Kaminski et al., 2008; Sloutsky et al., 2005; Stokes, 1997).

The *embodiment approach* put forth in this article borrows the progressive-formalization epistemological position that abstract notions are grounded in concrete situations yet also partially subscribes to the formalism-first ontological position that mathematical concepts should be grounded in generic images. On the one hand, as per *progressive formalization*, *embodiment* learning materials are pre-symbolic and informal. On the other hand, per *formalisms first*, the materials are generic or stark, that is, highly economical on any situated or narrative content.

Affordances and Constraints of Stark (Acontextual) vs. Rich (Situated) Manipulatives

Pedagogical approaches inspired by embodiment theory champion the principle of fostering opportunities for students to build new sensorimotor schemes prior to signifying the schemes in a discipline's semiotic register (Abrahamson, 2006; Nemirovsky, 2003). Our study considered from an embodiment perspective the effect of situatedness on the development of sensorimotor schemes. We thus sought a theory of situated perception and action that would enable us to model, anticipate, and analyze for effects of experimentally varying an activity's situatedness.

Our focus on the relationship between the properties of objects that students manipulate and their actions on these objects led us to consider the theoretical notions of affordances and constraints as relevant to the goals of this study. *Ecological psychology* (Gibson, 1977) theorizes an agent's potential actions on the environment as contingent on the agent–environment relations. An agent (e.g., a mathematics student) perceives opportunities for acting on objects in the environment (e.g.,

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classroom manipulatives) in accord to these objects' subjective cues; the agent tacitly perceives the object as *affording*, that is, privileging certain forms of goal-oriented engagement. Importing Gibson's interactionist views into educational research, Greeno (1994) modeled student learning as the process of attuning to constraints and affordances in recurring situations. Araújo and Davids (2004) further offer that an instructor can "channel" students' engagement in goal-oriented activity by controlling environmental constraints.

Still, to the extent that one subscribes to the constructivist thesis underlying this research, namely that sensorimotor learning mediates conceptual learning, *why might different degrees of the learning materials' contextuality afford different sensorimotor learning?* The answer, we believe, lies in the nature of these sensorimotor schemes vis-à-vis the particular features of the learning materials that the students mentally construct in the course of developing the materials' new perceived affordances. That is, a given situation may lend itself to different goal-oriented sensorimotor schemes. And whereas a variety of schemes may accomplish the prescribed task, some of these schemes may be more important than others for the pedagogical purposes of the activity. We hypothesize that *the situatedness (contextuality) of learning materials constrains which sensorimotor schemes the materials might come to afford*. Where particular contextual cues unwittingly constrain student development of pedagogically desirable affordances, the students' conceptual learning will thus be delimited.

In evaluating this hypothesis pertaining to the nature and quality of situated learning, we needed a theoretical construct that would both cohere with the embodiment perspective and enable us to implicate in our data which sensorimotor schemes students were developing. We realized we were searching for a means of determining how the students are mentally constructing the materials; what specifically they were looking at that mediated their successful manipulation. Such a theoretical construct already existed: an attentional anchor (see below).

An *attentional anchor* is a dynamical structure or pattern of real and/or projected features that an agent perceives in the environment as their means of facilitating the enactment of motor-action coordination (Hutto & Sánchez–García, 2015). Abrahamson and Sánchez–García (2016) demonstrated the utility of the construct, which originated in sports science, in the context of mathematics educational research. Abrahamson et al. (2016) studied the role that visual attention plays in the emergence of new sensorimotor schemes underlying the concept of proportion. They overlaid data of participants' eye-movement patterns onto concurrent data of their hand-movements. They found that the participants' enactment of a new bimanual coordination coincided with a shift from unstructured gazing at salient figural contours to structured gazing at new *non*-salient figural features. The participants' speech and gesture confirmed that they had just constructed a new attentional anchor as mediating their control of the environment.

For this study, we adopted the construct of an attentional anchor as a key component of our methods. We sought to characterize what attentional anchors students developed during their attempts to solve a motor-action manipulation task. By so doing we hoped to gauge for effects of varying the contextuality of learning materials (iconic vs. stark) on student development of the sensorimotor scheme mediating an activity's learning goal. We hypothesized that richer manipulatives would constrain the scope of attentional anchors students develop.

Methods: Designing Constraints on Sensorimotor Engagement of a Technological System

The Mathematics Imagery Trainer for Proportion (MIT-P; see Figure 2) sets the empirical context for this study. Students working with the MIT-P are asked to move two cursors along vertical axes so as to make the screen green and keep it green. Unknown to the students, the screen will become green only if the cursors' respective heights along the screen relate by a particular ratio, otherwise the screen will be red. For instance, for a ratio of 1:2, the screen will be green only when

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the right hand is twice as high along the monitor as the left hand. Students develop a variety of motor-action strategies to satisfy the task demand (Howison et al., 2011).
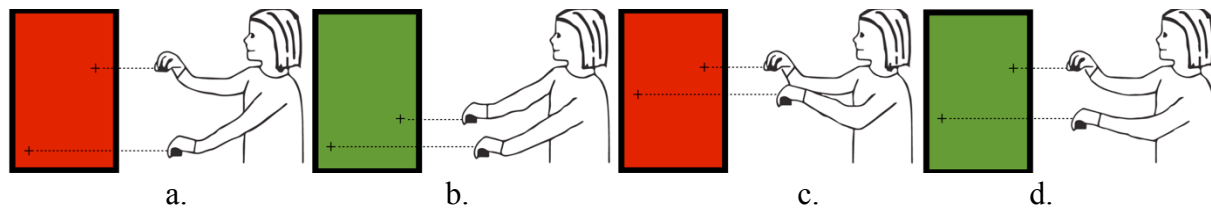
Figure 2. The Mathematical Imagery Trainer for Proportion (MIT-P) set at a 1:2 Ratio. Compare 2b and 2d to note the different vertical intervals between the hands and, correspondingly, the different vertical (or diagonal) intervals between the virtual objects. Noticing this difference is crucial to experiencing, then resolving a key cognitive conflict.

In the current study, images appear at students' fingertips when they touch the screen. These images are either stark crosshair targets (see Figure 3a) or iconic images (e.g., hot air balloons; Figure 3b).
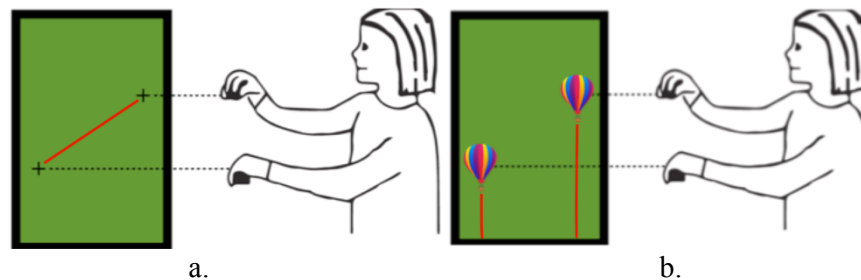
Figure 3. Experimental conditions and hypothesized attentional anchors: (a) stark crosshair targets cue the vertical or diagonal interval between the hands; and (b) iconic images (hot-air balloons) that cue the interval from each object to the bottom of the screen. In the actual experiments we used large touchscreens where the hands are on the interface.

In both experimental conditions (stark and iconic) students are led through a task-based semi-structured clinical interview. Following an unstructured orientation phase, in which the participants find several green locations, they are asked to maintain green while moving both hands from the bottom of the screen to the top. The interviewer and student then engage in a coordination challenge, where the interviewer manipulates the left image and the student manipulates the right image. The student is asked to predict the green locations. All along, the students are prompted to articulate rules for making the screen green.

We wished to investigate for attentional anchors that emerge during children's interactions with the technology. We reasoned that the attentional anchors would indicate what sensorimotor schemes the students developed. More specifically, we explored for an effect of experimental condition (stark vs. iconic cursors) on the types of attentional anchors students construct and articulate (via speech and/or gesture). We also looked at the effect of condition sequence.

Twenty-five Grade 4 – 6 students participated individually in the interviews, 14 in the "stark-then-iconic" condition and 11 in the "iconic-then-stark" condition. We exclusively interviewed students around the numerical item of a 1:2 ratio. These sessions were audio–video recorded for subsequent analysis. As our primary methodological approach, the laboratory researchers engaged in a micro-genetic analysis of selected episodes from the data corpus, focusing on the study

participants' range of physical actions and multimodal utterance around the available media (Ferrara, 2014). Our working hypothesis, to iterate, was that the virtual objects' figural elements may cue (afford) particular sensorimotor orientations and thus "filter" the child's potential scope of interactions with the device. Namely, we analysed for effects of the manipulatives' perceived affordances on participants' scope of interaction.

Results: Perceived Affordances of Stark vs. Iconic Situations Mediate Student Strategies
A main effect was found. Below, we report our findings in each experimental condition by first describing participants' typical strategies and then illustrating these behaviors through brief vignettes. The section ends with comparing observed student strategies under the two conditions.

Stark Targets Afford the "Distance Between the Hands" Attentional Anchor
In the trials where participants interacted with stark targets first, they began the activity by placing their left-hand- and right-hand fingertips on a blank touchscreen. Immediately they noticed crosshairs appear at the locations of their fingertips. In an attempt to make the screen green, the participants began moving their hands all over the screen with no apparent strategy, "freezing" their fingers as soon as the screen turned green. Eventually, participants oriented toward the spatial interval between their fingers, soon discovering that their fingers have to be a certain distance from each other at different heights along the screen. Finally they determined a dynamical covariation between the interval's size and height: the higher the hands, the bigger the interval must be (and vice versa). We turn to several vignettes (all names are pseudonyms). As we shall see, both participants will refer to an imaginary diagonal line connecting the cursors.

Luke (age 10). As he found various green-generating screen location, Luke commented about the space between his hands at these various locations: "It's the same angle. Well, I mean the line connecting them is the same direction" [4:53]. Later, he noted that the "[angle] is changing because my right hand is getting faster, so when this goes up that much (moves left hand approximately 2 in. on the screen) this one goes up at this much (moves right hand approximately 4 in. on the screen)" [11:10].

Amy (age 9). Amy reported her observation: "The diagonal [between the hands] at the top is different than [at] the bottom" [7:15]. Then later during the iconic challenge, she said: "You have to make them different diagonally from each other to make it change color" [7:42].

Thus during the stark-target trials the participants not only noticed that the interval between their hands was changing in size, they came to see this interval as an imaginary line between their hands. In turn, this imaginary line—its size, angularity, and elevation along the screen—apparently served the participants in finding and keeping green, ultimately enabling them to articulate a strategy for doing so. This imaginary line along with attributed properties is an attentional anchor: it is crafted out of negative space to mediate the situated coordination of motor intentionality; subsequently this mentally constructed object serves to craft proto-proportional logico–mathematical propositions. This spontaneous appearance of a self-constraint that facilitated the enactment of a challenging motor-action coordination is in line with dynamical-systems theory (Kelso & Engstrøm, 2006).

Of the 14 students in this stark-then-iconic experimental condition, 10 spoke about the interval between the hands still within the "stark" phase of their interview, with eight of them referring explicitly to its magnitude. Then during the "iconic" phase of the interview, 2 of these 10 students began to speak about the icons as separate entities, focusing on the speed of each respective icon, or reverting to a focus on the color feedback of the screen to determine where to place the hands. The remaining 8 of these 10 students continued to use the interval line between their hands as a guide for making the screen green. These students' attention to the diagonal line was consistent, suggesting that this imaginary "steering wheel" had become perceptually stable in their sensorimotor engagement with this technological system.

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Iconic Images Afford the "Distance From the Bottom" Attentional Anchor

Similar to the stark-then-iconic condition, in the trials where students interacted with rich icons first, they began the activity by placing their left-hand- and right-hand fingers on a blank touchscreen. However, in this condition they immediately saw hot-air balloon icons (not stark targets) appear on the screen. Thus, the virtual manipulatives in this condition are situational and not stark, even as the tasks are otherwise identical. Recall that these students worked first with the iconic images and then with the stark images. As we will now explain, beginning with the iconic images cued a narrative-based strategy that was based on a frame of reference that did not attend to the interval between the images but instead to each of these hot-air balloons vertical distance above the "earth" (the bottom of the screen). As we will see, this alternative sensorimotor orientation was so strong that it carried over to the stark condition, so that by-and-large these participants were less likely to attend to the interval and thus were less likely to benefit from its potential contribution to their problem-solving strategy.

Leah (age 11). Having generated green for the first time, Leah noticed that when she moves one hand, the greenness dulls out toward red. Later, she described her strategy for making the screen green referring gesturally to the hand's distance from bottom of the screen: "I would say what I said before, where one hand chooses a place and the other hand chooses a color based on where the hand is, and you can adjust it to keep it green. Once you find that, you just need to keep it the same height [from the bottom]" [8:40]. Then in the next task, she maintains her strategy, saying: "When you move one hand up you need to move the other hand up so it's the same distance [from the bottom], but higher" [12:22].

Jake (age 11). Jake described his initial strategy: "Try putting your hands together in the middle and then try moving one down or the other one up. One of the balloons should stay in the middle while the other moves" [4:47]. Note how "middle" refers to that balloon's location along a vertical axis irrespective of the other balloon. Jake perseverates with this strategy throughout the set of challenges, moving his hands up along the screen sequentially rather than simultaneously. When later tasked to make the screen green with the stark targets, he appeared disoriented, noting, "This is harder because I don't have a starting point" [24:12]. Jake refers to the absence of an "earth" as a grounding frame of reference for the cursors' vertical motion.

Of the 11 students who encountered the rich images first, 4 began to speak about the interval between the hands still during the rich condition, however these students did not elaborate about the line between the hands, and rather focused on each hand as a separate entity (e.g., stating that one hand controls color and the other controls brightness). During the second phase, in which they encountered the stark images, 2 of these 4 students as well as 3 of the 7 who had not attended to the interval demonstrated the emergence of this attentional anchor. The remaining students treated each of the two icons as separate entities throughout the entire interview, and hardly spoke about the interval between the hands. Collectively, these students were more inclined to treat the two objects on the screen as separate entities, focusing on the changing height of each object and the differing speeds of the two objects as they move upward.

In summary, participants who began in the stark condition oriented toward the distance between their hands as their attentional anchor, whereas participants who began in the iconic condition tended to treat the manipulatives as independent, untethered entities. It would appear that participants who began in the stark condition generated the interval as their attentional anchor because no other frame of reference was cued. Participants who began in the iconic condition, on the other hand, followed the cued narrative implicit to the familiar images and tended rather to visualize the two balloons as launching up from the ground.

It thus appears that objects bearing rich associative content introduce a new layer of baggage onto an interaction task, including forms, dynamics, hierarchies, and social conventions that guide the students' perception of the action space (on framing, see Fillmore, 1968; Fillmore & Atkins, 1992).

For instance, we typically think of hot-air balloons as "starting" at a point, such as the ground at takeoff, and these evoked frames implicitly constrain the scope of possible attentional orientations to a situation by privileging the interval from each object down to the bottom of the screen, at the expense of the interval between the hands. In contrast, when manipulating stark cursors, there is no "starting point" as such, enabling the possibility for students to attend to the interval between the hands.

Supporting our study's hypothesis, the findings suggest an effect of situatedness on the construction of sensorimotor schemes. The finding is relevant to mathematics pedagogy, because sensorimotor schemes mediate conceptual learning. It follows that situatedness of instructional materials is liable to impede mathematical learning by precluding the emergence of sensorimotor schemes pertinent to a cognitive sequence toward the generalization of rules. Future iterations of this intervention would avail of eye-tracking (e.g., Abrahamson et al., 2016) to corroborate students' verbal and deictic report of attentional anchors.

Conclusion

Mathematics education researchers have long debated the question of whether concepts best develop from rich or stark learning materials. We contributed to the debate by offering that the focus of such research should be not on the learning materials per se but on the sensorimotor schemes they may afford (cf. Day, Motz, & Goldstone, 2015, for a competing position). Richer materials, we demonstrated, constrain the scope of sensorimotor schemes students may develop through engaging with the materials. In particular, richer materials may diminish opportunities for conceptual development, because they draw students' attention toward less mathematically relevant ways of thinking about the situations. Students might even miss out on opportunities to think about the situation in ways that are critical for an instructional sequence.

Students, that is to say children, are highly imaginative. They readily engage in pretense with stark objects, visualizing them one way and then another way. It is the *low* situativity of stark manipulatives that lends them to a greater variety of narratives and consequently a greater variety of sensorimotor orientations. And so we agree with Uttal, Scudder, and DeLoache (1997) that sensory richness of manipulatives may derail certain forms of mathematics learning. But we stress that the issue here is not so much about sensory overload distracting from intended forms of engaging the objects. It is not about manipulatives but about *manipulation*—it is about task-oriented sensorimotor schemes students should develop in solving challenging bimanual motor-action problems. So the issue at hand is the hands' motion. Learning is moving in new ways, and we should ensure that the tasks we create facilitate this motion. Sometimes the objects children manipulate might be so perceptually stark that there are no objects at all—just imagined objects. One might speak of mathematics students' right to bare arms.

Acknowledgment

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A FRAMEWORK FOR EXAMINING HOW MATHEMATICS TEACHERS EVALUATE TECHNOLOGY

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Our mathematics cognitive technology noticing framework is based on professional noticing and curricular noticing frameworks and data collected in a study that explored how secondary mathematics teachers evaluate technology. Our participants displayed three categories of noticing: attention to features of technology, interpretation of the features, and response to these features. We believe this framework can potentially allow researchers to document the evolution of teachers' evaluation of mathematical cognitive technologies and examine how the inclusion or exclusion of particular features of technology influences their evaluation and subsequent selection of technology.

Keywords: Technology, Curriculum Analysis

As technology is an essential tool for learning mathematics in the 21st century (NCTM, 2014) and is becoming more ubiquitous in the classroom, mathematics teachers are being asked to evaluate, select and implement quality technological tools to use with their students. However, little is known about how teachers select technology, particularly mathematical cognitive technologies (MCTs), digital technological tools that can be used to teach and learn mathematical concepts "that [help] transcend the limitations of the mind (e.g., attention to goals, short-term memory span) in thinking, learning, and problem-solving activities" (Pea, 1987, p. 91). The purpose of this paper is to propose a framework for thinking about what teachers notice when they evaluate, select, and modify technology to use in their mathematics classrooms.

Background

As an everyday term, *noticing* is used to refer to particular observations that one makes (Sherin, Jacobs, & Philipp, 2011). In the field of mathematics education, researchers specifically use the phrase *teacher noticing* to refer to what teachers attend to in a classroom situation and how they make sense of what is observed using their knowledge of mathematics teaching and learning and the context being observed (Blomberg, Stürmer, & Seidel, 2011; Dreher & Kuntze, 2015; van Es & Sherin, 2002). Jacobs, Lamb, and Philipp (2010) and McDuffie et al. (2013) extended previous views of noticing to include instructional responses teachers make. Jacobs et al. conceptualized *professional noticing of children's mathematical thinking* as three interrelated and cyclic components: attending to students' thinking, interpreting their thinking, and deciding how to respond based on their understanding. Researchers in mathematics education extended Jacobs et al.'s noticing framework to connect to equity (Wager, 2014) and curricular materials (Males, Earnest, Dietiker, & Amador, 2015).

By drawing upon constructs of the study of Jacobs et al. (2010) for professional noticing of children's mathematical thinking, Males et al. (2015) developed the *curricular noticing* framework to provide support for examining what teachers notice in curriculum materials, how they make sense of what they attend to, and what actions they make based on their observation of curriculum materials. Providing findings of four exploratory studies by prospective elementary and secondary teachers, Males and colleagues demonstrated that mathematics method courses could support prospective teachers in developing noticing skills of curriculum materials. They claimed that the curricular noticing framework might provide a lens for describing the mechanisms for teacher decision-making with curricular materials.

Once considered an add-on to the curriculum, technology has a strong presence in many of

today's mathematics classrooms and teachers are being asked to regularly evaluate technology and integrate it into the curriculum. Previous research on teachers' evaluation of technology to teach mathematics has focused on the criteria created and analysis performed by prospective elementary teachers (e.g., Battey, Kafai, & Franke, 2005). Battey et al. (2005) studied prospective elementary teachers' criteria for evaluating rational number software. The researchers found the prospective teachers' criteria focused more on surface features of the software (e.g., clarity of directions, clear visual presentation, and ease of use) than on specific mathematics content or how technology could support students' learning. Johnston and Suh (2009) studied prospective elementary teachers' planning for mathematics instruction with technology. They found that only a few of their participants selected MCTs to use in their lesson plans and they considered technology to be a beneficial tool for developing students' conceptual learning or providing a visual representation. The prospective elementary teachers who selected review games or non-MCTs (e.g., digital cameras or Smart Boards) indicated that a benefit of using technology is that it is a fun, engaging, or motivating tool. The studies by Johnston and Suh and Battey et al. indicated that, in general, prospective elementary teachers seem to select and use technology based on student engagement, surface features of the software, and motivation rather than developing students' understanding of mathematical content. Although these studies provide evidence of the criteria prospective elementary teaches use to select technology and what the teachers believe to be the main benefits of using technology, the authors of the studies did not provide much insight into how the teachers actually evaluated these technologies to teach mathematics. In particular, what features did the teachers attend to? How did they interpret these features? What were their responses? We believe our framework builds upon and extends the actions that other researchers have identified in a way that specifically characterizes what teachers notice when evaluating MCTs as an important subset of their curricular noticing.

The Mathematical Cognitive Technology Noticing Framework

The framework is based heavily on the work of Jacobs et al. (2010) in professional noticing and the work of Males et al. (2015) in curricular noticing. In both works, the authors focus on three categories of teachers' action: Attending, Interpreting, and Responding. If we can agree the evaluating technology is a subset of curricular noticing, then it seems natural to use these same categories. However, the codes would be slightly different from those proposed by Males et al. because we focus specifically on the evaluation of MCTs while they provide a more general framework on curricular noticing. To develop the codes, we observed and analyzed four trios of teachers as they evaluated and compared four MCTs to use to teach the triangle inequality theorem to eighth grade students. Each of the four MCTs was an online pre-constructed dynamic geometry sketch. In the transcript, we noted when the teachers seemed to be attending, interpreting, and responding and created codes based on their words and actions within each of these categories. As we continued our analysis, we modified and collapsed codes in order to have meaningful codes and categories. The three categories of noticing, along with the codes of each, constitute the mathematical cognitive technology noticing framework (see Table 1).

Table 1: Mathematical Cognitive Technology Noticing Framework

	Attending		Interpreting		Responding
Interaction	Attends to the ways in which one can engage with the MCT	*Student Thinking*	Interprets how students' uses of the MCT may influence their thinking	*Choose*	Considers whether he/she would select the MCT to use in his/her classroom
Supportive Features	Attends to features that assist in operating the MCT or provide help with the content	*Student Engagement*	Interprets whether the MCT enhances or distracts from students' learning	*Redesign*	Considers how he/she may change the MCT to better fit his/her instruction
Instructions and Questions	Attends to questions or instructions included in the MCT or website	*Value*	Interprets whether a feature of the MCT is beneficial or a drawback	*Adapt*	Considers how he/she would adjust the given activity but not changing the MCT
Mathematical Features	Attends to the mathematical ideas displayed on the screen	*Design*	Interprets how the layout of the MCT influences student interaction and learning	*Incorporate*	Considers how the MCT could be included as part of the activities in the classroom without modification
Aesthetics	Attends to the layout of the MCT	*Differentiation*	Interprets whether all students would be able to use the MCT productively		
		Mathematics	Interprets the display and ways students interact with mathematical objects		

As an example, consider the following analysis performed by Mary, Abby, and Bob (all names are pseudonyms) who had earned undergraduate degrees in secondary mathematics education and were pursuing Masters degrees in mathematics education. All were currently in their first year of teaching mathematics at a middle school or a high school. In this example, Mary, Abby, and Bob were evaluating an internet-based pre-constructed sketch (Garrison, n.d.) that students could use to develop the triangle inequality theorem (see Figure 1). The website provides brief directions on how to use the applet and a question asks users to generalize a theorem based on their experiences using the applet. Two sets of objects are displayed at all times: a set of sliders (pink, green, and yellow) and a set of segments whose lengths and colors are linked to the corresponding sliders. Users are able to change the length of segments by using sliders that range from 0 to 10 with 0.5 increments. The applet's appearance is very different depending on whether a triangle can be formed with a given set of segment lengths. When a triangle can be formed, two additional objects appear: a smiley face and a tan triangle (see Figure 1a). The tan triangle shows the user where the pink and green segments would need to be dragged in order for the triangle to be created. The user can drag each endpoint of

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the two segments (pink and green) to make the triangle. The tan triangle also has markings of the interior angles of the triangle which are named angles alpha, beta, and gamma. However, the measures of these angles are not displayed and the user cannot measure them. If the conditions for a triangle are not satisfied, the applet does not make the tan triangle and the smiley face is not displayed (see Figure 1b). Users can drag each endpoint of the two segments to check the fact that a triangle cannot be formed. The applet's website does not provide any further assistance or directions other than the name of the applet's creator and a link to the Geogebra website.
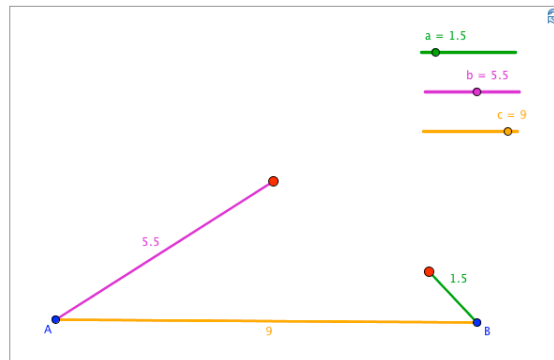
(a)   (b)

Figure 1: MCT Analyzed by Mary, Abby, and Bob when the length of the segments do (a) and do not (b) form a triangle.

Mary, Bob, and Abby spent over 12 minutes evaluating this applet. Given the space limitations, it is impossible for us to provide the transcript in its entirety. Instead, we provide the following narrative with direct quotations used as much as possible. Opening the website, Abby read the instructions aloud and Bob pointed out that the triangle is present and the angles are marked with Greek letters. Bob briefly dragged the sliders and said, "It gives you the awesome smiley face. It would be better if it gave you a frowny face if it didn't work." As they determined whether the applet engages students in meaningful ways, Abby said, "it's pretty straightforward, except for the smiley face and we'll just drag {it}." Bob responded, "Well it's unnecessary that it [instruction] tells you to move the line segments, since it already creates it, so it's kind of unnecessary." Bob continued to explore the applet and supported his previous claims stating,

> *Bob*: [Dragging segments] See here when they're the same, it does, it shows you if you move your triangle, and then it shows point C as, well, so it shows that. But then there's still no new, I guess, to see that you'd move it here.
> *Mary*: Yeah.
> *Bob*: But with the ones that actually make triangles, there's no need to move anywhere. It's very obvious.
> *Mary*: It's very lame.

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Changing the foci of the conversation to the specific benefits and limitations of the applet, Abby said, "Okay, yes the use of three colors distinguishes the different segments. Um, when a triangle is formed a smiley face appears encouraging students, encouraging students that they achieved a triangle. They create a triangle." Mary stated a benefit is that it provides instructions on how to use the applet. Abby considered whether the angle markings are beneficial. She said, "In terms of the mathematics. I think that in this case with the angle measures, since we're not really concerned with angle measures, it could potentially distract [from the triangle inequality theorem]." She continued, "They [Angle markings] are not, they are not essential for the development of this theorem."

Next, the teachers discussed whether the applet allows students to develop an appropriate conception of the triangle inequality theorem.

Mary: And I guess, this question down here at the bottom does make them [students] think about it [triangle inequality theorem], but unless I require that they submit some kind of answer for that, they're just going to ignore that.
Bob: Mm hmm.
Mary: And they're going to be like, ooh smiley, not smiley.
Bob: Yeah. There's going to be, yeah, they're going to see the smiley face or the no.
Abby: Yeah.
Mary: So we could say, if the questions at the bottom are required, and emphasized,
Abby: then the program
Bob: emphasis is placed on, on the end questions.
Mary: Yeah.
Abby: If the questions on the end are required and emphasized, then the tool does
Mary: could
Abby: could, yeah, provide
Mary: an appropriate thought process.

The teachers also stated that if these questions are ignored, the students will think very little about the theorem.

The teachers then considered the mathematics of the applet focusing on the restrictions on the segments lengths and the types of triangles can be formed. Abby asked, "Does it provide all cases? Yes." Mary responded, "The slider can be manipulated to any length." Bob interjected, Well, between zero and ten." Abby then asked, "Can it be an obtuse angle?" Bob responded, "Yeah, we can have an obtuse. Within. Yeah". As Bob created an obtuse triangle, the teachers discussed the merits of the segment lengths ranging from 0 to 10, specifically the length of a segment being zero. Abby said, "Um, yes the slider can be moved to any length from zero to ten. Students could potentially find a problem with side length of zero." She continued, "However, a discussion could be had about this issue." Mary and Bob agreed.

The teachers discussed whether the applet has a balanced difficulty level; whether all students would be able to use the applet to make appropriate meanings of the applet. Abby asked, "Does it have a balanced difficulty level?" Mary responded, "Yes, it's not too easy." Abby stated "Yes, not too easy not too hard."

When considering which applet they might select to use with students, the teachers compared the affordances and limitations of each of the four online applets.

Bob: If we were going to use an applet, we would use, what was it, the sec, the third one, the one that created the triangle for you. Even though it created the [tan] triangle for you, you could still discover the [triangle] inequality [theorem].
Abby: Overall if we wanted our students to develop the inequality on their own, tool three would allow, um, would, um, would allow

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Mary: for

Abby: freedom?

Mary: freedom in knowing when triangles can and cannot be created. And then possibly talking about why.

Abby: Alright, what'd you, I'm sorry. Okay.

Mary: Uh, knowing when triangles can and cannot be created, then a discussion could be had about why this is.

When asked which applet they would use with eighth grade students, Mary, Bob, and Abby strongly considered selecting this applet but chose to use a different applet.

Examples and Analysis

In the subsequent paragraphs, we provide examples of each of the framework's codes using Bob, Mary, and Abby's evaluation of the online applet and a brief analysis of their evaluation using the framework.

Attending

When evaluating the technology, teachers attend to particular features of the technology. We categorized these features into five codes (see Table 1). Upon opening the website, Abby attended to the instruction included in the applet by reading it aloud, and Bob continued to attend to it by saying, "Well it's unnecessary that it [instruction] tells you to move the line segments..." Mary, Bob, and Abby also attended to a given question. Abby said, "And I guess, this question down here at the bottom does make them [students] think about it [triangle inequality theorem]... " We coded these actions of attention as *instructions and questions* because the focus was on these particular features. At times, Mary, Bob, and Abby attended to *mathematical features* displayed in the applet such as angles and side lengths of the triangle. For example, Abby said, "In terms of the mathematics. I think that in this case with the angle measures... " Abby also attended to *aesthetics* when she noticed segments' color of the triangle by stating, "Okay, yes the use of three colors..." In addition, the teachers would attend to *interaction* by dragging or clicking as they explored the abilities and limitations of the applet. For example, dragging segments in the applet, Bob said, "See here when they're the same, it does, it shows you if you move your triangle, and then it shows point C as, well, so it shows that..." Attending to the *interaction* is unique to this particular framework because in other noticing studies, the analysis of teacher attending was mostly based on the teachers' verbal or written comments (e.g., Jacobs et al., 2010; Wager, 2014). In their evaluation, the teachers did not attend to *supportive features*, which is likely due to the fact that these features were not present in the applet. Other applets they evaluated included these kinds of features (e.g., a video demonstration) and the trio did attend to those features when they were present. While evaluating this particular applet, other trios did take note that the applet lacked supportive features.

Interpreting

When teachers provide their thoughts and ideas about features of the technology and the technology as a whole, we refer to these actions and statements as *Interpreting*. In this category, teachers reflect on their own uses of technology, consider how students may use the technology and anticipate how that affects students' learning of key mathematical concepts. Our analysis generated 6 codes for interpreting (see Table 1). Abby interpreted *student engagement*, when she said the angle markings could distract students from learning the theorem "because they are not essential to the development of the theorem." She wanted students to be engaged in learning the triangle inequality theorem and thought the angle markings may distract students from staying focused on the goal of the activity. Often Mary, Abby, and Bob would interpret whether a feature had *value*. When Bob realized there was no reason to move the segments because the tan triangle appeared, Mary said, "It's

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very lame." Mary was interpreting the value of the tan triangle or, in this case, the lack thereof. There were times when Mary, Abby, and Bob interpreted the *mathematics* of the technology including the display of mathematics and how students could interact with mathematical ideas. For example, Abby noted the lengths of the segments could range from 0 to 10 and "students could potentially find a problem with side length of zero." Abby was anticipating students' difficulty with this mathematical issue. In one instance, Mary and Abby interpreted whether the technology could be used for *differentiation*, when they were attempting to determine whether the technology had "a balanced difficulty level". Abby also interpreted the technology's *design* when she noted the use of color and the smiley face as features that could positively influence student learning. Finally, Mary, Abby, and Bob interpreted *student thinking* when Abby and Mary discussed whether they would select this applet. Abby said, "Overall if we wanted our students to develop the inequality on their own, tool three would allow, um, would, um, would allow…" Mary continued, "freedom in knowing when triangles can and cannot be created." Abby and Mary considered how students could develop conjectures related to theorem based on their uses of the tool.

Responding

Based on their interpretation about features of the technology, teachers make possible or potential curricular decisions. We refer to these actions as *Responding*. In this phase, we defined 4 codes for the activities in which teachers might engage: Choose, Incorporate, Redesign, and Adapt (see Table 1). When comparing four applets, Bob argued he would *choose* this applet to use with eighth grade students. He said, "If we were going to use an applet, we would use, what was it, the sec, the third one, the one that created the triangle for you. Even though it created the [tan] triangle for you, you could still discover the [triangle] inequality [theorem]." When playing with the applet, Mary, Abby, and Bob found that a smiley face appears when the segments formed a triangle. Bob said, "It would be better if it gave you a frowny face if it didn't work." We coded his idea as *Redesign* because Bob considered how he could change the tool itself. When Mary, Abby, and Bob discussed whether the applet helps students develop an appropriate mathematical conception of the triangle inequality theorem, Mary said teachers need to emphasize the bottom questions in order to facilitate students' deeper understanding of the theorem. Mary's decision to emphasize the question did not change the applet itself; rather she was attempting to *adapt* it by placing greater emphasis on the question in order to best meet the needs of her students and meet her goals as the teacher. Finally, when Abby discussed the feature that allows students to adjust the sliders such that lengths could be zero, Abby considered how she could *incorporate* the applet into her teaching. Abby said, "Students could potentially find a problem with side length of zero." She continued, "However, a discussion could be had about this issue." By including a discussion, Abby did not want to redesign the applet or adapt particular features to meet her or her students' needs. Rather, she wanted to incorporate the applet as it stands, but she recognized the need to have a discussion about this particular feature.

Analysis

Using the framework in our analysis of Mary, Abby, and Bob's evaluation of the applet, we noticed that the trio attended mostly to mathematical features, instructions and questions, and how they interacted with the applet. They seemed to initially focus on the features the stood out the most (e.g., the smiley face and the tan triangle) but progressed to less noticeable features such as the range of the segment lengths. The teachers' initial interpretations were the design of the applet in which they focus on how the layout and features of the tool would influence students' interaction and learning. Their interpretations evolved into anticipating and interpreting how students would think when engaging with the applet. Finally, when we coded a response as adapt or incorporate, we noticed that we had also coded the teachers' interpretations as student thinking. It seems when the

teachers were considering how to use the applet in their own classroom, they were taking into account how students may think when using the technology.

Conclusion

We propose this framework as a device that allows researchers to examine how teachers evaluate and select MCTs to use in their classroom. In our evaluation of our data, we have been able to use the framework to document the evolution of teachers' evaluation beginning with their initial attentions and ending with the selection, rejection, or modification of the tool. We believe this framework can potentially allow researchers to examine how the inclusion or exclusion of particular features of technology influences teachers' evaluation and subsequent selection of technology. By conducting such an analysis with this framework, mathematics education researchers would be able to provide a clearer understanding of the processes teachers employ to select MCTs to use with their students.

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EXPLORING STUDENTS' UNDERSTANDING OF THE LIMIT OF A SEQUENCE THROUGH DIGITAL AND PHYSICAL MODALITIES

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The purpose of this study was to investigate how a set of physical and digital instructional activities can serve as an example space to help further develop a concept image that is aligned with the formal concept definition for the limit of a sequence. In addition, the unique affordances and constraints allowed by using either a physical or digital modality in understanding the convergence of a sequence was analyzed. Results suggest that both of these activities served to help students conceptualize the arbitrary nature of the error bound, and for some students it further illustrated the relationship between an arbitrarily small error bound, the limit value, and the index of the sequence. The physical activity constrained students to think of the sequence as a finite terminating set of numbers whereas the digital activity provided additional information that student used in subsequent problem solving.

Keywords: Technology, Post-Secondary Education, Instructional Activities and Practices

Students' misconceptions of limits and infinity are a well-documented and researched area in math education literature (e.g., Cory & Garofalo, 2011; Roh, 2008; Tall & Vinner, 1981). Students often have difficulty advancing from an intuitive dynamic conception of limit to the more formal limit conception, and often fail to understand the relationship between a given epsilon error range and the index of the sequence (Roh, 2010; Tall & Vinner, 1981). Students may resort to memorization tactics and often confuse the formal definition of the limit of a sequence with the limit of a function or other limit ideas.

Roh (2010) developed instructional materials that utilize visualization to support student understanding of the formal definition of the limit of a sequence based on the vast amount of misconception literature on limits and calls for the increased role of visualization in calculus (Dreyfus, 1990). Using an instructional activity referred to as the epsilon-strip task, Roh was able to categorize and support student understanding of the logical structure between epsilon and the index of a sequence. Roh has also shown how students' understanding of the definition of the limit of a sequence are influenced by their images of limits as asymptotes, cluster points, or true limit points using the same instructional activities (Roh, 2008).

This research project was driven by the conjecture that a technologically enhanced epsilon-strip instructional activity would make the instructional materials more accessible and promote deeper engagement by the user. Research shows that technology can be utilized to help students make sense of calculus concepts by using multiple representations (Tall, 1994) and that when technology uses dynamic versus static visualizations there is an overall benefit for student learning (Hoffler & Leutner, 2007). In a recent study with preservice teachers, dynamic sketches for sequence convergence were utilized to help strengthen their understandings of formal limit ideas as they integrated the visual representation with the symbolic definition (Cory & Garofalo, 2011). In this study, I seek to address how physical and digital epsilon-strip activities relate to students' understanding of limits of sequences. Furthermore, I investigate what constraints and affordances are provided in using either the physical or the digital epsilon-strip activity.

Theoretical Framework

Tall and Vinner (1981) examined student understanding of limits and developed the constructs of *concept image* and *concept definition* to describe the cognitive process of learning mathematics.

Concept image refers to the "total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes," whereas the *formal concept definition* is the form of words which are accepted by the mathematical community to describe the concept (Tall & Vinner, 1981). Often times a student's *evoked concept image*, which is the image generated given a certain context or prompt, may not be globally coherent and may deviate considerably from the formal concept definition. In the context of this study the formal concept definition for the limit of a sequence is referred to as the ε-N definition presented in Figure 1.

$\epsilon - N\ definition$: A sequence $\{a_n\}_{n=1}^{\infty}$ convergeges to L if,
for any $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that for all $n > N, |a_n - L| < \epsilon$

Figure 1. Definition for the limit of a sequence referred to as the ε-N definition.

In addition to drawing on the concept image and concept definition framing, in this study I utilizes gesture theory from the field of embodied cognition as a means for illustrating student understanding through non-verbal communication. Preliminary research suggests that using dynamic visual environments can evoke metaphoric gestures to convey the temporal relationship in calculus concepts and are an essential element to effective mathematical communication (Ng, in press; Núñez, 2004). In this study I seek to answer the following research questions: In what ways do these instructional activities produce an evoked concepts image consistent with the concept definition for the limit of a sequence? Are there unique affordances or constraints in using either the digital versus the physical instructional activity?

Methods

The participants in this study were three undergraduate mathematics majors recruited from a Vector Calculus course at a 4-year public research university. At the time of the study, each student had completed a sequence of single-variable calculus and a course in analytic geometry and multivariable calculus. Additionally, each had previously been exposed to the definition of limit of a sequence and the concepts of convergence and divergence. The 45-60 minute semi-structured, individual interviews were video recorded and interview tasks were based on a modified sequence of instructional activities drawn from Roh (2010). Students were told they would explore a concept in calculus in a non-evaluative problem solving interview and were encouraged to think aloud as much as possible as they worked on each of the tasks.

In order to gain insight into each student's initial evoked concept image for the limit of sequence they were asked about their prior experience with limits of sequences and how they thought about of the convergence of a sequence. To further explore student conceptions of convergence they were asked to represent the sequence $\left\{\frac{n^3}{2^n}\right\}_{n=1}^{\infty}$ numerically in a table and prompted to determine what would happen to the sequence as the index *n* gets larger. After assessing students reasoning and prior background regarding the convergence of a sequence they were presented with the following digital and physical epsilon-strip activities.

The digital epsilon-strip activities were created with the design heuristic principles of manipulation of content and guided discovery (Plass, Homer, & Hayward, 2009). The *manipulation of content* principle asserts that student learning is improved if the learner is able to manipulate the content of a dynamic visualization and the *guided discovery* principle suggests students learn better when guidance is used in discovery-based learning in multimedia contexts. Using these principles I created digital activities using Desmos[©] online graphing utility as shown in Figure 2. A link to the resources and graphs used in this study are available at the following web address (goo.gl/Bm31eD). Students were given a brief tutorial explaining the onscreen graphs and the epsilon-strip. In

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particular, they were shown how to manipulate where the epsilon-strip was centered and how to adjust its width through the use of sliders, which provide dynamic changes of the values, or by manually entering a desired value. Based on the principle of guided discovery, students were given an activity sheet that instructed them where to center the epsilon-strip, and had them iterate through three decreasing values for the width of the epsilon-strip. For each given epsilon width, students filled out a table where they counted the number of terms inside the epsilon-strip and the number of terms outside the epsilon-strip, which is referred to as the *counting process*. Student were then asked to repeat the same procedure and counting process but with the epsilon-strip centered at a different value. There were a total of three graphs explored, each selected to illustrate a different type of sequence convergence or divergence: monotonically convergent $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$, oscillating convergent $\{(-.75)^n\}_{n=1}^{\infty}$, and oscillating divergent $\{(-1)^n\}_{n=1}^{\infty}$.
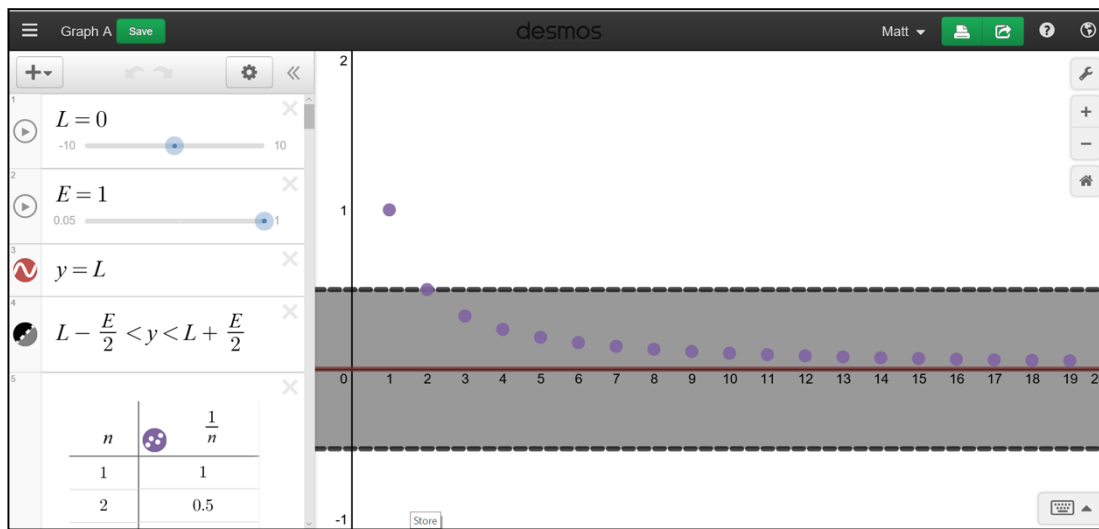
Figure 2. Digital epsilon-strip graph for monotonically convergent sequence.

The physical activity consisted of printed graphs of a sequence and strips of rectangular transparencies that represented the error bounds of a limit of a sequence, as shown in Figure 3. In accordance with the guidelines presented in Roh (2008) the strips had a constant width and each were marked through the center with a red line. The same procedure and counting process as described for the digital activity were used with the physical activity. There were a total of three graphs explored with this exact process: monotonically convergent $\begin{cases} \frac{1}{n}, & n \le 10 \\ .1, & n > 10 \end{cases}$, oscillating convergent $\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$, and oscillating divergent $\left\{(-1)^n \left(1+\frac{1}{n}\right)\right\}_{n=1}^{\infty}$.

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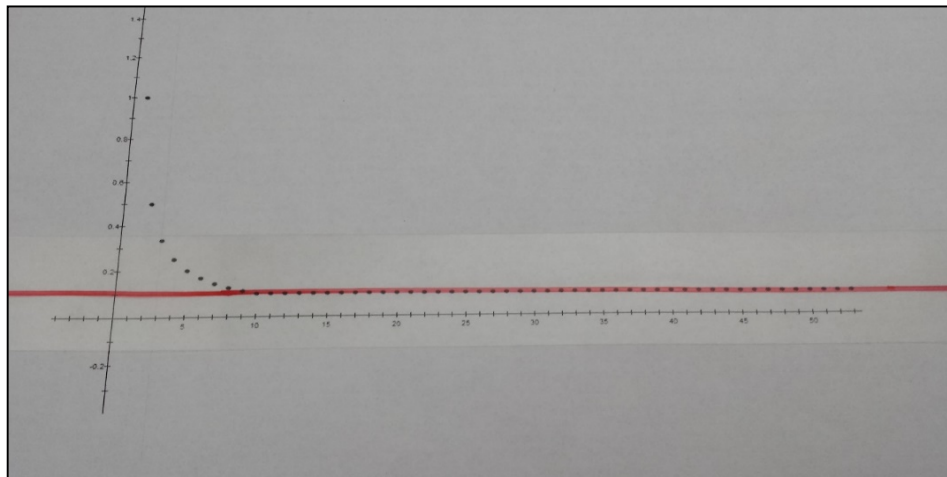
Figure 3. Physical epsilon-strip activity for monotonically convergent sequence.

After the digital and physical epsilon-strip activities, students were asked to evaluate the validity of two "student" generated definitions for the limit of a sequence, called ε-strip definition A and the ε-strip definition B:

- ε-strip definition A**:** *L* is a limit of a sequence when infinitely many points on the graph of the sequence are covered by any ε-strip as long as the ε-strip is centered a *L.*
- ε-strip definition B: *L* is a limit of a sequence when only finitely many points on the graph of the sequence are NOT covered by any ε-strip as long as the ε-strip is centered a *L.*

After evaluating these definitions students were given the formal ε-N definition and asked to explain each of the components of the definition and how they made sense of this definition. In the last part of the interview students were asked to explain how they saw the formal definition in relation to the ε-strip activities.

Results

Students' evolving concept image

Pedro's (all names are pseudonyms) initial concept image for the limit of a sequence was based on discrete representations, pattern recognition, and procedural computations. He recalled computing values for the limit of a sequence in prior mathematical classes, but was unable to provide a definition. When he was asked about the end behavior of the sequence $\left\{\frac{n^3}{2^n}\right\}$ he said it would depend on the relationship between the numerator and denominator. He stated that he would need to "plug in numbers and see the pattern" of values and then visualize those values to determine the limit. Pedro appeared to draw on a discrete representation of the sequence and pattern recognition to determine what the sequence was approaching as the index increased.

After the completion of the epsilon-strip activities Pedro was able to articulate several additional components related to his concept image for the limit of a sequence. He stated expressly the importance of where the epsilon-strip was centered, L, as crucial in determining if the sequence had a limit. He then linked the dependence on L with the idea that epsilon was allowed to vary and that for any epsilon there could only be finitely many terms outside of the strip. Although he didn't formalize the directional relationship between any given epsilon and the existence of some index *n*, he did articulate a relationship between the arbitrary epsilon and the finite number of terms outside

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the epsilon error range, and used this reasoning in rejecting ε-strip definition A. In describing the formal definition, he used the physical transparency showing how the sequence will be contained using gesture to show a motion like pattern in the strip, and then pointed to the digital epsilon-strip activity to show that it has to be bounded in that range. Although the activities elicited further components related to the limit of sequences, he was still unable to explain how definition B was valid, or articulate how each of the components in the formal definition related to the limit of a sequence.

Nick's initial concept image of the limit of a sequence was the most robust of all three students. Nick initially drew on the limit as an asymptote concept and descried convergence as "it hitting a particular number as the index n approaches infinity." When computing the limit of the sequence $\left\{\frac{n^3}{2^n}\right\}$, Nick used a graphing calculator to plot the sequence as a continuous function and used the visualization to assert that the function was converging to zero. Nick's initial evoked concept image of the limit of sequence drew heavily from ideas of continuous functions and asymptotes versus the discrete representation conception evoked by Pedro.

After the completion of the epsilon-strip activities Nick's evoked concept image was similar to Pedro's yet contained further refinement regarding the relationship of the index n in the formal definition. Nick expressed that epsilon was "arbitrarily small" and you could, "pick any one" that you wanted, and that the limit value L was crucial in determining if the sequence converged to that particular value. Nick was the only student to explain the importance of the index n after picking an epsilon-strip, explaining that n is important because "we can always find an $n+1$ such that it is inside our strip." Nick first rejected both definition A and definition B, using an oscillating divergent sequence as a counterexample to definition A. Nick further argued that definition B was an invalid definition because it was possible to have a finite terminating sequence and thus there would be finite number inside and a finite number outside the epsilon-strip. When invited to consider an infinite sequence, Nick utilized the physical graphs, and then stated that definition B would be valid.

Heng's initial evoked concept image for the limit of a sequence included categorization and computational techniques. When discussing the limit of a sequence he drew on ideas of divergence as infinity and convergence as "related to some sort of range" of values. When computing the limit of the particular sequence$\left\{\frac{n^3}{2^n}\right\}$, Heng said it would go to infinity because he used the ratio tests to compare the "values of a_{n+1} and a_n." However, since he only compared the first two values of the sequence he failed to see that the sequence increases for the first few terms and then converges to zero.

During the physical epsilon-strip activity Heng viewed the oscillating convergent sequence as divergent since it alternated between positive and negative values. However, when working through the digital epsilon-strip activity he stated that the oscillating convergent sequence did converge since it was getting closer to the value of zero. Heng had one of the least developed concept images prior to the activity, and in addition expressed difficulty as an English language learner in engaging with the definition-provoking-activities that followed the digital and physical epsilon-strip activities. Heng initially asserted that definition A and B were correct, but after being asked to explain them in relation to the oscillating divergent sequence, he stated that A didn't fulfill the requirement and thus we "need both definitions for the limit" to exist. He did articulate that the epsilon-strip in the formal definition was arbitrary and that for "anyone that I pick" we can bound the sequence. He did not express the epsilon as arbitrarily small, nor was he able to express a relationship between epsilon and the index n. Heng, although acknowledging that they were different, wrote out the formal delta epsilon definition for continuous functions to try and make sense of the ε-N definition for convergence.

Affordances and Constraints

In addition to examining how the overall activities helped the development of student concept images related to the limit of a sequence, I analyzed the unique affordances and constraints provided by either using the digital or the physical modality for the epsilon-strip activity. During this study there were two prevalent constraints observed in using the physical modality related to the finite nature and imprecision of the epsilon-strips. Due to the nature of the static paper, both Pedro and Heng conceived of the graphs as a terminating sequence with only a finite number of terms and therefore had to be instructed to think of them as infinitely long. Although Nick did not evoke this misconception, he did use the example of a terminating sequence as a counter example to definition B, arguing that you could have finitely many inside and outside the strip. The imprecision of the epsilon-strips also lead each of the students to have questions about the ambiguity of terms lying on the epsilon-strip during the paper activity. This could have been a result of human cutting error regarding strip width uniformity or difficulties with the physical act of aligning and using strips. I speculate that the latter is more probable, since the strips were measured and cut using a guillotine style paper cutter and were observably uniform.

The digital modality offered several unique interactions that were not observed during the physical epsilon activity, including students using ancillary information provided by the digital modality and an increase in the use of gestures. Both Heng and Nick used the ancillary information that was located in the graphing utility on the left hand panel to reason and complete the activities. Heng, when working on the digital activities, first computed the epsilon error range given the width and center, and then used this range of numbers to compare against the table of sequence values in order to determine which terms were outside and inside the strip. Heng used this information to correctly identify the convergence for each of the digital activities, yet he failed to do so for the physical oscillating convergent sequence. Although this strategy was accurate, it is still unclear how this may have impacted his visualization of limits as it relates to the formal definition. Nick also drew on the additional information and argued that since the function was programmed in as $\{1/n\}$, he knew from prior experience that this sequence would converge to zero. It should be noted that the same ancillary information could have been included in the physical graphs, but they were an automatic result of using the digital technology.

Both Pedro and Nick increased their use of gestures when interacting with the digital modality and during the definition activity, as illustrated in Figure 4. I attended to the use of gestures in this study since the prior research from Roh's studies have failed to address the role of gesture in mathematical learning. Moreover, using the dynamic digital modality one might expect an increase in the use of gestures because the visual movement parallels the movement of the gesture. Pedro utilized his thumb and index finger in a collapsing motion when highlighting the arbitrarily small nature of epsilon while using the digital modality. Nick used a similar gesture when talking about the arbitrary nature of epsilon but used his thumb and all of his remaining fingers to show a collapsing dynamic motion. While neither Nick nor Pedro explicitly stated the concept that epsilon values tend toward zero, their gestures indicate an understanding of this very relationship. Nick also utilized his left forearm and right hand in an up and down motion when discussing the importance of where the epsilon-strip was centered in determining the limit while interacting with the physical modality, suggesting that the tangible movement of the strip was more salient given the physical context.

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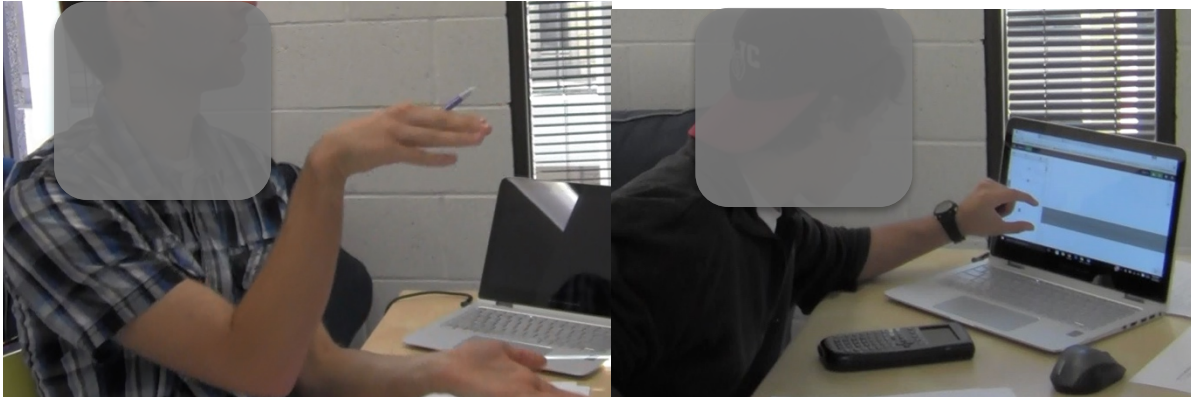
Figure 4. Nick (Left) and Pedro (Right) using gestures during the activities.

Conclusion

This study was driven by a desire to take existing innovative instructional materials and utilize technology in a controlled setting to ascertain how they may jointly help develop students' concepts images for convergence, and furthermore provide insight into the affordances and constraints in using the two different modalities. Both the physical and digital activities helped to illicit a more detailed evoked concept images regarding the limit of a sequence from all three of the participants. All of the students after the activity expressed the limit in relation to the arbitrary nature of the epsilon-strip, and two of the students conceived of epsilon as getting arbitrarily small. Students still had issues explaining the formal ε-N definition of the limit of a sequence, with only one student articulating the importance of the index *n* as it related to the chosen epsilon error range. Although students didn't have a globally aligned concept image with the formal concept definition, each expressed how the nature of these activities helped them think and reason about the formal ε-N definition of the limit of sequence.

In comparing the two types of modalities, it appears that the physical activity seemed to constrain students into thinking of the sequences presented as finite and terminating. In addition there were issues of ambiguity related to when a given term was located inside the epsilon-strip either because they were not cut uniformly or because they were not centered level to the x-axis. The digital activity provided students with a dynamic representation which may have resulted in the increased presence of gestures in describing the definitions. In addition, the ancillary information was used in unique ways during the counting processes in unanticipated ways, such as the formal examination of the table of values. Future research studies may examine each of these activities with a larger range of students and determine if the gestures used by students is a unique affordance of the dynamic nature of the digital representation of the epsilon-strip activity and address the effect of the ancillary information (i.e. table of values or programmed function) on students understanding.

There is clearly more room for the examination of innovative and improved curriculum even though mathematics education literature abounds with research on student misconceptions of limits and infinity. In this study, a technologically enhanced epsilon-strip activity proved beneficial to promote student understanding of the limit of sequence. It provided a dynamic visualization of the epsilon-strip as an arbitrary small error range for a sequence that appeared to continue on ad infinitum. It is the hope that as these instructional materials and technology further develop, they will serve to broaden the example space that students draw upon in order to understand the formal mathematical definition of the limit of a sequence.

Acknowledgments

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TEACHING WITH VIDEOGAMES: AN EXPLORATION OF EXPERIENCE

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Digital games have demonstrated great potential for supporting students' learning across disciplines. But integrating games into instruction is challenging. One factor that contributes to the successful use of games in a classroom is teachers' experience implementing the technologies. We explored this idea by comparing years 1 and 2 of a middle-school mathematics teacher's use of Boone's Meadow, a digital problem-solving game around ratio and proportion, in her classroom. While the two implementations were quite similar, the teacher was able to use students' gameplay time more productively in the second year, both for mathematical engagement and for immersing students in the narrative of the game.

Keywords: Instructional Activities and Practices, Middle School Education, Technology

Introduction

It is difficult to claim at a large scale that technology integration is associated with increased learning (Penuel, 2006). However, digital games in particular have demonstrated potential for supporting student learning across disciplines (Gresalfi & Barnes, 2016; Pareto et al., 2011; Squire, 2006). Much has been said about the potential of games to capture student attention (Dickey, 2007; Garris, Ahlers, & Driskell, 2002; Lepper & Malone, 1987), to situate disciplinary learning in realistic contexts (Barab et al., 2005; Clarke & Dede, 2009), and to offer consistent and substantive feedback (Gresalfi & Barnes, 2016; Mayer & Johnson, 2010; Nelson, 2007).

Despite their potential, integrating digital games into instruction also creates new challenges and requires a shift from current pedagogical practices to those afforded (or demanded) by new technologies (Ertmer, 2005; Straub, 2009). It is not simply a matter of making the tools available (Ertmer et al., 2012; Ertmer, Ottenbreit-Leftwich, & Tondeur, 2014). Instead, like every reform that has the potential to change classroom practice, it is important to develop a model of what successful integration looks like in that particular context. In particular, one factor that contributes to the successful integration of games is teachers' experience with the technology, which affects teacher practice in a number of ways (Ertmer & Ottenbreit-Leftwich, 2010; Mumtaz, 2000; Sheingold & Hadley, 1990). As with any new classroom technology, the more teachers use it, the more they understand how students interact with the technology and what aspects are difficult for students to understand, which can lead to more organized and focused classroom discussions based on students' needs.

This paper focuses on a videogame that was designed to incorporate teacher-student interactions rather than replace instruction. In this context, the teacher's role is central to implementing the game successfully. We explore how one teacher uses the game across two years and how integration of the game changes with her experience with the technology.

The Game

Boone's Meadow is an interactive problem solving experience that involves mathematical ideas of ratio and proportion. In the game, a veterinarian recruits players to help save an injured eagle using an ultralight (a small plane). Players must choose between three different ultralights to use based on fuel efficiency, speed, and payload. Players must also plan the route to pick up the eagle and safely return her to the veterinary clinic in time, choosing whether or not to stop for gas. Players have two attempts to save the eagle, and they can use the second try to find a more optimal solution (taking less time, using less gas, and spending less money) or test another route.

Methods

Setting/Participants

This paper focuses on Ms. Lynn, a 7th grade mathematics teacher at a school with 92% free and reduced lunch and 30% English language learners in a southeastern U.S. city. Ms. Lynn used the game in her classroom for 4 days during the fall of both years 1 and 2.

Data Collection

Data includes videos of gameplay and whole class discussions, and students' pre and post-tests about ratio and proportion. We also interviewed Ms. Lynn informally after the game implementations, and we use her responses to triangulate our findings.

Analysis

The teacher's talk was coded by utterance. We coded the teacher's talk for four different types of mathematical engagement (from Gresalfi & Barab, 2011): (1) procedural – following procedures, (2) conceptual – conceptual explanations of procedures, (3) consequential – examining how procedures relate to outcomes, and (4) critical – questioning why one procedure should be used over another. We also categorized talk as defining terms from the game or narrative immersion (when the teacher makes explicit connections to the story).

Findings

Table 1: Average Pre and Post Test Scores for Years 1 and 2 in Ms. Lynn's Classes

Year	Average Pre Score	Average Post Score	Significance (p value)
1	3.46	6.77	p<0.004
2	11.17	12.39	p<0.04

We first analyzed the pre and post-test changes for Ms. Lynn's classes (Table 1). The year 2 scores start out higher because items were added to the tests. Both years showed significant pre to post-test change, with a larger gain occurring in the first year. However, Ms. Lynn devoted much more time to the implementation in year 2 (262 minutes total of gameplay and discussion time in year 1 versus 404 minutes in year 2), taking less time to transition between classes or talk about other issues unrelated to the content of the game (like field trips).

The number of utterances coded as conceptual, consequential, or critical mathematical engagement was similar in both years, with less than 30 utterances in each of those categories. Most of Ms. Lynn's math talk with her students afforded procedural engagement. While the overall counts are similar, more of the procedural engagement occurred during gameplay (and less in discussion time) in year 2 (see Figure 1). Similarly, the amount of teacher talk involving narrative immersion increased during gameplay time in year 2 (see Figure 2). The amount of talk around defining terms from the game (such as fuel capacity or endangered) also increased.

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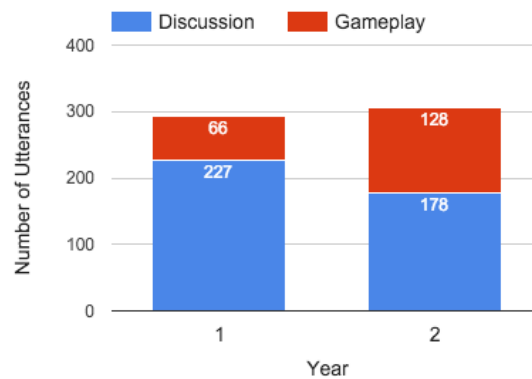
Figure 1. Graph of number of utterances coded as procedural engagement during discussions versus gameplay in years 1 and 2.
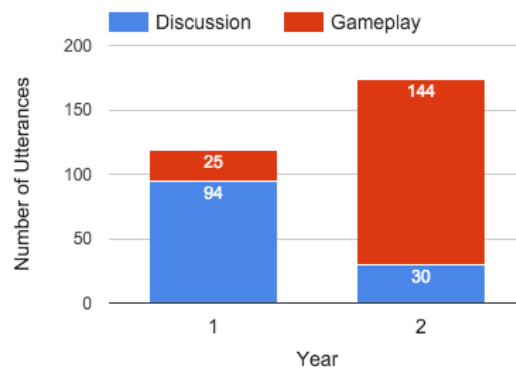
Figure 2. Graph of number of utterances coded as narrative immersion during discussions versus gameplay in years 1 and 2.

Discussion and Conclusions

In this paper, we examined changes in Ms. Lynn's math and game talk during discussion and gameplay time in her first two years of implementing Boone's Meadow. In our informal interview with Ms. Lynn after year 2, she reported her that the second year went much better, specifically because students were much more engaged in the game and the mathematics in year 2. "That's one thing that I liked this year versus last year…they would come in and they were working and trying to understand and very rarely did they need an adult."

The biggest differences in teacher talk between years 1 and 2 can be seen in the context of the talk, that is, whether it occurred during class discussions or gameplay time. Much more of the mathematical discourse took place while students were actually playing the game in year 2. Similarly, Ms. Lynn engaged her students in much more narrative immersion during gameplay in year 2. This might be due to her increased familiarity with the game or with her increased value of the narrative, which she felt created an important context for her students' mathematical thinking. We believe this shift in the context of discourse reflects Ms. Lynn's increased experience with integrating the technology into her classroom.

The case we examined in this paper was unique in a number of ways. First, the students in Ms. Lynn's class were far below grade level in their mathematics achievement; most students had very little multiplicative reasoning. This might help to account for the time spent on procedural engagement. However, it is interesting that the teacher saved her procedural mathematics talk for

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gameplay times, rather than "pre-teaching" before game play commenced. Second, the teacher was strongly committed to providing her students with an opportunity to engage in mathematical problem solving. She saw the game as an important and unusual opportunity for her students, despite comments she received from the mathematics coordinator that the game had caused her class to fall behind schedule: "…to me, I feel like the experience is so valuable, that it is worth the time, and we will skip something else that is less valuable."

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STUDENTS' USE OF A COMPUTER PROGRAMMING ENVIRONMENT TO REPRESENT DISTANCE AS A FUNCTION OF TIME

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I used the framework of Realistic Mathematics Education to examine students' reinvention of representations of distance through the use of a visual programming environment. With the resources of the environment, most students initially created pictoral or pseudo-symbolic representations of the distance traveled by a moving car. Although students struggled to create graphs, their use of variables foreshadowed connections between the motion of the car and a graphical representation. This study is part of a design experiment aimed at privileging students as active creators of mathematics through the use of technology, while also scaffolding connections between different representations of function in algebra.

Keywords: Algebra and Algebraic Thinking, Technology, Middle School Education

Education within STEM fields often reveals boundaries between different subjects, driven in part by subject-specific content standards that students are expected to learn. Computing and programming technologies support the work of doing mathematics professionally in the 21st century (Lockwood, Asay, DeJarnette, & Thomas, 2016); and the increasing relevance of computer science instruction for students in secondary grades makes it necessary to consider how computing may be used as a resource for learning mathematics (Grover & Pea, 2013). This creates an opportunity to identify openings in the boundaries between mathematics and technology education, specifically in the use of computer-programming technologies.

I report on the first phase of a design research project aimed at using computer-programming technologies to support students' conceptual understanding of linear functions. I examined students' creation of representations of distance as a function of time through the use of a visual programming environment. I considered two research questions: (1) What representations did students create through their use of a visual programming environment to represent distance traveled by an object? (2) How might students' creation of representations be scaffolded so that students can discover connections between these representations? The findings of this study have implications for the design of a learning experience to support students to make connections between representations of function in algebra through the use of programming technologies.

Theoretical Framework

The approach of Realistic Mathematics Education (RME) places students at the center of recreating mathematics through explorations of relevant contexts (Freudenthal, 1973; Gravemeijer & Doorman, 1999). This principle, known as *guided reinvention*, posits that students should be able to participate in the process through which mathematical ideas and procedures were created, thus foregrounding mathematics as an ongoing human activity (Freudenthal, 1973; Gravemeijer, 2004a). To take as an example the case of multiple representations in algebra, this means that students should have the opportunity to conceive of the use of tables, graphs, and symbolic representations of a function, rather than being presented with these representations as predetermined techniques or resources for particular problems. In fact, there is evidence from prior research of students using a Logo-like environment to program real-life motion and inventing graphing as a way to keep track of that motion (DiSessa, Hammer, Sherin, & Kolpakowski, 1991). Students' use of programming–environments provide a setting for students to recreate mathematics as they formulate and solve problems through the concrete representations afforded by the environment.

An important aspect of the principle of guided reinvention is that the instructional environment plays a critical role in providing the conditions for students to recreate mathematical ideas and activity (Kaput, 1994; Gravemeijer, 2004a, 2004b). To support students in actively reinventing important mathematics, teachers need instructional activities that have been designed and revised through empirical evidence of student work (Gravemeijer, 2004a). Design research allows for instructional activities to be developed through a cyclical process of instructional experiments that home in on students' learning processes (e.g., Gravemeijer & Doorman, 1999; Rasmussen & King, 2000). This study reports on the first phase of a design experiment intended to understand students' thinking about multiple representations through the use of a programming environment. I report on students' work on a task of representing distance a moving car has traveled as a function of time within the environment.

Data and Methods

I conducted this study as part of an after-school club in a large suburban junior high school serving grades 7–8. During the club, students used Scratch (http://scratch.mit.edu) to learn about computer-programming techniques such as the use of if-then statements and loops. Scratch is a visual programming environment, which means that users are able to program objects through a collection of drag-and-drop tiles, rather than through the inputting of a programming syntax. In this study I focus specifically on weeks 3-4 of the club. During this time students needed to program a car to race another car around a track, and then they needed to create dynamic representations of their car's distance traveled and speed as functions of time. For the analysis, I include only pairs for which at least one student in the pair was present each of the two weeks. This includes data from 11 pairs of students, each pair working together at one computer. I recorded students' work on the computer through the use of screen capturing software. The software continuously recorded all activity happening on the computer screen while recording audio, producing a video of students' work.

Following the data collection each week, members of the research team used video recordings to produce timelines of students' work at the computer. Using the timelines, I identified increments during which students worked on the task of creating a representation of the distance traveled by their car. I used a coding template to identify the types of representations proposed by students and how the representations students created were connected to the motion of the car. Finally, because there were times when students proposed representations that they did not create, I noted any barriers that students came across in their attempts to create a specific representation.

Findings

Table 1 summarizes pairs of students who created different types of representations in Scratch. In cases where students were satisfied with their creation of an initial representation, the instructor encouraged them to try to create a second representation of distance traveled. Given the different paces at which different pairs worked, not all students worked on a second representation. Pictoral and pseudo-symbolic representations (described further below) were the most common among students' choices. Some pairs attempted to create graphs and tables, although these were more common in in students' second representations.

Table 1: Students' Creation of Different Representations of Distance

	Picture	Pseudo-Symbolic	Graph	Table	Measurement
1st Representation	5	4	1	0	1
2nd Representation	0	3	3	2	0

The most prevalent initial representation was the pictoral representation (included on the left of Figure 1). Scratch is equipped with a collection of drag-and-drop tiles that can be used to program a

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moving object to trace its path. In the picture in Figure 1, students programmed the red car to trace its path with a blue curve. With the pictoral representations, students seemed to view the curves drawn by the moving cars as an artifact of the distances those cars had covered. The tracing commands could be used independently of how the car's motion had been programmed, which made this representation an accessible entry point in students' work.

   I use the term "pseudo-symbolic" to describe those representations in which students created a variable to keep track of the total distance their car had traveled. For example, the students whose work is pictured on the right side of Figure 1 created a variable in Scratch called "distance (number of steps)", which increased by 10 each time the car moved 10 steps. To create this representation, students used a variable to represent total distance and implicitly made use of the formula *distance = rate×time*, where the rate was measured in number of steps the car moved per iteration of the loop, and time was measured in the number of iterations of the loop. The representation was pseudo-symbolic in that it required the creation of a variable and some explicit connection between that variable and the motion of the car, but the representation did not require any explicit connection with symbols to represent rate or time.
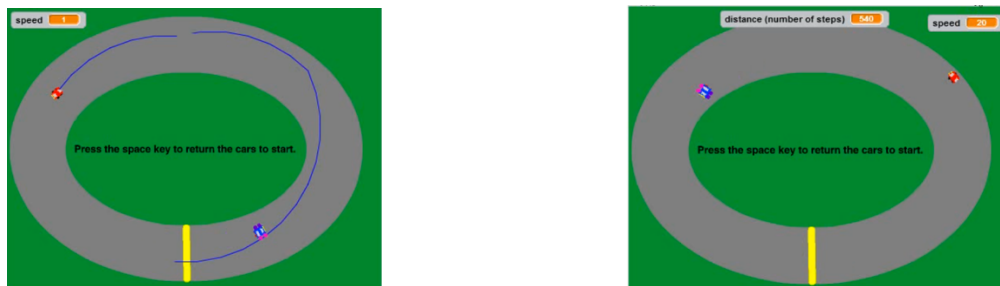
Figure 1. Pictoral (left) and pseudo-symbolic (right) representations of distance.

   Although graphical and tabular representations were less common among students, they became more common after students had produced a pseudo-symbolic representation. Consider one example of this, from the work of Nick and Greg, who controlled the speed of their car by a variable labeled "speed". After creating a pseudo-symbolic representation of distance by aggregating the car's rate over time, Nick and Greg used their speed variable in an effort to produce a graph of the distance the red car had traveled as a function of time (Figure 2). Although Nick and Greg's graph contained errors, their solution used their pseudo-symbolic representation as a precursor to graphically representing distance traveled as a function of time. Students' use of symbolic tiles provided a concrete way to connect the speed of the moving object with a dynamically created graph representing that object. In a future iteration of the design research project, these connections will be made explicit to guide students towards the reinvention of graphical and tabular representations of distance as a function of time.
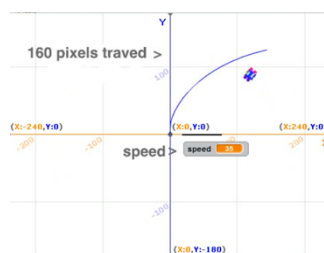
Figure 2. Nick and Greg's graphical representation of distance as a function of time.

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Discussion and Conclusion

This study has provided the foundation for a sequence of instructional activities designed to support students' guided reinvention of representations of function through the use of a computer–programming environment. Findings from students' work indicate that an environment such as Scratch provides the resources to create pseudo-symbolic representations with explicit connections between the motion of an object and the total distance it has traveled. Although graphical representations may require more explicit guidance from an instructor (DiSessa et al., 1991), pseudo-symbolic representations can serve as a link between a physical phenomenon (created or modeled through the use of Scratch) and a dynamic representation of that phenomenon. In the next iteration of this work, students' creation of pictoral and pseudo-symbolic representations will be used to scaffold the introduction of graphs and tables.

The use of Scratch for creating physical motion through the use of a computer program, and then representing the distance that object travels over time, represents a blurring of the distinction between computer science and mathematics education. The use of a computer-programming environment, which provides a concrete, experientially real context for students to explore, provide a setting for their guided reinvention of mathematical ideas and processes. Thus, guided reinvention can be inspired not only by the ways that mathematics has been done historically (e.g., Gravemeijer & Doorman, 1999), but also by the ways that mathematics can be done through the use of increasingly robust and accessible technology tools.

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OPTIMIZING TEACHER AND STUDENT AGENCY IN MINECRAFT-MEDIATED MATHEMATICAL ACTIVITY

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This brief research report describes a project we undertook to design and implement mathematical tasks for use in Minecraft. At several points in the implementation of these tasks we were mindful that what we refer to as teachers' and students' spheres of agency were in interference, with teachers' agency to pursue instructional goals in competition with students' agency to pursue self-directed activity. By acknowledging that agency is a powerful force that should be honored but perhaps also negotiated, we sought to identify the essential elements of the design and implementation of tasks that generate a space for mathematical activity in which both teachers and students are afforded agency and have the capacity to exercise it. In this report we present excerpts from the data that illustrate varied enactments of agency by teachers and students that point to a way forward in identifying a productive balance between the two.

Keywords: Technology, Design Experiments, Affect, Emotion, Beliefs, and Attitudes

There has been a recent surge of interest in the potential of video games as immersive, interactive, and creative spaces for students to learn mathematics. For students, video games offer the promise of familiar and enjoyable modes of creative play; for teachers, they offer the promise of an opportunity to leverage that play in pursuit of the advancement of mathematical knowledge. Unfortunately, classroom teachers have historically found it difficult to implement these environments in their own classrooms due to what seem to be inevitable conflicts that arise between the objects of students' self-directed activity and teachers' own instructional goals (Jenkins, 2014). Our research seeks to produce a framework that teachers can use to implement gaming environments that sustain the joy and creativity of gaming, while emphasizing the purpose and utility of mathematics (Ainley, Pratt & Hansen, 2006) and harnessing the potential for students to act as agents of their own learning (Stroup, Ares & Hurford, 2005). Recognizing that what we refer to as teachers' and students' *spheres of agency* must be negotiated and are bound to intersect, we sought to identify the essential elements of the design and implementation of tasks that generate a space for mathematical activity in which both teachers and students are afforded agency and have the capacity to exercise it.

Theoretical Framework

Our vision of Minecraft as a tool for learning is informed Vygotsky's (1978) notion of mediated activity and Verillon and Rabardel's notion of instrumental genesis (1995), each of which express the dialectic relationship between tools and sense-making. Minecraft is a sandbox-style video game in which players explore and build within a world that looks similar to our own. Players assume an individual, first-person view of the world and experience it through the eyes of their avatar. The graphics are purposefully "blocky" and provide a "visual allusion to LEGO™" that "suggests a space in which the player is given free rein to create whatever he or she wishes from the pieces provided" (Duncan, 2011, p. 4). Indeed, as Kuhn (2015) suggests, agency is the unique affordance of sandbox-style games like Minecraft, which generate a space of possibilities in which users can exercise this agency to craft their own narratives.

Our perspective on Minecraft-mediated mathematical activity (MMMA) is informed by Resnick's (2014) approach to Constructionist design, which features four core elements of creative learning: meaningful projects, peer collaboration, a passion for the work, and playful

experimentation. We designed mathematical tasks with a "Freirean intention" of generating a space for MMMA characterized by an environment of mutuality and trust among students and teachers that encourages students to assume active roles (Freire, 1970). Within that space, "willing and doing are unified. Students are central to what expressive artifacts and insights are produced" (Stroup, Ares, & Hurford, 2005, p. 192). They decide not just "to play along, but also to play a part in what the activity is about [and] what matters" (p. 192). Accordingly, we focused on agency as a significant feature of MMMA and explored it by taking a socially situated perspective that views learning as legitimate participation (Lave & Wenger, 1991) in communities where students' collaborative activity involves them in shaping classroom practice. As a consequence of that involvement, students come to feel powerful and engaged. Their sense of ownership is advanced and they gain legitimacy by exercising the agency afforded to them.

Mindful of the conflicts that have been seen to arise between the objects of students' self-directed activity and teachers' own instructional goals when gaming environments are introduced classrooms, we sought to answer the question: How might a teacher implement a gaming environment in a mathematics instructional setting so that both teachers' and students' agency are honored and leveraged in the development of mathematical knowledge?

Methods

We took a design experiment approach (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) to determining principles for the design of tasks that generate a learning ecology wherein students engage in enjoyable, agentive, and productive mathematical work that could engender changes in their mathematical knowledge. Two of the authors assumed the roles of teacher-researchers engaged in systematic inquiry in relation to developing a framework for the design and implementation of agentive MMMA tasks. Three students in grades 4 and 5 who were enrolled in a STEM summer camp that took place at our university participated in the study. We chose these students because they had worked with Minecraft before and were familiar with the environment.

We collected video data of classroom discourse and computer interactions in addition to reflective journals in which we recorded observations on students' mathematical work at the conclusion of each of the five class meetings. We conducted a thematic analysis (Braun & Clarke, 2006) by transcribing the video data and looking for relationships that emerged from an iterative re-reading of that data by each of the authors. Specifically, we sought to identify interactions between the design of tasks, their implementation, and their generated spaces of agency. As our findings are preliminary, the instances of negotiated and intersecting spaces of agency that we identified from the video data formed the basis of the analysis that is presented here.

Preliminary Findings

The theme across all of the tasks was that the students played the role of town developers who were being asked to renovate and build structures in a Minecraft environment called CraftLand. In the first of these tasks, we pre-installed a pile of stone blocks in the shape of a rectangular prism (Figure 1, Stone Pile task) and challenged the students to find the number of blocks in the pile. Our goal for this task was to engage students in the experience of volume measurement in three dimensions and they did so by leveraging the inherent agentive nature of Minecraft and also the space of agency generated by a task featuring multiple pathways and an agreed upon endpoint (Stroup, Ares, & Hurford, 2004). We saw a diversity of approaches to finding the volume, ranging from simply counting the blocks on one level and multiplying by the height, to digging inside the pile to confirm that it was solid, to building additional nearby structures that allowed students to gain a perspective on the pile from above.
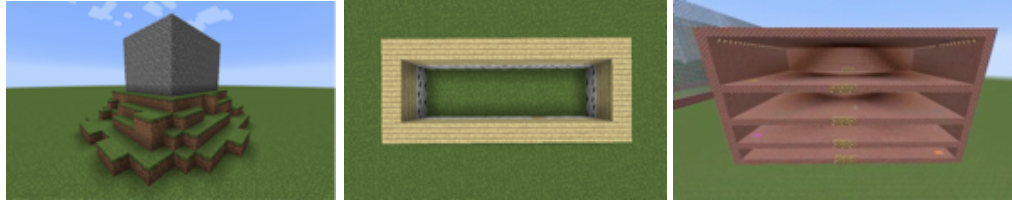
Figure 1. (left to right) Stone Pile, House Design, and Staircase tasks.

The Stone Pile task also called on students to obtain a set of tools with which to build a mine cart and a track that would carry the demolished stones to another location. The outcomes of this task were not as we envisioned. As there was just one set of hidden tools to be found, students were expected to share those tools in collaboration with each other. However, when one student found the tools, he was reluctant to share them with the others. The teacher then assumed her obligations not just to the target mathematical content but to her commitment that every student engage in agentive and collaborative MMMA by intervening to alert them that the task required collaboration, not the kind of competition-style gaming they were used to in Minecraft. In traditional gaming environments, such competition for resources is to be expected and this expectation seems to have been the rationale for the student's behavior.

The House Design task (see Figure 1) called on students to build a house with a base that has a perimeter of exactly 36 blocks. This task was generative (Stroup, Ares, Hurford, 2004) in that students had free reign with respect to how they would accomplish it to satisfy the single constraint. Students were afforded agency that they could exercise in the completion of the task, but the boundaries of that agency were not well understood. When the teacher exercised agency to initiate a discussion in which students would share their strategies, not all students were interested in participating. Doing so would require that they give up what they perceived as their own Minecraft time, something that the students could be reluctant or unwilling to do. At one point when the teacher asked "Jacob" how he determined that the perimeter of his house was 36, he responded, "Let me show you after." The teacher's request came at a time when Jacob was engaged in a task he had set out for himself.

A similar conflict occurred during the Staircase task (see Figure 1). Students were given the opportunity to either work together or on their own as they constructed staircases to fit each of the four levels of a building. One student was intent on exploring another area of the world, climbing in and out of the gardens and chopping down any bushes he encountered. The teacher noticed what this student was up to and insisted, "This isn't where you're supposed to be. You're supposed to go find this building and work with Jacob." Ignoring the teacher's redirection, the student continued to go about his business. In response, the teacher used Minecraft's admin tools to teleport the student to the building where he was meant to be working. He was rattled by the interruption of his play when his screen darkened and his avatar was suddenly transported from the gardens to the building. With respect to both of these conflicts, we were cognizant of our need as teachers to assume our obligations to instructional goals, but nonetheless we were dismayed by the felt need to interfere in activity that students assumed they had been afforded more agency over.

Conclusion

Throughout the implementations of the tasks of the design experiment, we witnessed the negotiations and interactions that were provoked between the instructional goals of the teacher and the desire of students to be agents of their own activity. At ideal points, such as when students exercised self-directed activity in the completion of a given task, teachers were able to leverage that agency toward the development of mathematical knowledge. The Minecraft environment proved to be a suitable space wherein students could construct "objects to think with" (Papert, 1980), thereby

providing them opportunities to explore and advance that knowledge. In contrast, when teachers' and students' spheres of agency were in interference, such as when students exercised the agency afforded to them in the pursuit of their own task-irrelevant goals, teachers exercised their own agency in the pursuit of instructional goals by shutting down students' activity and redirecting them to the task at hand. At these points, the environment proved to be well suited to students' intentions as they leveraged their agency toward game-based play but ill-suited to teachers' intentions to leverage their own agency in the support of mathematical learning.

Regardless of the nature of the ways that teachers and students exercised the agency afforded to them, their actions were reasonable with respect to their expectations for Minecraft play and the target goals of that play. We hope that the task design and implementation framework for Minecraft-mediated mathematical activity we are constructing will honor both teachers' and students' intentions to the extent that they align with and support learning. We believe that such a framework will be beneficial to teachers who wish to engage their students in game-based mathematical activity and also to the developers of such gaming environments who wish to support it.

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ROLLY'S ADVENTURE: DESIGNING A FRACTIONS GAME

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New digital medias, such as games, provide the opportunity for radically re-envisioned mathematics representation and engagement in both formal and informal contexts. This paper aims to present a game, Rolly's Adventure, that was specifically designed for and built on the PlayStation 3 platform LittleBigPlanet 2, and combines best practices for game design, research on supporting fractions learning, and the affordances of this unique platform.

Keywords: Technology, Design Experiments, Number Concepts and Operations

"I claim that using video games is the way Euclid would have taught basic mathematics had that technology been around in ancient Greece" (Devlin, 2011, p. 47).

New digital medias provide new opportunities for mathematics education, but – perhaps most importantly – new challenges. This paper proposes an investigation of some of those challenges that aligns with the PME-NA theme of *Sin Fronteras: Questioning Borders with(in) Mathematics Education*, by presenting a game built at the intersection of mathematics education and videogame design. From with(in) this intersection, *Rolly's Adventure* (Williams, 2015) was designed to take full advantage of these new interactive spaces, instead of following the unfortunate trend of creating digital versions of flash cards, worksheets, and activities where the mathematics is disconnected from the context. In particular, the author combined the learning affordances of videogames (e.g., Salen & Zimmerman, 2003; Schell, 2008) with the ways that students best learn particular mathematics concepts (in this case, fractions), as identified by leading mathematics education researchers (e.g., Hackenberg 2007, 2010; Steffe & Olive, 2010; Tzur, 1999). This process of *combining* is by no means simple, as game design principles, mathematics learning research, and even the vagaries of chosen digital platforms are occasionally prone to irreconcilable differences. However, digital media such as videogames provide the opportunity to design new representations and interactions for learning mathematics in powerful ways, and such a border is worth investigating.

Mathematics Learning Games

A considerable number of games ostensibly intended to teach mathematics concepts have recently flooded the market, and many of them are merely digital duplications of non-digital mathematics activities, such as flashcards and other skill-and-drill activities. Other math games have richer digital worlds, but the in-game activity and the mathematical activity are often deeply disconnected. For example, Gillispie (2008) describes *DimensionM*, a rich game world designed to support mathematics learning, in which the players are stranded on a beautifully rendered desert island. A nearby gate must be unlocked by finding nearby nautilus shells that have prime numbers on them. In other words, the task (unlocking a gate) requires the very unrealistic task of identifying prime numbers on objects that do not normally have prime numbers on them, requiring the player to engage in non-authentic mathematical activities.

Other math games are designed by learning scientists or math educators and researchers, and strive for a more organic design that pairs the activity and the mathematics more deeply. For example, the virtual world *Quest Atlantis* has a mission that uses the theory of transformational play to position player, content, and context together for powerful statistics learning (Barab, Gresalfi, & Ingram-Goble, 2010). *Candy Factory* (see, for example, Norton et al., 2014) is a fractions game with deep roots in research and theory, designed to assist the player in progressing through five schemes

(Steffe & Olive, 2010). Both games offer legitimate engagement with mathematical concepts that are naturally embedded within mathematical activities, amid game structures to support engaged learning.

However, few math games are able to take full advantage of the new opportunities by re-envisioning what doing mathematics can *look like*. This partially due to the constraints of participating in a field where a considerable amount of design for mathematical activities are conducted within formal educational environments that have their own limits in terms of time and technology access, and partially because the field of videogames – and educational videogames in particular – is still nascent. It is difficult to simultaneously invent new digital contexts for mathematics learning *and* invent new representations within those digital contexts.

Learning and Design Goals

Fractions

Smith (2002) declares that "No area of elementary school mathematics is as mathematically rich, cognitively complicated, and difficult to teach as fractions, ratios, and proportionality" (p. 3). The *Common Core State Standards for Mathematics* (CCSSM) (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010) in the United States also recognize fractions as a complex and important area of mathematical learning. For the purposes of this paper, I define fractions as "a *relationship* or multiplicative comparison between two quantities" (Lobato & Ellis, 2010, p. 58, italics original). Quantities are characteristics of objects that can be measured prior to any actual measuring, or even a determination of *how* such a characteristic could be measured (Thompson, 1995, 2010).

Of particular importance is that quantities, even prior to measuring, can still represent a fraction – in other words, a glass half full is always half full, even when the liquid or glass have not been measured or numerically labeled. I highlight this because Thompson (1995) warns us to be on our guard when using notation: "...it is easy to confuse convention and principle unthinkingly. There is nothing principled about standard algorithms. They are standard only because they have become customary within a community" (p. 201). Thus, the familiar notation of ½ is a convention designed to abstract from, for example, a glass half full – but both ½ and the glass half full legitimately represent a fractional relationship.

Game Design Learning Principles

The field of games studies regularly discusses possible definitions of *game*, but has yet to settle on a single overarching definition. My definition is one of the many outlined by Schell (2008): *a problem-solving activity approached with a playful attitude*. In short, I developed a problem-solving activity using quantitative representations of fractions, and relied upon the game context to support players approaching it with a playful attitude. In doing so, I relied upon learning principles commonly used in games (e.g., see Gee, 2005), but given space constraints, I share only one here: the opportunity for failure.

The term *failure* nicely illustrates the contrast between mathematics classrooms and mathematics games. In classrooms, the word is associated with a variety of high stakes issues that may result in a student re-taking a class or possibly staying back a grade – there is nothing positive about failure. In well-designed games, however, players learn *through* failure: as they enter this new imaginary world and begin testing ideas, the game indicates an incorrect hypothesis about how to behave through failing the player. The player then revises her hypotheses and ideas in response, and tries again. As Gee (2005) notes, a good game "makes failure part of the fun and central to learning" (p. 13). Mathematics games have the opportunity to incorporate the useful parts of failure in a way that

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classrooms may not, given the negative associations with the term (and the experience of *failing*) in formal education.

LittleBigPlanet 2

LittleBigPlanet 2 (LBP) is simultaneously a *game* and a *platform*: the disc comes with a game to play, and provides the opportunity to unlock the tools used to design and build the same game, or one of the player's own devising. After creating their own game, players can hit "publish" to share with the entire LBP community via the PlayStation Network – and millions of levels have been created and shared in this fashion, including *Rolly's Adventure*. LBP provides sophisticated tools that allow for complex design with only an understanding of logic, requiring no background in programming, and its own coherent aesthetic approach. However, LBP also has its own peculiar and unique constraints – for example, it is impossible to design a game using LBP which supports the concept of multiplication as scaling, because there is no way to design the game such that players can transform the attributes of quantities in such a way.

Game Design

In the following section, I highlight some of the main design elements in *Rolly's Adventure*. First, however, I must highlight that as this game was designed to be played *in the wild*, the goal was to create a *provocative object* (Williams-Pierce, accepted) that caused player-learners to discover the mathematical content and relationships on their own, with no instruction or guidance other than the game itself. Furthermore, the game never states what the (digital) objects are, what the relationship of those objects are to each other, or what the goal of the game is. Rather, unlike most traditional mathematics learning contexts, player-learners must test, hypothesize, and discover those elements themselves directly through interaction and failure.

Figure 1, on the left, shows the very first puzzle presented to the player-learners. They must discover their goal (to get Rolly, the blue circular character at the top left, across the hole), and endeavor to determine what the secret relationship is between the buttons (green and gold circles on the ground), the button labels (blue and black underneath the buttons), and the gold bar in the hole. Through testing and failure, player-learners quickly discover that activating the button labeled with two black dots results in two gold bars filling the hole, and Rolly is able to continue.
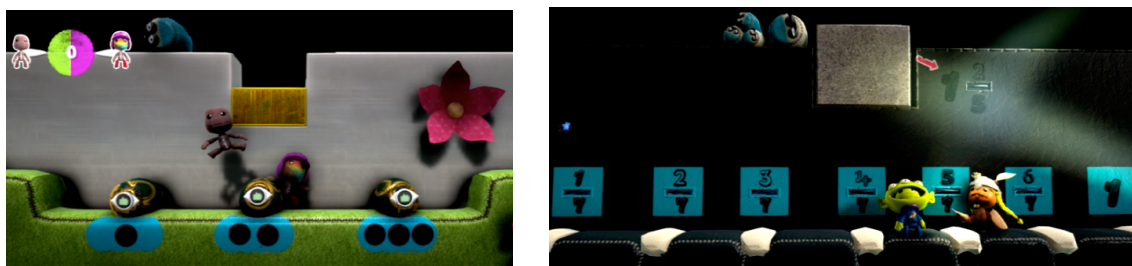
Figure 1: The first (left) and the last (right) puzzles in *Rolly's Adventure*.

As the player-learners progress through the game, traditional fraction notation begins to be introduced, as is apparent in the final puzzle (Figure 1, on the right). However, long before such notation appears, player-learners have to determine the relationship between the button labels and the resulting number of gold blocks: the presented gold block is the unit, and the circles are the number of those units that result. While the first puzzle is a gentle introduction, later puzzles that present – for example – a gold block that is three-fourths of the hole, requires the player-learner to mentally partition the gold block in relationship to the amount missing from the hole, in order to realize that the correct choice is one and one-third of the unit in order to fill the hole.

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The difficulty present in such complex activity – without paper and pencil, or any way to otherwise quantify the relationship – was intentionally designed throughout the game trajectory, in order to use the limitations of visual perception to provoke a desire for the accuracy and efficiency of traditional notation. As the gold blocks become smaller or more complicated (as in Figure 1, on the right), and player-learners can no longer rely upon their visual perception, the fraction notation is immediately useful, aligning both with Harel's (2007) necessity principle and Gee's (2005) info-just-in-time principle.

Conclusion

The goal of this paper is to present a new game, *Rolly's Adventure*, that was designed specifically to provoke fractions learning. Although analysis of gameplay and interview data is ongoing, preliminary results indicate that player-learners produced mathematical language during gameplay that reflected their own growing understanding of fractions knowledge. The final results of analysis will be briefly presented at the conference, alongside discussion of the design process and final game.

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SUPPORTING EMBODIED GAMES FOR MATH LEARNING WITH WEARABLE TECHNOLOGY

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Keywords: Technology, Problem Solving, Measurement, Geometry and Geometrical and Spatial Thinking

We had students use cell phones and watches use on their wrist that act as "wearable tutors", that provide instructions and support while playing math games. Students search for objects such as geometric pieces with identifying tags. Students physically manipulate shapes and measurement tools to complete tasks, receiving feedback and hints/scaffolds via a display, buzzers and lights, learning while interacting with artifacts and students (Arroyo et al., in press).

The TANGRAMS RACE is a team-based game for 3rd grade, a relay race where students take turns to run and retrieve shapes out of a basket at the end of the room as indicated by the watch (Liu, 2014). The final goal is to construct a Tangram puzzle, at the starting line. Children run one at a time to find a basket full of shapes, and need to discern which geometric shape fits the description on their watch (e.g. "*I am a quadrilateral with two sets of parallel sides*").

ESTIMATE IT! is a measurement estimation and number sense game for 4th–6th grade students (Rountree, 2015). Children search for objects around a physical space. For example, the display may show the following: *"Find a cube with a 6" edge"*. Hint 1 may say: *"Use your 12" dowel to measure"*. Students are given unmarked measurement tools to encourage estimation.

The results presented in Tables 1 and 2 suggest that students learn and improve their expectation of personal success in math, as well as their appreciation of mathematics.

Table 1: Cognitive and Affective Outcomes, Before and After Playing the *Tangrams Race*

Variable (N=15)	Pretest	Posttest	% Increase
MCAS Standardized 7-item Test	0.65 (0.13)	0.74 (0.14)*	+9%
Self-concept in ability to do math (min=1; max=5)	4.2 (0.80)	4.9 (0.36)*	+14%
Liking of Mathematics (min=1; max=5)	4.5 (0.94)	4.8 (0.42)	+6%

a. * Significant paired-samples t-test difference, $p<0.05$

Table 2: Cognitive and Affective Outcomes, Before and After Playing *Estimate IT!*

Variable (N=13)	Pretest	Posttest	% Increase
MCAS Standardized 7-item Test	0.65 (0.22)	0.70 (0.23)	+7%
Self-concept in ability to do math (min=1; max=5)	4.37 (0.50)	4.67 (0.41)**	+6%
Liking of Mathematics (min=1; max=5)	4.55 (0.89)	4.64 (1.15)	+2%

a. ** Significant paired-samples t-test difference, $p<0.005$

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MATHEMATICS STUDENTS' EXPERIENCES WITH PROGRAMMING

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Keywords: Technology, Research Methods, Post-Secondary Education

For many topics in mathematics, the use of computer programs is essential or, at the very least, may enhance student experience. While teaching such university-level topics, the mathematicians I interviewed (Broley, 2015) all faced the same question: "To what extent will my students develop, examine, and operate the programs?" Our analysis of the interviews shows that professors choose to address this issue in significantly different ways. A qualitative approach allowed us to identify *six kinds of programming interactions* they may have offered to their students:

L0: Strict observation of the results of a computer program, often directed by the professor;
L1: (Extracurricular) manipulation of an interface of an existing program;
L2: Observation (and perhaps interpretation) of the code of a program;
L3: Modification of an existing code to accomplish something new;
L4: Construction of code, with aspects (e.g., the algorithm) specified by the professor; and
L5: Creation of a program, including algorithm development, coding, and validation.

Each of our participants seemed to favour a subset of these interactions. Philippe, for example, perceived his use of programs (L0) in teaching probability as an effective way to challenge students' intuitions about important theorems; after his illustrations, he posts his programs online and encourages students to play with them (L1) with the hope that they become more active in observing the phenomena and gain a better understanding of the material. He avoids other interactions because, in his view, they require programming skills that are not related to his course. Paul, also a probability professor, agrees that "The purpose of the programming is to help [students] learn probability." Nonetheless, he's convinced that "If they're doing it themselves, they learn it way better." His students interact with programming in the form of L4 or L5.

Bearing witness to such debates, I am led to wonder about (a) the affordances of each interaction type, and (b) the opportunities taken and obstacles faced by students who actually engage in them. To address these issues, I propose a case study approach (Yin, 2013). For (a), I will construct a reference model of affordances, building on previous findings reported in the literature (e.g., Marshall et al., 2014). I will then solicit characterizations from professors (information-rich cases) who offer programming interactions to their students. Finally, to gain an in-depth understanding of (b), I will follow the journey of some of these students.

At my poster, I expect to discuss and receive feedback on my proposed research direction. Focussing on pertinent issues such as the variables I should consider when choosing my cases or the methods I could use to collect and analyse rich data, I hope to advance in my pursuit of a reliable and meaningful study of mathematics students' experiences with programming.

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TECNOLOGÍAS DIGITALES Y LA RESOLUCIÓN DE PROBLEMAS: DISCUSIONES MATEMÁTICAS MÁS ALLÁ DEL SALÓN DE CLASES

DIGITAL TECHNOLOGIES AND PROBLEM SOLVING: MATHEMATICAL DISCUSSIONS BEYOND THE CLASSROOM

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Palabras clave: Álgebra y Pensamiento Algebraico, Educación Media Superior, Resolución de Problemas, Tecnología

El objetivo de esta investigación es analizar a qué nivel el uso de herramientas como YouTube y Padlet ayudan a los estudiantes a discutir y compartir ideas matemáticas en un ambiente de resolución de problemas fuera del salón de clases. Las preguntas que guían está investigación son ¿cómo extender y promover la discusión matemática fuera del salón de clases con el muro digital Padlet y los videos educativos de YouTube? ¿De qué manera impacta el uso coordinado de videos que explican y promueven la comprensión conceptual de contenidos matemáticos seleccionados de YouTube y muros digitales de Padlet en las discusiones matemáticas fuera del salón de clases?

Marco conceptual

En necesario tener presente que un aspecto importante para aprender matemáticas en un ambiente de resolución de problemas es "identificar y promover las formas de razonar que los estudiantes pueden construir a partir del empleo sistemático de varias herramientas digitales como enciclopedias, software, tabletas electrónicas, etcétera" (Santos, 2014, p.24). Dentro de la caracterización de la cognición de los individuos al atender situaciones problemáticas que demanden el uso de estrategias y conocimiento matemático, Schoenfeld (1985) propone un modelo de análisis el cual se caracteriza por cuatro elementos: recursos, heurísticas, control y sistema de creencias.

Resultados

Los estudiantes resuelven de forma correcta los problemas planteados (Véase Figura 1), sólo con ayuda de los videos sugeridos, la información e ideas compartidas en Padlet.

Figura 1. Comentarios de los estudiantes en el muro de Padlet.

Con el uso de las tecnologías digitales es posible extender las discusiones matemáticas más allá del salón de clase; sin embargo, el papel del profesor es fundamental para guiar las discusiones que permitan llegar a la reflexión y solución de los problemas. Esto abre la oportunidad de hacer nuevos cuestionamientos: ¿Cómo seleccionar material audiovisual que presenten información relevante e incentive al estudiante a cuestionarse sobre el contenido matemático? Y ¿cómo elaborar material que promueva la resolución de problemas?

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Keywords: Algebra and Algebraic Thinking, Higher Secondary Education, Problem Solving, Technology

The purpose of this research is to analyze to what degree the usage of tools such as YouTube and Padlet helps students discuss and share mathematical ideas within a problem-solving environment outside the classroom. The questions that guide this research are these: How can we extend and promote mathematical discussion outside the classroom using the Padlet digital wall and the YouTube educational videos? How are mathematical discussions outside the classroom impacted by the coordinated usage of videos selected from YouTube and the Padlet digital walls that explain and promote conceptual understanding of mathematical content?

Conceptual Framework

It is necessary to keep in mind that an important factor to learn mathematics in a problem-solving environment is "to identify and promote the ways of reasoning that students can construct from the systematic use of various digital tools, such as encyclopedias, software, tablets, etcetera" (Santos, 2014, p.24). Within the characterization of an individual's cognition when addressing problematic situations that require the use of mathematical strategies and knowledge, Schoenfeld (1985) proposes a model of analysis characterized by four elements: resources, heuristics, control and beliefs system.

Results

The students correctly solve the presented problems (See Figure 1) only using the help from the suggested videos and the information and ideas shared on Padlet.

By using digital technologies, it is possible to extend mathematical discussions beyond the classroom; nevertheless, the teacher's role is fundamental to guide such discussions to promote reflection and solving problems. This opens up the opportunity to generate new questions, how to select audiovisual material that presents relevant information and encourages the student to question mathematical content? Moreover, how to elaborate material that promotes problems solving?

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TECHNOLOGY PROFESSIONAL DEVELOPMENT: MATHEMATICS TEACHERS IN THE STATES

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The professional development (PD) of teachers has been studied in various ways. Avalos (2011) stated, "at the core of such endeavors there is the understanding that professional development is about teachers learning, learning how to learn, and transforming their knowledge into practice for the benefit of their students' growth" (p.10). The United States Department of Education's 2010 National Education Technology Plan underlines the importance of teacher development in technology integration in the classroom by stating that, "technology can help us better prepare effective educators and increase their competencies throughout their careers while building the capacity of our education system to deliver effective teaching." (p.39). Although the Common Core and the International Society for Technology in Education (ISTE) Standards emphasize technology-supported pedagogy in mathematics classrooms, there are differences in state policies about teacher's professional growth regarding technology use in mathematics instruction. Also, there is scant documentation on the differences in math teachers' technology professional development (TPD) activities across the states.

The purpose of this study is to highlight the variations in teachers' PD in technology use in mathematics instruction across nine states in the United States. Those states are the benchmark states in the Trends in Mathematics and Science Studies (TIMSS) 2011. The National Assessment of Educational Progress (NAEP) secondary data was used. The items were selected from the teacher questionnaire. The question guiding the study was: What are the differences in fourth grade students' teachers PD participation to learn technology integration into their mathematics classes across the selected states from the years 2005 to 2015? The NAEP Data Explorer was used to examine the differences in teacher TPD for mathematics instruction.

The results highlighted the disparities of the teacher TPD participation rates over the years and across the states. Specifically, in 2005, 32% of the fourth grade math teachers reported that they never participated in TPD in the last two years. By 2015, the participation to the TPD increased relatively as the percent of the teachers who reported that they never participated in a TPD decreased to 24%. Furthermore, 8% of the teachers in 2005 reported large extent participation in using technology for math instruction compared to 10% in 2015. As access improved overtime, there was a slight increase in teachers' attempts to learn how to use technology in their math instructions. In particular, the teachers in Florida, Alabama, Indiana, and Minnesota continually exceeded the national average in their participation in PD activities for teaching math using technology. Further investigation will be conducted to examine the relationship between teacher TPD and fourth grade students' algebra achievement.

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COMMON CORE MATHEMATICS BECOMES A POP CULTURE ENTITY

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It is not often the Fox News and MTV agree on something, but in the case of Common Core Mathematics the two media outlets have found common ground; Common Core math is something to be ridiculed. This commonality is demonstrated in both outlets' coverage of Doug Herrmann's, now famous (infamous to some), "common core check" posted on Facebook. The message said: "Wrote a check to Melridge Elementary using common core numbers. I wonder if they'll take it? #YouFigureItOut" (Herrmann, 2015). The check depicted the amount to be paid using a ten-frame. For days following his post the headline "Frustrated dad writes hilarious Common Core check to child's school" was shared, reshared, tweeted and retweeted by countless media sources and individuals. Facebook and Twitter feeds were filled with parents and teachers sharing these articles. Has mathematics curriculum became part of popular culture? The media frenzy around Herrmann's check, the creation of twitter accounts like @thankscommoncore, and now even a beer commercial mocking common core mathematics make it clear this is a conversation happening in social and mass media, as much as at the dinner table. What does this mean for the teaching and learning of mathematics? How is this impacting the work that we do as mathematics teachers, mathematics teacher educators, and young mathematicians engaging in learning and doing of mathematics in schools?

A Discourse Analysis

Utilizing discourse analysis to guide my work this poster will look at how these conversations are being shaped both in the mass media and in social media, looking critically at who (and what) is privileged by discourses created as common core mathematics has found its place in popular culture. While complexities and nuances exist my analysis shows that many of the same power structure that benefited from previous discourses about mathematics continue to benefit from these newly created discourses. That in many ways we are watching the "math wars" of our past be played out in commercials and comedy sketches. But now that this has been taken up outside of the mathematics education community even more voices are being silenced, especially those of teachers and students.

Questioning Borders With(in) Mathematics Education

As Common Core mathematics is taken up in popular culture we as mathematics educators can choose to let our voices be silenced by not participating in the conversation, or we can take this opportunity to engage with those outside of the mathematics community. In this poster I advocate for the later, that we use this opportunity to dismantle the borders that have existed between mathematics educators (and mathematics education researchers) and come together. To work together to simultaneously critique and find the benefits of common core mathematics. To take this time that mathematic curriculum is so widely being acknowledged to open the conversation to parents and students. If ever there was a time to take on a revolution in mathematics education, it is when Stephen Colbert is taking about mathematics curriculum!

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EXAMINING THE QUALITY OF INTEGER "PINS" ON PINTEREST

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Keywords: Number Concepts and Operations, Technology

In this poster we report results from a research study examining negative integer resources available on the social media site Pinterest. Pinterest is a virtual space similar to a real-life bulletin board. The site allows users to organize and share online content (webpages, images, videos, links, etc.) by creating "pins" that link to specific resources. Once created, pins can be discovered by other users using keyword searches, "repined" (i.e., reshared), collected onto "boards," and "liked." The ease of finding content on Pinterest means that teachers, both preservice and inservice, can use the site to find ideas for activities, projects, and lessons. Use of Pinterest in this way is akin to a kind of border crossing in that individuals leave traditional sources (e.g., books, colleagues) and journey a into a new space in search of resources.

Although pins are easily accessible, they represent an educational resource of unknown quality. Pins are created by users without review or scrutiny from peers. Once created, the only data provided related to a specific pin are the number of repins and the number of likes it has accumulated. At the same time, a recent survey by Darleen, Kaufman, and Thompson (2016) found that 87% of elementary and 62% of secondary teachers reported using Pinterest as a source for instructional materials. In our own classrooms, we have seen a trend of preservice teachers using Pinterest as a primary source for mathematical lessons, activities, and projects.

To take some initial steps in investigating pins as an educational resource, we conducted a project focusing on negative integers, which are a challenging topic for students of all ages (Piaget, 1948; Stephan & Akyuz, 2012). Our investigation focused on the following questions:

1. What is the scope of mathematical content in pins related to negative integers?
2. What is the accuracy of mathematical ideas in pins related to negative integers?
3. What patterns are present in the source material of pins related to negative integers?

Using these questions as an analytical lens, we created a coding scheme that considered a variety of topics including mathematical operations, mathematical language, mathematical notation, use of manipulatives, use of real-world context, and source material. We then gathered a dataset of 200 pins using keyword searches and analyzed it using the coding scheme.

Results indicated mathematical errors in more than one-third of pins, a dominance of addition over other operations, strong connections to the MathTwitterBlogosphere for source material, and infrequent use of real-world context. Additionally, results raised questions about the appropriateness of using repins or likes as a measure of quality. We will provide a breakdown of results and highlight implications for use of pins as an educational resource.

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EPISTEMOLOGICAL AND DIDACTICAL VALIDITY ISSUES OF ONLINE TEACHING AND LEARNING OF MATHEMATICS

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Keywords: Post-Secondary Education, Teacher Education-Inservice/Professional Development, Technology

Here (see more information in Hoyos, 2016) it is proposed a discussion about main characteristics of new online educational applications for the learning and teaching of mathematics, namely MOOCs (Massive Open Online Courses) and how it could affect the teaching and learning of mathematics. It was based in revisiting and applying the theoretical notions of epistemological and didactical validity of learning computational environments, now to the case of distance teaching and learning of mathematics mediated by ICT. This discussion is supported by the analysis of new data on the resolution of mathematical complex tasks accomplished by in-service secondary teachers enrolled in an online program for professional development, both on mathematics content and the use of digital technology in teaching.

First discussion on the epistemological and didactical validity issues of computational environments was due to N. Balacheff and R. Sutherland (see Balacheff 1994, 1999, 2004; Balacheff and Sutherland, 1994; and Sutherland and Balacheff, 1999), to illustrate different contributions that bring the use of certain software when they are incorporated in particular computational learning environments. Now the time is ripe to revise what could be the potential of new MOOCs (see, for example, http: //www.coursera.org), what this type of online courses are bringing to the field of mathematics education, to describe and analyze their characteristics under the light of the theoretical constructs from Balacheff & Sutherland.

While using computational environments for the teaching and learning of mathematics raises questions of complex character from an educational point of view, particularly focusing in the active role of the student and the teacher as an administrator or manager of these media in learning situations (Sutherland & Balacheff, 1999), new educational trends now materialized in the massive open online courses (MOOC) may call for free online education without any tutorial intervention, occurring outside of school and when perhaps pedagogical means are automatically managed, for example by digital tutorials. Also through these courses, users are expected to assume the responsibility of their own appropriation of knowledge, and their learning and possible development is promoted primarily through a series of opinion exchanges among peers. All this considered, in this work it is shown and discussed by data obtained from an online course for the professional development of teachers, which in fact simulate some MOOC characteristics, how it is not enough to have free access to mathematics digital technology and/or Internet free resources to achieve expertise on a complex mathematical content that is online being addressed.

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FOSTERING KNOWLEDGE AQUISITION IN AN ASYNCHRONOUS LEARNING ENVIRONMENT: A POTENTIAL STRUCTURE

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Keywords: Classroom Discourse, Technology, Geometry and Geometrical and Spatial Thinking

Fully online courses have become more widespread for graduate programs that are designed for in-service teachers due to the flexibility of the time commitment. In this environment most interaction among students, and between students and instructors occur on discussion forums. That is why there is a growing interest researching the nature of this interaction and how it contributes (or does not contribute) to individual knowledge construction.

Using Weinberger and Fischer's (2006) framework, we analyzed three in-service mathematics teachers' knowledge construction in a fully online graduate level geometry course. We purposely selected Kevin, Dana, and Helen since they showed high presence on the discussion board and they were along the spectrum of course grade levels (low, average, and high, respectively). We focused on the participants' discussion board engagement. Weinberger and Fischer's framework to provide us four dimensions that may "extend and refine our understanding of what kind of student discourse contributes to individual knowledge acquisition" (p. 73): participation, epistemic, argumentative, and social mode of co-construction.

Our preliminary results reveal that in the participation dimension all three participants' statistics of course content and discussion board access were consistently above class average throughout the semester. However, Kevin stood out when we run the statistics for discussion board posts read and posted. Unlike Dana and Helen who *read* all 789 posts, Kevin read only 72 posts. Additionally, Kevin posted 10, Dana posted 26 and Helen posted 16 threads. These threads were coded under the epistemic dimension. Kevin and Dana had all, but one, *construction of conceptual space (CCS)* posts and another one was coded as *construction of adequate relationship between conceptual and problem space (CARC-P)*. Helen, on the other hand, presented more diverse threads with 10 *CCS*, 5 *CARC-P* and 1 *construction of inadequate relation between conceptual and problem space (CIRC-P)*. In argumentative dimension, Kevin only had 9, Dana had 33, and Helen had 22 threads posted. Unlike Kevin, Dana and Helen had all, but one, threads as valid claims. Kevin, on the other hand, posted only 3 valid claims. Lastly, within social mode of co-construction dimension, all three participants' main mode of interaction was externalization (Kevin=16, Dana=33 and Helen=25). However, Helen (7) and Dana (5) also utilized elicitation much more than Kevin (2), and Helen utilized non-argumentative moves significantly more than Kevin and Dana (23 vs 8 and 8, respectively).

Based on our findings, we argue that although the quantity of the time spent on discussion board is important, the "quality" of the time spent must be closely monitored. Also, it has been reconfirmed that discussion board engagement plays a significant role in knowledge construction. Lastly, developing a discussion board engagement rubric based on Weinberger and Fischer's framework might potentially increase meaningful engagement and a stronger knowledge acquisition in an online environment.

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DESIGN OF A VIDEO GAME FOR PROMOTING EMBODIED MATHEMATICAL REASONING

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Keywords: Cognition, Geometry and Geometrical and Spatial Thinking, Technology

An emerging literature on embodied cognition (Wilson, 2002) shows that mathematical ideas can be learned through action-based interventions that ground understanding of abstract principles in concrete, perceptual and motor experiences. As theoretical perspectives like embodied cognition emerge in education, so do alternative ways of using technology to support teaching and learning. Our video game, *The Hidden Village,* utilizes *directed motions*, which embody the geometric relations that form the basis of players' proofs and justifications for subsequent geometry conjectures. The game tracks players' movements using rapid visual processing of images through the built-in camera found in common laptop computers and uses no other specialized hardware (see Figure 1).

Justification and proof are central in mathematics education (Stylianides, 2007). Yet students struggle to construct viable and convincing mathematical arguments and valid generalizations of mathematical ideas (e.g., Healy & Hoyles, 2000). Past research (Nathan, Walkington, Boncoddo, Pier, Williams, & Alibali, 2014) showed that directing undergraduates to perform actions where they modeled triangles and turning gears with their bodies aided generation of correct insights and valid proofs. Notably, successful participants were largely unaware the beneficial actions were related to their subsequent math performance. In a recent pilot study, proof success for a range of geometric conjectures was correlated ($r = 0.348$) with learners' production of spontaneous *dynamic gestures*, which are hand movements that reveal learners' abstract and generalizable thinking about imaginary mathematical entities as though they were real objects.

In this newest study, grade 6-11 students ($n = 18$) played *The Hidden Village* for 6 geometry conjectures. Performance was initially lower than for college students, but increased significantly ($p = 0.02$) after pedagogically directed hints relating the game directed actions to the conjectures. Video analyses show use of spontaneous gestures illustrating how the directed motions from the game left a *legacy* in the dynamic gestures learners made that contributed to their proofs. Having established a viable platform for introducing embodied forms of mathematical reasoning through game play we plan to study the effect of this game on proof practices in high school classrooms.

If one angle of a triangle is larger than a second angle, then the side opposite the first angle is longer than the side opposite the second angle. **[TRUE]**

Figure 1. An example conjecture that is first acted out by *The Hidden Village* game characters and copied by players is more apt to elicit a valid proof.

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A FRAMEWORK FOR HOMEWORK IN FLIPPED MATHEMATICS CLASSES

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One defining characteristic of flipped instruction is the homework teachers assign, which typically consists of an instructional video rather than problem sets (Bergmann & Sams, 2012). We present a framework for flipped homework that categorizes types of homework and draws on existing literature to discern quality for each type (see Figure 1). This framework allows for the distinction between different implementations of flipped instruction with respect to the homework assigned, thus moving away from the assumption of flipped and non-flipped teaching as a binary distinction as in past studies (Clark, 2015; DeSantis et al., 2015).
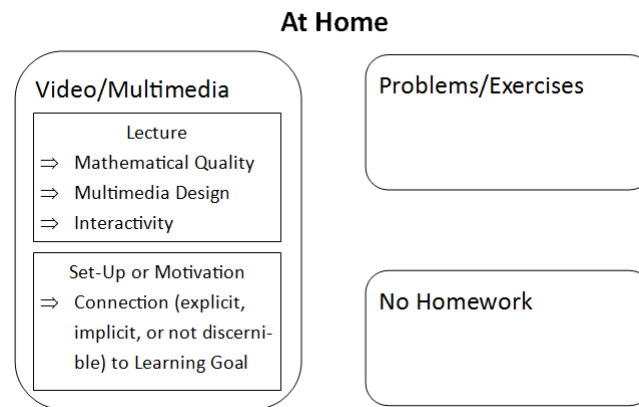
Figure 1. A framework for "homework" in flipped mathematics classrooms.

Video/multimedia homework is separated into *lecture* and *set-up or motivation* categories based on the purpose of the homework, and for each category we provide illustrative examples from a study of flipped mathematics classes. We show how the quality of instructional videos can vary according to specific criteria. We also discuss how, in our study, teachers seldom included interactive features in their lecture videos and the teachers more frequently assigned lecture videos than set-up/motivation videos. Looking beyond homework, it is likely that the in-class implementation of flipped instruction is just as (or more) important than the homework.

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DESIGNING ACTIVITIES USING DIGITAL MULTIMODAL TECHNOLOGIES FOR THE EARLY LEARNING OF CONGRUENCE AND SIMILARITY

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Keywords: Geometry and Geometrical and Spatial Thinking, Technology, Early Childhood Education

This poster presents results of the first stage of a larger project aimed to contribute to the recent ongoing research on the potentialities of integrating dynamic geometry environments and multi-touch technologies in the strand K-2 to foster young children's geometrical thinking. The goal of the first stage of the study was to design a sequence of seven mathematical activities in a learning environment that combines Geometer's Sketchpad® (Jackwic, 2009) and the iPad to help children informally understand congruence and similarity in dynamic ways. Grounded on prior research on the early implementation of dynamic geometry (Hegedus, 2013; Sinclair & Moss, 2012) and on the development of shape (Clements & Sarama, 2014), it is conjectured that young learners could link properties of congruence and similarity to continuous motion and multiple representations of triangles in different positions and orientations and to informal measuring actions. The use of iPads could improve gestural expressivity and collaboration in small teams (Hegedus, 2013). The main goal of the second stage of the study was to implement the learning sequence with second-grade students to examine the ways in which they can develop intuitive understandings of congruence and similarity while collaboratively working in small groups with the technologies. The poster is focused on the first stage related to activity design, but examples of children's forms of reasoning illustrate the learning potentialities of the tasks.

The sequence of activities was designed following a student-centered, inquiry-based approach from a sociocultural view and design principles for the use of dynamic geometry software and multi-touch interfaces with young children such as: continuous motion, connectivity and communication (Sinclair & Crespo, 2006) and, executability of multiple representations, co-action and multiple manipulation and interaction (Hegedus, 2013). The sequence includes two exploratory activities and one problem-solving situation for congruence and, three exploratory activities and one-problem solving situation for similarity. The Cognitive Task Analysis of the activities includes the description of the tasks, mathematical structure, cognitive demand, learning goals and ways of reasoning that they could potentially promote.

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INTERPRETACIÓN FIGURAL DE LA GRÁFICA EN 3D

FIGURAL INTERPRETATION OF GRAPHICS IN 3D

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Palabras clave: Educación Secundaria, Tecnología

El objeto de estudio de esta investigación es la interpretación de las gráficas en 3D de ecuaciones paramétricas usando los recursos figurales asociados a éstas, para ello proponemos a los estudiantes *dispositivos gráficos* para usarlos como herramientas en la construcción del conocimiento y dar sentido a incidencias sobre ellas. Estos dispositivos pueden ser puntos, curvas o planos adecuadamente proyectados sobre los planos *xy. yz* y *xz* los que permitirán interpretar las propiedades gráficas de la forma en 3D en términos de la gráfica en 2D, tales como concavidad, continuidad, puntos críticos entre otros.

El procedimiento de proyección paralela se apoya en un uso inicial de lápiz y papel para luego usar la computadora con fines de exploración. Las propiedades figurales pueden ser visualizadas matemáticamente (Duval, 1998) y con apoyo de herramientas histórica y culturalmente determinadas (Radford, 2014) podremos anticipar y dar sentido a las gráficas 3D con dispositivos particulares en 2D.

Por ejemplo, podríamos usar una parábola en 3D como la de la figura 1, la que será proyectada convenientemente sobre el plano *xy* en 2D que nos sugiere continuidad, puntos mínimos, concavidad, entre otras propiedades de la primera. Además, si asociamos una recta sobre la superficie que hace las veces de dispositivo gráfico que se desplaza a lo largo de la proyección, recuperamos la gráfica en 3D. Tanto la parábola proyectada como la recta desplazada por esta pueden ser dispositivos gráficos usados para dar sentido a las gráficas mencionadas.

Las aproximaciones figurales pueden permitir incrementar la dificultad tanto de las curvas estudiadas como de los dispositivos gráficos adecuados para analizarlas.
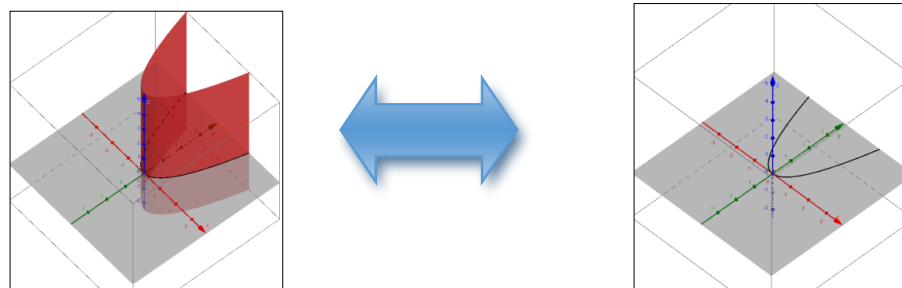
Figura 1. Parábola en 3D con su dispositivo gráfico en 2D.

Keywords: High School Education, Technology

The object of study of this research is the interpretation of graphs in 3D of parametric equations using the figural resources associated to these; to do so, we propose *graphics devices* to students so they can use them as tools in the construction of knowledge, and to give a sense to incidences about the graphs. These devices can be points, curves or planes, adequately projected on the planes *xy, yz*

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and *xz*, which will allow to interpret the graphic properties in 3D shape in terms of the graph in 2D shape, such as, concavity, continuity, critical points, among others.

The process of parallel projection is supported by an initial use of pencil and paper, to proceed later with the use the computer to be able to explore. The figural properties can be visualized mathematically (Duval, 1998), and with the support of historical and culturally determined tools (Radford, 2014), we can anticipate and make sense to 3d graphs, with particular devices in 2D.

As an example we could use a parabola in 3D, like the one in Figure 1, which will be conveniently projected on the *xy* plane in 2D, which suggests continuity, minimal points, concavity, among other properties of the first one. Moreover, if we associate a straight line on the surface, which sometimes acts as a graphic device that moves along the projection, we recover the graph in 3D. Both, the projected parabola, and the associated straight line may be graphic devices used to make sense of the mentioned graphs.

The figural approaches may allow to increase the difficulty of the curves under study, as well as of the appropriate graphic devices used to analyze them.
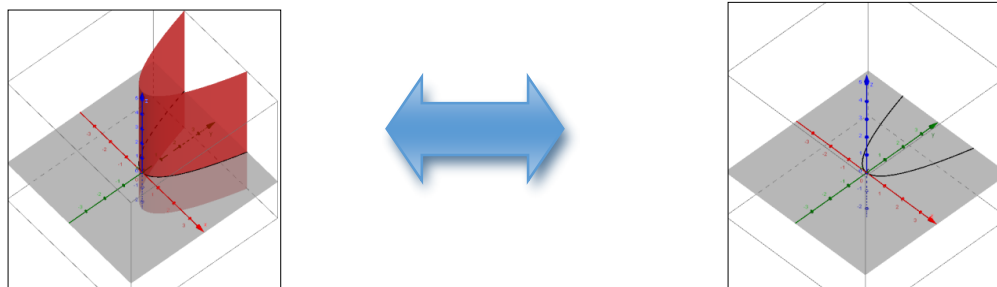
Figure 1. Parabola in 3D with its graphic device in 2D.

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Chapter 13

Theory and Research Methods

Research Reports

Brief Research Reports

PRAXES PROXIES: REVISITING EDUCATIONAL MANIPULATIVES FROM AN ECOLOGICAL DYNAMICS PERSPECTIVE

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The recent proliferation of technological devices with natural user interfaces (e.g., touchscreen tablets) is regenerating scholarship on the role of sensorimotor interaction in conceptual learning. Some researchers of mathematical education have adopted views from constructivism, phenomenology, enactivism, and ecological dynamics to interpret implicit sensorimotor schemes as both forming and manifesting disciplinary competence. Drawing on these views, this theoretical paper returns to the enduring question of what it means to develop a new skill by way of task-oriented interaction with objects. Beginning with sports then moving to mathematics, we focus on a subcategory of pedagogical artifacts that serve students only during training activities yet constitute proxies for developing target schemes toward normative application. We argue for the contribution of these views to conceptualizing the design of effective mathematics instruction.

Keywords: Instructional Activities and Practices, Learning Theory, Technology

Background and Motivation: Teaching for Learning Mathematics as an Activity

Hans Freudenthal (1971, p. 435) once remarked that learning mathematics is better understood as learning how to swim. This remark could be understood as metaphorical, that is, Freudenthal is indicating that learning mathematics is an *activity*, rather than something passively done to the student. And yet one way to teach mathematics as an activity is to have students *physically* engage in the mathematical process, much like swimming necessarily involves an initial period of flailing one's arms and legs in water.

This theoretical paper is based on the speculative premise that learning mathematics is literally, not only metaphorically, much like learning to swim. Accordingly, we will be devoting the first portion of this paper to introducing, from the sports science, a framework for modeling how people learn to perform physical techniques, such as martial-arts maneuvers. Only once we have established this analytical model will we then turn to discuss the case of mathematics, which we perceive as significantly analogous.

To allay the reader's concern, let us stress that this project is not construed naively: We are well aware of patent differences between sports and mathematics. To an onlooker, freestyle swimming appears quite different from solving for *x*. The overt physical actions, tasks, goals, media, and semiotic register and tenor across these disparate cultural practices are indeed incongruous. And yet we submit that these apparent differences need not undermine a project to investigate for meaningful similarities across the activities.

A comprehensive comparison of sports and mathematics is probably beyond the scope of this modest paper. And so our tack here will be to carry out the comparison by way of selecting a particular prism as a means of narrowing and focusing the comparison. We have chosen to examine what it means to learn a skill with the aid of an object, what is often referred to in the mathematics-education literature as a manipulative. We are thus proposing here an inter-disciplinary comparative analysis of the didactical function of manipulatives in sports and in mathematics, because we believe that this comparison bears potential rewards for the theory, design, and practice of mathematics education more generally.

Praxes Proxies: Pedagogical Artifacts for Practicing Novices

To even further hone this paper, we will look at an exclusive class of manipulatives, those that are used not in enacting an actual goal practice in its normative trappings but only in training toward enacting that goal practice. For example, a soccer player might develop prowess by way of dribbling in slalom path along a line of cones, even though there would never be cones on the field during an actual game. Here the cones are standing in for people, and they are positioned in a form that exaggerates and distils the core coordination challenge facing the novice.

We are tentatively calling this class of objects *praxes proxies*, because we believe they do not quite fall squarely under any current definition of materials or activities that are employed in the service of learning a new skill. For example, one might think of this specialized class of artifacts as scaffolds, in the sense that they constitute auxiliary elements or structure temporarily deployed into a learning environment and then later removed (faded) once the learners are prepared to engage in normative practice, as in the case of bicycle trainer wheels (Bakker, Smit, & Wegerif, 2015). And yet our artifacts of interest are not really scaffolds, because they are used not only to support the student's physical execution of otherwise overwhelming actions but also at times to simulate ecologically authentic environmental features actually encountered in normative experiences—our artifacts of interest might *pose* rather than preempt problems. One might therefore think of our artifacts of interest as enabling what Kirsh (2010) would view as a form of rehearsal, in the sense that they support practitioners in reducing, decomposing, or schematizing a complex practice by way of focusing on some circumscribed aspect. And yet these artifacts actually supplement structure onto the activity—structure that does not exist in the goal activity. Moreover, the student has not yet experienced the goal practice and thus cannot have any agency in decomposing it. One might therefore think of these artifacts as enabling a form of simulation of the goal activity that is less dangerous, costly, or otherwise resource consuming (Schwartz, 2007). And yet we are looking at cases where the learner need not or even cannot be aware of how the performance of these unique exercises is related to the goal practice, so that it does not make sense to call them simulations, at least not in view of the student's own phenomenology.

By calling our focal class of pedagogical objects *praxes proxies* we are considering them as basic-training substitutes for critical aspects of "the real thing." As we will be arguing, praxes proxies serve to entrain learners toward operating effectively within target activity contexts by way of creating for the students opportunities to develop sensorimotor coordinations relevant to, and applicable in enacting the goal practice. These new sensorimotor coordinations are elicited from students as they attempt to overcome performance challenges that emerge in the course of satisfying a task objective. From the systemic perspective that we will be evaluating, learning is developing new sensorimotor coordination oriented on newly constructed aspects of the environment: As they practice, students reconfigure the environment to afford new interactions.

Our focus herein on praxes proxies could be instrumental in considering educational phenomena. Namely, praxes proxies are quite unique in that they have been historically designed or selected explicitly for teaching and learning a skill, not for actually performing the skill in its goal contexts. The very existence of this class of objects should be interesting to educational researchers, because it helps us isolate and thus examine "purely" pedagogical affordances of cultural practice as they relate to the target skills they foster. The sections below clarify and demonstrate our thesis, arguing for its broad utility.

Theoretical Framework

Why might researchers of mathematics education care how people learn to perform sports techniques? This intellectual orientation warrants some explaining.

We grant that the learning of physical skills is often seen as substantially different from the learning of conceptual notions. Yet we ask the reader to keep an open mind, as looking to physical

disciplines for inspiration may not be as radical as it appears at first glance. Indeed, over thirty years ago, in his address to the International Group for the Psychology of Mathematics Education, when von Glasersfeld (1983) criticized research in mathematics education for having under-delivered, he added,

> this disappointment—I want to emphasize this—is not restricted to mathematics education but has come to involve teaching and the didactic methods in virtually all disciplines….There is only one exception that forms a remarkable contrast: the teaching of physical and, especially, athletic skills. There is no cause for disappointment in that area. (p. 42)

von Glasersfeld argued that we ought to learn from those physical domains that have been exceptional, because

> the primary goal of mathematics instruction has to be the students' conscious understanding of what he or she is doing and why it is being done…[W]hat the mathematics teacher is striving to instill into the student is ultimately the awareness of a dynamic program and its execution—and that awareness is in principle similar to what the athlete is able to glean….from his or her performance. (pp. 51-52)

We agree with von Glasersfeld that analogizing competence in sports and mathematics might go more than skin deep. Our work has been an attempt to implement this radical-constructivist epistemological position in the form of pedagogical activities designed for mathematics students to ground targeted curricular content in new forms of physical activity they learn to enact. Accordingly, in this paper we consider content learning as sprouting from 'dynamic programs' then maturing via guided reflection into 'conscious understanding.' The heart of this paper is on artifacts that both elicit and shape said dynamic programs. Yet just before we focus on these artifacts, we explain how our epistemological position on grounded mathematics is not as arcane as might at first appear.

This paper is situated within a current turn in mathematics-education research toward theorizing conceptual knowledge as grounded in sensorimotor schemes. Per this view, learning new concepts requires the development of new spatial–dynamical motor-action coordinations. This notion, which is shared by cognitive developmental psychology (Piaget, 1968), enactivism (Varela, Thompson, & Rosch, 1991), and neuro-educational research (Norton & Deater-Deckard, 2014), orients mathematics education researchers on the action patterns themselves—how a physical task is accomplished—equally or perhaps more so than on the material or semiotic products of these actions (Nemirovsky, 2003). We hope with this paper to contribute to the field's discourse specifically around designing activities that foster targeted action patterns.

Several reasons account for our somewhat unusual program of research that has led us to collaborate with scientists who specialize in modeling how people develop and control motor action. To begin with, converging reports from empirical research studies in diverse branches of the cognitive sciences have been putting forth claims to the effect that cognitive activity is grounded in the tacit enactment of perceptually guided physical motor action, even when no overt corporeal action is manifest to the on-looking observer (Barsalou, 2010). In fact, it has been claimed that sports psychology is essential for understanding cognition (Beilock, 2008). It thus stands to reason, at least per our judgment, that researchers of mathematical cognition should have a firm grasp of the relation between sensorimotor and mathematical activity (see Nemirovsky, 2003).

Granted, the thesis that concepts are grounded in sensorimotor action is foundational to constructivism (Piaget, 1968) and carries through to neo-Piagetian scholarship (Kalchman, Moss, & Case, 2000; Norton & Deater-Deckard, 2014). And yet, by-and-large the community of mathematics-education researchers is not equipped to capture, document, analyze, and model these sensory perceptions and physical motor actions in which concepts are allegedly grounded (Abrahamson & Sánchez–García, 2015, in press). For example, research articles on mathematics learning process

rarely offer micro-ethnographic analyses of the minute sensorimotor schemes students develop via solving interaction problems. When articles do offer these analyses, we witness careful transcriptions of multimodal activity that include descriptions of students' and teachers' instrumental and representational gestures (Nemirovsky & Ferrara, 2009). However, these descriptions are not framed by, and therefore do not attend to, kinesiological properties of these actions. That is, the studies might espouse an embodied-cognition framework to explain why a concept is challenging and how teachers and students interact with artifacts to construct and elucidate the inherent mathematical notions, yet the studies are not founded on the premise that *the physical actions themselves are challenging and that this challenge is explanatory of the conceptual challenge of grasping and applying the new notions*.

To be sure, lifting a fist-sized red block and placing it on a nearby blue block is certainly within the motor skills of normally developing kindergarten students. For this population, the physical motor operation of re-positioning a block is not designed in and of itself to be challenging, and so performing this rudimentary isolated physical operation per se is not conceived as fostering the development of a new sensorimotor scheme. Rather, to the extent that we view mathematics learning as contingent on overcoming sensorimotor challenges, this view is situated in a relatively new design genre for STEM education, namely in technologically enhanced embodied learning environments (TEELE, see Lindgren & Johnson-Glenberg, 2013).

Pioneered by Nemirovsky and his collaborators (e.g., Nemirovsky, Tierney, & Wright, 1998), TEELE involve a dynamical interaction task that turns out to demand a challenging sensorimotor coordination. Performing the task is enacting a new form of reasoning. Further classroom discourse on the solution, which may elicit reflection, description, representation, interpretation, and argumentation, makes to consolidate the new form of reasoning in normative semiotic registers, such as vocabulary, diagrams, graphs, and symbolic notation (Abrahamson & Lindgren, 2014). It is these environments—and in particular their central focus on dynamical solutions to sensorimotor coordination problems—that have been motivating our research program to step aside from mainstream mathematics-education scholarship and seek the wisdom and practice of disciplines dedicated to the investigation of motor-action learning and control.

In sum, we are interested in a class of cultural artifacts we call praxes proxies. Unique about these artifacts is that they are not used in performing some goal practice per se, but rather to *train* towards performing this goal practice. For example, consider the speedbag used in boxing. This artifact is viewed as invaluable for training boxers yet is clearly *unlike* what is enountered in an actual boxing match. As von Glasersfeld stipulates, the pedagogical utility of these artifacts is that through training students become aware of, or attuned to, some aspect of practice. We argue that mathematics education, particularly the design of manipulatives, stands to benefit from leveraging what other fields have discovered about the pedagogical utility of these artifacts.

In the remainder of this paper we turn directly to an emerging theoretical framework from sports sciences, ecological dynamics, so as better to understand the contribution of praxes proxies to students' development of competence in a physical activity. We will then illustrate an analogous case from mathematics education.

Ecological Dynamics & Non-Linear Pedagogies: Introducing Adequate Constraints

Relations between student, task, and available artifacts can be modeled after an ecological-dynamics perspective (Abrahamson & Sánchez–García, 2015). From this view, the student, task, and artifacts comprise a system. As the student attempts to perform some goal task, we say that the artifacts constitute productive constraints on these efforts. By "productive constraints" we mean just that—the artifacts usefully negate a vast spectrum of possible yet culturally irrelevant routes of action (degrees of freedom) leaving only a narrow range of possible ways to go about completing the task, where these ways are in line with the pedagogical intention of the activity. A child building a

castle from interlocking plastic blocks will learn through exploration to adjoin the blocks according to the designer's intention: For this child, the blocks will come to privilege (afford) particular modes of interacting. As researchers, we look to understand, build, and evaluate learning environments that foster new dynamical interaction patterns by productively constraining how a learner might engage an activity in seeking to satisfy some task objective.

Ecological dynamics originates in the sports sciences. The application of dynamical systems to ecological psychology enables sports scientists to explain the learning of physical activities as the complex self-organizing of subject–environment dynamical systems (Vilar, Araújo, Davids, & Travassos, 2012). In this systemic approach, learning is modeled not as generating a sequence of disembodied symbolical propositions, such as abstracted inferences and decisions, but as intrinsically emergent from and tuned to the agent's embedded action structures within a non-linear system (Araújo, Davids, Chow, Passos, & Raab, 2009). Thus, the unit of analysis is not the isolated individual but the indissoluble pair of individual–environment in interaction.

The self-organizing behavior of this agent–environment system can be affected or "channeled" by different kinds of constraints. Newell (1996) identified three sources of constraints affecting the behavior of the system: organismic (biochemical, biomechanical, neurological), environmental (gravity, temperature, light), and task (goals and rules). In this paper we are theorizing the role of supplementary artifacts as introducing task constraints appropriate to the pedagogical objectives. Notably, the introduction of praxis-proxy artifacts changes the task, and so we may consider them as task constraints.

From a didactical point of view, the introduction of adequate constraints becomes a paramount issue. Non-linear pedagogy (Chow et al., 2011) is based on supplementing and modifying constraints in the learning environment. Coaches, similar to constructivist mathematics educators, adopt a strategy of discovery-based learning, where students' learning process is constrained in deliberate ways. By assigning what should be done and constraining how it might be accomplished, instructors generate "fields of promoted action" (Reed & Bril, 1996). Therein, learners are encouraged to engage in explorative behavior by which to find personal solutions to the task at hand. Such constraints can be introduced either by changing the game's rules/conditions, changing/restricting physical space, modifying equipment, increasing gradually the complexity of the task, or simplifying it (Davids et al., 2008, pp. 161-167). The main aim of this cluster of activities is to foster self-discovery by providing enough variability for individual learners to find their own solutions to varying situations that bear for the learners emergent contingency (Bernstein, 1967).

The introduction of constraints must always follow the principle of "representative design" (Brunswik, 1956): Activities created specifically for training purposes should not distort or change some key information of the environment that learners would find in the actual conditions of the real game (Renshaw, Davids, Shuttleworth & Chow, 2009). Rather, learners are to engage in behaviors that foster attunement towards the key perceptual information sources pertaining to the ultimate physical performance in the authentic goal context. Chow et al. (2009) stress that practice activities must be representative of performance demands so as to lead to transfer of skills between practice and performance environments.

Artifacts that create representative-design task constraints might be either complex or simple and even mundane familiar objects. Consider a well-familiar object: a wall (see Figure 1).
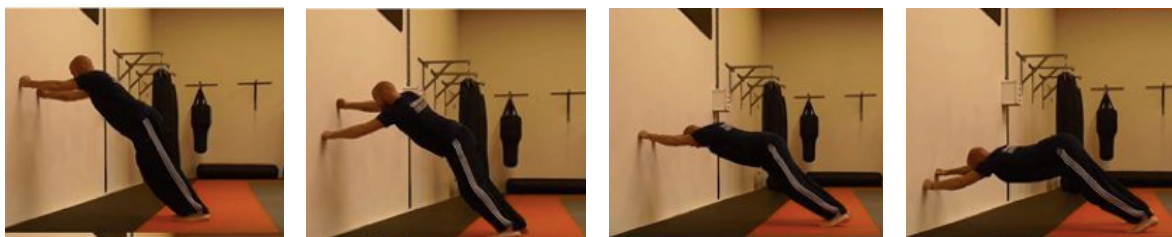
Figure 1. In Systema, a Russian self-defense method, a wall serves in an exercise, per the principle of representative design. Stand opposite a wall. Lay your hands on it. Now "walk" your hands down the wall, one hand at a time. All the while, your feet, too, must walk backwards so as to maintain a more-or-less straight body. Once low, you would walk your hands into a push-up position on the floor. You would then reverse the sequence back up to the starting position.

The introduction of this simple artifact (a wall) channels your activity into dynamically maintaining a corporeal structure able of sustaining a line of force transmitted from feet to hands, thus resulting in a constant pushing-forward action. Such performance is crucial when facing a fighting opponent who pushes you backwards with her attack. Thus, the wall exercise fosters the student's attunement to key information coming from the haptic (dynamic touch) sense. The wall acts as a praxis proxy, substituting an opponent yet maintaining representative-design principles.

We thus view praxes proxies in terms of their systemic role. As students attempt to perform the given task under the constraints poised by the artifact, the students learn to move in new ways. That is, the environment comes to afford new ways of moving, namely new motor-action coordinations. (By way of cultural reference, the reader might remincise about the Karate Kid.)

Praxes proxies are a unique class of pedagogical artifacts in the sense that they serve students in the deliberate absence of actual goal contexts, creating intact worlds that idealize or essentialize specified aspects of the normative tasks. Various gym devices serve in this capacity by way of demarcating a space and dedicated equipment for engaging in an activity that, to the naked eye, bears little to no ecological authenticity (e.g., we rarely waddle down a wall). Yet this activity nevertheless focuses attention and effort on developing and exercising a targeted corpus of motor action highly relevant and environmentally attuned to ecologically authentic tasks.

As we turn from sports to math, we stress that across these domains praxis proxies

foster not only local skill development per se, that is, becoming better at some particular movement, but are methods for gaining insight into disciplinary knowledge. In other words, the central argument is that, through their practice, these routines serve as *methods* of developing disciplinary knowledge. [They are]….not practiced for their own sake, but for what is gained through practice. (Trninic, 2015, p. 24)

Praxis Proxies in Mathematics Education: The Mathematical Imagery Trainer

From the epistemological perspective of embodiment theories, we have argued, learning mathematical and athletic skills is similar, in that both require the construction of new sensorimotor schemes as a condition for competent performance. Creating representative design for mathematics learning hinges on determining, perhaps inventing, particular sensorimotor schemes that arguably capture the dynamical cognitive substrate of reasoning about and toward a target concept. Thus, if we the designers wish to create praxes proxies for proportional reasoning, we must first account for what proportional reasoning looks like, feels like, moves like for us—we must 'phenomenalize' (Pratt & Noss, 2010) proportional reasoning in the form of a sensorimotor coordination and then build embodied-interaction tasks that foster the development of these very coordinations by way of solving some performance challenge.

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a.   b.   c.   d.
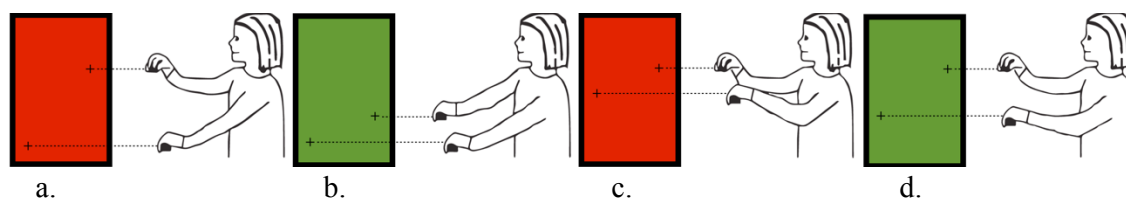
Figure 2. The Mathematical Imagery Trainer for Proportion (MIT-P) set at a 1:2 ratio, so that the favorable sensory feedback (a green background) is activated only when the right hand is twice as high from the monitor's bottom line as the left hand. This figure sketches out our Grade 4 – 6 study participants' paradigmatic interaction sequence toward discovering and then practicing an effective operatory scheme: (a) while exploring, the student first positions the hands incorrectly (red feedback); (b) stumbles upon a correct position (green); (c) raises hands maintaining a fixed interval between them (red); and (d) corrects position (green). Compare 1b and 1d to note the different vertical intervals between the hands and, correspondingly, between the virtual objects.

An example for a mathematics praxis proxy is the Mathematical Imagery Trainer for Proportion (see Figure 2; Abrahamson & Howison, 2010). We first designed a bimanual motor-action scheme that enacts proportional equivalence, and then we engineered conditions for students to move in a new way that would require developing this scheme. Our two-step activity plan was for students to: (1) develop a target motor-action scheme as a dynamical solution to a situated problem bearing no mathematical symbolism; and (2) describe these schemes mathematically, using semiotic means we then interpolate into the action problem space.

Empirical evaluation of the design, including the integrated micro-genetic analysis of clinical, electronic, and eye-tracking data, suggests that indeed students are devising target sensorimotor schemes in solving the interaction problems (Abrahamson et al., 2016).

Conclusion

We are only just beginning to understand the relation between bimanual coordination and mathematical cognition. Researching the emergence of this digital-cum-digital prowess, from manual action to symbolic notation, is contingent upon thinking out of the curricular box to design opportunities for witnessing and investigating this emergence. And yet for all these efforts ultimately to contribute to education in its authentic ecology, we must strive to remove borders between research and practice. Per the PME-NA 38 call, "We see borders as potentially productive as well as potentially problematic." We agree: We seek partners across the border to engage in productive, synergistic discourse so that mathematics learning can move in new ways.

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DUAL ANALYSES EXAMINING PROVING PROCESS: GROUNDED THEORY AND KNOWLEDGE ANALYSIS

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This report presents dual analyses of an undergraduate student, Cassie, whose work provides nice contrasts between Grounded Theory (GT) analysis and Knowledge Analysis (KA). The analyses highlight particular methodological differences, such as grain size of findings, positioning of novices and more general implications about expert–novice studies. The combination of the two methods results in a more complete and nuanced description of Cassie as a prover, while mediating many of the methodological concerns from the individual analysis.

Keywords: Cognition, Equity and Diversity, Reasoning and Proof, Research Methods

At the higher academic levels (graduate and professional mathematics), proving can be considered to be a way in which the truth of a claim is established or realized (Hanna, 2000; Weber, Inglis, & Mejia–Ramos, 2014). Proving, and more generally, justifying is a process by which mathematical knowledge can be furthered throughout the K–12 grades and in higher mathematics education. Selden, McKee, and Selden (2010) stated that the proving *process* "play[s] a significant role in both learning and teaching many tertiary mathematical topics, such as abstract algebra or real analysis" (p. 128). Various empirical studies have ventured to provide insight into the process of proving separate from the product of proof. For instance, previous work involving the comparing of expert and novice productions of proof have examined issues such as differences in strategic knowledge in the construction of proofs in abstract algebra (Weber, 2001), private and public aspects of proof (Mejia–Ramos & Tall, 2005; Raman, 2002), and use of examples by doctoral students in evaluating mathematical statements (Alcock & Inglis, 2008). This current work adds to this corpus of literature in order to better understand the process of proving, without necessarily associating the process to the finished product of proof.

The study of the process of proving as an unfamiliar phenomenon benefits from the use of qualitative methods that focus on explaining the phenomenon being studied instead of validating existing theory about the phenomenon. Grounded Theory (GT) research is a qualitative research method that is defined by the generation of theory that is inductively derived from the study of data (Glaser & Strauss, 1967). GT research can be used to investigate how individuals go through a process and to identify the different steps in that process (Charmaz, 2000, 2006; Creswell, 2007). More specifically the types of questions routinely investigated using GT methods are of the type: "What was the process?; …What was central to the process? (core phenomenon); … What strategies were employed during the process? (strategies); [and] What effect occurred? (consequences)" (Creswell, 2007, p. 66).

The goal of a GT analysis is to develop a substantive theory based on categories that are generated from iteratively gathering and examining data until the categories generated have been sufficiently saturated. This often implies that any theory developed through GT analysis is expected to be coarser and may necessarily sacrifice details and nuances about the phenomenon. Parnafes and diSessa (2013) asserted that claims about cognitive processes using GT typically use "time–scale and meaning–resolution [that] are rather large and indefinite" compared to other types of analyses of learning that are more microgenetic in nature (p. 15).

One such method is Knowledge Analysis (KA) (diSessa, Sherin, & Levin, 2016; Parnafes and diSessa 2013). KA is a methodological approach associated with the Knowledge in Pieces (KiP)

theoretical framework (diSessa, 1993) to "study the content and form of knowledge for the purpose of understanding learning" (diSessa et al., 2016). KA aims to describe details of models for mental representations of students' knowledge through micro–assessments and tracking of an individual's learning in real time. In principle, KA explicitly rejects the notion that novices' knowledge is a subset of experts' knowledge, and thus prioritizes the investigation of how mathematical knowledge emerges out of naïve thoughts.

In this paper, we employ both GT analysis and KA to explore the proving process of Cassie (a pseudonym), a "novice" prover. We anticipate that the two methods to complement each other in some aspects but to diverge in others. The goal is to provide a more complete and nuanced account of Cassie's proving process with the two methods, and to simultaneously contribute to the more general discussion about combined methods in qualitative research.

Methods

Data Source

Data was drawn from a larger study (Karunakaran, 2014), which examined the similarities and differences in the usage of knowledge by expert and novice provers of mathematics. The larger study involved individual semi–structured interviews where 5 undergraduate mathematics students and 5 mathematics Ph.D. students validated or refuted the truth of five mathematical statements. At the time of data collection for the larger study, Cassie was a female undergraduate student majoring in mathematics. Cassie was selected because her work offers a palpable illustration of the affordances of using the individual methods of GT and KA. Cassie's proving work was not unique compared to that of others within the novice prover group. In fact, the use of the GT analysis necessitated that the resultant claims about Cassie were consistent with her peers. Five mathematical statements were presented to Cassie over the course of two 90-minute interviews, which were audio and video recorded for subsequent transcription and analysis. The analysis in this paper focuses on Cassie's engagement with the first task.

Grounded Theory Analysis

GT analysis uses methods such as *constant comparison*, and coding strategies such as *open coding*, *axial coding* and *selective coding* (Charmaz, 2000, 2001, 2006; Straus & Corbin, 2008). These phases of coding can be described as examining the data in order to develop categories of information (open coding), examining these categories to develop them further and to interconnect them (axial coding), and using these developed categories and their interconnections to build a theory that explains the existence of the categories (selective coding).

To instantiate this coding and analysis process, consider the following excerpt from Cassie's first interview, where she was addressing the task shown in Figure 1. For space considerations, the transcript has been abridged to exclude non–mathematical language and probing questions from the interviewer.

Cassie: Let's see. So if a_n were 1, uh a_1 were 1 then that would have to be less than or equal to a_2, a_3 um so and then that would have to be um that would have to be less than or equal to a_4 plus a_5 plus a_6 plus a_7 /…/ Ok. (Pause) um I'm inclined to say that it's false. /…/ Because the, I mean the only constraint is like in the future so the a_{2n}. a_{2n+1} can grow arbitrarily large and like all that matters is that the next two, like the next a_{2n} and a_{2n+1} are bigger than that. … So like a_n is only constrained by a_{2n} and a_{2n+1}. But uh nothing before it so my initial inclination is to say that it's false.

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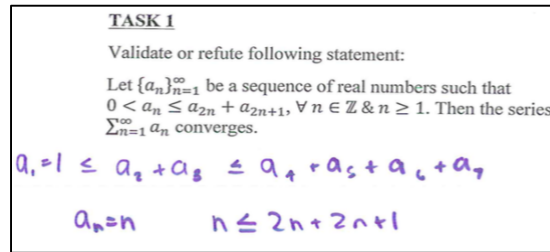
Figure 1: The task statement for Task 1 and Cassie's counterexample to the statement.

The first level of coding involved the identification of the mathematical objects used by Cassie (*resources*) and the acts performed on or with them (*actions*). For instance, the inequality condition from the task statement was identified as a resource to perform the action of generating an example sequence. The second level of coding was to infer the intention behind Cassie's use of the resources and actions. In the above transcript, all the identified actions and resources seemed to be expressly used with the intention of generating an example sequence to refute the statement. Thus, the second level of coding generated *bundles* of actions and resources based on their common intention. This was done iteratively for every task worked on by every participant. The final level of coding involved comparing the bundles identified across every task for each single participant and across every participant for each single task. For more details about data collection and of the GT phases of analysis see Karunakaran (2014).

Like typical GT studies, the claims generated for the expert and novice groups in this study were solely derived and supported by the data. This was achieved by constant and repeated examinations of the data for confirming and for disconfirming evidence. Since the emergent claims during the GT process were constantly evolving based on the confirming or disconfirming evidence found with continuing analyses, the final claims put forth by this method were necessarily consistent with the corpus of data collected within each group.

Knowledge in Pieces (KiP) and Knowledge Analysis (KA)

KiP models knowledge as a system of diverse elements and complex connections. One of the main principles of KiP is that knowledge is context sensitive (Smith, diSessa and Roschelle, 1993). This means that the productivity of a piece of knowledge is highly influenced by the context in which it is used. In contrast to studies that focus on identifying students' misconceptions, KiP focuses on the ways that students build new knowledge onto their prior knowledge. Adopting this theoretical framework implies that one analysis in this paper focuses on ways that Cassie builds on her prior ideas while suspending judgment about her correctness.

There are different types of KA studies. *Microanalytic* studies are a type of KA studies, which focuses on identifying knowledge elements/resources and how they are used in real–time reasoning. Knowledge resources considered in KA do not have to be mathematical, and can be intuitive in nature. KA tends to focus on a short segment of thinking, and document moment–by–moment changes in the process by which different ideas develop in students' engagement with the topic. The KA in this paper segmented Cassie's engagement with the task into thematic episodes. It exploited any relevant data (e.g., gestures, other parts of transcripts) to optimally understand the activation of knowledge resources in various contexts. The analysis then generated multiple models (interpretations) of Cassie's argument in each episode, and put these models in competition with one another. This process of *competitive argumentation* (VanLehn, Brown, & Greeno, 1984) was used to refine interpretations of Cassie's thinking. In this paper, we illustrate the use of the counter models with the first episode of Cassie's engagement with task 1. For the rest of the episodes we only present the final model from the analysis due to space constraint.

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Analysis

Grounded Theory Analysis

The GT analysis found, through comparison of bundles across the tasks and students, that if the novice provers (NPs) searched for a counterexample to invalidate a given task statement then they seemed to require an earlier rationale (intuitive, inductive, deductive, or otherwise) for the invalidity of the statement. That is, the NPs needed to already believe that a statement is invalid, before they searched for a counterexample. An alternative to this would be the strategy of using the search for a counterexample as an investigative tool, without any earlier rationale for the invalidity of the statement. This alternative strategy was more descriptive of the expert provers in the larger study.

For instance, when given the statement in Task 1 (see Figure 1), Cassie stated that, "with sequences I usually start just by like counting numbers and then seeing if it seems like it's gonna converge or not." She then assigned the first term of the sequence to be 1. Cassie went on to realize that since the inequality condition presented in the task statement did not place a strict upper bound on the terms of the sequence. She justified this by stating, "the only constraint is like in the future so a_{2n} and a_{2n+1} can grow arbitrarily large and like all that matters is that the next two, like the next a_{2n} and a_{2n+1} are bigger than that." She then made her initial conclusion that the statement of Task 1 was invalid. Cassie then explicitly expressed her intention of finding a counterexample. She proceeded to generate the sequence $a_n = \{n\}_{n=1}^{\infty}$ (where n is a positive integer) as a valid counterexample to the statement of Task 1. That is, the sequence $a_n = \{n\}_{n=1}^{\infty}$ satisfies the inequality condition $0 < a_n \leq a_{2n} + a_{2n+1}$, but the corresponding series diverges.

Cassie also routinely used examples that were generated only by using the constraints or assumptions present within the task statements. These types of examples are termed as *constructed examples*. For instance, when Cassie contended with a modified version of the statement in Task 1, she used only the constraints within the task statement to generate more examples. The modified task statement was identical to the original statement, except for the last word changed from "converges" to "diverges." Cassie seemed to focus on constructing an example of a sequence that expressly satisfied the inequality condition $0 < a_n \leq a_{2n} + a_{2n+1}$. She successfully did this by generating the sequence {1, ½, ½, ¼, ¼, ¼, ¼, … } with the goal of constructing a sequence with converging terms, and where the corresponding series diverges.

The previously described instances are exemplars of Cassie's proving behavior. This behavior was consistent with the proving behavior of other members of the greater NP group. In the next section, we present the KA of Cassie's proving process separate from her group. We analyzed Cassie's engagement with Task 1, which was split into three episodes. The analysis was organized chronologically.

Knowledge Analysis

In episode 1, Cassie came up with the sequence $\{n\}_{n=1}^{\infty}$ as a counterexample for the statement in Task 1. She relied on a transitive property of inequalities and inferred the strictness of the problem's constraint.

> *Cassie:* So if a_n were 1, uh a_1 were 1 then that would have to be less than or equal to a_2, a_3. So and then that would have to be, that would have to be less than or equal to a_4 plus a_5 plus a_6 plus a_7, a triangle inequality.

Cassie started by plugging in numbers for n and described the nature of the sequence based on the inequality, $a_n \leq a_{2n} + a_{2n+1}$. She immediately saw that if the terms of the sequence satisfied the inequality for all integers n, then $a_1 \leq a_2 + a_3$ meant that $a_1 \leq a_2 + a_3 \leq a_4 + a_5 + a_6 + a_7$. She called this a triangle inequality. We posit that she mistakenly referred to the transitivity property of inequality as a triangle inequality.

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Cassie: I'm inclined to say that it's false because the, I mean the only constraint is in the future. So the a_{2n}, a_{2n+1} can grow arbitrarily large and all that matters is that the next two, like the next a_{2n} and a_{2n+1} are bigger than that. So. /.../ So like a_n is only constrained by a_{2n} and a_{2n+1}, but nothing before it. So my initial inclination is to say that it's false.

At this point Cassie saw that each term of the sequence could grow to become arbitrarily large insofar as it was smaller than its associated $a_{2n} + a_{2n+1}$. Her use of the phrase "in the future," "all that matters," and "nothing before" suggests that Cassie did not see the constraint as particularly strict, or as immediately affecting the terms of the sequence. The inequality only required that the sum of the future terms of the sequence had to be larger than the current term. Thus, given that the sequence could grow to become arbitrarily large, and there was not an immediate constraint on the rate of growth, Cassie posited that a sequence that satisfied the inequality could diverge, and thus the statement was false.

Cassie: So let's see (pause) um (pause) yeah I mean it, it seems like the sequence just a_n equals n wouldn't converge because obviously we have the n's less than or equal to $2n$ plus $2n+1$. So that meets the criteria but /.../ which means the a_n's are going to infinity. So the sum wouldn't converge.

Cassie used the sequence a_n as a counterexample to disprove the claim in task 1.

Model. By plugging in values for n, Cassie immediately recognized the behavior of the terms of the series. Applying the transitive property, she concluded that $a_1 \le a_2 + a_3 \le a_4 + a_5 + a_6 + a_7 \le \cdots$. This seemingly "nested" quality of the terms did not play a major role in the process of coming up with the counterexample. Cassie asserted that the terms of the sequence could grow to become arbitrarily large, and she deemed the constraint of the terms' needing to be smaller than a sum of two consecutive future terms to be relevant, but not immediately consequential to the growth of the terms. With this she came up with the sequence $a_n = \{n\}_{n=1}^{\infty}$ as a counterexample to disprove the claim in Task 1.

Counter Model. Cassie initially considered the sequence $a_n = 1$ ("if a_1 were 1,"), which she would return to later. However, she noticed that the terms were only constrained by a_{2n} and a_{2n+1}, but "nothing before it", and so any a_i's before a_{2n} could behave without constraint. Thus, the unpredictability of the terms before a_{2n} and a_{2n+1} led her to believe that it could diverge. So she came up with the $a_n = \{n\}_{n=1}^{\infty}$ as a counterexample because it satisfied her initial thought of $a_1 = 1$ and a_n grew to become really large before it reached a_{2n}.

We posit that students' taking $a_1 = 1$ is a common practice. After that, Cassie worked with an arbitrary sequence a_n, and did not revisit $a_1 = 1$. She later came up with the sequence $a_n = 1$, but only after she recognized the lack of strict inequality in the constraint. The counter model suggests that Cassie did not treat the sum of a_{2n} and a_{2n+1} as the bound, but instead the individual terms. Her acknowledgement that $n \le 2n + 2n + 1$ with her counterexample refutes the counter model. Cassie understood that the inequality constrained the terms, but attributed the looseness of the constraint to the possibility of the terms diverging, which motivated her counterexample.

In episode 2, Cassie constructed an example of a sequence that satisfied the inequality, and where the corresponding series diverges, by recognizing and utilizing the lack of strict inequality in $a_n \le a_{2n} + a_{2n+1}$. Using the transitive property from earlier, she constructed a sequence $\{a_n\}$ where $a_1 = 1$, $a_2 = a_3 = \frac{1}{2}$, and $a_4 = a_5 = a_6 = a_7 = \frac{1}{4}$ and so on, such that $1 = a_1 = a_2 + a_3 = a_4 + a_5 + a_6 + a_7$. She generalized the pattern associated with this example, where the 1's were made of 2^{n-1} many terms of $(\frac{1}{2})^{n-1}$. She effectively used a sequence $b_n = 1$ as a counterexample, where b_n followed the pattern $b_1 = a_1$, $b_2 = a_2 + a_3$, $b_3 = a_3 + a_4 + a_5 + a_6$, etc. Cassie asserted that her example was a defining example for the claim that any series satisfying the task's criteria must always diverge.

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In episode 3, Cassie justified that claim by arguing that it was not possible for any series that satisfied the given criteria to be convergent. Cassie explained that with two general related sequences (say, a_n and b_n), if b_n was a strictly increasing sequence, then $\sum b_n$ (and thus, $\sum a_n$) would be infinity (The Divergence Theorem), and therefore the series would not be convergent. Cassie asserted that the best–case scenario for the series to be convergent would be if all the b_n's were equal, like with her counter example. Cassie proved the statement by using her example as a boundary case.

In summary, Cassie's engagement with the task and the modified task is sophisticated. She productively inferred the implication of the loose constraint of the inequality to the rate of growth of the sequence, and used it to find the first counterexample. She attended to relevant details about the task, like the lack of strict inequality to construct her example. She recognized $\sum 1$ as an example, and using the transitive property constructed and generalized the sequence $1, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \dots$. She recognized her example as a boundary case and worked with two different, albeit related sequences, a_n and b_n to prove the modified statement.

Discussion and Implication

The results from the two analyses provide insights into Cassie's proving process, albeit at different grain sizes. The Grounded theory (GT) analysis provided broader patterns about Cassie's process as a representative of the novice prover (NP) group across the five tasks. Particularly pertaining to the need to find a counterexample to disprove a statement, Cassie needed prior conviction (grounded in a particular rationale) about the fallacy of the statement. To then construct it, Cassie solely focused on the explicit constraints set in the problem statement.

Knowledge Analysis (KA) focused on the details of Cassie's engagement with one task to identify knowledge resources that were influential in her proofs. While the GT analysis also started with identifying actions and resources, albeit strictly mathematical, the ultimate result of that analysis necessarily removed details that do not apply across the NP population. KA remained at that level of detail in providing specific insights about Cassie's proving process.

KA also highlights the power of Cassie's productive inferences in constructing a counterexample, and to generalize from an example. Cassie did need the prior conviction before constructing a counterexample. However, that prior conviction came from a productive inference she made about the loose constraints of the inequality on the growth rate of the terms of the sequence. In addition to providing the prior conviction, her attention to task constraints (lack of strict inequality) also allowed her to construct a counter example ($b_n = 1$). She then productively inferred that a sequence satisfying the equation, $a_n = a_{2n} + a_{2n+1}$, was a boundary case, and used her example to prove the modified claim. KA confirms the findings from the GT analysis, but allows us to observe Cassie's sophistication through the details of her proving process.

These two analyses together provide a more complete picture of Cassie's proving process. On the one hand, this is common sense. Employing complementary methods results in richer analysis. On the other hand, to our knowledge this might be one of the first studies that uses GT analysis in concert with KA, thereby crossing the boundaries between the two methods set out in Parnafes & diSessa (2012). We now discuss the importance of considering these two methods together, and the danger of privileging only one of the analyses.

Combining the two methods provides a more accurate positioning of Cassie as a prover. The result of the GT analysis positioned Cassie as a consistent member of the novice prover group, whereas KA was able to uncover Cassie's sophistication. Adiredja (2015) has argued that theoretical perspectives of cognition hold the power in determining *what* counts as productive mathematical practice, and who are deemed as "successful" learners, highlighting the connection between cognition and equity issues. While KA and KiP value and prioritize knowledge and sense making of novice learners, and are against treating novices' knowledge as a subset of that of experts, GT is not tied to any such particular theoretical perspectives. In fact, the GT analysis in this paper was

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particularly mindful of the danger of positioning the novice group in a deficit way. However, by grouping students a priori as novices and experts, the study was in danger of beginning with a particular positioning of students before the analysis even started. This discussion highlights the importance of framing for GT studies, and how framing, in addition to theoretical perspectives also contribute to students' positioning. We were able to mediate that concern by combining the two methods.

Combining the two methods also mediates some generalizability concerns of KA and problematizes the common novice–expert dichotomy. By favoring depth and richness of analysis of cognition, one of the potential limitations of the findings from the KA done in this paper is the lack of immediate generalizability of its findings to other subjects. In a typical microanalytical KA study, the analysis would continue to identify particular kinds of resources or knowledge elements that Cassie used. Then it would examine the generalizability of those theoretical entities with other students. As is, in this paper, while KA was able to show Cassie's sophistication in her proving process, that sophistication is unique to Cassie. Little can be said about how undergraduates generally prove or construct counterexamples.

The GT analysis grounds Cassie's sophistication in her membership in the NP group as implicated by her proving process. Without the GT analysis, any resemblance of Cassie's proving process to those of an expert, or any sophistication could be attributed to her being an exception. The fact that Cassie's proving processes were consistent with those of the NP group, allows her sophistication to challenge the novice–expert dichotomy. Researchers have argued against the over–privileging of experts' knowledge, and suggested the shift in focus to understanding novices' knowledge in their own terms (diSessa et al., 2016; Smith, diSessa & Roschelle, 1993). In fact, Weber et al. (2014) have shown the continuity between novice and experts proving behaviors.

In summary, putting the two methods in communication with each other proved productive. The varying grain sizes in the results of the analyses provide a more complete reporting of the patterns we observed about Cassie's proving process. Mindful of the role of cognitive studies in positioning students, the two methods together also provide a more accurate positioning Cassie as a student. Related to that point, we problematize the common expert–novice framing of studies. Our dual analyses highlight how framing, independent from methods and theory, also contribute to the positioning of students. At the same time, the analyses also highlight the power in combining the two methods in challenging the subset model of novices' knowledge to that of experts.' By only focusing on Cassie, we were able to see the power of her proving process and the important inferences she was able to make.

Acknowledgments

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DESIGNING A STAGE-SENSITIVE WRITTEN ASSESSMENT OF ELEMENTARY STUDENTS' SCHEME FOR MULTIPLICATIVE REASONING

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In this paper we examine the application of Tzur's (2007) fine-grained assessment to the design of an assessment measure of a particular multiplicative scheme so that non-interview, good enough data can be obtained (on a large scale) to infer into elementary students' reasoning. We outline three design principles that surfaced through our recent effort to devise a sequence of items to assess at which stage—participatory or anticipatory—a child might have constructed the multiplicative double counting (mDC) scheme (Tzur et al., 2013). These principles include the nature and sequencing of prompts, number choice, and authenticity of responses.

Keywords: Number Concepts and Operations, Assessment and Evaluation

Introduction

In this paper we articulate a design process, focusing on assessing upper-elementary students' mathematical reasoning through a written instrument to develop an assessment tool sensitive to students' multiplicative reasoning. Through our project[1], we intended to assess over a 1,000 students, with future follow-ups likely to scale-up. Therefore, an immediate challenge presented itself: how to gain a trustworthy inference into a child's reasoning without conducting interviews with each participating student.

Following the approach of Norton and colleagues (e.g. Norton & Wilkins, 2009), we decided to try producing a written, pencil-and-paper assessment tool[2]. The written format would have to reflect the theoretical distinctions—schemes and stages in students' multiplicative reasoning—that we intend to measure. Our project team decided to assess two of the six schemes articulated by Tzur et al. (2013). In this paper, we focus on the design process, and design principles gleaned from the production of assessment items for the first of those six schemes (further explained in the next section), which marks a child's conceptual leap from additive to multiplicative reasoning. As this is not an empirical study, sections of the paper do not follow a canonical format of a research report.

We contribute an important elaboration on the work of Norton and colleagues (e.g. Norton & Wilkins 2009)—the need to distinguish a stage at which a learner's scheme might be constructed. Their work demonstrated how written assessments could provide a good enough substitute to inferences made from data collected in time-consuming, effort-intensive interview settings. A written assessment constrains and/or complicates the possibility for a researcher (or teacher) to make inferences about students' reasoning, because data obtained from written items hardly indicate language and actions the child might be using (e.g., moving fingers, counting to oneself). To overcome these challenges, assessment items Norton & Wilkins (2009) created for children's fractional schemes were highly correlated with interview-based inferences. Norton & Wilkins (2009)

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tested a correlation against the underlying schemes a child might be using while successfully solving assessment items—first written and then through interviewing. The premise of their correlation seems to be that a child's ability or inability to solve certain items reflects having or not having constructed the underlying scheme, respectively.

In our project, we take a compatible, yet more nuanced approach. We premise our project on the recognition of gradation in successful solution to assessment items that is contingent upon the stage at which a learner has constructed the measured (inferred) scheme. Specifically, we articulate our design process and posit design principles that attempt to distinguish children who constructed a new scheme only at the participatory stage, which limits a learner's use of an evolving scheme to items only if somehow being prompted (Simon, Placa, & Avitzur, 2016; Tzur & Simon, 2004).

Conceptual Framework

We ground our conceptual framework in constructivist scheme theory (Piaget, 1985; von Glasersfeld, 1995), meaning that we define learning as a conceptual change brought forth through a learner's own goal directed activity (Simon, Tzur, Heinz, & Kinzel, 2004). We use "scheme" to indicate a three-part conceptual 'building block': a situation ("assimilatory template") that sets the learner's goal, an activity triggered to accomplish that goal, and a result that can turn into a prediction. We distinguish two stages in a learner's development of a new scheme: participatory and anticipatory (Tzur & Simon, 2004). Broadly, we classify a learner as "participatory" if she needs a prompt to elicit a new scheme, and "anticipatory" if she does not need a prompt to elicit a new scheme. In the context of assessment, we use "prompt" to refer to a written statement or diagram used to elicit the use of a new scheme that, otherwise, would not be brought forth.

Tzur and Simon (2004) proposed the participatory/anticipatory stage distinction to explain the "next day" phenomenon, which entails the inconsistent accessibility learners seem to have to an evolving, new way of reasoning. Initially, a learner might connect the last two parts of a scheme, as she might notice novel effects that the activity brings about. While the activity and its effect(s) are enacted, the learner can anticipate future 'mental runs' of the activity to yield the same effects. However, at the participatory stage the newly noticed activity-effect is yet to be linked with the situation/goal. Thus, at a later time ("next day"), if/when the activity has not been triggered and 'ran' to bring the newly noticed effects—the learner is likely to 'revert back' to using previously established, anticipatory schemes (Tzur & Lambert, 2011).

To distinguish between participatory and anticipatory stages, our assessment of a learner's way of reasoning needs to be sensitive to a learner's independence (or lack thereof) in using a newly constructed scheme (Tzur, 2007). To make participatory/anticipatory stage distinctions, we follow his recommendation to use fine grain assessment that proceeds from prompt-less items indicative of the anticipatory stage to prompt-dependent items indicative of the participatory stage of a new scheme. We provide a twofold reason for our choice, First, when/if prompted, a learner at the participatory stage can solve assessment tasks in ways that provide 'false positive' data—as if the learner is properly using the new scheme. Therefore, providing prompts too early may prohibit distinguishing a learner at a participatory stage from a learner at an anticipatory stage of the same scheme. Second, if assessment items avoid prompting for an evolving scheme altogether, then the learner's prompt-independent schemes (anticipatory stage) are expected to trump the use of prompt-dependent schemes (participatory stage) (Tzur & Lambert, 2011). In such a case, the learner's responses to assessment items will provide 'false negative' data—as if unable to use the newly constructed scheme, whereas she can actually reactivate and properly use the new scheme once prompted. This possibility for a 'false negative is a key aspect of our elaboration on Norton & Wilkins (2009) work.

Our assessment draws on findings about students' constructions of six schemes for multiplicative reasoning (Tzur et al., 2013). A core goal of our assessment is to identify stages in learners'

transition from additive to multiplicative reasoning. We use learners' coordination of composite units—single "things" a learner can conceive of as being made up of sub-parts (Steffe & von Glasersfeld, 1985) —to distinguish between additive and multiplicative reasoning (Clark & Kamii, 1996; Steffe & Cobb, 1998). Consider the problem: *Maria is making treat bags for her 3 friends; each treat bag will have 8 candies; how many candies does Maria need in all?* A child using additive reasoning preserves the type of composite unit (e.g., 8 candies + 8 candies + 8 candies = 24 candies). In contrast, a child using multiplicative reasoning transforms the type of composite unit, and for the candy bag problem there would be two types of composite units (8 candies per bag; 3 bags). A child using multiplicative reasoning coordinates two different types of composite units by distributing items of one composite unit over the items of another composite unit; the result of this distribution is a third, different kind of unit (e.g., 8 *candies per bag* distributed into each of 3 *bags* = 24 *candies*). Tzur et al. (2013) termed this scheme of multiplicative reasoning *multiplicative double counting* (mDC), to reflect the way a child determines the total number of 1s in the coordinated quantity by operating simultaneously on two counting sequences (e.g., 1-is-8, 2-is-16, 3-is-24).

Design Process and Principles

In this section, we articulate our process for designing a written assessment distinguishing between participatory and anticipatory stages of the mDC scheme. We chose the mDC scheme because it reflects a child's crucial conceptual leap from additive to multiplicative reasoning. A minimum of 3 items on a written assessment is considered necessary to provide a valid and reliable score of a theoretically grounded construct (Hinkin, Tracey, & Enz, 1997; Raubenheimer, 2004). To play it safe, our team decided to begin by developing five items – up to 2 of which could later be eliminated. This paper focuses on the creation of one item—a sequence of questions—the design of which proved most challenging. This item, initially asking (with drawing like Fig. 1) "How many small, gray boxes are needed to fill the large box?", required most of our effort and recurrent revisions, which assisted in our explication of design principles.

The goal we set for each assessment item was to provide data to distinguish the stage, participatory or anticipatory (Simon et al., 2016; Tzur & Simon, 2004), at which a child had constructed the mDC scheme. Following the fine-grained assessment rationale (Tzur, 2007), each item had to progress from prompt independent ("hard") to prompt dependent ("easy") questions, so that a child at the anticipatory stage can demonstrate reasoning without any prompts at the start of each item. The assessment item on which we focus here was created as a task of figuring out, in a context of two-dimensional shapes (here rectangles – see Figure 1), how many small units fit into an entire larger unit[3].

Figure 1. A small and a large rectangle providing a context for mDC.

For this item, we took caution to counteract giving a lower-level solution to a child already at the commencing, prompt independent version. For example, a child may simply draw all small rectangles and then count them one by one, which gives no information about a child's mDC scheme. We identified the need for this caution through informal interviews that one of our team members had conducted with her 7- and 9-year old daughters. The girls' use of this "easy-way-out" strategy made us realize that such a solution can yield the correct answer even for a child who is still operating on units of one (1s) and is yet to construct numbers as composite units—let alone

coordinate such units multiplicatively.

During an interview, an interviewer could present and monitor constraints to make "easy way out" strategies less likely. Furthermore, an interviewer could also probe (or avoid probing) to figure out if the child would spontaneously supply, say, columns (here, a composite unit of 7) and rows (here, a composite unit of 4) to organize and then execute a simultaneous count of the items that constitute each of those units (e.g., 1-has-4, 2-have-8, etc.). However, how might a written format steer students away from the "easy-way-out" strategy, and instead elicit data indicating an anticipatory, simultaneous operation on (coordination of) composite units?

Our tentative solution, which at the time of writing this paper is yet to be tested with more children, brought us face-to-face with the reading and writing capabilities, as well as known or unknown multiples of particular numbers (e.g., 5s vs. 7s), of 3rd and 4th grade students whose reasoning we intend to assess. Thus, on one hand, we strive to reduce the number of words in each assessment item as much as possible, and use active-voice phrases with simple-yet-precise mathematical terminology (e.g., the drawing is of rectangles, not of a cube or a box – which we initially used). We also want to allow a student to present her or his solution (answer + reasoning) while having to write as little as possible. On the other hand, we must provide clear directions about constraints we might have introduced in an interview (e.g., use numbers for which a child does not have a memorized sequence of multiples, such as 7). Several rounds of comments and suggested revisions from our project team members, led to changing the item's first question into: "How many small, gray rectangles fill the big rectangle? To find the answer, do NOT draw small rectangles." A drawing like in Figure 1 is presented below this revised question, and below that drawing three more probes are provided in large font:

A: Do you understand the question? (circle one)   Yes   No

B: Your Answer (fill in the blank): It takes _____ small rectangles to fill the big rectangle.

C: Show how you got your answer. Do NOT draw small rectangles.

With this first question to assess if a child has an anticipatory stage of the mDC scheme, we turned to creating subsequent questions that provide prompts for solving the problem by identifying and coordinating units a child may not yet be able to bring forth spontaneously. For each of those questions, we presented a drawing of the two original rectangles with some changes (see Fig. 2a-b-c), each figure followed by three probes like the ones above.

As our team discussed how to sequence the follow-up questions/drawings, we took notice of three design principles:

1. Sequencing gradually more explicit prompting for the mathematical reasoning at hand (here – units coordination);
2. Selecting "non easy" numbers to mitigate students' use of memorized facts and to engineer opportunities for students to simultaneously use composite units;
3. Eliciting authentic student responses under the given constraints.

To address the first design principle, we created the first of three prompts (Fig. 2a) for this assessment item to possibly elicit a child's organization of the rectangular space into two kinds of composite units. To this end, we drew (not fully!) 7 columns and just one row, using the same size as the small rectangle's dimensions. We considered this as the least explicit prompt of all three, because the child has not only to notice the dimension compatibility but also supply the organization of a unit of four 1s, and then coordinate the accrual of 7 units of 4 units of 1.

The second of three prompts further explicated 4 small units of 1 that would constitute a single, 4-unit 'column' (Fig. 2b), to possibly trigger the child's constitution of an anticipated, similar

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composite unit for each of the 6 other columns. Because adding those 4 small rectangles could confuse children about the reference in the written question, we ended up extending it as follows: "Maria drew four (4) small gray rectangles in the big rectangle. How many small, gray rectangles fill the big rectangle? To find the answer, you can draw up to 6 more small rectangles."

As seen in Fig. 2c, the third of three prompts explicated almost all small units of four 1s, while still leaving two columns for the child to supply such units and successfully complete the "filling" task. Here, we wanted to ensure a majority of the students would be successful in solving the problem while still having to anticipate the interjection of a composite unit of 4 into the two empty columns, hence showing at least a rudimentary form of an assimilatory scheme for such a unit. For students who may still be unable to solve the problem, this last prompt of the assessment item could provide an indication that, perhaps, they are yet to construct number (e.g., 4) as a composite unit.

The three prompts shown in Figures 2a-c illustrate our second design principle. For each assessment item we produce, we strategically choose numbers to mitigate students' use of memorized facts and engineer opportunities for students to use composite units. First, we chose "non-easy numbers" (4 and 7) for which many students may not have a memorized sequence of multiples that can mitigate the need to keep track of the simultaneous accrual of composite units and 1s that constitute them. Second, we chose to introduce the first 4 rectangles and allow drawing up to 6 more, so the child is still constrained to not drawing each and every unit of 1 (small rectangles) while having a possibility of producing a "mental marker" for the two sets of composite units (4 along the 'width' and 7 along the 'length').

Figure 2a. First prompt   Figure 2b. Second prompt   Figure 2c. Second prompt
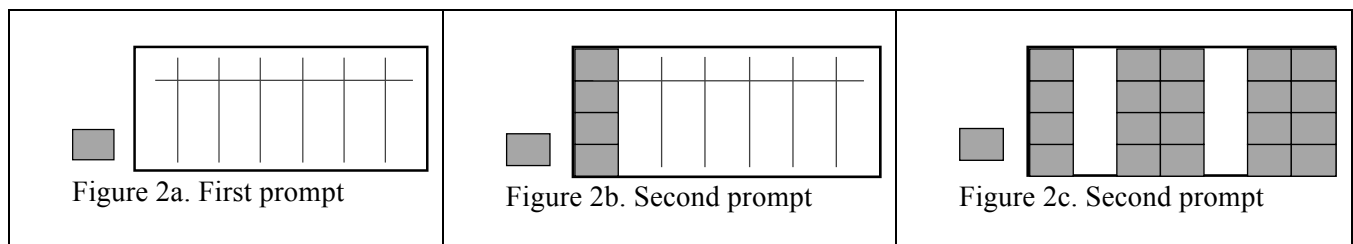
Figure 2. Three, diagram-based prompts for simultaneously coordinating 7 units of 4 (1s).

To address the third design principle, we faced a challenge of how to encourage children to express to others, with only a pencil and paper available, what they did in their mind to solve the item. In an interview, the child's bodily motions (e.g., fingers, head, eyes) provide an interviewer with hints as to follow-up probing questions to ask about the reasoning that took place. In a paper-and-pencil format, while trying to eliminate the need for the child to produce written, linguistic explanations, we instead guided the child as openly as possible ("Show how you got your answer") while maintaining the initial constraint ("Do NOT draw small rectangles"). We presumed that a child's drawing of some sort, likely using the figure provided as a basis, could provide a window onto the mental units and operations she or he used. It should be noted that administration of the assessment would include an explicit request to use drawing as a way to help us understand the student's thinking, as if she or he "shows it." We also decided to administer each assessment item, with its gradual-prompting versions, one question at a time. This way enables to invite the child to show how she solved a problem by approaching the person who administers the assessment. That person would document, in an abbreviated form, what the child said (e.g., "Used fingers; counted 4 and 4 = 8; then 8+8 = 16; + 8 = 24; +4 = 28). As soon as the assessment is over, that person would then select and record a code developed for conceptually distinct solutions.

Discussion
Guided by Tzur's (2007) constructivist notion of fine grain assessment, we identified three

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design principles that can guide the design of a written assessment for distinguishing, without conducting an interview, between a participatory and an anticipatory stage in students' reasoning (Simon et al., 2016). The principles we identified included (a) gradual sequencing of prompt-less to prompt-heavy assessment items, (b) selecting "non easy" numbers, and (c) eliciting authentic student responses. We illustrated how each of these principles was applied to the creation and refinement of items for assessing the stage at which a child might have constructed a particular mathematical idea—the multiplicative double counting (mDC) scheme (Tzur et al., 2013). In this section, we discuss two main contributions the paper can make.

First, we assert that providing guidance for assessing if a student is at the participatory or anticipatory stage in the construction of a new way of reasoning (scheme) seems complementary to the design of instruction so that a classroom zone of proximal development (ZPD) can be promoted (Murata & Fuson, 2006). We root our assertion in Jin & Tzur (2011) and Tzur & Lambert's (2011) postulation of a theoretical linkage between the participatory/ anticipatory stage distinction and Vygotsky's (1986) core notions of ZPD/ZAD (zone of actual development), respectively. Written assessments that allow such a distinction seem key to effectively selecting (a) goals for what each student should learn next based on what they already know and (b) instructional activities that may facilitate such learning. Specifically, instructional activities would attempt to foster two types of reflective processes proposed by Tzur (2011) as suitable for promoting the construction of each stage. Simply put, the design process and principles depicted in this paper explicate important considerations for assessing cognitive correlates of ZPD.

Second, this paper also contributes to the ongoing efforts in the field to figure out ways to both infer into and promote key developmental understandings (KDUs, Simon, 2006) in students. Specifically, our paper focused on the work devoted to creating a stage-sensitive assessment of the first scheme (mDC) a child may construct when embarking on the conceptual leap required to transform one's additive into multiplicative reasoning about whole numbers. As noted in the Conceptual Framework, such a shift in reasoning is considered central to children's mathematical progression (e.g., to fractions and later to algebra). Illustrating how a scalable assessment can help identify students whose initial way of reasoning multiplicatively is still prompt-dependent, and thus prone to being left behind if instruction moves along prematurely, may provide a model for creating such an assessment for other KDUs.

Endnotes

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[2]Testing the tool for validity and reliability would involve a sequence of steps that includes administering the assessment items in a written format and a follow-up interview format to a sample of ~25 students, and figuring out the correlation between each student's work on each item in both format.

[3]Andy Norton suggested this item to us, based on his use of a similar item to assess, without the stage distinction, a fraction scheme.

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ALGORITHMIC THINKING: AN INITIAL CHARACTERIZATION OF COMPUTATIONAL THINKING IN MATHEMATICS

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Computer scientists have reported on "computational thinking," which Aho (2012) defines as "the thought process involved in formulating problems so their solutions can be represented as computational steps and algorithms" (p. 832). We wanted to investigate such thinking in mathematics. To study this, we interviewed five mathematicians about the role of computation in their work, and the notion of "algorithmic thinking" developed from these interviews. We found that mathematicians value this thinking, and in this paper we define algorithmic thinking and present a number of applications and affordances of such thinking.

Keywords: Post-Secondary Education, Cognition, Reasoning and Proof

Introduction and Motivation

Suppose we asked you how many ways there are to arrange the letters in the word COMPUTER, and, even more, we asked you to sketch out an algorithm that would enumerate an organized list of all the arrangements. Or, considering other contexts, suppose we asked you to write a program to compute the standard deviation of a data set or to shift a graph of the function $y = sin(x)$ to the right by π radians. Is there a certain way of thinking, a type of knowledge, or a set of skills that you would use to successfully complete such tasks?

These kinds of questions motivate our work, and broadly we seek to explore the ways that technological tools shape the way that people think about and do mathematics. One component of this is to consider the role of procedures and algorithms in doing mathematics, as well as the kinds of thinking that might underpin the design and implementation of such procedures. In the literature, there are ongoing conversations about the nature of procedural knowledge (e.g., Hiebert & Lefevre, 1981; Baroody, et al., 2007; Star 2005, 2007), and there have been explicit calls to foreground the study of procedural knowledge so that it may be better understood (Star, 2007). As a preliminary step toward understanding such thinking, we interviewed mathematicians about the role of computation and algorithms in their work, whether they think there are certain ways of thinking or activity to help students learn and successfully implement computation, and whether or not such thinking is important. By interviewing mathematicians, our goal is to provide evidence that members of the mathematical community value such thinking and that it may be beneficial to foster such thinking among students. We consider these findings a proof of concept, supporting the idea that such thinking is of value to members of the mathematical community and is worthy of further investigation. Specifically, our research questions are: a) How do mathematicians characterize the kind of thinking that facilitates computation in their work, and b) what are applications and affordances of such thinking?

Background Literature

In computer science, there is a construct called *computational thinking* (CT) (Grover & Pea, 2013; Wing, 2006; 2008), which Wing initially described as "taking an approach to solving problems, designing systems and understanding human behaviour that draws on concepts

fundamental to computer science" (2006, p. 33). Wing went on to characterize CT broadly and as encompassing many kinds of thinking and activity, such as "thinking recursively" (p. 33), "using abstraction and decomposition when attacking a large complex task or designing a large complex system" (p. 33), "using heuristic reasoning to discover a solution" (p. 34), and "making trade-offs between time and space and between processing power and storage capacity" (p. 34). This broad characterization makes it difficult to pin down a precise definition, but it provides a start for identifying common threads among computational activity, suggesting that perhaps there are common ways of thinking that underlie reasoning about computation and algorithms. Aho (2012) built on these definitions, and he considered "computational thinking to be the thought process involved in formulating problems so their solutions can be represented as computational steps and algorithms" (p. 832). These computer scientists seem to be articulating a way of thinking that they view as essential to the kind of work they do in their field, and these initial definitions of computational thinking as inspiring this study. There has also been some work in STEM education related to computational thinking (e.g., Weintrop, Beheshti, Horn, Orton, Trouille, Jona, & Wilensky, 2014; Wilensky & Reisman, 2006). We are motivated to understand whether there is some version of computational thinking that might be applied in a mathematical context.

Theoretical Framework

Conceptual and Procedural Knowledge

Hiebert and Lefevre (1981) elaborate an important distinction between conceptual knowledge and procedural knowledge, and since then many more mathematics education researchers have contributed to the discussion on the topic (e.g., Baroody, Feil, & Johnson, 2007; Star, 2005, 2007). Hiebert and Lefevre characterize *conceptual knowledge* as "knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information" (p. 3-4). They define *procedural knowledge* as being made up of two parts: "One part is composed of the formal language, or symbolic representation, of mathematics. The other part consists of the algorithms, or rules, for completing mathematical tasks" (p. 6). Hiebert and Lefevre (1986) also made a distinction between meaningful and rote learning. They claim that "meaning is generated as relationships between units of knowledge are organized or created," (p. 8) and noted that by their definition, conceptual knowledge must be learned meaningfully (p. 8). Rote learning "produces knowledge that is notably absent in relationships and is tied closely to the context in which it is learned" (p. 8), and they state that conceptual knowledge cannot be developed by rote learning, while procedural knowledge can. Although procedural knowledge and rote knowledge ought not to be conflated, there has been a tendency to characterize procedural knowledge as being shallow. As Star (2005) points out, Hiebert and Lefevre characterize the sequential nature of relationships in procedural knowledge, and "by this definition, procedural knowledge is superficial; it is not rich in connections" (p. 407). Star (2005, 2007) argues that this characterization, in part, has contributed to perhaps a devaluing of procedural knowledge and resulted in a lack of empirical research on procedural knowledge.

We acknowledge that there is some debate as to the nature of procedural knowledge, and that procedure knowledge has typically been related to students' implementation of known procedures. The mathematicians we interviewed talked about not just the implementation of procedures, but also the development and designing of procedures and algorithms. We are interested in studying the kind of thinking and knowledge involved in designing procedures, and we wonder whether such knowledge is related to (or is an extension of) procedural knowledge as it is currently characterized. With these ideas in mind, the constructs of conceptual and procedural knowledge helped to frame our study. We hope that our findings can contribute to Star's narrative that procedural knowledge can be deep and important, ultimately providing insight about why and how such thinking might develop.

Methods

To accomplish these goals, we interviewed five mathematicians in single 60-90 minute semi-structured interviews. Four of the mathematicians were professors in mathematics departments (M1 was a mathematical biologist, M2 an applied mathematician who specialized in modeling, M3 a numerical analyst, and M4 a geometer), all holding PhDs in mathematics, and one worked in industry and had a Master's degree in statistics. All interviews were audio-recorded. We were motivated by the literature on computational thinking (Wing, 2006, 2008), and we initially framed the interviews in terms of better understanding the role of computation. Thus, we asked the mathematicians to reflect upon various aspects of computation, including computation in their own work, the value of computation for themselves and for students, how they might teach computation, and whether there were particular kinds of thought that might support computation. Because terms like "computational thinking" or "algorithmic thinking" were not likely to carry much meaning with the mathematicians initially, and because of the various areas of mathematical expertise, we sought to establish some common language for each interviewee.

The first half of the interview was thus spent asking mathematicians about their own particular work, and especially for examples of computation in their own work. The interviewer could then follow up with appropriate questions that could target particular questions about (what we now term as) algorithmic thinking. For example, after asking some preliminary demographic questions, we asked the following: *Do you use computation in the work that you do? How so? How are you defining 'computation'?* and *What are some specific ways (or contexts) in which you use computation in your work?* This enabled us to get a sense of how they viewed and might use computation. When we had an example, we could ask questions such as *Is there a certain type of thinking/behavior/activity that needs to be carried out when performing such a mathematical computation?* This was the crux of what we wanted to study in the interview, and we could adapt this question based on specific examples a certain mathematician would give. Once this kind of language and discussion was established, we could ask them for preliminary definitions of "computational thinking" and "algorithmic thinking." We also asked whether they thought computation is important for mathematicians to learn, and why or why not. We concluded with discussions about whether and how they had taught computation before, and for them to weigh in on how students might learn computation.

To analyze the data, we transcribed the interviews, and members of the research team organized responses to the various questions. Then, using a constant comparative method (Strauss & Corbin, 1998), we generated categories of relevant phenomena that emerged from passes through the transcripts, resulting in themes that shaped a narrative to describe the results.

Results

We remind the reader that our main research question was to investigate how mathematicians characterize the kind of thinking that facilitates computation in their work, and we were especially interested in studying whether or not *computational thinking* might be a meaningful term for mathematicians. Throughout the interviews, though, it became clear that the mathematicians found computational thinking to be too broad to characterize meaningfully, but that the term *algorithmic thinking* provided more appropriate language to describe the kind of thinking we targeted in the interviews. It is important to emphasize that the term algorithmic thinking is something that emerged from, and seemed to resonate with, the interviewees. In this results section, we first define and characterize algorithmic thinking and describe how it emerged from the interviews with the mathematicians. Then, we outline two reasons why mathematicians seem to value such thinking: first, because it has a variety of practical applications that they use in their work, and second, because such thinking offers broader implicit affordances that are desirable for doing mathematics. With

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these findings, we hope to convince the reader that algorithmic thinking is a specific, useful construct that is worth trying to engender in students.

Algorithmic Thinking

As noted in the methods, in each interview there was some effort required in getting on the same page with the mathematicians, having them articulate the kinds of computation they do in their work and the kind of knowledge, skills, or training might be necessary for such work. These conversations gave rise to the following preliminary, working definition of algorithmic thinking: *A logical, organized way of thinking used to break down a complicated goal into a series of (ordered) steps using available tools*. We note that our definition has similarities with Aho's (2012) characterization of computational thinking, although we emphasize using procedural steps to solve a given goal, and he emphasizes formulating problems so they may be solved with computational steps. To see how this definition emerged from the data and to elaborate various aspects of this definition, we highlight some of the mathematicians' responses. Due to space constraints, we report on M1's response in detail and only briefly mention comments from some of the other mathematicians.

The term "algorithmic thinking" first emerged in our interview with M1, whose work involves analyzing biological data. In describing how computation arises in his work, he noted that after familiarizing himself with data, he will ask himself a question like "Okay, what sort of math tools or what sort of algorithmic approaches would be fit for this approach?" Then, after implementation on test data, he might engage in an iterative process, asking himself "Okay, how can we modify this algorithm to make it work a little bit better?" M1's work involved not just computation, but in particular the development, implementing, and refining of algorithms.

We eventually asked M1 if he could define computational thinking or algorithmic thinking. He said, "computational thinking makes me think of somebody that has an eye to computation" and went on to say that, "algorithmic thinking is something I think that I could try to describe a little bit more." He gave the following definition of algorithmic thinking: "Yeah, so, I have only just today verbally articulated, but really like, the idea of how to practically solve a goal by breaking it down from a series of steps that consist only of the tools that you have at hand."

To clarify what he meant by "only the tools you have at hand," he noted that there are some restrictions depending on the context or environment in which one is creating an algorithm. In one context, like Matlab, there may be powerful packages that one can evoke, whereas in Mathematica the commands might need to be more specific. Thus, the algorithm must be created given whatever restricted set of tools is available. We have incorporated some of his ideas into our definition above, highlighting the notion of a "break down" and the idea "available tools" from M1's definition. Prior to our interview with M1, the term "algorithmic thinking" had not been something we were necessarily targeting (rather, we were interested in "computational thinking"). Through his interview, however, we realized that this might be an appropriate and meaningful term for the mathematicians. Because a working definition of "algorithmic thinking" emerged from the first interview, we subsequently asked about it in the remaining interviews.

M3 highlighted two features of algorithmic thinking – compartmentalization and order. He talked about compartmentalization as being able to "see the big picture but also recognize the smaller blocks." He went on to say, "It's the same thing when you're looking at code. It might be written linearly, but each chunk does different things." He also emphasized the importance of putting things in the proper order. He said, "The big picture has maybe three blocks. Those obviously have to be in the proper order, but the farther you zoom in, you still need to make sure each subsequent subsection is in the proper order." M5 also articulated the series of steps and the compartmentalization that M3 alluded to.

M5: I guess the algorithmic thinking is something I can relate to, based on the conversations we just had. Is that you have an idea of the series of steps that you need to go through to analyze a data set. And each one of those steps could then be decomposed into more steps.

The language of "ordered steps" in our statement is meant to convey the structured process that M3 and M5 articulate, and M3's compartmentalization echoes M2's "break down" language.

In our interview with M4, we were discussing the activity of writing code (specifically designing, as opposed to simply implementing, a package in Geometer's Sketchpad). We asked him, "What does writing the package bring that is beneficial?" and he responded, "So one of the things that it brings is these general critical thinking for programming skills." We then pressed him on what "critical thinking for programming" might mean, and he said, "Um, it's a very linear structure, a linear representation that I'll let, for the lack of a better name I'll call coding, um, that requires a particular form of creative, logical reasoning." Later, when discussing what the term "algorithmic thinking" might mean, M4 noted that such thinking was closely tied in to his notation of critical thinking for programming. We interpret then that his sense of algorithmic thinking involves a linear structure that requires logical reasoning.

The mathematicians also indicated that algorithmic thinking transcended particular programming environments or particular contexts. For example, M1 said, "Yeah, I want them to gain the skills, and I don't really care what the language is. Because once—it's really surprising. So once you've steeped yourself in one language and really understand the process of algorithm design and approximating mathematical quantities and practically implementing it, those same skill sets transfer over real quickly to some other language." M4 had a similar sentiment about the overarching nature of algorithmic thinking: "What I care about is that they develop both an appreciation for, and an ability with, interacting with a machine to problem-solve. And that they learn how to write some code, not just ask one-line questions, so that they have an experience with debugging code. Once they've done that for any language, they can build on that in any context they need it later." We feel that these comments are important because they highlight algorithmic thinking as a way of thinking that may be used and applied in a variety of contexts. They also underscore the fact that there is some kind of broader way of thinking that may be associated with engagement with designing and implementing procedures and algorithms.

To summarize, the mathematicians articulated some key features of the kind of thinking that might be involved in computation that entails writing, implementing, and checking algorithms. We address further issues related to the nature of algorithmic thinking in the Discussion section.

Practical Applications of Algorithmic Thinking

In the interviews, the mathematicians articulated several of the ways that they use computation and algorithms in their work. In this section, we use the term computation to mean the kind of computation that occurs in conjunction with algorithmic thinking (i.e., computation involving the design and implementation of procedures and algorithms). Without exception, the mathematicians saw algorithmic thinking as being useful. The following comments by M1 are representative of the opinions of a number of the mathematicians, which are that the ability to design and implement algorithms is a vital part of being a mathematician:

M1: Honestly with how data-driven [it] is and [is] increasingly becoming, and how everything Is electronics- and computer-based. I think it's pretty vital to have that, even if you're going to be some totally pure mathematician or field—like number theory or something like that— eventually there will be a real-world application. … Even being aware and having a little bit of exposure and practice of that sort of theory to practice algorithmic implementation and computationally thinking is, I think, critical and/or it would be an incomplete education without having it.

On the whole, the mathematicians articulated that such computation is useful because it allows them to accomplish some things that cannot be done efficiently by hand. Here we briefly mention examples of practical applications of thinking that emerged in the interviews.

First, some of the mathematicians seemed to frame such computation as a tool allowing them to do mathematics, which is perhaps not surprising. Here, computation is framed as a means to an end, something that facilitates the exploration of a conjecture or the generation of examples that ultimately serve the purpose of informing a theoretical mathematical exploration. M4 discussed his use of Mathematica, and he said, "the more Mathematica I have learned, the easier it has been to perform the complicated computations. But all of that elegance in programming is just to make my life easier. There's nothing inherently, um, there's, there's no direct contribution to the research project or the results that comes from my ability to program in Mathematica, beyond the fact that there is a way for me to check these difficult computations." His comment that the "elegance in programming is just to make my life easier" suggests a perspective of programming as a tool to streamline the mathematical exploration he needs to conduct. He then made the following comment, which again highlights the relationship between the role of computation and how it relates to his mathematical work.

> *M4*: So it's an interaction. I'm, I'm using Mathematica as a crutch to do my computations for me as I am thinking through which calculations to do…So I try things and see what happens. And Mathematica gives me a way to tell what's happening. And sometimes it's productive and sometimes it isn't. Um, there's a little bit of each. There are times when I know the answer and I'm just checking that it's true.

M1 and M5 offered a different kind of practical application, one in which the algorithm is itself an integral part of the work of analyzing data. M1 described a practical application of needing to refine a particular algorithm for the purpose of analyzing biological data: "Yeah, yeah, so basically there's this math quantity that I want to compute, called the earth-mover's distance, and there exists algorithms to compute it exactly, but they're too slow. So I'm going to approximate that actual metric in a heuristic, algorithmic fashion." Here the act of developing an algorithm is not just for facilitating mathematical exploration, but rather it itself is a fundamental aspect of the mathematical activity. M5 similarly used algorithms in order to analyze data statistically: "Well, yeah, I do use computation in that I analyze data for a living. And most of that is through a series of algorithms that I'll run on a set of data that can range from doing simple statistics to executing predictive models to graphics." We consider these examples as practical ways that algorithmic thinking can be used in both pure and applied mathematics, highlighting that mathematicians do draw upon such thinking that allows for a variety of applications across a number of contexts.

Implicit Affordances of Algorithmic Thinking

In addition to the practical applications for which algorithmic thinking might be useful, the mathematicians also suggested a number of other potential benefits of algorithmic thinking. This emerged through explicitly asking the mathematicians about whether and why students should develop this kind of thinking and why such computation is important for students to learn. There are three major affordances that emerged from the interviews.

First, algorithmic thinking seems related to mathematical practices such as proving and problem solving. For example, M1 said the following of algorithmic thinking: "It's problem solving. And it's, at its core, the same problem solving that mathematicians do all the time, which is "Here are the constraints, my assumptions. This is the goal, my theorem. Now what steps can I break it down to get over there?" And mathematicians do this all the time. It's just that the set of tools that they have are totally different than when you want to apply that same process to a computational problem." We infer that M1 is suggesting that the kind of thinking involved in designing and implementing an

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algorithm is closely aligned with mathematical problem solving, and thus that reinforcing algorithmic thinking could help to strengthen problem solving.

Second, M3 made interesting comments about the relationship between algorithmic thinking and communicating mathematical ideas with others. He said, "you couldn't make clear arguments to convince people of other types of conclusions that you're trying to make if you didn't take your arguments through logical steps. So if you're trying to convince anybody of something, you need to tell them that your solution, or your idea, does what you think it does and nothing else. And that's exactly what a code is supposed to do, too." He went on to say, "I'm saying that it, it not only helps with doing math. It also helps with communicating math." For his students, then, he saw that beyond whatever mathematical insight they were gaining, such thinking could help improve their mathematical communication skills.

Finally, a number of the mathematicians also talked about the importance of having a certain disposition toward being wrong and fixing mistakes, and they suggested a relationship between this idea and algorithmic thinking. For example, M2 said, "Well, of course, and there is the other ability of being able to recover from mistakes. Which, in computing, is fundamental. And not being too frustrated and just keep going back and forth. And trying to morph something that you know worked to something you know should work." Other mathematicians also suggested that debugging code is a key activity in working with algorithms, and we contend that debugging may help to engender the ability to recover from mistakes that M2 thinks is important.

In sum, the mathematicians articulated a number of affordances of algorithmic thinking that extend beyond practical applications. These results provide additional reasons for why mathematicians value such thinking and believe it is important for students to develop.

Discussion

An important note about our definition of algorithmic thinking is that in some ways it is similar to the kind of procedural knowledge that Hiebert and Lefevre define, as they highlight procedural knowledge as involving a sequential nature of steps: "A key feature of procedures is that they are executed in a predetermined linear sequence. It is the clearly sequential nature of procedures that probably sets them most apart from other forms of knowledge" (p. 6). However, our definition of algorithmic thinking goes beyond just the implementation of a procedure, or even the explanation of why a procedure works as it does. It involves planning and designing the steps, and having an overall sense of what the algorithm can do, as well as having the details in place to successfully implement the algorithm.

In this way, the mathematicians' characterization of algorithmic thinking may be related to Star's (2005) definition of deep procedural knowledge, which involves "knowledge of procedures that is associated with comprehension, flexibility, and critical judgment" (p. 408). In particular, algorithmic thinking seems to involve a decision-making process about what tools to use and how to organize them, and this necessarily involves a flexible way of thinking. We suppose, then, that this notion of algorithmic thinking may be conceived of as overlapping with procedural knowledge. Indeed, we argue that an understanding of how to design (and not just implement) an algorithm may be a hallmark of procedural knowledge that could be considered to be more than just superficial. We also see our work as contributing because we highlight what might be entailed in algorithmic thinking at a more advanced level and as a disciplinary mathematical practice. This offers another perspective on procedural knowledge, allowing another perspective to flesh out nuances of this important construct.

Conclusion and Avenues for Future Research

Although this work is preliminary, we believe that the mathematicians' responses suggest that it may be promising to pursue and refine this idea of algorithmic thinking and what it might entail, as mathematicians seem to believe that algorithmic thinking is a valuable component of the practice of

doing mathematics. In light of this, it is important for researchers to think through how we might help students develop this way of thinking. In terms of next steps, we hope to continue to investigate the nature of algorithmic thinking. In particular, we want to gather more evidence about algorithmic thinking from both mathematicians and students at a variety of levels, with the ultimate goal of better understanding the relationship between (deep) procedural knowledge and algorithmic thinking as well as how to engender such thinking among students.

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(IR)REVERSABILITY IN MATHEMATICS

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In this theoretical paper, I consider reversibility as a defining characteristic of mathematics. Inverse pairs of formalized operations, such as multiplication and division, provide obvious examples of this reversibility. However, there are exceptions, such as multiplying by 0. If we are to follow Piaget's lead in defining mathematics as the science of reversible mental actions, such exceptions must be examined. We consider the case of multiplying by 0 by adopting Davydov's model of multiplication as a transformation of units and by investigating the underlying mental actions. Results of this investigation have implications for breaking down the barriers between various domains of mathematics.

Keywords: Cognition, Learning Theory

As mathematics educators, our ultimate goal is to support students' development of mathematical knowledge. We can define this body of knowledge culturally, as the processes and products in which mathematical communities engage (cf., Harel, 2008), but this definition is circular (how do we recognize these communities as mathematical?). Dictionaries define mathematics as "a group of related sciences, including algebra, geometry, and calculus"; or the like ('mathematics', 2016). Such definitions defy the unity of mathematics and the emersion of new branches of mathematics, such as algebraic topology and game theory. Compare those definitions to the definition of biology: "the study of living organisms" ('biology', 2016). Like mathematics, biology has numerous branches, but those branches are unified under the umbrella of living organisms. If mathematics is a science, we should be able to define its objects of study in a unified manner—one that crosses the boundaries of its various branches.

Piaget (1970) defined mathematical objects as products of coordinated mental actions. In particular, logico-mathematical actions (operations) are characterized by their composability and reversibility. Composability empowers mathematical reasoning with the possibility of combining chains of mental actions. For example, students who *count on* can take a result from counting and combine it with further acts of counting to reach a new result. Reversibility guarantees perfect reliability in mathematics: "Because every operation is reversible, an 'erroneous result' is simply not an element of the system" (p. 15). Indeed, mathematics education researchers have studied reversibility as a critical aspect of students' development of mathematical reasoning (Greer, 2011; Hackenberg, 2010; Simon, Kara, Placa, & Sandir, 2016). By inverting mental actions, such as those involved in counting, students can return to the previous result, from which they can count on, again, with assurance that they will reach the new result again. Note that no other science has perfect reliability because the objects of study (e.g., living organisms) are derived from experimental observations (rather than mental actions)—experiments that cannot be repeated with perfect precision.

As examples of mathematical objects, consider the cube and the number 5. According to many definitions of mathematics, these objects are categorically different (shape and number), but Piagetian theory demonstrates that both arise from the coordination of composable and reversible mental actions. No one has ever seen a cube; what we see are two-dimensional projections. However, we know the three-dimensional object perfectly, through coordinated mental actions, such as rotations (Piaget & Inhelder, 1967; Roth & Thom, 2009). "Children are able to recognize and especially to represent, only those shapes which they can actually reconstruct through their own

actions. Hence, the 'abstraction' of shape is achieved on the basis of co-ordination of the child's actions and not, or at least not entirely, from the object direct" (Piaget & Inhelder, 1967, p. 43).

Mental rotations are composable and reversible, and by coordinating them, we can imagine the whole cube at once. Likewise, 5 arises through the coordinated activity of pointing and reciting a verbal number sequence, "one, two, three, four, five" (Piaget, 1942). This activity generates a one-to-one correspondence between items that are taken as units of 1 and elements of the verbal number sequence. On that basis, students develop the mental action of iterating a unit of 1 "five" times—a mental action that can be reversed by partitioning the collection into the constituent five units of 1 (Steffe, 1992).

As mathematical objects, students can act on the cube and the number 5 in new ways—ways that might not be possible with physical objects. For example, they can reflect a cube about a plane through its center (see Figure 1). This action is an involution; it reverses itself. Acting on mathematical objects in new ways provides the basis for constructing new mathematical objects, at a higher level, and this is the sense in which mathematics builds upon itself.

Consider the number 5 again. As a unit containing five units of 1, 5 is an object that can be acted upon through mental actions associated with multiplication (Steffe, 1992). By coordinating these mental actions, products of two numbers can become objects in themselves.
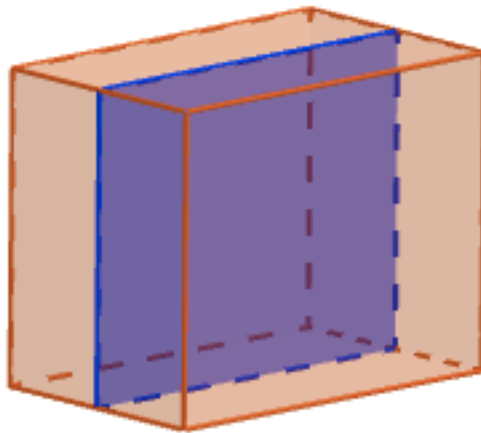
Figure 1. Reflecting the Cube about a Plane through its Center.

When we consider formalized operations, such as addition and multiplication, it may seem obvious that mathematical operations are reversible (via subtraction and division, respectively). However, reversibility does not refer to formalized operations but, rather, to the mental actions that undergird them. Moreover, some formalized operations are not reversible; consider non-invertible matrix transformations, constant functions, and multiplying by zero. If we are to define mathematical objects—including matrices and functions—as the coordination of reversible mental actions, these examples must be examined. Here, we consider the simplest example: multiplying by zero.

The purpose of this paper is to investigate the apparent irreversibility in the case of multiplying by 0. We use that example to consider other cases of apparent irreversibility. Finally, we consider the educational implications of Piaget's definition of mathematical objects. As such, we address the conference theme of questioning "borders between mathematical content areas" and how those borders "limit access to mathematical content."

Multiplying by 0

Multiplying any real number by 0 yields a product of 0. From that product, we cannot uniquely

determine the real number with which we began. In this sense, multiplying by 0 is an irreversible formalized operation. We investigate this example further by considering the mental actions that undergird the formalized operation. We begin by defining multiplication as a transformation of units. Then, we relate that definition to the mental actions involved in units coordination. Thus, our investigation involves examining the mental actions associated with units coordination and the roles they play in transforming units.

Multiplication as a Transformation of Units

Davydov (1992) defined multiplication as a transformation from one unit of measure to another. Building from this definition, Boulet (1998) too sought to break down boundaries—boundaries between the various contexts in which multiplication arises. She demonstrated that, by defining multiplication as a transformation of units, researchers could understand the principal commonality in multiplying whole numbers, integers, rational numbers, and irrational numbers. Whereas repeated addition fits concrete models often used to introduce multiplication, the transformation of units explains what distinguishes multiplicative reasoning from additive reasoning.

Consider the product, B×A=C, as illustrated in Figure 2. We know that C is B measures of the unit A, but we want to transform the relationship to determine C as measured in units of 1 (Davydov, 1992). Thus, we are interested in a transformation of units, from units of A to units of 1. This model explains the efficacy of repeated addition when A and B are whole numbers, but the transformation also highlights what is essentially multiplicative about multiplication. Namely, to reason multiplicatively, students need to coordinate three different *levels* of units (C in units of A, and A in units of 1), not just two levels (A as A 1s) being repeated at level of 1s. Research on students' development of units coordination helps us to understand the distinction, as well as the mental actions associated with transforming units.
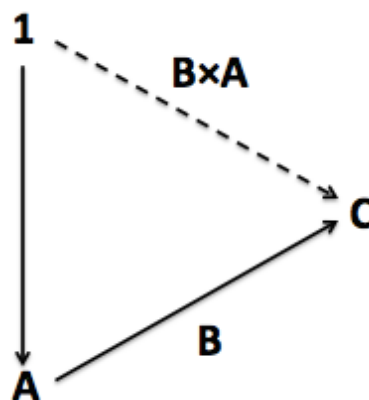
Figure 2. Davydov's Model of Multiplication.

Units Coordination

Units coordination refers to the number of levels of units that a student maintains when acting in a numerical situation (Steffe, 1992). Levels refer to the way numbers are embedded within each other. At the lowest level, there are 1s, from which all other numbers are constructed, primarily through counting. From units of 1, students can construct composite units—units containing other units. For example, students who coordinate two levels of units can consider 5 as a composite unit, made up of five 1s, any of which can be iterated (repeated) five times to produce 5. As a composite unit, 5 as five 1s is immediately available to the student in conceptualizing 5. Furthermore, students

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can act on 5 as an object. For example, students can iterate the composite unit, 5, to produce a unit of units of units.

A student who has constructed composite units can also determine the number of 5s in 35 without relying on memorized facts and without having to start from 1s. She can iterate 5 seven times, simultaneously keeping track of these iterations as iterations of 5 and iterations of five 1s— two different levels of units—while building up the third level (35 as seven 5s). As such, she knows that when she has produced seven 5s, she has also produced 35 1s (see Figure 3). Thus, by iterating the composite unit, 5, she has produced 35 as a unit of seven units of 5, each of which contains five 1s—three levels of units.

Figure 3. 35 as a Unit of Seven Units of Five 1s.

Mental actions that support units coordination include unitizing, disembedding, partitioning, iterating, and distributing (Steffe, 1992). *Unitizing* refers to the mental action of taking an item, or collection of items, as a whole unit that can be further acted upon. *Disembedding* refers to the mental action of taking a sub-collection of items without destroying the whole; the sub-collection exists simultaneously as part of the whole and as a part out of the whole. As such, unitizing and disembedding can be organized as inverse actions within a structure for assimilating and coordinating units: a collection of parts can be unitized as composite unit and, inversely, any number of those parts can be disembedded from the composite unit while maintaining a part-whole relationship.

Partitioning and iterating from another pair of reversible operations (Wilkins & Norton, 2011). *Partitioning* refers to breaking a whole into equally sized parts; *iterating* refers to making connected copies of a part. These operations are inverses because iterating a part can reproduce the whole, and partitioning an iterated part reproduces that part.

Distributing refers to inserting the units within one composite unit, into each of the units in another composite unit (Steffe, 1992). With regard to Figure 3, the five units of 1 within 5 are inserted into each of the seven 1s within 7. This can involve iterating the composite unit, 5, seven times, or recursively partitioning each of the seven parts (1s) within 7 into five new parts. Either way, distributing relates to Davydov's definition of multiplication, as illustrated in Figure 4.

Figure 4. Coordinating Units of Measure.

When A and B are positive integers, we can consider the yellow bar as a composite unit containing A units of the blue bar, and we can consider the red bar as B iterations of that composite unit. Thus, there are three levels of units to consider: the blue bar as a unit of 1; the yellow bar as a composite unit; and the red bar as the result of iterating that composite unit. As such, the yellow bar

acts as an intermediate unit for measuring the red bar. We know the measure of the red bar in units of the yellow bar, but we want to know its measure in units of the red bar. To determine this measure, we can transform the yellow bar as the unit of measure into the blue bar as the unit of measure by partitioning each iteration of the yellow bar into A units of the blue bar. If the partitioning were done sequentially, it would amount to nothing more than repeated addition. However, if the partitioning occurs across all of the yellow bars at once, we have a multiplicative transformation of units.

Mental Actions that Comprise Multiplying by 0

In Davydov's (1992) definition of multiplication, commutativity is not taken for granted. Thus, the case of multiplying by zero occurs as two subcases: $0 \times A$, where 0 is the multiplier; and $B \times 0$, where 0 is the multiplicand (see Figure 5). We consider these subcases separately.
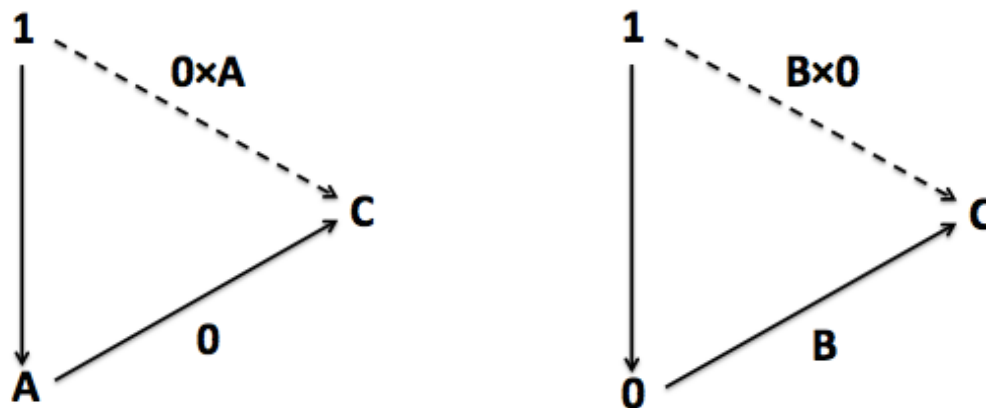
Figure 5. Zero as Multiplier (left) and Multiplicand (right).

When 0 is the multiplier, the transformation of units (between units of 1 and units of A) is reversible. This reversibility relates to partitioning and iterating as inverse mental actions: the composite unit, A, is A iterations of 1, which can be partitioned into A parts to reproduce 1. Whether we iterate A zero times or iterate 1 zero times, we produce the same result (0). This kind of many-to-one mapping is the root cause for irreversibility of formalized operations, in general, and we return to that issue in the discussion section. However, in the subcase under consideration, the transformation of units itself is reversible.

When 0 is the multiplicand, the transformation of units is irreversible because a unit is lost; the unit of 1 cannot be recovered from 0. In fact, strictly speaking, there is no transformation of units because 0 is not a unit of measure; it has no quantity, measure, or extent. We must rely upon other forms of logico-mathematical reasoning to determine the product, thus completing the formalized system of multiplying two non-negative integers. We know that C is B units of 0 and, from that, we deduce that C is 0 units of 1. Here, reversibility takes the form of reciprocity, rather than inversion.

Piaget (1970) distinguished two forms of reversibility: inversion and reciprocity. The integration of these two forms undergirds children's construction of number (Piaget, 1942). Up to this point, we have been considering inversion alone, wherein one mental action undoes another (or itself, in the case of involution). Reciprocity constitutes a form of reversibility wherein one mental action compensates for another. For example, among ordering relations, "5 succeeds 4" is reversed by way of reciprocity, in the form, "4 precedes 5." In the subcase at hand, reciprocity demands that $B \times 0 = 0$ because $B \times 0$ must map 1 to 0 again in order to compensate for the unit lost in the initial transformation. The mental action represented by each mapping/transformation is *projection*, which

generally conflates units but also can annihilate them. Thus, in the second subcase, we do not have a transformation of units, but an annihilation of units, by way of projection.

Discussion

According to Piagetian theory, mathematical objects arise through coordinated mental actions, and actions on mathematical objects can become coordinated as new objects (Piaget, 1970). This is the sense in which mathematics builds upon itself. Composite units become objects for students, through coordinated mental actions of unitizing, disembedding, partitioning, and iterating (Steffe, 1992). Once students have constructed composite units, they can act on those objects through new mental actions, such as distributing. By transforming units, this mental action supports the construction of a formalized operation for multiplication, as well as its products (Davydov, 1992).

A key aspect of coordinating mental actions is the reversibility of those actions. Reversibility provides for perfect reliability in mathematics (Piaget, 1970). In some cases, this reversibility is apparent in the formalized operation. However, in cases like multiplying by 0, the formalized operation is irreversible. Nonetheless, when we consider the underlying mental actions in one of the subcases (0×A), the transformation of units is reversible, through inversion of partitioning and iterating. In the other subcase (B×0) there is no corresponding transformation of units. Rather, we must rely on a different form of reversibility—reciprocity—to generate the product.

Determining Bx0 without a transformation of units is a means of completing the formal system of multiplication for non-negative integers. The system can be further extended to all integers, and even complex numbers, by considering directed quantities (Boulet, 1998). In the subcase of B×0, the trouble in transforming units arises from the fact that the primary unit of measure, 1, is lost. Geometrically, we can think about this subcase as a projection of the entire continuum to a single point, 0. This is precisely what happens in the case of non-invertible matrix transformations and constant functions; an independent unit is lost by way of projection. In fact, the same projection can be represented by the 1×1 matrix, $[0]$, or the constant function, $f(x)=0$.

When we consider how projection affects units and values based on those units, the result is irreversible, as it is for any many-to-one relation, because there is no way to uniquely recover any of the many values from the one value. However, the mental action of projection, itself, is reversible. In geometry, we see the inverse action in the form of sweeps: A point (P) can be swept to produce a line segment (l); that line segment can be swept in another direction (l'), to produce a square area; and that square area can be swept in yet another direction to produce a cubic volume (see Figure 6). Inversely, projections in those directions collapse the cube into a square; the square into a line segment; and the line segment into a point.

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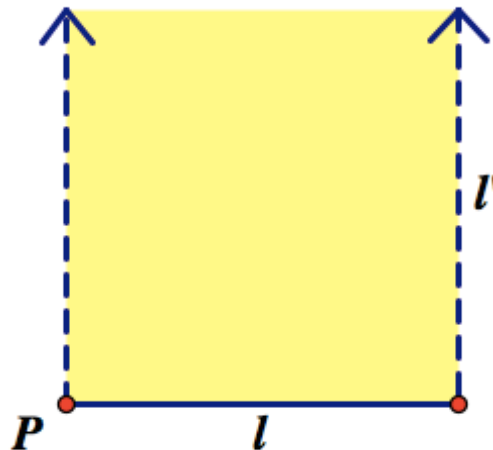
Figure 6. Sweeping Point *P* and Segment *l*.

Implications

Borders between content domains limit access to mathematical content by focusing students' attention on superficial features within each domain, rather than supporting students' development of mental actions that cross those domains. As the study of reversible mental actions, mathematics is a unified science, and we can teach it as such. For example, units coordination involves the coordination of several mental actions that undergird students' knowledge of whole number (Steffe, 1992), fractions (Steffe, 2002), integers (Ulrich, 2012), algebra (Hackenberg & Lee, 2015), and geometry (Battista & Clements, 1996; Wheatley, 1992). Moreover, we can support students' construction of higher levels of units within various domains (Boyce & Norton, in press; Norton & Boyce, 2015). In the case of multiplying by 0, we find further connections to projections in geometry.

If reversible mental actions are the objects of study in mathematics, then students' mental actions need to be the focus of research within mathematics education. What mental actions are available to students? What activities will help students to reverse those actions and to compose them in new ways, in order to construct mathematical objects? Conveying mathematics as a unified science to students can support their creativity, as it has for professional mathematicians. Many of the intractable problems in mathematics have been solved by crossing domains. Specifically, proofs concerning the geometric construction problems of antiquity (e.g., the impossibility of trisecting angles with compass and straightedge) came millennia later, in the form of Galois theory, within abstract algebra.

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UNDERSTANDING THE KNOWING-TO ACT OF MIDDLE SCHOOL MATHEMATICS TEACHERS IN THE MEXICAN BORDERLAND

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This qualitative study sought to understand the practices (actions) enacted by mathematics middle school teachers in Mexico during mathematics instruction. We explore the "knowing-to act" of the teachers. According to Mason (1998), Knowing-to act refers to "the kind of knowledge which enables people to act freshly and creatively." (p.245). A case study was conducted with two participating teachers. Classroom observations were performed to identify the "knowing-to act" of these teachers. Mainly teachers' actions identified were asking for student's justification, revealing students' mistakes, and providing an explanation. Individual interviews with the participants allow them to describe and justify the way he or she acts. These actions and reasons to act in certain way are relevant to know in order to become aware of the factors that influence their teaching, and how it might impact on student learning.

Keywords: Teacher Knowledge, Teacher Education-Inservice/Professional Development, Mathematical Knowledge for Teaching, Instructional Activities and Practices

Introduction

Since considering what knowledge or knowing have a direct influence on teaching practices that could be used to enhance teacher education programs and teaching practices, a growing number of studies focusing on teacher knowledge has been conducted (e.g. Shulman, 1986; Tchoshanov, Cruz, Huereca, Shakirova, Shakirova, & Ibragimova, 2015). This study aimed to provide insights that will provide awareness to teacher education programs and policy makers to make important decisions in regards of what teachers need to do to know how to act desirably during teaching mathematics.

This study focused on "knowing-to act" which is the process where "knowledge [that] enables people to act creatively rather than merely react to stimuli with trained or habituated behavior" (Mason and Spence, 1999, p.136). According to Mason and Spence (1999), there is an absence of "knowing-to act" that leads mathematics teachers not to be able to respond creatively in the moment even when they possess mathematical and pedagogical content knowledge. Consequently, this absence may limit the learning opportunities that teachers can offer. Research on this kind of knowing may help teachers develop the active knowledge needed to respond creatively in the moment. The following questions guided the research: how do the teachers act in the KtA situations occurring in mathematics classroom? And how do middle school mathematics teachers describe and justify their knowing-to-act?

Theoretical Framework

This paper analyzed the teachers' "knowing-to act" (Mason & Spence, 1999) possessed by in-service teachers in mathematics classrooms at the middle school level in Mexico. Following Mason and Spence (1999), in this research "knowing-to act" refers to "active knowledge which is present in the moment when it is required" (p.135). They mentioned that this construct depends on the structure of attention in the moment, thus, "knowing-to act" depends on what one is aware of (Mason & Spence, 1999). Also, Mason and Spence (1999) identified different forms of knowing which

composed knowing-about: factual knowledge, which is the knowing-that; knowing-how, which refers to knowing the technique and skills; and knowing-why, to be able to have a story to "account for phenomena and actions" (Mason & Spence, 1999, p.137).

More specifically, "knowing-that" refers to the factual knowing, to know about facts, topics, among others. "Knowing-how" means to know how to do something, techniques or skills utilized in accomplishing a particular task. "Knowing-why" is about having an argument in order to structure actions and from which to reconstruct actions (Mason & Spence, 1999). Jong and Ferguson-Hessler (1996) categorize these kinds of knowing as: situational knowledge, conceptual knowledge, procedural knowledge, and strategic knowledge. Mason and Spence (1999) recognize that "knowing-about" is immersed in the Shulman's categories (1987). However, they considered that more than these categories are necessary to enable a teacher to act at the moment required. "Knowing-about" is considered as static knowledge possessed by a person, but it does not mean that she/he is able to act creatively in a particular situation. Mason et al. (1999) refers to the term of "knowing-to act" (KtA) as "knowledge that enables people to act creatively rather than merely react to stimuli with trained or habituated behavior" (p.136).

According to Skemp (1979b), "knowing-to act" is to be able to use the knowledge or technique in a novel situation. This type of knowing implies more than possessing ability or knowledge. The essence of "knowing-to act" is the use or call of the knowledge when required. In Rowland, Huckstep and Thwaites (2005) a categorization of the kinds of knowledge needed to teach mathematics is provided. This framework is called "knowledge quartet". One of its components is contingency, which is similar to "knowing-to act". This component is concerned about how the teachers perform when students ask questions in a particular way. According to Mason and Spence (1999) the absence of "knowing-to act" blocks students and teachers to respond creatively in the moment, thus, there is a need for distinguishing "knowing-to act" from "knowing-about" and their elements. Studies that focus on how teachers act during their instruction in the mathematical classroom is critical considering "knowing-to act" as a fundamental part of teacher knowledge, professional development activities could be implemented to promote and help in-service teachers to develop this kind of knowledge.

Methodology

This qualitative study sought to understand the practices (actions) enacted by mathematics middle school teachers in Mexico during mathematics instruction. Qualitative research places a holistic emphasis (Jackson & Verberg, 2007) on the phenomena under study. It means to conduct the research of mathematics middle school teachers in natural settings where their "knowing-to act" is enacted. As Marshall and Rossman (2011) state a particular setting influences the human actions performed in it. Therefore, to study the participating teachers' actions during instruction in the mathematics classroom was crucial in order to understand and describe the "knowing-to act" of mathematics middle school teachers.

The "knowing-to act" of the teachers was captured through face-to-face interactions with the participant and the natural setting where the actions were enacted. We were in the middle schools in Mexico where the mathematics teachers teach every day. Since the purpose of this study was to understand and explore the teachers' performance during mathematics instruction in which their "knowing-to act" was involved, the observation and analysis of teachers' "knowing-to act" was ensured. A case study method was performed to understand the "knowing-to act". To achieve this purpose and answer the research questions of the study, three classroom observations and a video-recorded interview were conducted for each participant. Another data source consisted of field notes taken during the observations. The observations were conducted using a designed protocol that focused on classroom situations (KtA situations) where the knowing-to act can be enacted, and the actions performed in those situations. Each classroom observation lasted 45 minutes. An hour-long

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interview was later conducted with each participant to allow describing and justifying their actions. Fragments of videos from the observations were shown to them during the interview.

Participants

The case study included two participants: Omar and Maria (pseudonyms). Omar was a middle-aged teacher at the eighth and ninth grade. He had 27 years of teaching experience in mathematics. He went through a Normal School preparation. The most advanced mathematics course that he took was Integral Calculus. Omar's middle school has an enrollment of 401 students for the current academic year. This school has a dropout rate of 11.6%. However, more alarming is the fail grade rate which is 38.9%-- meaning that 38.9% of the students in this school do not pass at least one course (SNIEE, 2013b). There are approximately 25 students per classroom. Maria has 7 years of teaching experience. She holds a bachelor's degree in engineering. She taught mathematics at the eighth grade, and science at that time. The student population of the middle school where Maria worked was 845. There was a dropout rate of 9.95%. The fail grades rate is 13.31% (SNIEE, 2013b) in this school. Her class was large with 40 students.

Data analysis

The data collection process started with the classroom observations, field notes, and video recordings of the classroom observations. The observation protocol was designed with classroom situations classified as student misconceptions (Shulman, 1986; An, Kulm, & Wu, 2004); student difficulties (Fennema & Franke, 1992; Ball et al., 2008); situations that are a challenge for the teacher (Mason & Spence, 1998; Rowland et al., 2005); and emerging situations (Mason & Spence, 1998; Rowland et al., 2005). The situations included are KtA situations. The interview was designed to allow getting insights about teachers´ thinking in that situations.

 Subsequently, the analysis of these data was performed using a coding process that was developed to identify the teachers' actions as based on the data sources. The analysis of the classroom observations informed the design of the interview questions. The interview was video recorded and transcribed for further analysis. Triangulation analysis of the observations, field notes, and video recordings yielded themes that described the teacher's reasons for acting in certain ways in given situations during mathematic instruction.

Findings and Discussions

During the observations, there were four KtA situations that were common among the participants. One situation was *when a student solves a particular mathematical problem and asks if the answer or steps are correct*. In this situation, Omar and Maria acted most of the time asking for student's justification. Another KtA situation was *when a student makes a mistake.* In this situation, Omar acted reveling the students' mistakes several time while Maria was asking for students' justification of their mistaken task. *When a student is unable to see an obvious pattern in the problem was a situation that took place in Omar and Maria´s classrooms*. During this situation Omar was providing hint to their students to identify the pattern. Maria revealed the students´ mistakes when they were trying to identify the obvious pattern as well as she asked some students to keep trying. Additionally, she was also providing an explanation when she was on this last KtA situation. Another KtA situation that we focused on was *when students were having hard time completing an assigned activity/task.* When Maria and Omar were on this situation, they were providing explanations to their students to help them to complete their tasks.

During the interviews, the teachers described their actions and provided their reasons to act as they did. We found that teachers' actions are influenced by several factors such as the curriculum, class activity, students (Student's attitude; student's mistake; learning expectation; students' capabilities; student's answer), time, content, and teaching practice.

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Exploring the Knowing-to act of these mathematics teachers allows understanding the teacher actions undoubtedly have an impact on student learning. The knowledge added by this study may help to restructure mathematics teacher education programs, engaging pre-service teachers to focus on the "knowing-to act" during the observations and activities that pre-service teachers are asked to do in their preparation as well as during their teaching practice. And hopefully it impacts the teaching and learning process in Mexican middle schools. Educators need to educate in-service and pre-service teachers about the "knowing-to act" processes that characterize their teaching in mathematics classrooms. This could be achieved through the implementation of professional development that includes activities that allow teachers to identify their "knowing-to act" and reflect about their actions enacted in "knowing-to act" situations. Teachers need to pay attention in particular situations that arise in their classroom that either limit or promote critical thinking.

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CHALLENGING THE REFRAMING OF EQUITY IN MATHEMATICAL SYSTEMS: ACCESS, AGENCY, AND ALLIES

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In this theoretical paper, we discuss fairly novel conceptual tools for thinking about and carrying out equity-oriented work in mathematics education. Specifically we discuss the framing of an equitable mathematical system, and the role of three key features in this system: access, agency, and allies. In doing so, we also problematize their use and frame the tensions that arise as we have begun to use them in our research. We intend to re-examine and challenge these three core features by using specific concepts from race-critical theories (e.g., interest convergence, centrality of race and difference) as well philosophical perspectives and theories of justice, social contracts, and epistemic injustice. We conclude by discussing implications for our related research and professional development in an urban school district.

Keywords: Teacher Education-Inservice/Professional Development, Equity and Diversity

In recent years, equity has emerged as a "legitimate object of study for mathematics educators [that] can potentially move the field into new and significant directions" (NCTM Research Committee, 2005, p. 94). Yet, despite the increased attentiveness to issues of equity (and inequity) in the field, there have been very few efforts to revisit and rearticulate goals, agendas, or new conceptual tools for equity-oriented scholarship, policy, and practice. The purpose of this paper is to challenge the boundaries of how the conceptualization of new tools for thinking about and carrying out our work in mathematics education as researchers and practitioners can occur with deep and renewed attention to equity.

According to Atweh (2011), "concerns about equity are about who is excluded from the opportunity to participate and achieve in mathematics within our *current practices and systems*, and about how to alleviate their disadvantage" (p. 65). Typically these current practices and systems include a focus on the achievement gap or testing. As an alternative to this view, we suggest that the disparities should be re-framed in terms of the unfair distribution of opportunities to learn (OTL) (Gresalfi & Cobb, 2006). We argue that addressing equity in mathematics education as a function of OTL shifts the focus of mathematics education reform away from the remediation of particular groups of students towards ensuring powerful mathematics learning for these groups within classrooms, schools, and districts (Martin, 2009; Moschkovich, 2010).

Theoretical Framework and Concepts

In the proposed framework, we define an equitable system as the qualities of mathematics education that empirical studies have shown afford opportunities for groups of people who have experienced longstanding marginalization to participate powerfully in mathematics teaching and learning. We introduce and elaborate three features of equitable systems in mathematics education: *access, agency,* and *allies*. Drawing on a framework developed by Gutiérrez (2012) that emphasizes the importance of access and agency in equitable mathematics teaching and learning, the framework is augmented by research from Tatum and colleagues (1994) and others (e.g., Katsarou, Picower, & Stovall, 2010; Swalwell, 2012) on the role of allies in disrupting systems of privilege and oppression in education.

Access

Access relates to the resources (tangible and otherwise) and practices that enable students and teachers to participate in ways that are viewed as competent by standards shared within the same community of teachers and students and influences of the community at large (Gutiérrez, 2007). Access for *students* relates to, for instance, opportunities to engage in "high level" tasks (Stein & Smith, 1998) that have multiple entry points (e.g., Boaler, 2006) and support students to use mathematical practices of explaining, reasoning, justifying, and arguing in multiple "communication contexts" (Herbel-Eisenmann, Steele, & Cirillo, 2013). Access for *mathematics teachers* requires opportunities to learn in relationship to problems and dilemmas that are real within the context of their practice (Lampert, 2001).

Agency

Agency in mathematics learning relates to the ability for individuals to take action or operate upon objects (tangible or otherwise) in a mathematical learning environment (Boaler & Greeno, 2000; Gresalfi, Martin, Hand, & Greeno, 2008). Agency for *students* involves being able to utilize mathematical concepts and procedures to solve problems in sensible ways (Gresalfi & Cobb, 2006), as well as being able to participate in mathematics learning in ways that are meaningful to them (Nasir & Hand, 2008). Agency for *mathematics teachers* involves having teachers define (real) problems of practice in order to generate solutions to these problems that result in productive learning for students.

Allies

Our view of an equitable mathematical system also requires collaboration among allies who are invested in negotiating a locally situated sense of what equity means for their contexts. An ally is a person who helps, supports, or acts in solidarity with another in a particular effort. In the case of racism in the U.S., for example, Tatum (1994) notes, "there is a history of white protest against racism. A history of whites who have resisted the role of oppressor and who have been allies to people of color" (p. 470) and who have "support[ed] their efforts to swim against the tide of cultural and institutional racism" (p. 472).

Challenging Boundaries of Theoretical Perspectives on Access, Agency, and Allies

Although we see promise in incorporating new conceptualizations of access, agency, and allies within a reframing of equity, we also see a continual need to raise critical questions. For instance, what does it really mean to focus on developing and sustaining an equitable mathematics system with a focus on these three concepts? And as this work progresses, what does it mean for equity-oriented discourse to move from the margins toward the center? Who will benefit most? What are the tradeoffs?

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As a challenging framework, we draw on race-critical theories, philosophical perspectives on justice (Rawls, 2009; Mills, 1997), and recent perspectives on the nature of epistemic injustice (Fricker, 2007). Critical Race Theory is perhaps the most comprehensive of the broader collection of race-critical theories (see Essed & Goldberg, 2002), and we draw particularly on the centrality of race, the salience of counter-narrative and voice, and interest convergence. From the broader constellation of race-critical theories, we also draw on aspects of everyday racism, racial formation theory, and the intersections of racialization, class, and gender. In an effort to continue to strengthen the framing of equitable mathematical systems, we will continue to interrogate our framing to challenge the boundaries where tensions arise. Toward locating tensions that may also emerge from the enactment of our framework, we explore the following questions in relation to the three major concepts in the presentation:

Access: Access for whom and for what purposes? In what ways does access for one group of stakeholders converge with and possibly conflict with the interests of other stakeholders? Does access guarantee agency? What is the relationship between access and agency?

Agency: What is agency and who decides? Can one enact agency without access to resources and practices? Drawing on race-critical theories and recent scholarship in in philosophy regarding epistemic injustices, one might conceptualize agency as the capacity for individuals to counter supposed credibility deficits and assert their own counternarratives and testimonies (e.g., Fricker, 2007).

Allies: What does it mean to co-negotiate a social contract for change? Who benefits from the alliance? How will the alliance function such that the voices of those with the least relative privilege to enact are amplified? How will the idea of alliance intersect with the principle of interest convergence, such that the interests of white allies do not outweigh the strategic interests of allies of color?

Discussion and Implications for Future Work

The framing of an equitable mathematical system—and particularly, the triad of access, agency, and allies—is positioned as an attempt to reframe of equity discourse in mathematics education, but it is also the theoretical basis for a new project in which we aim to co-construct a more equitable mathematical system in an urban school district. In the presentation, the implications of this framing for that ongoing work will also be discussed.

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BACK TO THE DRAWING BOARD: ON STUDYING INTERACTION WITH MECHANICAL DESIGN

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In a pilot study of an experimental calculus activity centered on the CalcMachine—a concrete manipulative—subjects visually "projected" the anticipated results of their actions before executing them. From these empirical findings, we tentatively argue for integrating two theoretical models: distributed cognition (Kirsh, 2009) and instrumental genesis (Vérillon & Rabardel, 1995). Emerging from a study in the concrete domain, this theoretical development may bear implications also for digital interactive educational technologies.

Keywords: Cognition, Design Experiments, Learning Theory, Technology

Education technology research in the 21st century increasingly focuses on electronic media (Hourcade, 2015), particularly in mathematics education (e.g., see Confrey et al., 2010). Yet this research draws on educational theory often based on interactions with now-antiquated media. We assert that new forms of interaction warrant re-conceptualization of learning, teaching, and educational design (Papert, 2004). In this paper, we take a step back in hopes of taking a few steps forward, turning our attention away from virtual manipulatives and toward concrete ones (Sarama & Clements, 2009). From studying how students learn to operate tangible devices within a concrete context, we hope to contribute to digital realms of interactive technology.

Reporting on a modest study from a design-based research practicum, we first explain the design problem that inspired this project. We then propose a theoretical contribution of the pilot study and introduce two case studies as applications. We end by reframing the study as a case of our larger argument for the value of dabbling in "low tech" to inform innovation in "high tech."

A Pilot Study

The problematic role of calculus as an academic gatekeeper (Steen, 1988) motivated us to improve students' first encounters with calculus. Inspired by arguments for the inherently embodied nature of mathematics (Nemirovsky, 2003), we created an embodied learning environment (Abrahamson, 2014) to motivate and steer students' goal-oriented actions and descriptions thereof toward normative disciplinary practices (Abrahamson & Trninic, 2015).

Design

We designed, built, and pilot-tested the CalcMachine (see Figure 1a, next page), a 1-foot square frame containing: (I) a metal curve approximating a parabola; (II) a drawing bar; and (III) two magnetic points attaching the drawing bar to the curve. The points slide along a slit in the drawing bar, allowing it to be adjusted along the curve at various locations and angles. Users trace against the bar to draw secant and tangent lines to the curve.

Students' activity with CalcMachine centers on a set of target images (see Figure 1b, next page). Students are asked to recreate these images with the device. For each image, they are to set the drawing bar at an appropriate location on the curve and then use a pencil so as to trace a line on a sheet of paper placed under the device. The images were designed to promote motor-action schemes presumed as relevant to reasoning about secants and tangents. The rationale was to orient subjects toward relationships between the curve, action schemes, and resulting shapes.

Methods

The subjects, one mid-20s male and one mid-60s female, do not reflect the ultimate target population but were auspicious for this paper, as they could articulately reflect on their work. Interviews began by introducing the CalcMachine, demonstrating how to draw, and then inviting the subject to explore. Once subjects drew comfortably, the target images (Figure 1b) were introduced. Subjects were asked to draw a target image of interest and explain his/her process.
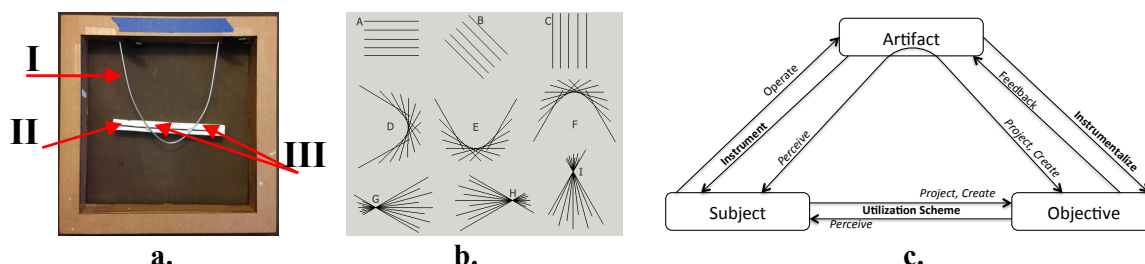
Figure 1. a. CalcMachine prototype. b. Target Images. c. Diagram for theoretical integration of IAS (bold font) & Distributed Cognition (italicized font).

Theory Interlude: Toward Integrating Seminal Frameworks

Subjects spent significant time orienting themselves toward operating the CalcMachine and applying it as a drawing tool. We believe this orienting work profoundly shaped their learning. Here we present two relevant theoretical frameworks as well as a proposed integration thereof.

Instrumental Genesis

Vérillon and Rabardel's (1995) *instrumented activity situations (IASs)* capture the multi-directional interactions among a Subject who learns to use an Artifact to facilitate an Objective (see Figure 1c, above). Examples for IASs include using an abacus to do sums or using written words to communicate. As the subject learns to operate the artifact, the artifact **instruments** the subject, limiting the subject's actions to those within the artifact's constraints. As the subject learns to use the artifact toward the objective, the artifact is **instrumentalized**, becoming a tool for the task. Feedback from the objective during instrumentalization surfaces new capabilities and constraints of the artifact, further instrumenting the subject. Through instrumented action and in keeping with cultural norms, subjects develop **utilization schemes** (USs) through which they interact with and imagine the objective. USs reflect instrumentation and instrumentalization, narrowing the subject's actions and perceptions to those afforded by the artifact. We use the IAS framework to analyze how subjects: (1) learn to operate the CalcMachine; (2) instrumentalize it to achieve the drawing objective; and (3) become instrumented as users of the CalcMachine.

Distributed Cognition

In addition to the system-level IAS framework, we also seek a perspective on moment-to-moment work. We observed nonstandard instrumentalizations of the CalcMachine that suggested the subjects' thinking to be distributed through the artifact onto the task environment. Kirsh's (2009) construct of 'projection' helps us clarify such subject–environment interactions.

Projection captures features of the environment that we visualize despite their not being physically present. For example, when solving a geometry problem, we might visualize a line bisecting an angle even though such a line is not yet physically in the environment. Projection is often paired with *creation*, either by gesturing or constructing something in the environment to materialize a projection. The former projection now perceptible, the freed cognitive resources can be put to other uses, allowing iterated projection perhaps by visualizing greater detail (as in the geometry problem). In social settings, creation may occur as much, or more, for an observer as for

the subject him/herself, transforming a private projection into a shared discursive element.

A Proposed Integration

We propose integrating project–create into the IAS triangle (see Figure 1c, previous page). We offer that 'project' and 'create' illuminate particular instances of sensorimotor feedback that shape broader instrumentation, instrumentalization, and utilization schemes. Thus subjects can project–create onto the Object, either via the Artifact, thus further instrumentalizing it, or directly onto the Object. In the latter case, implicit utilization schemes likely mediate the direct projection–creation. We find support for this theoretical integration in the task-based interviews.

Findings

In these excerpts project–create seems to co-occur with, and serve, novel instrumentalization. We also interpret moments of project–create as suggesting nascent utilization schemes.

Projected Parabolas
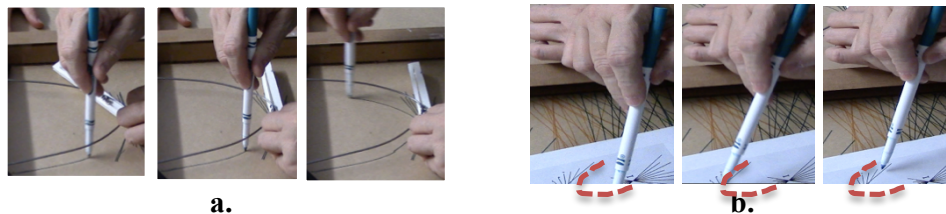
Figure 2. a. Traced parabola against the curve. b. Gestured parabola on Target Image F.

Working on Target Image F, Karen traced a parabola against the curve (Figure 2a) and then indicated an analogous parabola on Target Image F (Figure 2b), where drawing curves thus was a novel instrumentalization of the CalcMachine. Karen's drawn and gestured creation of parabolas not present in the environment suggests she had projected them and also indicates a utilization scheme; Karen may view the target images as things in which to recognize parabolas.

Projected Tangents
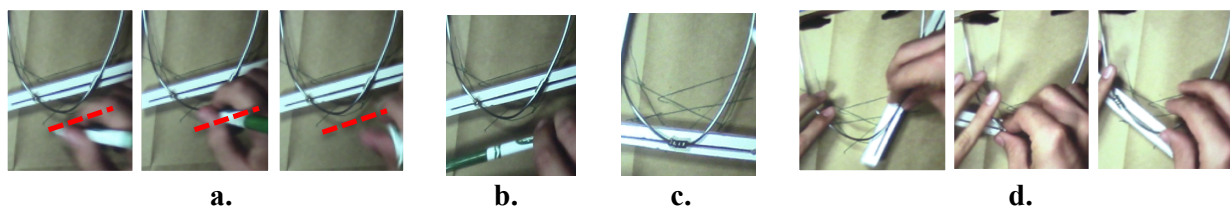
Figure 3. a. Gestured tangent. b. Marker as tangent. c. Drawing bar as tangent. d. Drawing bar itself utilized spontaneously as constituting a tangent line at various locations.

Drew considered moving the drawing bar with the points adjacent. He gestured a line roughly tangent to the curve (Figure 3a), indicating a line projected there, then materialized this projection with the marker (Figure 3b). This direct creation suggests a conception of the space as comprising such lines, a utilization scheme in step with Drew's academic background in calculus. He also recreated his projection and achieved novel instrumentalization by placing the drawing bar along the formerly gestured path (Figure 3c) and other tangent locations (Figure 3d).

Summary

From these pilot cases, we tentatively advance two propositions integrating project–create into

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IASs. First, project–create cycles may facilitate novel instrumentalization. The temporal link between actions/utterance indicating both project-create and novel instrumentalization suggests that such instrumentalization may be a specific case of project–create, where an intended action with the artifact (or an intended product of that action) is projected before it is materialized by using (elements of) the artifact. Second, direct projection–creation onto the environment carries subjects' developing conceptions of legitimate Subject–Objective interactions in that space—a utilization scheme for both artifact and environment. These tentative conjectures require further evaluation, particularly in capturing novel instrumentalizations and nascent utilization schemes.

Based on these proposals, we submit that projection, from cognitive science, can contribute to education research, specifically to micro-analytic investigations of mathematics learning.

Conclusions

Calculus's role as an academic gatekeeper (Steen, 1988) motivated us to undertake a design-based research study using embodied learning. We designed, built, and pilot-tested the CalcMachine, a dynamic tangible device for exploring derivatives. Pilot study observations prompted us to integrate theories of instrumental genesis (Vérillon & Rabardel, 1995) and distributed cognition (Kirsh, 2009) to better understand subjects' work to operate a novel artifact.

We submit that our insights from this project emerged because we stepped back into the concrete. We likely would not have included in a virtual CalcMachine any function for tracing against the curve or using the marker as a line rather than a drawing tool, actions that—in the task context— represent failures. Yet these very failed attempts at normative operation rendered transparent the artifact's embedded functionalities, in turn offering us glimpses into authentic and ultimately productive learning processes en route to normative actions. Even as we advance with the innovations of interaction technology, we also pause to consider interaction affordances of physical artifacts—exploration and transparency among them—that we risk leaving behind. These affordances, in turn, also enabled us to step back and integrate theoretical models.

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MATHEMATICAL GESTURES: MULTITOUCH TECHNOLOGY AND THE INDEXICAL SIGN

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This paper is about the threshold between gesture and touch in mathematical activity, focusing on the role of multitouch technology. Drawing on the work of gesture theorist Jürgen Streek, we discuss the notion of indexical gestures, in the context of young children's mathematical explorations with a multitouch iPad environment designed to promote counting on and with the fingers.

Keywords: Technology, Cognition, Number Concepts and Operations

With the advent of multitouch technologies, new media gestures have become an essential feature of user interface. The *Touch gesture reference guide* (Villamor, Willis, & Wroblewski, 2010) shows the basic actions needed for screen interaction; core gestures are listed as tap, double tap, drag, flip, pinch, spread, press and rotate. In this paper, we draw attention to the distinctive gestures new media elicit and the way these new manual activities are changing the way we perform mathematics. We also interrogate the taken-for-granted distinction between the touchscreen gestures common in the technology world and the in-the-air gestures that have been the focus of much study in mathematics education research over the past decade. At first blush, they may seem quite distinct— albeit having the same name—but we follow Streeck (2009) in treating all these gestures as part of a continuum, which allows us to study gestures as "media events" in mathematics learning (de Freitas, 2016). We thus seek to expand definitions of gesture so as to address the changing ways in which hand and media interact in contemporary digital culture. In order to illustrate how these new media gestures operate as expressions of numeracy, we draw on research involving a novel multitouch App in which fingers and gestures are used to count (Jackiw & Sinclair, 2014).

Gesture as movement

Researchers often distinguish between hand movements *in the air* and hand movements *that make graphic marks*, where the former is deemed a gesture and the latter an act of inscription. However, such distinctions become fuzzy when we study the movement of the hand across and through media, where 'media' can be more or less receptive of trace or mark. In other words, all hand movements traverse and incorporate media. We see a trace in certain media, and not in others. Since the logic of new media is to break with current conventions of perception, this distinction is provisional. New media allow for new kinds of traces. This insight allows for new ways of studying numeracy and mathematical sign-making in relation to multitouch technologies.

The predominant line of research in gesture studies focuses on movements of the body (especially the hand) and their interactions (i.e. correlations) with speech in communication (Kendon, 2004). Researchers have identified different categories of gestures (iconic, metaphoric, deictic and beat) so as to distinguish different relations between gesture and speech. McNeill has drawn on Peirce's (1994) semiotics in which signs (icons, symbols, and indices) differ in terms of the nature of the relationships between the signifying sign and the signified. Much of this research focuses on the *linguistic* potential of gestures, and tends to overlook the physicality of the hand movement, except insofar as such movement contributes to or obscures linguistic meaning (Kita, 2003; Rossini, 2012). As Streeck (2009) indicates, "it is common to treat gesture as a medium of expression, which meets both informational and pragmatic or social-interactional needs, but whose "manuality" is accidental and irrelevant" (p. 39). Instead, Streeck (2009) defines gesture:

… not as a code or symbolic system or (part of) language, but as a constantly evolving set of largely improvised, heterogeneous, partly conventional, partly idiosyncratic, and partly culture-specific, partly universal practices of using the hands to produce situated understandings. (p.5).

Thus Streek studies gesture for how it is "communicative action of the hands", with emphasis on the term *action* (p.4). This focus on action allows him to show how gesture couples with and intervenes in the material world, and indeed how gesture reconfigures the matter-meaning relationship. Peirce's indexical signs are relevant here, as they emphasize the *material* link between signifier and signified. Unlike icons and symbols, indexical signs are bound to the context in important ways, or in Peirce's words they "show something about things, on account of their being physically connected with them" (Peirce, 1894/1998, p. 5). As in the case of smoke billowing from a chimney indicating that the fireplace is in use, the smoke *indexes* the fire. In other words, smoke is *produced by and contiguous with* the fire. What is distinctive about the indexical sign is that it is a sign that is materially linked or coupled to "its object". According to Peirce (1932), an index "refers to its object not so much because of any similarity or analogy with it, (…) as because it is in dynamical (including spatial) connection both with the individual object, on the one hand, and with the senses or memory of the person for whom it serves as a sign, on the other" (2.305). In this paper, we are particularly interested in the role that indexical signs play in mathematics. For instance, the chalk drawing of a parallelogram on a blackboard is often considered to be an *iconic* reference to a Platonic conception of parallelogram, but it is (also) an *indexical* sign that refers to the prior movement of the chalk. This latter indexical dimension emphasizes the manual labour that produced the sign or trace. This focus on the indexical resonates with our work on inclusive materialism in attending to the material activity of the body in mathematical activity (de Freitas & Sinclair, 2014).

A case study

In this case study, the hand actually operates very close to the surface of a screen: pointing to objects on the screen by tapping them; sliding objects along on the screen so as to leave visual and aural traces of the finger's path; pinching objects together in order to make new ones. These gestures of pinching and pointing both communicate meaning and inscribe marks. *TouchCounts* is an iPad application that permits young learners to coordinate simultaneously various forms of number: number names like 'three', number of taps on the screen, number of discs on the screen and number symbols like 3. It enacts a multimodal correspondence between finger touching, numeral seeing and number-word hearing (a one-to-one-to-one correspondence of touch, sight and sound). The App has two worlds: the Enumerating and the Operating worlds. In this paper, we focus on the Enumerating world, which is the one that children usually first experience.

In this example, a five-year-old girl named Katy is interacting with *TouchCounts* for the first time. The room is quiet. Without prompting, Katy's hand approaches the screen, and her finger touches the top of it and slides down to the bottom. A yellow disc appears under her finger with the numeral '1' on it and the sound 'one' is made. The index finger moves back to the top of the screen, slowly swimming downwards. A chorus of 'two' comes both from her mouth and the iPad. This happens repeatedly, although sometimes only the iPad can be heard announcing the new numbered disc while Katy's lips move in synchrony (see Figure 1a). The appearance of '10' on the tenth yellow disc attracts attention, perhaps because of its double digits, and Katy bends over to look closely. Now only the iPad counts the numbers (see Figure 1b). After 'seventeen', several fingers fall on the screen at once, and then 'twenty-one' is heard. This produces a pause in the action, and Katy's lips spread into a smile. All but the index finger are tucked away, as the rhythmic tapping continues along with the chorus of named numbers. At 'twenty-seven' Katy looks up, no longer watching the screen (see Figure 1c), and she continues swiping and saying numbers. This continues until a finger seems to accidentally land on the Reset button.

Katy's finger – as the main organ of touch in this encounter –takes on new capacities through the reset button. No longer the organ that can only move or drag the yellow circle, it now becomes a finger that resets. The power of the reset button to recalibrate the tempo and rhythm of this encounter, becomes part of the finger's potentiality, thereby redefining what is currently entailed in the sense of touch.
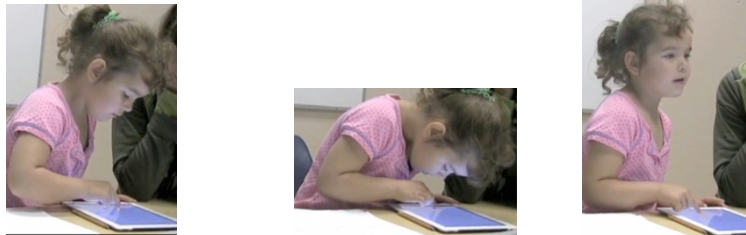
Figure 1. (a) Katy swiping; (b) Following the yellow disc; (c) Tapping while looking up.

Discussion

Katy's hand actions change over the course of the episode, not only in the particular muscular form they take, first sliding down the screen as if lingering on the yellow discs to produce or partake in their falling off the screen, and then tapping impetuously so that each new touching of the screen follows the end of the sounds of the voiced numerical. The swiping gesture seems more exploratory while the tapping gesture seems to concatenate into a unit the touch-see-hear bundle of sensations involved in making a new disc-numeral-name. As Streeck writes, tapping is also "characteristic of ritualized behavior" (p. 76), which suggests that Katy has moved from exploration to practice. In both the swiping and the tapping, the finger can be seen as making an indexical gesture, with the trace being both visible and audible, not to mention tangible for Katy. Although the initial movement and touch of her finger is what produces the disc, it is the disc that drives the swiping movement of her finger. Indeed, both her finger and her eyes *follow* the yellow disc as it heads down the screen. In shepherding the numbered disc off the screen, Katy is able to see when it's time to lift her finger and start making a new disc. But with the tapping, the eyes attend to the numerical sign on the disc— indeed, when "10" appears, Katy notices the change from the previous one-digit numerals. In this sense, the eye and the finger do very similar things in the swiping, the visible trace is followed closely by Katy's eyes as the swiping takes place, so that the hand is subordinated to the watchful eye. With the tapping, the hand seems less subordinated, with the eye only interested when a novel situation comes up. When Katy looks up, the hand is no longer subordinate at all. Finally, when Katy's eyes close, her fingers do the seeing and touching as they repetitively tap.

One might question whether Katy's actions on the screen, which we might think of as touch-pointing, can really be thought of as gestures. Streeck argues that such touch-pointing gestures (and indeed all gestures) emerge from the touching and handling of things—the tracing (or other "data-gathering devices" such as caressing, probing, cupping) of objects that allows one to discover its texture and temperature (and, for young children, for instance, the difference between a cylinder and a pyramid). When the hand has done its exploring of the object, which fulfills an epistemic function in gathering information, it may then be lifted off the object and inclined to repeat the same movements 'in the air': "the hands' data-gathering methods are used as the basis for gestural communication" (p. 69). Streeck identifies such gestures as being communicative, which for him is the characteristic feature of a gesture. So perhaps Katy's touch-pointing becomes a gesture once she lifts her hand form the screen to do her tapping.

However, distinguishing hand movements that explore from ones that communicate is problematic. As Streeck writes, exploratory actions can become communicative when they are made visible to others, who may join in the action or infer tactile properties. If we look at Katy's swiping

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and tapping gestures, we might say that they are both exploratory, with the swiping gestures involving prolonged tactile contact that enables her to discover what would happen when her finger touches the glass—that a yellow disc would appear, with a numeral on it; that the disc would move down the screen; that the iPad would speak the numeral's name aloud, and that this could all be repeated as often as she wished. But Katy's swiping and, later, her tapping, are also communicative inasmuch as they tell *TouchCounts* what to do and say. The same might be said for clicks of the mouse or key presses of the keyboard, with the difference that the touchscreen is acted upon by direct hand motions. Instead of disentangling the tracing from the pointing (the exploration from the communication), we suggest that re-assembling them into an indexical enables us to see how Katy's hand movements can tap into the potentiality of the body by reconfiguring the relationships between sensations of touch, sight and sound that are at play. This potentiality mobilizes new mathematical meanings as Katy uses her fingers to count on, to count with and to count out one by one and indefinitely. Streeck recognises that hand-gestures "enable translations between the senses" (p. 70) as tactile discoveries provide visual information for interlocutors. With Katy though, the tactile discoveries provide visual and auditory information to herself. She is her own interlocutor.

Conclusion

We have shown that the gestures made by Katy in *TouchCounts* entail ritual and inventive movements of the hand. Streeck (2009) argues that hand-gestures cannot be taken only as components of a language system, which are cast apart from the material world, and used only to communicate about the world. Rather, they are *of* the world, and part of how we *feel* the world around us. This perspective requires us to see the moving hand as "environmentally coupled" (Goodwin, 2007), that is, as indexical and inextricable from the things it touches and engages with. Like Streeck (2009) who suggests a link between the exploratory hand action and the communicative hand-gesture, our case study reveals how the hand becomes part of a ritual production of ordinal numeracy in a new media event. This new kind of gesture is possible in large part because of the feedback mechanism of digital technologies, which can talk, push and show back, but also significant is that the fingers are gesturing and materially bringing into existence a new concept of number. With the touchscreen interface, and particularly the multitouch interface, the hand is involved in a process of communicating that is also a process of inventing new forms of numeracy.

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PRESSING METHODOLOGICAL BOUNDARIES: ANALYZING PCK USING FRAME ALIGNMENT PROCESSES

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Conceptual tools from frame analysis (Benford & Snow, 2000; Goffman, 1974; Snow & Benford, 1988) have been empirically linked with productive analyses of a situated model of teacher learning (Bannister, 2015) in a community of practice (Wenger, 1998). However, a limitation of this work is that is does not make claims about individual teacher learning, though some scholars question the utility of an individualist approach (Barab, Barnett, & Squire, 2002). This paper builds upon and extends this prior work using literature linking frame alignment processes (Snow, Rochford, Worden, & Benford, 1986) with teacher development of pedagogical content knowledge (Bannister, 2015). By pressing upon established boundaries for understanding the "black box" of teacher learning (Little, 2003), this papers opens up the scholarly mathematics education community to a novel theory for studying teacher learning.

Keywords: Learning Theory, Mathematical Knowledge for Teaching, Research Methods

Pedagogical content knowledge (PCK) refers to a "special amalgam of content and pedagogy" that teachers use in their professional work (Shulman, 1986, p. 226). PCK for mathematics teachers includes knowledge of subject matter, curriculum, teaching strategies, and how these facets work together to support student understanding (An, Kulm, & Wu, 2004; Hill, Ball, & Shilling, 2008; Marks, 1990). PCK is a widely used heuristic in studies of mathematics teacher learning, likely due to strong empirical evidence linking teacher PCK development with improved student outcomes (Depaepe, Verschaffel, & Kelchtermans, 2013). Despite its utility and broad reach, there is disagreement among scholars about whether or not PCK is a static entity—stemming from a cognitivist theoretical perspective—or if PCK is a dynamic classroom-based process, a perspective based upon a situated cognition framework. What is more, while the (mostly agreed upon) components of PCK have been identified, we do not yet understand fully how novice mathematics teachers develop PCK (Ball, Thames, & Phelps, 2008).

The consequences of a nuanced debate about the definition of PCK—"Where is the Mind?" (Cobb, 1994)—unintentionally leave room for unclear theoretical foundations and inconsistencies within research studies on mathematics teacher PCK development (Depaepe, Verschaffel, & Kelchtermans, 2013). This paper addresses these problems in part with a proposed method for analyzing PCK using conceptual tools from frame analysis literature. We argue in our paper that frame alignment processes (Snow, Rochford, Worden, & Benford, 1986) provide productive conceptual tools for explaining mathematics teacher PCK development, thereby contributing new theoretical insights capable of extending current methodological boundaries for analyzing individual teacher learning.

Where is the Mind?

Studies in which PCK is grounded in a cognitivist perspective tend to describe a fixed set of components that compose a category system for teacher knowledge that can differentiate PCK from other categories such as content knowledge or pedagogical knowledge. On the other hand, proponents of a situated perspective of PCK "acknowledge that the act of teaching is multi-dimensional in nature and that teachers' choices simultaneously reflect mathematical and pedagogical deliberations" (Depaepe et al., 2013, p. 22). These perspectives are consequential on the way PCK is empirically investigated:

> Advocates of a cognitive perspective on PCK believe it can be measured independently from the classroom context in which it is used, most often through a test. They typically focus on gaps in individual teachers' PCK, on how PCK is related to and distinguished from other categories of teachers' knowledge base, on how PCK is related to students' (cognitive) learning outcomes, and on how PCK can be improved through a training. Adherents of a situated perspective on PCK, on the contrary, typically assume that investigating PCK only makes sense within the context in which it is enacted. Therefore, they often rely on classroom observations…and typically aim at unraveling the nature of PCK within a particular context, as well as its development through reflection on one's own classroom practices and through participation within professional communities (e.g., discussion groups, mentoring). (Depaepe et al., 2013, p. 22)

While different theoretical perspectives necessarily influence empirical research on PCK, we—like Cobb (1994)—question assumptions that force a choice between perspectives. Rather than "adjudicating a dispute between opposing perspectives" (p. 13), Cobb (1994) instead made a case for the coordination of complementary perspectives because "sociocultural analyses involve implicit cognitive commitments, and vice versa. It is as if one perspective constitutes the background against which the other comes to the fore" (p. 18). However, even we make the blanket assumption that different theoretical perspectives for PCK are complementary, questions remain about how this assumption translates into empirical methods for studying teacher development of PCK.

Table 1: Types of Frame Alignment

Frame Bridging	"linkage of two or more ideologically congruent but structurally unconnected frames regarding a particular issue or problem" (p. 468)
Frame Amplification	"clarification and invigoration of an interpretive frame that bears on a particular issue, problem or set of events" (p. 469)
Frame Extension	extend "boundaries of its primary framework so as to encompass interests or points of view that are incidental to its primary objectives but of considerable salience to potential adherents" (p. 472)
Frame Transformation	redefine "activities, events, and biographies that are already meaningful from the standpoint of some primary framework, in terms of another framework" (p. 474)

Frame Alignment Types as a Means for Capturing Components of PCK

We rely on borrowed frame alignment concepts from the frame analysis literature for this work. Snow, Rochford, Worden, and Benford (1986) define frame alignment as the linkage between individual and larger interpretive frameworks, such as the linkage between individual teacher perspectives on practice and PCK. The authors proposed four types of frame alignment processes, including: (a) frame bridging, (b) frame amplification, (c) frame extension, and (d) frame transformation (see Table 1). "The underlying premise is that frame alignment, of one variety or another, is a necessary condition for movement participation, whatever its nature or intensity, and that it is typically an interactional accomplishment" (p. 464). In keeping with this reasoning, and by extension through Bannister's (2015) prior research, we propose studying shifts in the varying components of PCK using these frame alignment processes. In so doing, we theorize that these processes provide a method for analyzing and generating empirical evidence of individual teacher PCK development, thereby documenting empirical evidences of teacher learning.

Research Strategies

We conducted a meta-analysis of data collected as a part of a 5-year funded research study of novice mathematics teacher PCK development. After taking a grand tour of the data, we began our inquiry of teacher learning with analysis of how beginning mathematics teachers orientations toward and professional visions for teaching shift over time. The data archive includes 4 years of data about novice mathematics teacher participants, beginning with their pre-service phase and then

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transitioning to the teacher induction phase. In this study, teachers in the pre-service phase were enrolled in a 1-year graduate-level teacher-training program, while teachers in the induction phase had a teaching job and up to 3 years of experience as a teacher. More specifically, the data set includes:

1. Pre-service mathematics teacher responses to an instructional planning task upon their entry and exit to their teacher-training program (Van Der Valk & Broekman, 1999). Participants were interviewed after completing the task, where researchers asked participants questions specific to four PCK categories to document participant knowledge of students' understandings, instructional strategies, assessment, and curriculum.
4. Pre-service and induction classroom observation cycles, conducted twice for each participant per year—once in the fall and again in the spring. These 2-day observation cycles produced the following data: (a) 2 days of consecutive mathematics lesson plans; (b) a pre-observation interview in which participants discussed the mathematics lessons, the knowledge they drew on when designing these two lessons, their mathematics teaching orientations, their knowledge of learners, instructional strategies, assessment, and curriculum; (c) observations, video recordings, and field notes for the two consecutive lessons taught to the same class; and (d) two stimulated recall interviews conducted after each day of observation (Pirie, 1996; Schempp, 1995) in which participant knowledge was probed via playback of critical parts of the lesson.

The archival dataset, however, focused on the individual learners rather than the grain-size of teacher community inherent to Bannister's (2015) prior work, and as such, extended this line of research to include frame alignment processes in an effort to conceptualize our inquiry of individual teacher learning from a situative perspective (Collins & Greeno, 2010). We take cues from analogous research in social movement theory that links together "social psychological and structural/ organizational factors and perspectives in a theoretically informed and empirically grounded fashion" (Snow et al., 1986, p. 464). So conceptualized, Snow and colleagues argue that frame alignment is a necessary condition for movement participation, and by extension through Bannister's (2015) prior research, a necessary condition for learning (Wenger, 1998).

Initial Conclusions

After selecting a case of teacher learning that would allow our team to investigate related research questions about teacher learning in context of our proposed theory for analyzing teacher learning, we reduced our dataset using the different components of PCK. From there, we examined our data for evidence of frame alignment processes, which included the teacher's continued alignment with her own framework or movement toward the frameworks build upon the bedrock of the National Council of Teachers of Mathematics (NCTM, 2000) vision. Preliminary findings suggest that the beginning teacher is making sense of her new role and reform-oriented perspective on teaching through her existing framings of teaching. As she learned more effective methods in her teacher training program, she also gained awareness around the absence of sensemaking her prior learning and teaching experiences. As a beginner, she struggles to move beyond her perspective of herself as a learner, and seems less able to take on the perspective of a student, making it hard for her to help students gain entry to and support their learning of this kind of work. Overall, she is struggling with getting her emergent framing of teaching and learning to align with her practice of teaching and learning. For the purposes of this paper, our ongoing analysis of teacher learning allowed us to further refine our theory as a method that effectively links PCK and frame alignment processes.

Importance

This research proposes novel methods for analyzing beginning mathematics teacher learning using processes of frame alignment. In so doing, this work presses existing methodological boundaries and opens up our scholarly community to novel interpretive frameworks for understanding teacher learning.

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A HYPOTHETICAL LEARNING TRAJECTORY FOR LOGARITHMS

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Keywords: Learning Trajectories (or Progressions)

Logarithms plays an important role in the high school curricula, particularly as a foundation for further studies in the fields of science, technology, engineering, and mathematics. Unfortunately, the logarithm concept gives students considerable difficulty. Researchers and educators alike have recognized that traditional instructional methods used to teach logarithms do not lead to a productive understanding of logarithms (Weber, 2002; Wood, 2005).

In this poster, I propose a hypothetical learning trajectory (HLT) for logarithms, based on a view of logarithms as an arithmetic operation independent from exponentials. A learning trajectory consists of: (1) a mathematical learning goal, (2) a hypothesized model of students' thinking and learning, and (3) an instructional sequence. I made use of APOS Theory as a framework for the design of the HLT. I found this theory to be useful because it is "principally a model for describing how mathematical concepts can be learned; it is a framework used to explain how individuals mentally construct their understandings of mathematical concepts" (Arnon et al., 2014, p. 17). The poster will also display a series of proposed activities specifically designed to move learners through the HLT.

The main learning goal of the HLT is for students to understand $\log_b m$ as the number of factors of b that are in the number m. Other goals are for students to correctly use and understand logarithmic notation, evaluate logarithmic expressions in which the result are whole numbers, estimate between which two whole numbers the logarithm of a number is, derive the addition and subtraction laws of logarithms, and derive the identities $\log_b 1 = 0$ and $\log_b b = 1$. It is **not** a goal of the HLT for students to find the exact value of a logarithm expression when the result is not a whole number. It is **neither** a goal of the HLT for students to make a connection between logarithms and exponentials.

In line with APOS Theory, I argue that students need to develop and move from an Action, to a Process, to an Object understanding of logarithms in order to have a productive understanding of logarithms. An Action understanding of logarithms would allow students to solve problems that implicitly call for repeated division by the same factor as a solution method, in which the number of repetitions would be the answer. With a Process understanding of logarithms, students can make generalizations about logarithms by deriving and making sense of some of their properties. An Object understanding of logarithms allow students to conceive a logarithmic expression as static, not as an instruction to do something. This particular type understanding is essential for the manipulation of operations with logarithms and generalizations about the laws that these operations entail.

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DEVELOPING A HYPOTHETICAL LEARNING TRAJECTORY FOR MULTIPLICATIVE STRUCTURE

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This study presents a hypothetical learning trajectory (HLT) for prospective elementary teachers' (PTs') understanding of multiplicative structure. Simon (1995) describes an HLT as consisting of a learning goal, instructional activities, and the ways in which students' thinking may evolve. Zazkis and Campbell (1996a) define multiplicative structure as "conceptual attributes and relations pertaining to and implied by the decomposition of natural numbers as unique products of prime factors" (p. 541). Research shows that an understanding of multiplicative structure can facilitate the transition from arithmetic to algebra and support work with divisibility, fractions, and decimals (Campbell, 2006; Brown, Thomas, & Tolias, 2002).

Methodology

Qualitative analysis of classroom video transcripts from a mathematics content course for PTs was conducted. The focus of analysis was the interactions of one small group of five PTs during a three-week number theory unit focused on factors, prime factorization, and divisibility. Analysis of the video transcripts consisted of partitioning each lesson into *pedagogical episodes*. Each pedagogical episode was then analyzed for "subtle shifting in thinking" as characterized by Simon and colleagues (2010). Once these shifts were identified, an HLT for PTs' understanding of multiplicative structure was constructed using Simon's (1995) three components.

Hypothetical Learning Trajectory

The *learning goal* is that PTs will understand the relationship between the multiplicative structure of a natural number and its factors, and will be able to use this relationship to identify divisibility and indivisibility. The *instructional activities* consist of three lessons: factors, prime factorization, and divisibility. Analysis resulted in the identification of four mathematical constructs that describe *the ways in which PTs' thinking may evolve*: 1) Magnitude-divisor count relationships; 2) uniqueness of prime factorization; 3) divisibility; and 4) indivisibility.

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LATINX CRITICAL THEORY IN MATHEMATICS EDUCATION

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Keywords: Research Methods, Equity and Diversity

History

LatCrit's roots are closely tied to Critical Race Theory (CRT). Both LatCrit and CRT began as critical legal studies that "challenge dominant liberal ideas such as colorblindness and meritocracy and show how these ideas operate to disadvantage people of color and further advantage whites" (Bernal, 2002, p. 108). Most, if not all, of the law professors involved with the creation of LatCrit identified as Critical Race Theorists. These intertwined histories leave many to mistake LatCrit for a subset of CRT, when in fact LatCrit was born from the tension many of its founder held with CRT. From these tensions the three guiding principle of LatCrit were born (Valdes, 1996):

1. The possibility to exist as a theoretical genre that focuses Latinxs as both the creators and subjects of scholarship.
2. The possibility of being rooted in both the individual and shared lived experiences of Latinxs, utilizing modern and post-modern theories together to move away from the essentialism that divides Latinxs while also purposefully working to challenge the oppression Latinxs face via social constructs.
3. The possibility of commitment to collaborative work amongst scholars who identify as members of communities traditionally dehumanized by the majoritarian status quo.

Just as many other OutCrit theories have, LatCrit's theoretical frame has begun to be applied to scholarship outside of legal studies. At the turn of the 21st century scholars such as Bernal, and Solorzano brought LatCrit to education research. And since then scholars such as Gutiérrez have brought it to the sociopolitical turn in mathematics education.

Borderlands

Gloria E. Anzaldúa's (2007) teaches us that borderlands are the places where la mezcla is most likely to occur. The border of LatCrit and Mathematics Education being a prime example of this. Many still feel LatCrit (and other theories like it) do not have a place in mathematics education, but when we allow the two to meet conceptions of who can be a mathematician, and what it means to do mathematics are allowed to grow. In this poster I will share examples from my work with young emergent bilingual mathematicians, sharing their testimonios using a LatCrit lens to honor their stories.

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USING NARRATIVES ALONGSIDE CULTURAL HISTORICAL ACTIVITY THEORY: A DOCTORAL PROJECT IN THE MAKING

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Keyword: Research Methods

In the context of a doctoral project, this poster presents a work in progress that examines how one could use narratives alongside Cultural Historical Activity Theory (CHAT) in a project that aims to understand phenomena in a particular institution. First, I present a description of the theoretical framework and the methodology as they are used in the associated project. The second part consists of explaining how those two concepts can blend together, both internally and practically, using epistemological and pragmatic arguments.

Cultural Historical Activity Theory (CHAT) is used to frame human thinking and behaviour as parts of an activity system (LaCroix, 2010). The participants of activity systems are seen as constantly being transformed through interactions with other individuals and with their natural and social environment. CHAT emphasizes that every transformation is socially, culturally and historically situated and caused by some tensions, that an individual is trying to resolve. Hence, the best way to understand an activity system through a period of time is to study the tensions (Cole & Engeström, 1993).

Narrative inquiry is interested in relating or giving meaning to one's life experiences. Rooted in Dewey's vision of the nature of experience (social and individual, along with the notion of continuity), the history of any experience and the continuous state of change in any situation are at the center of a narrative endeavor (Clandinin & Connelly, 2000). In my project, I am particularly interested in Desgagnés' (2005) view of narratives, where teachers tell stories about problems they solved and the researcher creates an associated narrative. Desgagnés' stories are motivated by the need to account for the internal complexity inherent to the professional practice of teaching.

The connection between the theory and the methodology presented above is rooted in their emphasis on *significant events*, *tensions* for the former, *problems to solve* for the latter. Both conceptions consider these significant events as very useful tools to accurately understand experiences and changes in a specific social, cultural and historical context, *activity system* for the former or the *professional practice* for the latter. Therefore, we hypothesize that CHAT and Desgagné's narratives are internally compatible, as both emphasize the importance of crucial events and see the individuals in similar manners, and practically compatible, as they allow the researcher to focus on significant events in real life to understand a phenomenon. This poster will present more thoroughly both conceptions and the connections and differences between them.

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MATHEMATICAL ACTS AND THE RAL FRAMEWORK: UNDERSTANDING ENGLISH LEARNERS' PARTICIPATION IN MATHEMATICAL DISCOURSE

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Keywords: Classroom Discourse, Equity and Diversity, Research Methods

To provide adequate support for ELs in mathematics classes, educators must help ELs become fluent in *mathematical Discourse* (Merrill 2015). A "big D" Discourse (Gee, 1996) focus attends to the language use *necessary for* authentic (Brown, Collings, and Duguid 1989) mathematical activity, and situates its learning *within* authentic mathematical activity. Based on a study of college-aged ELs with basic English fluency and their work on secondary-level mathematics tasks in pairs (Merrill 2015), I suggest two ways to improve and focus research on ELs' participation in mathematical Discourse.

First, researchers should study ELs' participation in mathematical Discourse by studying their mathematical acts. A *mathematical act* consists of 1) the use of symbol systems such as spoken English, and written equations or gestures; 2) the meaning associated with that use; and 3) the purpose(s) the EL engaged in with the symbol use. This unit of analysis attends to the crucial elements for participation in mathematical Discourse. I will describe how attending to each component improves understanding of ELs' participation in mathematical Discourse.

Second, researches should study how ELs are able to use many symbol systems that do not rely heavily on English (*non-English language (NEL) symbol systems*) to participate meaningfully in mathematical Discourse. My research shows that ELs use NEL symbol systems to compensate for lower literacy with spoken English; ELs use NEL symbol systems to *replace* spoken English, to *augment* spoken English, and to *learn* spoken English. These categories of moves make up the Replace Augment Learn (RAL) Framework, shown in Table 1.

Table 1: RAL Framework

REPLACE	AUGMENT	LEARN
Predominantly use symbols	Use symbol systems as referents	Use own symbol use to improve
Symbol replaces operation	Correctly represent something being said incorrectly	Interpret another's symbol use to improve English
	Legitimize non-standard uses of English	
	Illustrate words or phrases	

Studying ELs' mathematical acts and analyzing how ELs use NEL symbol systems to support spoken English augments the field's understanding of how ELs should be supported in mathematics classes by developing a more thorough understanding of how ELs participate in mathematical Discourse. I will also discuss possible implications these research methods have for classroom practice, such as presenting problems using multiple symbol systems.

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USING A TEACHING SIMULATION TO EXPLORE HOW TEACHERS RESPOND TO STUDENT THINKING

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Keywords: Research Methods, Teacher Knowledge, Affect, Emotion, Beliefs, and Attitudes

Though there are numerous teaching practices that impact student learning (e.g. facilitating whole class discussions, giving explanations, etc.), one ubiquitous practice embedded within many of these is the mathematics teaching practice of managing students' (apparently) incorrect responses. Enacting this practice skillfully necessitates being "responsive" to student thinking and ideas (Pierson, 2008) which is no small feat. Managing students' responses involves complex cognitive processes and resources utilized by mathematics teachers in-the-moment to hear, interpret (hence, the qualification that responses are *apparently* incorrect), and respond to what a student has said (Jacobs, Lamb,&Philipp, 2010). These needed cognitive resources include both specialized skills and specialized content knowledge, such as mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008), that a teacher must use while simultaneously navigating her obligations to the mathematical discipline, individual students, the class as a whole, and the institution in which she works (Herbst & Chazan, 2012). It is no wonder, given these obligations and this complexity, that this practice can be incredibly stressful, even for veteran teachers. But how does stress really impact this practice and how might stress be mitigated by various factors (such as teaching experience, MKT, and beliefs)?

To explore answers to these questions, I have designed a teaching simulation that asks participants to place themselves in a 6th grade mathematics classroom, teaching a lesson on solving equations using fact families. I present participants with a series of student responses and ask them to respond to each student as if they were the teacher in the classroom. All of the student responses contain some incorrect thinking as well as elements of sense making. In this design, I have also incorporated the collection of physiological data (electrodermal activity) that has been used in other fields as a measure of stress (Boucsein, 2012). Prior to the teaching simulation, participants also fill out questionnaires about their teaching experience, trait anxiety, and beliefs about teaching and learning mathematics (Stipek, Givvin, Salmon, & MacGyvers, 2001) and answer some MKT items about relevant algebraic content. This poster will present the simulation design and preliminary results from the analysis of the physiological data and participant responses in the simulation and on the questionnaires.

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WRITTEN ASSESSMENT OF UNITS COORDINATION IN WHOLE NUMBER CONTINUOUS CONTEXTS

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Units coordination (Steffe, 1992) is a scheme in which a student distributes the elements of one composite unit (whole number greater than one) across the elements of another composite unit in order to carry out, for example, multiplication. Units coordination (UC) can take place in a variety of contexts, including fractional and whole number continuous contexts. UC has been studied in teaching experiments and been determined to serve as a constraint in developing key ideas in secondary mathematics (e.g., Hackenberg & Tillema, 2009). Written assessments of UC would allow larger-scale data generation. Current written assessments measure UC indirectly in fractional contexts (e.g., Wilkins, Norton, & Boyce, 2013). We are developing a written assessment that includes a series of units coordination tasks (the bars tasks) in whole number continuous contexts. The first iteration of our assessment was administered to 93 sixth-grade students. We had five bars tasks that require the following, in order of increasing hypothesized difficulty: iteration of a small composite unit with the number of iterations shown as partitions (B1), iteration of a large composite with partitions shown (B2), iteration of a composite without partitions shown (B3), the reverse of B2 (B4), and the reverse of B3 (B5). We hypothesized the following correspondences between the bars tasks and other tasks assessing three stages of students' unit construction and coordination (referred to as S1 to S3; Ulrich, 2016): B1 would be possible for some students who had not yet constructed composite units (S1), B2 would be possible for students who had constructed composite units (S2), B3 and B4 would be possible for students who could assimilate using composite units (advanced S2), and B5 would be possible for students who had constructed iterable units (S3).

The data supported the hierarchy of task difficulty we had hypothesized; the overall percentages correct were as follows: B1, 82%; B2, 72%; B3, 71%; B4, 55%; B5, 46%. However, B1 corresponded in difficulty to the tasks that identify S2, and B2 corresponded in difficulty to tasks to identify advanced S2. While we thought that students would be able to construct a composite unit in activity to iterate, it appears that assimilation with composite units is necessary to recognize a single partition as representing a composite. Because students develop the ability to anticipate using small composite units before developing a general assimilatory composite unit construct, B1 was equivalent to easier S2 tasks, while all other UC tasks were at least as hard as advanced S2 tasks. The presence of visible partitions did not decrease the difficulty. We have utilized our findings from the first administration of the assessment to revise our bars tasks.

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Chapter 13

Working Groups

EXPLORING CONNECTIONS BETWEEN ADVANCED AND SECONDARY MATHEMATICS

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The second meeting of this Working Group continues to explore questions about the connections between abstract algebra and school mathematics. Our goal is to focus in on questions around the way in which teachers' practice might be influenced based on their understanding of such connections. In particular, we will gather interested individuals in an effort to deepen our understanding of existing connections between abstract algebra and secondary mathematics and which of these connections are important for secondary teachers to know and understand. Moreover, we aim to further research in this area by first considering connections between abstract algebra and school mathematics present in tasks and curricular materials. Second, we will discuss efforts to design and implement professional development focused on these connections. Through our working group meetings, we plan to develop on-going collaborations to further research around the mathematical content preparation of teachers.

Keywords: Advanced Mathematical Thinking, Teacher Education-Preservice, Teacher Education-Inservice/Professional Development, Teacher Knowledge

Brief History of the Working Group

The first working group on connections between advanced and secondary mathematics was held during PME-NA 37 (Bartell, Bieda, Putnam, Bradfield, & Dominguez, 2015). During this working group, facilitators shared definitions of connections, and presented current work about important connections between abstract algebra and secondary mathematics. Through facilitator presentations and whole and small group discussions, the working group addressed the following questions: (1) What are the important connections between abstract algebra and secondary mathematics? (2) How does knowledge of connections between advanced and secondary content impact instruction in secondary classrooms? (3) How can we better support teachers to understand connections between advanced and secondary content and to use pedagogy that employs these connections? (4) How do we determine the depth of teacher knowledge of the connections between abstract algebra and secondary mathematics? (5) What are indicators that teachers have gained particular understandings? (6) What do we want teachers to be able to do with this knowledge?

After considering these questions, the group identified concrete next directions for collaboration. These include both examining tasks or curricular materials and designing and implementing professional development with secondary teachers around connections between abstract algebra and secondary mathematics. These projects were born from participants' interests in next steps as well as the group's desire to better understand teachers' knowledge of connections, how this knowledge could be assessed, and how mathematics teacher educators and teacher leaders might be able to support secondary teachers' knowledge of connections.

To ensure that our work was immediately beneficial to the community as well as of sustained duration, our initial working group proposal contained three primary follow-up activities: (1) submitting an article to the Notices of the American Mathematical Society, the most widely-read journal by professional mathematicians; (2) preparing an entry for the American Mathematical

Society's blog on the Teaching and Learning of Mathematics; and (3) longer-term work via continued meetings at other conferences and establishing a collective research agenda. We are actively working to write an article for the Notices, and we anticipate its completion by the end of the Spring 2016 semester. We completed our blog entry, which focused on providing several examples of connections between abstract algebra and high school mathematics, and it appeared in December 2015 (Baldinger, Broderick, Murray, Wasserman, & White, 2015). Due to its ongoing nature, we are continually working on the third item. Thus far various subgroups have met virtually, via regular email correspondence, and in person at the Joint Math Meetings and the Association of Mathematics Teacher Educators Annual Meeting.

Background and Theoretical Perspective

Connecting Abstract Algebra and High School Algebra

This working group explores connections between advanced mathematics content (in particular, abstract algebra) and secondary mathematics content. We are particularly interested in considering those connections that influence in some way a teacher's instructional practice. Recent policy documents (e.g., Conference Board of the Mathematical Sciences, 2012) advocate for including abstract algebra as a fundamental part of secondary teacher preparation. This provides motivation for better unpacking the connections between this advanced mathematics and school mathematics. Research focused on these connections has begun to gain traction in the mathematics education community. Such research addresses (a) the connections that exist between particular advanced content courses and secondary mathematics (e.g., Baldinger, 2014; Cofer & Findell, 2007; Suominen, 2015; Usiskin, 2001; Wasserman, 2014) and (b) the impact learning connections between advanced and secondary content has on teachers and their instruction (e.g., Baldinger, 2013; Wasserman, 2014).

Researchers have identified and listed connections between abstract algebra and school mathematics using a variety of approaches. For example, using the Common Core State Standards to represent the school mathematics curriculum, Wasserman (2015) considers teachers' potential practices for teaching specific elementary, middle, and secondary topics, before and after learning about introductory concepts in abstract algebra. He developed a framework that considered the K-12 content areas for which teaching might be influenced by teachers' knowledge of abstract algebra. This provides a different perspective than the traditional listing of connections between abstract algebra and secondary mathematics (e.g., $(Z, +)$ is a group). Suominen (2015), in contrast, considered connections by analyzing commonly used abstract algebra textbooks and interviewing abstract algebra professors. She found that the connections made in textbooks could be categorized in different types of connections (e.g., generalizations, real-world applications). After developing a list of connections, she found that the mathematicians interviewed prioritized different connections. Baldinger (2015) took yet another approach, and explored connections as they related to pre-service teacher engagement in mathematical practices. She found that in an abstract algebra course designed for pre-service secondary teachers, participants had opportunities to learn about and engage in mathematical practices such as justification, attending to precision, and communicating mathematical ideas. Following the course, participants were able to better engage in these practices while solving high school algebra tasks.

Influencing Instructional Practice

The connections identified through previous research are particularly important to the extent that they influence teachers' instructional practice. For example, Hill (2003) describes how a secondary teacher was able to build on the axiomatic approach to abstract algebra in a unit on complex numbers for her secondary students. Wasserman (2014) investigated teachers who studied abstract algebra

topics from a perspective emphasizing algebraic structures and their connections to school mathematics. He found that they began to change their beliefs and practices around teaching content such as number operations and solving equations. Cofer (2015) explored how pre-service secondary teachers were able to incorporate ideas from abstract algebra when explaining topics in school mathematics. She found that some participants were able to give explanations that incorporated an "advanced mathematical argument" and clearly drew on their abstract algebra knowledge. In other cases, participants relied on analogy or rules in their explanations, showing a lack of integration of abstract algebra knowledge. Cofer's framework provides a lens for investigating how knowledge of connections might be manifested during instructional practice.

Supporting Teachers to Learn about Connections

Building on the importance of connections between school algebra and abstract algebra, the question comes up about how we can support teachers to learn about such connections. Research suggests that pre-service teachers struggle in traditional abstract algebra courses, and these courses tend to feel disconnected from school algebra (Clark, Hemenway, St. John, Tolias, & Vakil, 1999; Zazkis & Leikin, 2010). Additionally, many do not understand the purpose of taking advanced mathematics courses (e.g., Cuoco, 2001), especially the relevance to teaching secondary mathematics (e.g., Cuoco & Rotman, 2013). For example, Broderick (2013) interviewed prospective secondary teachers about the usefulness of their college math courses. He found their comments were consistent with the literature (e.g., Cuoco, 2001; Hill, 2003), with one caveat. One participant had not passed abstract algebra the first time and went through it again. She found more relevance the second time through and was more satisfied with taking the course. Such findings have led to several efforts to make abstract algebra more accessible and applicable.

At the pre-service level, researchers have documented different strategies for better supporting teachers to learn about these connections (Murray & Star, 2013). Some courses, for example, intentionally utilize cooperative learning environments (Barbut, 1987; Cnop & Grandsard, 1998; Grassl & Mingus, 2007) or include a component of technology (Clark et al., 1999; Leron & Dubinsky, 1995). Through emphasizing non-traditional pedagogy, these courses provide pre-service teachers with a new model of teaching mathematics. Such courses support learning connections by helping pre-service teachers more deeply understand the abstract algebra content, and their non-traditional pedagogical approach may support changes in practice. However, the connections across abstract algebra content and school mathematics are largely implicit. Other courses intentionally address connections with school mathematics (e.g., Baldinger, 2013; Cuoco & Rotman, 2013; Wasserman, 2014). In these cases, the connections are an intentional part of the course design, and pre-service teachers are supported to learn about connections alongside learning about abstract algebra content.

For in-service teachers, research suggests that professional development driven by a focus on mathematics content tends to have a stronger impact on practice than professional development with, for example, an activity focus (Garet, Porter, Desimone, Birman, & Yoon, 2001; Marra et al., 2011). Furthermore, Sowder and colleagues (1998) found that increasing teachers' content knowledge could have impact on teachers' practice. This helps motivate a need to focus explicitly on connections between abstract algebra and school mathematics within professional development, as doing so helps make the content more relevant to classroom practice. This can help mitigate the tension teachers feel when professional development is disconnected from their work with students (Nipper et al., 2011).

Taken together, this research suggests important starting points in considering how to support teacher learning around connections between school algebra and abstract algebra. In this working group, we plan to build on this research to discuss strategies to support teacher learning of these connections in ways that help teachers to make positive changes in their practice based on their deepened understanding.

Theoretical Perspectives

We view teacher knowledge of the connections between school algebra and high school algebra as part of their mathematical knowledge for teaching. We build on the work of other scholars (Ball, Thames, & Phelps, 2008; McCrory, Floden, Ferrini-Mundy, Reckase, & Senk, 2012) in taking a practice-based approach to conceptualizing teacher knowledge, since we are interested in connections that have an impact on instructional practice. Though there is debate about whether some of the categories of mathematical knowledge for teaching identified by Ball and colleagues (specialized content knowledge, in particular) can be appropriately transferred to secondary level mathematical knowledge for teaching (Speer, King, & Howell, 2015), we draw on the approach of considering knowledge in practice. In thinking about connections, we are particularly interested in what Ball and colleagues call "horizon content knowledge". Wasserman and Stockton (2013) unpack this category of knowledge, and explore how it can be applied to the work of teaching.

In considering how teachers develop mathematical knowledge for teaching, and horizon content knowledge around connections in particular, we are guided by the construct of "key developmental understandings (KDUs)" (Simon, 2006). A KDU is a "conceptual advance that is important to the development of a concept" (Simon, 2006, p. 365). Silverman and Thompson (2008) use this construct in their framing of teachers' mathematical knowledge, by proposing that teachers develop mathematical knowledge for teaching of a topic if: (a) they have achieved a KDU that encompasses mathematical understanding of the topic; and (b) they have an understanding of how the topic may evolve instructionally in support of students' reasoning in the K-12 classroom. We apply this notion to understanding of connections, by investigating in particular how understanding those connections contributes to teacher practice.

Connections to PME-NA

For many teachers, the border between high school algebra and abstract algebra is significant. However, the connections between these two content areas that we focus on here serve as key bridges across that border. Our working group this year seeks to address questions that support teachers in questioning this border themselves—beyond just identifying connections between the two areas, we want to support teachers to apply knowledge of these borders to their classroom practice.

Working Group Goals

A long-term collective goal is to fundamentally reconsider the teaching of more advanced content courses for teachers in ways that are useful both generally for the teaching of all mathematics majors and also specifically for the unique considerations and professional practices of secondary teachers.

The goals for the second meeting of the working group are to share the work done for the two projects begun at PME-NA 37, to discuss next steps for these projects, and to consider new research that could advance our collective understanding of connections between advanced and secondary content and the role of connections in secondary mathematics teaching and learning.

The overall goal for this working group remains to strengthen our collective understanding of connections between advanced and secondary content and their role in secondary mathematics teaching and learning. We focus on abstract algebra as an entry into this domain, particularly for its rich connection to secondary mathematics topics such as algebra and functions, while also anticipating that the impact of these conversations will prompt research into other content areas (e.g., calculus, linear algebra, and analysis).

Plan for Working Group

Across the three sessions, participants will have the opportunity to interrogate the current state of the research through brief presentations and guided discussions.

Session 1: Examination of Tasks and Curricula

In the first session, group members will present work about the analysis of tasks and curriculum around the connection between abstract algebra and secondary mathematics. Leading up to the 2016 working group, a subset of participants from the previous year have been exploring this idea from two perspectives. The first perspective is to interrogate mathematics courses taught for prospective secondary teachers (e.g., capstone courses; content courses; content-focused methods courses) for connections, including where the connections are located, which ones are addressed, and how they are addressed. The intent of this analysis is to inform what is going on currently in mathematics teacher preparation and compare the nature of these connections to those that are highlighted in mathematics courses.

The second perspective is to merge the frameworks developed by Wasserman (2015) and Suominen (2015) discussed above. The goal of this work is to see if there are commonalities in the connections noted in both the Common Core State Standards and higher education textbooks. Moreover, the group is interested in determining how meaningful the connections are to instruction in secondary school, particularly those connections that are less commonly noted in the literature. By merging these two approaches, the group is attempting to narrow down what to look for in tasks and curricular materials for secondary mathematics. This attempt will further the group's understanding of what sections, courses, and topics in secondary mathematics could be further explored for connections.

After the presentation, facilitators will lead small group discussions so that participants can provide their own input on connections embedded in current tasks for secondary mathematics students, regardless of whether these tasks are being used for high school students or prospective teachers. At the end of this session, the facilitators will create a summary of important connections and tasks discussed to help organize and motivate the final session.

Session 2: Designing Professional Development

In the second session, group members will present work regarding the design and implementation of professional development with secondary teachers around connections between abstract algebra and secondary mathematics. Based on ongoing discussions among participants of the original working group and others at various professional meetings, the professional development is being planned to have a strong pedagogical component, which involves using classroom-based scenarios or vignettes that highlighted some aspect of high school mathematics connected to abstract algebra. As we design this professional development, we consider the following questions: (1) What are the objectives, goals, and outcomes of professional development focused on the connections between abstract algebra and secondary mathematics? (2) How we can use the professional development to examine curricula and tasks and to solicit teachers' ideas about the connections?

In the second part of this session, we will engage in small group discussions of this particular professional development design. Through this conversation, we will collect ideas and feedback for further professional development workshops and consider the following questions: What are the research questions aligned with this professional development? How do we answer these questions, or measure what we are interested in? How might the internalization of connections between abstract algebra and secondary mathematics changes teachers? At the end of this session, the facilitators will create a summary of ideas generated during the discussions to help organize and motivate the final session.

Session 3: Determining Concrete Next Steps

In the final session of this working group, we will begin by sharing our summaries from the previous two sessions. We will engage in a whole group discussion of possible research questions

that were generated in previous conversations. We will conclude the session by splitting up into groups, based on interest, to begin planning next steps.

Anticipated Follow-up Activities

We will actively continue our work together beyond the working group meetings at PME-NA 38. The formation of new connections among scholars, as well the deepening of the current connections, will positively impact post-conference activities. Our two subgroups (the tasks and curriculum subgroup and the professional development subgroup), along with any new subgroups arising out of the meetings at PME-NA 38 will work on specific research questions generated through our meetings at PME-NA. We also plan to collaborate on grant and journal submissions; an NSF DRK-12 grant is a likely starting point. In addition, the subgroup focused on professional development will take direct action-steps toward producing, testing, and investigating professional development focused on connections between abstract algebra and school mathematics. Through participation in the working group, several members are already sharing and providing feedback on the work of other members, and we will expand and deepen these efforts. We anticipate that these connections among scholars can be maintained and potentially lead to collaboration on related research.

We anticipate that members of this working group will continue to meet virtually, correspond regularly about the work via email, and meet in person at various professional meetings.

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EXPLORING AND EXAMINING QUANTITATIVE MEASURES

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The purpose of this working group is to bring together scholars with an interest in examining the research on quantitative tools and measures for gathering meaningful data, and to spark conversations and collaboration across individuals and groups with an interest in synthesizing the literature on large-scale tools used to measure student- and teacher-related outcomes. While syntheses of measures for use in mathematics education can be found in the literature, few can be described as a comprehensive analysis. The working group session will focus on (1) defining terms identified as critical (e.g., large-scale, quantitative, and validity evidence) for bounding the focus of the group, (2) initial development of a document of available tools and their associated validity evidence, and (3) identification of potential follow-up activities to continue the work to identify tools and developed related synthesis documents (e.g., the formation of sub-groups around potential topics of interest). The efforts of the group will be summarized and extended through both social media tools (e.g., creating a Facebook group) and online collaboration tools (e.g., Google hangouts and documents) to further promote this work.

Keywords: Assessment and Evaluation, Research Methods

Introduction

There is value in the knowledge that large-scale quantitative research can bring to the field in terms of generalizability to educational practice when appropriately conducted (American Statistical Association, 2007; Hill & Shih, 2009). The American Statistical Association's report (2007) on *Use of Statistics in Mathematics Education Research* states:

> If research in mathematics education is to provide an effective influence on practice, it must become more cumulative in nature. New research needs to build on existing research to produce a more coherent body of work… Studies cannot be linked together well unless researchers are consistent in their use of interventions; observation and measurement tools; and techniques of data collection, data analysis, and reporting. (pp. 4-5).

As education has shifted more towards data driven policy and research initiatives in the last 25 years (Carney, Brendefur, Thiede, Hughes, & Sutton, 2016; Hill & Shih, 2009), the data for policy-related aspects are often expected to be quantitative in nature (e.g., end-of-course assessments and numerical value of reform-oriented teaching). Funding agencies encouraging research (i.e., National Science Foundation and Institute of Education Sciences) often request proposals to employ quantitative measures with sufficient validity evidence (see http://ies.ed.gov/ and http://www.nsf.gov/).

Measure (instrument) quality strongly influences the quality of data collected and relatedly, findings of a research study (Gall, Gall, & Borg, 2007). Measures with a clearly defined purpose and supporting validity evidence are foundational to conducting high quality large-scale quantitative work (Newcomer, 2009). There are few syntheses of quantitative tools for mathematics educators to employ and even fewer discussions of the validity evidence necessary to support the use of measures in a particular context. Syntheses of measures for use in mathematics education can be found in the literature but these are typically not intended as a comprehensive analysis. For example, Carney et al.

(2015) conducted a brief review of self-report instructional practice survey scales applicable to mathematics education. Boston, Bostic, Lesseig, & Sherman (2015) conducted a review of three widely known classroom observation protocols to assist mathematics educators in determining the appropriate tool for their particular research question and context. Both reviews provided a background on existing measures and their associated validity evidence in relation to a new measure under development. It is important that this type of work continues and is encouraged by the field. Thus, this working group aims to increase conversation around quantitative tools for use on a large-scale with this working group. We share three goals for this proposed working group: (a) To bring together scholars with an interest in examining the research on quantitative tools and measures for gathering meaningful data; (b) To spark conversations and collaboration across individuals and groups with an interest in large-scale tools and those conducting research on student- and teacher-related outcomes; (c) To generate products to disseminate widely across the field of mathematics education scholars.

Related Literature

Historical Context, Terms, and Rationale for Working Group

The National Mathematics Advisory Panel (2008) found that only a "small proportion of those [reviewed] studies have met methodological standards. Most ….failed to meet standards of quality because they do not permit strong inferences about causation or causal mechanisms" (pp. 2-7). Sound methodology is guided by appropriate measure or instrument choice. Good research takes on quantitative, qualitative, and at times both methodologies to become mixed-methodologies (Hill & Shih, 2009; Cresswell, 2012). Our focus for this proposal is quantitative-inclusive methodologies, specifically focusing on measures and tools associated with them, to support mathematics educators use of and need for quantitative tools that may be used in large-scale studies.

Near the core of any methodology is the measure or instrument used to collect data (Newcomer, 2009). The American Psychological Association, National Council on Measurement Education, and American Educational Research Association ([APA, NCME, AERA] 2014; 1999) provide clear guidelines regarding measurement validity and reliability. At a minimum, sufficient evidence for five variables must be shared related to validity: (1) content evidence, (2) evidence for relationship to other variables, (3) evidence from internal structure, (4) evidence from response processes, and (5) evidence from consequences of testing (AERA, APA, & NCME, 1999, 2014; Gall et al., 2007). Unfortunately, "evidence of instrument validity and reliability is woefully lacking" (Ziebarth, Fonger, & Kratky, 2014, p. 115) in the literature. Validation studies of quantitative measures are noticeably absent from mathematics education journals, which present the challenge of determining whether an instrument is appropriate for a given study much less whether it will generate valid and reliable data for analysis (Hill & Shih, 2009). Hill and Shih (2009) reported that eight of 47 studies published in the *Journal for Research in Mathematics Education* provided any evidence related to validity and the majority provided only psychometric evidence. Our goal for this literature review is to present a need for a working group at PME-NA 38 that will bring individuals from around North America to conduct more syntheses and further explore needed areas of tools that can be used to study both student- and teacher- related measures in large-scale research by mathematics educators.

Examining Student-focused Measures

Quantitative measures of student's mathematics content knowledge, problem solving, beliefs, and other factors have been employed across various contexts. We share an initial set of literature to frame the thinking for working group participants. Moreover, we welcome those that have interests not necessarily listed in this section.

Mathematics content knowledge. Students' mathematics content knowledge has been assessed in large-scale studies using end-of-course (high-stakes) measures during the last decade, Trends in Mathematics and Science Study (TIMSS), and National Assessment for Educational Progress (NAEP). Researchers who developed the PISA and NAEP report the validation process; however, the end-of-course measures are often shrouded by commercial entities (e.g., American Institutes of Research and Pearson). The latter group makes examining the quality of the measures for content knowledge problematic. Broadly speaking, it is challenging for researchers aiming to make decisions regarding use of items (or previously used measures) without syntheses describing measure qualities as well as similarities and differences across measures. Thus, a measure may claim to measure students' (at one grade- or developmental-level) content knowledge but how is content knowledge defined for each measure?

Beliefs. Students' beliefs of mathematics, mathematics teaching, and usefulness of mathematics for the real world have been examined in various ways. Students taking the NAEP assessment also responded to questions designed to measure their perceptions of mathematics (Dossey, Mullis, Lindquist, & Chambers, 1988). In the survey created by Dossey and colleagues, students responded to several Likert scale items regarding their attitudes and beliefs about mathematics. Similarly, Lazim, Osman, and Salihin (2004) created a mathematics belief questionnaire that had four belief dimensions: "[about] the nature of mathematics, about the role of teachers, about teaching and learning mathematics, and about their competency in mathematics" (p. 5). Again, the instrument consisted of Likert scale items self-reported by the students. The authors claim they achieved high reliability after the development of the survey but it was not reported. Hence, greater examination of these instruments is needed to benefit mathematics education research.

Examining Teacher-focused Measures

A couple articles have provided syntheses of the literature related to quantitative teacher-focused measures. We explore three sets here: observation protocols (of instruction), teachers' content knowledge, and teachers' beliefs. Again, we use this as a starting point and welcome interests within teacher-focused measures that are not necessarily represented within this frame.

Observation protocols. In 2015, Boston and colleagues compared the Reformed Teaching Observation Protocol, Mathematical Quality of Instruction, and Instructional Quality Assessment. A key finding of the study was that these three unique large-scale teacher-related observation protocols provided three unique lenses into teachers' instruction (Boston et al., 2015). The authors encouraged the field of mathematics education to execute further work to closely examine other observation tools and share syntheses of relevant literature.

Teachers' content knowledge. The components of the Mathematical Knowledge for Teaching (MKT) construct (Ball, Thames, & Phelps, 2008) can serve as a useful tool for exploring and examining quantitative measures of teachers' knowledge. Quantitative measures designed for teacher certification purposes (e.g., the Praxis series) tend to focus on the component of common content knowledge, ignoring other important components of the MKT framework often deemed important to mathematics educators. Other assessments are designed specifically with the intent of measuring teachers' knowledge of particular content areas (e.g., Knowledge of Algebra for Teaching measure, McCrory, Floden, Ferrini-Mundy, Reckase, & Senk, 2012) or grade bands (e.g., Diagnostic Teacher Assessment in Mathematics and Science, Saderholm, Ronau, Brown, & Collins, 2010). The most commonly used quantitative measures for teachers' content knowledge in mathematics come from the Learning Mathematics for Teaching (LMT) project (2005). The LMT assessments aims to measure teachers' content and pedagogical knowledge for teaching and are parsed into different content areas (e.g., K-6 geometry, 6-8 Number and Operations, and 4-8 proportional reasoning; LMT, 2005). A review of the NSF database for measures of teachers' math content knowledge for teaching (a) generating quantitative data, (b) with reliability and validity evidence, and (c) could be

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used in large-scale studies resulted in 16 measures, 11 of which were part of the set from the LMT series. While tools such as the NSF database or the National Council for Teachers of Mathematics Handbook Chapter "*Assessing teachers' mathematical knowledge: What knowledge matters and what evidence counts*" (Hill, Sleep, Lewis, & Ball, 2007) provide a brief summary of some potential measures a mathematics education researcher could use to examine teachers' knowledge, it does not provide a comprehensive synthesis that might aid in determining which measure to use for a given research question, much less describe the validity evidence associated with the measure. Again, there is no available synthesis of available tools to measure teachers' knowledge of mathematics.

Beliefs. Philipp (2007) defines beliefs as "held understandings, premises, or propositions about the world that are thought to be true. …Beliefs, unlike knowledge, may be held with varying degrees of conviction and are not consensual" (p. 259). Beliefs and attitudes are different; they are related and at times have been discussed synonymously in the literature (Philipp, 2007). One of the oldest and still used measures is the Fennema-Sherman Mathematics Attitude scale (see Fennema & Sherman, 1976). This measure uses a Likert-scale to assess respondents' attitudes towards several domains. The study describes four Likert-scale self-report measures and accurately suggests the limited scope of self-report measures with regards to validity evidence. The Integrating Mathematics and Pedagogy (IMAP, 2004; see also Ambrose, Clement, Philipp, & Chauvot, 2004) is a web-based survey with open-ended items. This measure overcame the challenges of Likert scales, the lack of context for an overall score, and that respondents may give an opinion when one is not naturally held (Ambrose et al., 2004). A search of academic journals for measures of mathematics teachers' beliefs provided numerous hits but few are found in mathematics education journals, much less a synthesis of those available with validity and reliability evidence to be used in studies with large data samples. Put simply, no syntheses of measures in this are shared.

Session Organization and Plan for Engagement

The purpose of this working group is to gather individuals across North America interested in synthesizing the literature on quantitative tools in mathematics education that can be used in studies with large samples to examine student- and teacher-related outcomes. When considering the process for conducting a synthesis of quantitative tools and measures, it may be helpful to think of identifying and compiling tools and measures and their associated evidence separately from summarizing and evaluating the quality of the evidence. A synthesis includes both compilation and evaluation. The sequencing of the activities for the purposes of a working group will begin with compilation followed by evaluation in subsequent follow-up activities. It is important for the group to come to consensus on the parameters and frameworks for the synthesis. We recognize that the scope of the working group sessions proposed for PME-NA 2016 must be greatly narrowed. Therefore, we primarily focus on our first two of the three goals for the conference, which are shared here:

1. Bring together scholars with an interest in examining the research on quantitative tools and measures for gathering meaningful data.
2. Spark conversations and collaboration across individuals and groups with an interest in tools for large-scale studies and those conducting research on student- and teacher-related outcomes.

Prior Work

The idea for this working group proposal started at PME-NA 2015. We explored interest across the field from potential attendees before writing this proposal. We sought feedback from colleagues using the Association Mathematics Teacher Educators' (AMTE) bulletin board feature as well as the Service, Teaching, and Research (STaR) list-serv. An interest survey was shared broadly with both groups (i.e., AMTE and STaR members) to gather an idea of the level of interest in this idea.

Twenty-six people expressed interest, including from individuals who could not attend AMTE's 2016 annual meeting. We held a follow-up meeting at AMTE to meet with fourteen individuals who expressed interest and were attending AMTE's annual meeting. A majority of those at the AMTE follow-up meeting shared that they planned to attend the working group if accepted for PME-NA 2016. To that end, we plan on organizing the sessions in the following manner to address our two primary goals for the PME-NA 2016 working group session.

Session 1

The first session will begin with introductions, in conjunction with discerning the interests and areas of expertise of those in attendance. This will be followed by a group discussion about the stated purpose and aims of the group and the following guiding questions: (a) What do we mean by the term quantitative tools? (b) What do we mean by the term 'large-scale'? (c) How will we define these terms within the working group? We anticipate this discussion will elicit several additional topics that can be further explored during session 1 and potentially sessions 2 and/or 3. Ideally we will conclude by summarizing the discussion from session 1 including potential definitions for the terms identified as critical (e.g., at-scale, large-scale studies, and quantitative) that will be necessary for bounding the subsequent discussion of currently available tools. At the conclusion of session 1, we will present a tentative framework (see table 1 below) for organizing our subsequent discussions around quantitative tools that can be used with large samples to examine student- and teacher-related outcomes. We will request that session participants return to sessions 2 and 3 with ideas for tools that potentially fit within different areas of the framework.

Session 2

The second session will begin with a discussion on current perspectives in validity related to the argument-based approach (e.g., Kane 2001, 2016). Finbarr Sloane, an NSF-program officer with expertise in mathematics education, measurement, and evaluation has offered to provide a brief overview and facilitate discussion regarding the argument-based approach to validity. Following Dr. Sloane's presentation, the remaining part of session 2 will involve whole-group discussion around potential measures that address the identified areas using the organizational framework for student- and teacher-related outcomes. A brief overview of the organizational framework will be used to ignite the discussion of specific instruments. Table 1 presents the initial organizational framework that will be presented with the full expectation that the group may modify it during sessions 1 and 2. Group facilitators and attendees may begin by placing some relatively well-known tools within the framework to ensure we have a common understanding of the process.

Table 1: Initial Organizational framework for discussion of measures

	Knowledge	Beliefs	Practice
Teachers			
Students			

Session 3

The third session will primarily focus on placing tools within the organizational framework including any associated citations related to publically available or published validity evidence. Depending upon the size of the group, this work may be conducted in small-groups with a whole-group share-out towards the end of session 3. While a long-term aim is to develop syntheses of the literature related to available tools, we see the primary aim of the working group's meeting at PME-NA 2016 as bringing together individuals interested in this conversation and working together on

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

future collaborative efforts in this area. By the end of the third session, we intend to have an initial draft document of some available tools and their associated validity evidence but we do not anticipate this will be a comprehensive document. We will conclude session 3 with a discussion of anticipated follow-up activities to determine the level of interest and commitment from the group in continuing with this work.

Anticipated Follow-up Activities

As a result of our working group discussion and document development, we anticipate several potential follow-up activities. Participants will greatly influence the specific follow-up activities; however, we outline a potential progression of activities to guide discussion of potential 'next-steps'.

One outcome of the working group sessions is a draft document outlining some of the available tools and their associated validity evidence. An anticipated outcome will be to determine how this document should be further refined and later distributed. This will include explicit discussion of next steps to develop a comprehensive synthesis of the literature for wide dissemination to the mathematics education community.

We see several possible venues for further conversations and work related to developing syntheses of the literature on quantitative tools in mathematics education that can be used with studies of large-scale samples to examine student- and teacher-related outcomes. First, we anticipate using both social media tools (e.g., creating a Facebook group) and online collaboration tools (e.g., Google hangouts and documents) to promote these syntheses. Second, we anticipate using mathematics education conferences venues to further the conversations and synthesis work around the project. More specifically, we plan on proposing to continue the PME-NA working group at the 2017 conference. In addition, we anticipate submitting for a symposium at either the 2017 or 2018 conference of the Association of Mathematics Teacher Educators. Lastly, there is potential to apply for grant funding through a NSF CORE Research proposal to support a conference with a focused outcome of a monograph synthesizing the research literature within a particular area.

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MODELS AND MODELING WORKING GROUP

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Modeling has received a great deal of attention from mathematics educators in recent years: for example, the Common Core State Standards for Mathematics have identified it as one of eight core mathematical practices, and the 2016 APME yearbook is dedicated to the theme of modeling. The Models and Modeling Working Group has provided a venue for discussing and collaborating to execute research fundamental to this area since the first PME-NA conference in 1978. In convening this Working Group at PME-NA 38 we propose to lay the groundwork for a shared writing project, producing an edited volume on the research questions and opportunities around learning through modeling over extended (course-length) time periods.

Keywords: Modeling, Problem Solving, Design Experiments

The Models and Modeling Working Group has been a significant presence at PME-NA since the Conference was inaugurated in 1978. It has supported substantive research efforts, and it has acted as a vehicle for convening a diverse set of researchers to work collaboratively on larger projects and research problems. An important historical purpose of the Working Group has been to discuss and extend the ways in which a focus on models and modeling can be used both to support learning in mathematics, science and engineering, and to study such learning processes in action. Early in its history, the Group focused heavily on particular activities, elaborating design principles for these activities as learning environments and documenting the idea development they promoted. Gradually, however, researchers associated with the Group have expanded their perspective to consider larger curricular sequences, organized around such activities. Moreover, researchers affiliated with the group have connected in a variety of ways with other lines of inquiry in mathematics education. These broader perspectives open both exciting opportunities and significant challenges for research.

We propose convening the Group at PME-NA 38 as a springboard for articulating and clarifying a shared research agenda to study learning and idea development through modeling at these larger timescales. As described in detail below, the three sessions at the conference will frame a shared writing project on the subject, exploring a range of research themes that have been identified as this proposal has emerged and been framed.

The Models and Modeling Perspective (M&MP)

For nearly forty years, M&MP researchers and educators have engaged in design research directed at understanding the development of mathematical ideas. A fundamental principle underlying this work has been that learners' ideas develop through, and in relation to conceptual entities called *models*, which we define as follows:

> conceptual systems (consisting of elements, relations, operations, and rules governing interactions) that are expressed using external notation systems, and that are used to construct, describe, or explain the behaviors of other system(s)—perhaps so that the other system can be manipulated or predicted intelligently (Lesh & Doerr, 2003, p. 10)

As conceptual systems expressed through representational media, models can provide illumination into how students, teachers, and researchers learn, develop, and apply relevant mathematical concepts (Lesh, Doerr, Carmona, & Hjalmarson, 2003). Further, under appropriate conditions, these models can be evoked and expressed: in such settings, they can become objects of reflection by learners and collaborative groups, and they can form the basis for rich communication. In particular, when individuals and groups encounter problem situations with specifications that demand a model-rich response, their models are observed to grow through relatively rapid cycles of development toward solutions that satisfy these specifications. A focus on learners' models can therefore provide an impetus for design and a lens for research focused on assessment and/or on the fine-grained texture of learning processes.

Originally, the M&MP tradition was focused squarely on local conceptual development: that is, on investigating the micro-evolution of ideas and knowledge in teachers and students. Thus, the resources and tools produced were first and foremost designed to *study* idea development (as opposed to serving teaching or curricular goals). However, by producing materials that fostered the rapid development of ideas, M&MP designers also laid the foundation for extremely effective instructional sequences addressing big ideas in important mathematical domains. Over time, however, the M&MP community has refined its techniques for creating situations that provoke students to express and improve their models. The results of this work include a body of *Model-Eliciting Activities* (MEAs), in which students are presented with authentic, real-world situations where they repeatedly express, test, and refine or revise their current ways of thinking as they endeavor to generate a structurally significant product—that is, a model, comprising conceptual structures for solving the given problem. These activities differ markedly from some "problem-solving" settings, which emphasize applications. In contrast, MEAs give students the opportunity to create, apply and adapt scientific and mathematical models in interpreting, explaining, and predicting the behavior of real-world systems (Zawojewski, 2013). Extensive research with MEAs has produced accounts of learning in these environments (Lesh, Hoover, Hole, Kelly, & Post 2000; Lesh & Doerr, 2003), design principles to guide MEA development (Hjalmarson & Lesh, 2007; Doerr & English, 2006; Lesh, et. al., 2000; Lesh, Hoover, & Kelly, 1992) and accounts and reflections on the design process of MEAs (Zawojewski, Hjalmarson, Bowman, & Lesh, 2008).

Example MEA: the Volleyball Problem

Students are usually introduced to the Volleyball Problem by reading a "math rich newspaper article" that describes a summer sports camp specializing in girls' volleyball. The newspaper article explains how issues arose in the past because it was difficult for the camp councilors to form fair teams that could remain together throughout two weeks of camp. The students' challenge in the volleyball problem is to develop a procedure that the camp councilors can use to form teams that are as equivalent as possible—based on information that is gathered during try-out activities that occur during the first day of the camp. After breaking into groups, students are presented with the problem statement below (Figure 1).

Along with the introduction and statement of the problem, student groups are given tryout data for a sample of 18 players. This data includes some information that is easily represented in tabular form (player's height, measured vertical leap, 40-meter dash time, etc.), as well as some that is not (players' performance in a spiking trial, and brief summative comments about their strengths and weaknesses from the coach of their home team). These data elements are chosen so as to present fundamental challenges to students, involving the nature of data (categorical versus numerical); scaling (e.g., the vertical leap where "large is good" versus the 40-meter dash where "small is good"); units (vertical leap is given in raw inches; height in feet-and-inches); and so on.

> **The Volleyball Problem.** Organizers of the volleyball camp need a way to divide the campers into fair teams. They have decided to get information from the girls' coaches – and to use information from try-out activities that will be given on the first day of the camp. The table below shows a sample of the kind of information that will be gathered from the try-out activities. Your task is to write a letter to the organizers where you: (1) describe a procedure for using information like the kind that is given below to divide more that 200 players into teams that will be fair, and (2) show how your procedures works by using it to divide these 18 girls into three fair teams.

Figure 1. The Volleyball Problem.

Student groups iteratively develop solutions to this problem in the time allotted—usually 50-60 minutes for this MEA. Afterwards, the teacher may choose to organize a structured "poster session" event. In one version of this activity structure, one member of each 3-person group hosts a poster presentation showing the results of their group. The other two students use a Quality Assurance Guide to assess the quality of the results produced by other groups in the class. These instruments are submitted to the teacher and contribute to assessment in various ways, providing evidence for the achievements of both individuals and groups.

A key outcome of the discussion unpacking this and other MEAs is the notion that when a construct such as "volleyball-playing ability" is operationally defined, a variety of reasonable definitions may be possible. Though none of these are objectively "correct," all of them involve different assumptions and may lead to markedly different conclusions or judgments. In particular, in real life situations where statistics procedures are useful, results often vary significantly depending on how the various types of data are quantified and combined. On the other hand, it is also quite possible for solutions to be "incorrect" – either through miscalculation or other errors, or by not adequately addressing the Client's problem or needs.

MEAs like the Volleyball problem present learners with situations in which familiar procedures and constructs are both applicable and insufficient. That is, on the one hand such problems are accessible to learners from a wide range of levels of ability, experiences, or knowledge (from upper elementary school through graduate school). On the other hand, learners encountering these problems find that they have no ready-made solution they can apply to address the client's needs. As a result, learners engage in solution-construction processes that put them off balance in comparison to typical school-mathematics tasks. Indeed, this uncertainty is part of the design of the MEA, illuminating fundamental conceptual issues associated with the core mathematical structures involved.

MEA Design Principles

Historically, as individual modeling activities emerged, an intense period of design research ensued to establish MEAs as a compelling genre of learning tasks that would (a) stimulate mathematical thinking representative of that which occurs in contexts outside of artificial school settings (Lesh, Caylor, & Gupta, 2007; Lesh & Caylor, 2007); (b) enable the growth of productive solutions through rapid modeling cycles; and (c) leave behind researchable traces of learners' ways of thinking during the process. This line of work produced the notion of Thought-Revealing Artifacts and Model-Eliciting Activities (MEAs) (Kelly & Lesh, 2000; English et al., 2008; Kelly, Lesh & Baec, 2008). The success of MEAs as research tools was both enabled by and illustrated through the articulation of a set of six design principles for such activities (Lesh & Harel, 2003; Lesh et al., 2000; Hjalmarson & Lesh, 2007); these principles indicate the key structural and dynamical elements in MEAs as contexts for problem solving. Table 1, below, also indicates "touchstone" tests for whether each of these six principles has been realized in a given implementation setting.

Table 5: Six Design Principles for MEAs

Principle	Touchstone Test for its Presence
Reality Principle	Students are able to make sense of the task and perceive it as meaningful, based on their own real-life experiences.
Model Construction Principle	To solve the problem, students must articulate an explicit and definite conceptual system (model).
Self-Evaluation Principle	Students are able to judge the adequacy of their in-process solution on their own, without recourse to the teacher or other "authority figure".
Model Generalizability Principle	Students' solutions are applicable to a whole range of problems, similar to the particular situation faced by the "client" in the MEA.
Model-Documentation Principle	Students generate external representations of their thinking during the problem-solving process.
Simplest Prototype Principle	The problem serves as a memorable representative of a kind of mathematical structure, which can be invoked by groups and by individuals in future problem solving.

Multi-Tiered Design Research

In parallel with learner-focused research using MEAs, researchers also have observed that teachers' efforts to understand their students' thinking involve yet another process of modeling: In this case, teachers engage in building models of student understanding. Although these teacher-level models are of a different category from student-level models, students' work while engaged in MEAs does provide a particularly rich context for teachers' modeling processes. Following this line of inquiry, the M&MP community has also produced tools and frameworks that can be useful to teachers in making full use of MEAs in classroom settings, while also providing *researchers* with insights into *teachers'* thinking.

Finally, at a third level of inquiry, researchers' own understandings of the actions and interactions in curricular activity systems (Roschelle, Knudsen, & Hegedus, 2010) involving students, teachers, and other participants in the educational process can also be studied through the lens of model development. Multi-tier design experiments in the M&MP tradition have done precisely this, involving researcher teams in self-reflection and iterative development as well (Lesh, 2002). Therefore, multi-tier design research involves three levels of investigators— students, teachers, and researchers—all of whom are engaged in developing models that can be used to describe, explain, and evaluate their own situations, including real-life contexts, students' modeling activities, and teachers' and students' modeling behaviors, respectively.

From Single Activities to Curricular Materials Supporting Modeling at Larger Timescales

Over the past 10 years, M&MP researchers have continued this direction of work in their own teaching and in partnerships with K-12 classroom teachers. Within the domain of statistical thinking in particular, this effort has produced resources and tools sufficient to support *entire courses* in several versions and including accompanying materials related to learning and assessment aimed at both student and teacher levels. Because the courses supported by these materials were designed explicitly to be used as research sites for investigating the interacting development of students' and teachers' ways of thinking, the materials were modularized so that important components could be easily modified or rearranged for a variety of purposes in different implementations. In particular, by selecting from and adapting *the same basic bank of materials*, parallel versions of the course have been developed for: (a) middle- or high-school students, (b) college-level elementary or secondary education students, and (c) workshops for in-service teachers. When these courses have been taught

by M&MP researchers familiar with the underlying theory, they have produced "six sigma" gains (Lesh, Carmona, & Moore, 2009).

One Approach to Longer-Timescale Modeling Research: Model Development Sequences

In the process of developing such curricular materials to focus on modeling, research in the M&MP has investigated ways in which MEAs can be integrated within larger instructional sequences or Model Development Sequences or *MDSs* (Lesh, Cramer, Doerr, Post, & Zawojewski, 2003; Brady, Lesh, & Sevis, 2015; Brady, Eames, & Lesh, 2015). MDSs offer classroom groups opportunities to unpack, analyze, and extend the models they have produced in MEAs, as well as to connect their ideas with formal constructs and conventional terminology. This unpacking work helps to ensure the lasting retention of concepts at a level of generality required for flexible use and application in novel situations. It also sets the stage for the critical connection between conceptual development (the centerpiece and focus of MEAs), and procedural knowledge that is also required for students to achieve well-rounded competence in any subject area.

Within an MDS, **reflection tools** support students in stepping back from their modeling processes and reflecting on this work as critical observers of both individual and group modeling behavior. We consider these tasks to be core to the learning processes in MEAs: when students interpret situations mathematically, M&MP research expects that they don't simply engage interpretation systems that are purely logical or mathematical in nature. Their interpretations also involve attitudes, values, beliefs, dispositions, and metacognitive processes. Moreover, the M&MP does not treat group roles and functioning as if they were fixed characteristics that determined students' behaviors. Instead, students are expected to develop a suite of problem-solving personae or profiles they themselves can learn to apply as the situation demands. Reflection tool activities encourage groups of learners to turn their attention to describing individual- and group-level processes, functions, roles, conceptions, and beliefs. Tools to support these activities include Ways of Thinking Sheets, various surveys and questionnaires, Concept Maps, Observation Sheets, Self-Reflection Guides, and Quality Assurance Guides for the products created in MEAs.

In **product classification** and **toolkit inventory** activities students continue the work of abstraction, characterizing and classifying the thinking they have done and identifying links among their solutions to different MEAs and between these solutions and the "big ideas" of the course. **Model exploration** or **Model extension activities** (MXAs) provide model-rich environments for introducing core concepts and skills from the broader curriculum that students need in order to formulate sophisticated models and present them to a mathematical community. In addition, these activities provide students with a vital opportunity to unpack the work they have done in the MEAs. These may use a combination of pointed YouTube videos and interactive simulations in dynamic mathematics software. Finally, **model adaptation activities** (MAAs) allow students to transfer ideas and techniques developed in MEAs to situations calling for similar performances. These MAA activities also provide smaller-timescale modeling scenarios that exercise concepts they have explored in other components of the MDS. They may be pursued individually or in small groups, depending on the nature of the task and the teacher's instructional or assessment goals. All of these elements of an MDS are designed to be highly modular, to accommodate (as well as to reveal) the needs and intentions of the teacher as they appropriate and adapt the materials for their own use. An example of how this variety of activity types might be laid out in a given unit is shown in Figure 2, below.

Figure 2. Example Structure of a Model Development Sequence (MDS). MEAs and other activities represent modular, re-orderable blocks of instructional time.

A Course-Sized Resource Repository for Research in Data Modeling and Quantification

Early successes in teaching entire courses with MDS units (e.g., Lesh, Carmona, & Moore, 2009) also demonstrated that these MDSs themselves were highly reconfigurable and re-orderable. The "big ideas" in statistics, data modeling, and quantification could be foregrounded in different orders, leading to a variety of possible longer-term modeling experiences for the students and teacher. In light of these findings, a course-sized repository becomes an important enabler for design research on modeling over more extended timeframes. The shared writing project of the Working Group will thus also provide research rationale to support the ongoing construction of this repository.

Thematic Clusters for Ongoing M&MP Research at Larger Timescales

In clarifying a shared research agenda, participants in the Models and Modeling Working Group have identified an initial set of ten thematic clusters that they feel would need to be addressed in an edited volume describing this work. For each cluster, we imagine that two to four short chapters would be dedicated to articulating perspectives on the associated research issues. These tentative clusters are as follows:

1. **What is the Models and Modeling Perspective?** Chapters in this cluster will provide a nuanced description of the main theoretical and philosophical thrusts of M&MP research, along with examples of extended projects in the tradition, with an emphasis on current work and findings. These chapters will also identify the key questions that have motivated and continue to motivate M&MP research.

2. **How do we situate the M&MP tradition within the broader field of education research?** This cluster will describe potential relationships and relevance of M&MP work to important non-M&MP research traditions in Mathematics Education (cf., Kaiser & Sriraman, 2006). Thus, it not only will seek to distinguish M&MP research from other work via contrast and counterpoint; it will also identify commonalities and affinities, along with examples of fruitful connections made across traditions in prior and ongoing work.

3. **Teaching Practice, Teacher Preparation, and Professional Development.** These chapters will describe work to understand the forms of expertise and the practices of teachers that make them successful in the kinds of learning environments demanded by the M&MP approach. An important outcome of research in this area is the elaboration and description of

approaches to professional preparation and professional development that can support sustained communities of teachers in their practice.

4. **Assessment & Measurement.** This cluster will focus on the use of MEAs and other modeling activities as environments for producing evidence to tell a rich and compelling story of what learners know and can do (Katims & Lesh, 1994; Eames, Brady, & Lesh, 2016). In particular, students' work in modeling activities can be used to augment the assessment picture of learners and groups of learners, with evidence of growth and change in conceptual understandings. This cluster will also work to identify the connection between (measures of) conceptual understanding and (measures of) procedural competency.

5. **Intuitions and Other Resources.** Chapters in this cluster may explore how modeling activities provide a medium in which intuitive understandings can build toward more formalized learning (Ferri & Lesh, 2013; Lesh, Haines, Galbraith, & Hurford, 2010). They may also work to demonstrate the value of activity design and assessment design that is sensitive to the ways in which learners recruit intuitions and other tacit/personal learning resources in their modeling work.

6. **Affect, Feelings and Beliefs.** Modeling work invokes more than simply logical/analytical thought. Often, important insights or aspects of students' solutions are rooted in empathic moves, feelings, or beliefs that they hold about the problem situation. Chapters in this cluster would investigate how to document and support the affective dimension of this form of learning.

7. **Learning Beyond School and Learning that does not look or feel like "Doing School."** This cluster will explore connections between the learning in MEAs and other modeling environments to mathematical thinking and acting as it appears beyond school. These chapters will engage the challenge of supporting students in *real-world* problem solving as a vibrant part of their *in-school* experience of mathematics. This is related to the challenge of providing school encounters with mathematics that demand the kind of creative interpretation and "seeing-as" work that is required by creative mathematical work in the real world.

8. **Methodology.** This cluster will deal with various methodological issues and questions associated with new directions and work in the M&MP tradition. Authors of these chapters will outline the need for methodological expansion and/or specialization of frameworks from the broader field of design research to approach questions central to the M&MP tradition, especially as we attempt to characterize idea development at longer timescales.

9. **Representations, Representational Competencies, and Thinking with Representations.** This cluster will investigate the role of representations and the construction (and/or design and invention) of representations in the M&MP tradition. One facet of this question has to do with the role and value of *computational* models and computational modeling activity, including the role of simulation, computational thinking, and even 'programming' of various kinds in the modeling toolkit that we want learners to have.

10. **The Situated-ness of Models and Modeling Research.** In this cluster, chapters investigate the variety of ways in which M&MP research and learning designs must be sensitive to the learning context. This might include discipline specificity (e.g., similarities and differences among math, science, engineering contexts); age or grade-level specificity (e.g., similarities and differences among elementary, middle-school, high-school, college, and graduate level models and modeling); or cultural specificity (e.g., Borba, 1990; and also the impact of institutional or cultural factors on the kinds of practices that are observed or encouraged).

Working Group Session Outline: Advancing the Agenda while Building Capacity

The concrete research product that this Working Group aims to present is a shared writing project consisting of a variety of empirical and theoretical pieces organized under the thematic clusters listed

above (or an iterative refinement of this set of clusters). In addition, however, Working Group meetings generally attract newcomers to the Models and Modeling Perspective, and it is essential to provide the means for these newcomers to engage productively in the sessions. In this way, the working group not only succeeds in setting and advancing a shared research agenda; it also serves to build capacity among junior researchers, paving the way for future work and collaborations.

The working group will meet in three sessions over the course of the conference. As preparatory work continues, the precise contents of each of these sessions will be more clearly defined, but the broad outlines are as follows:

Session One. The capacity-building objective for this session is to introduce the M&MP tradition and its approach to research. Newcomers to the Group will thus act as an authentic audience for a first iteration of material that would form part of Cluster #1, above. The Working Group facilitators will present on this topic, also in the process communicating and 'field testing' the logic of the cluster-based organization of the proposed edited volume.

The agenda-advancing objective of this session is to identify areas for collaborative breakout-group work. Because not all of the important voices to the M&MP may be present at the conference, it is not the objective of the Working Group to "cover" all of the thematic clusters in face-to-face discussions during PME-NA 2016. Instead, the goal will be to identify topics and themes where the participants exhibit diverse or contrasting perspectives. These participants will be grouped to allow them to engage in intense but still open-forum discussions in Session 2. Newcomers will also have the opportunity during this period to identify which breakout group they wish to affiliate with on Day 2. Finally, to support newcomers in preparing for Day 2, facilitators will provide two examples of MEAs, along with transcripts of students' small-group work on these problems, as well as related MDS activities that illustrate how follow-on work after an MEA can serve to "unpack" the thinking that emerged in the course of the MEA.

Session two. The capacity-building objective for Day 2 involves newcomers and junior researchers in frank and open discussions of key issues in the thematic clusters that they identify as personally interesting. The agenda-advancing work of this session involves pairs or small groups of experienced researchers in a thematic cluster in presenting their own research and in reacting to the research of others engaged with the themes of that cluster.

Based on the discussion that ensues, the breakout groups will formulate recommendations for the structure and contents of the cluster in the edited volume, identifying additional questions or topics that should be included, and generating a first-draft plan for the cluster. Each breakout group will also document their discussion, to enable them to report back to the whole Working Group at the end of Day 3. Based on past experience, it is expected that the discussions and debates of Day 2 will continue beyond the time limit of the session.

Session three. The first component of this session will give the breakout groups time to re-assemble and revise their accounts of their thematic clusters, based on discussions that have occurred between the two sessions. Then, the whole Working Group will come back together for the breakout groups to present. Discussion will allow commentary and questions across breakout groups, as well as reflections on next steps for refining ideas both within and across thematic clusters.

A subset of the facilitators for the Working Group will be the Editors of the proposed volume, and the third session will close with an outline of work to develop the thematic clusters into publishable chapter sequences. In the wake of the Conference, a wider set of researchers from the M&MP tradition will be involved in extending the work done in the in-person setting of PME-NA to all of the thematic clusters. It is anticipated that a full first draft of the volume can be produced within the year following the conference, providing a rich basis for discussion at a future iteration of the Working Group, at PME-NA 39.

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ADDRESSING EQUITY AND DIVERSITY ISSUES IN MATHEMATICS EDUCATION

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This Working Group has a dual focus on issues of mathematics teaching and learning and issues of equity and diversity. Following on the topics discussed at the Working Group in 2009–2013 and 2015, this year's focus is on three lingering "borders" in mathematics education research: the achievement gap, the "Where's the math?" question, and the hegemony of psychological approaches to investigating mathematics learning. Each session will consist of a "fish bowl" among scholar-experts followed by breakout discussions among participants and ending with a large-group discussion of the focal issue.

Keywords: Equity and Diversity, Gender

Brief History

This Working Group builds on and extends the work of the Diversity in Mathematics Education (DiME) Group, one of the Centers for Learning and Teaching (CLT) funded by the National Science Foundation (NSF). DiME scholars graduated from one of three major universities (University of Wisconsin-Madison, University of California-Berkeley, and UCLA) that comprised the DiME Center. The Center was dedicated to creating a community of scholars poised to address critical problems facing mathematics education, specifically with respect to issues of equity (or, more accurately, issues of inequity).

The DiME Group (as well as subsets of that group) has engaged in important scholarly activities, including the publication of a chapter in the *Handbook of Research on Mathematics Teaching and Learning* which examined issues of culture, race, and power in mathematics education (DiME Group, 2007), a one-day AERA Professional Development session examining equity and diversity issues in mathematics education (2008), a book on research of professional development that attends to both equity and mathematics issues with chapters by many DiME members and other scholars (Foote, 2010), and a book on teaching mathematics for social justice (Wager & Stinson, 2012) that also included contributions from several DiME members. In addition, several DiME members have published manuscripts in a myriad of leading mathematics education journals on equity in mathematics education. This working group provides a space for continued collaboration among DiME members and other colleagues.

The authors desire to continue DiME's tradition of discussing current work, hearing from leading scholars in the emerging field of equity and diversity in mathematics education, and opening up this space beyond DiME members in this Working Group. Specifically, the Center historically held DiME conferences each summer. These conferences provided a place for fellows and faculty to discuss their current work as well as to hear from leaders in the emerging field of equity and diversity issues in mathematics education. Beginning in the summer of 2008, the DiME Conference opened to non-DiME graduate students and new faculty with similar research interests from other CLTs such as the Center for the Mathematics Education of Latinos/as (CEMELA), as well as some not affiliated with an NSF CLT. This was initially an attempt to bring together a larger group of emerging scholars whose research focuses on issues of equity and diversity in mathematics education. In addition,

DiME graduates, as they have moved to other universities, have begun to work with scholars and graduate students including those with connections to other NSF CLTs such as MetroMath and the Urban Case Studies Project in MAC-MTL whose projects also incorporate issues of equity and diversity in mathematics education. Funding for the DiME project has ceased and the PMENA Working Group has become a major way in which to keep the conversation going.

It is important to acknowledge some of the people whose work in the field of diversity and equity in mathematics education has been important to our work. Theoretically we have been building on the work of such scholars as Marta Civil (Civil, 2007; Civil & Bernier, 2006; González, Andrade, Civil, & Moll, 2001), Eric Gutstein (Gutstein, 2003, 2006; Gutstein & Peterson, 2013), Jacqueline Leonard (Leonard, 2007; Leonard & Martin, 2013), Danny Martin (Martin, 2000, 2009, 2013), Judit Moschkovitch (Moschkovich, 2002), Rochelle Gutiérrez (2002, 2003, 2008, 2012, 2013) and Na'ilah Nasir (Nasir, 2002, 2011, 2013; Nasir, Hand & Taylor, 2008; Nasir & Shah, 2011). We have as well been building on the work of our advisors, Tom Carpenter (Carpenter, Fennema, & Franke, 1996), Geoff Saxe (Saxe, 2002), Alan Schoenfeld (Schoenfeld, 2002), and Megan Franke (Kazemi & Franke, 2004), as well as many others outside of the field of mathematics education.

Previous iterations of this Working Group at PMENA 2009–2013, and 2015 have provided opportunities for participants to continue working together as well as to expand the group to include other interested scholars with similar research interests. Experience has shown that collaboration is a critical component to this work. These efforts to expand participation and collaboration were well received; more than 40 scholars from a wide variety of universities and other educational organizations took part in the Working Group each of the past five years.

Focal Issues

Under the umbrella of attending to equity and diversity issues in mathematics education, researchers are currently focusing on such issues as teaching and classroom interactions, professional development, prospective teacher education (primarily in mathematics methods classes), factors impacting student learning (including the learning of particular sub-groups of students such as African American students or English learners), and parent perspectives. Much of the work attempts to contextualize the teaching and learning of mathematics within the local contexts in which it happens, as well as to examine the structures within which this teaching and learning occurs (e.g. large urban, suburban, or rural districts; under-resourced or well-resourced schools; and high-stakes testing environments). How the greater contexts and policies at the national, state, and district level impact the teaching and learning of mathematics at specific local sites is an important issue, as is how issues of culture, race, and power intersect with issues of student achievement and learning in mathematics. There continues to be too great a divide between research on mathematics teaching and learning and concerns for equity.

The Working Group has begun and will continue to focus on analyzing what counts as mathematics learning, in whose eyes (and for whose benefit), and how these culturally bound distinctions afford and constrain opportunities for traditionally marginalized students to have access to mathematical trajectories in school and beyond. Further, asking questions about systematic inequities leads to methodologies that allow the researcher to look at multiple levels simultaneously. This research begins to take a multifaceted approach, aimed at multiple levels from the classroom to broader social structures, within a variety of contexts both in and out of school, and at a broad span of relationships including researcher to study participants, teachers to schools, schools to districts, and districts to national policy.

Some of the research questions the Working Group will continue to consider are:

- What are the characteristics, dispositions, etc. of successful mathematics teachers for *all* students across a variety of local contexts and schools? How do they convey a sense of purpose for learning mathematical content through their instruction?
- How do beginning mathematics teachers perceive and negotiate the multiple challenges of the school context? How do they talk about the challenges and supports for their work in achieving equitable mathematics pedagogy?
- What impediments do teachers face in teaching mathematics for understanding?
- How can mathematics teachers learn to teach mathematics with a culturally relevant approach?
- What does teaching mathematics for social justice look like in a variety of local contexts?
- What are the complexities inherent in teacher learning about equity pedagogy when students come from a variety of cultural and/or linguistic backgrounds all of which may differ from the teacher's background?
- What are dominant discourses of mathematics teachers?
- What ways do we have (or can we develop) of measuring equitable mathematics instruction?
- How do students' out-of-school experiences influence their learning of school mathematics?
- What is the role of perceived/historical opportunity on student participation in mathematics?

Specific to this year's Working Group focus, we will interrogate three existing borders in mathematics education research:

- The achievement gap: Questioning its framing, interpretations, and responses
- "Where's the math?": Questioning what constitutes mathematics education research
- Defining mathematics learning: Questioning the hegemony of psychological approaches

Plan for Working Group

The overarching goal of the group continues to be to facilitate collaboration within the growing community of scholars and practitioners concerned with understanding and addressing the challenges of attending to issues of equity and diversity in mathematics education. The PMENA Working Group provides an important forum for these scholars to come together with other interested researchers who identify their work as attending to equity and diversity issues within mathematics education in order to develop plans for future research. The main goal for this year is utilize the conference theme, "Sin Fronteras," as a means to interrogate lingering tensions in key areas of mathematics education research. We will do this by bringing together scholars to share their perspectives on the field, and then provide space and time for smaller groups to discuss, reflect on, and amplify ideas from the presentation.

Our plans for PMENA 2016 will proceed as follows. Each session will follow a similar format, beginning with a facilitated conversation around the session's central theme. The format for all 3 sessions will include:

- A fishbowl conversation with 4 experts on the focal topic
- 4 break-out groups with audience members and 1 expert to continue the conversation
- Panel discussion on the 4 main questions that emerged during the break-out conversations

Session 1 Focal Topic: "The Mathematics-Achievement Gap"

Session 2 Focal Topic: "Where's the Math?" New Perspective and Possibilities from Emerging Scholars

Session 3 Focal Topic: "Defining Mathematics Learning: Who Decides?"

Previous Work of the Group

The Working Group met for six productive sessions at PMENA 2009–2013, and 2015. In 2009, participants identified areas of interest within the broad area of equity and diversity issues in mathematics education. Much fruitful discussion was had as areas were identified and examined. Over the past five years subgroups met to consider potential collaborative efforts and provide support. Within these sub-groups, rich conversations ensued regarding theoretical and practical considerations of the topics. In addition, graduate students had the opportunity to share research plans and get feedback. The following were topics covered in the subgroups:

- Teacher Education that Frames Mathematics Education as a Social and Political Activity
- Culturally Relevant and Responsive Mathematics Education
- Creating Observation Protocols around Instructional Practices
- Language and Discourse Group: Issues around Supporting Mathematical Discourse in Linguistically Diverse Classrooms
- A Critical Examination of Student Experiences

As part of the work of these subgroups, scholars have been able to develop networks of colleagues with whom they have been able to collaborate on research, manuscripts and conference presentations.

As a result of the growing understanding of the interests of participants (with regard both to the time spent in the working group and to intersections with their research), we began to include focus topics for whole group discussion and consideration and continued to provide space for people to share their own questions, concerns, and struggles. With respect to the latter, participants have continually expressed their need for a space to talk about these issues with others facing similar dilemmas, often because they do not have colleagues at their institutions doing such work or, worse yet, because they are oppressed or marginalized for the work they are doing. These concerns, in part, informed the focus topics for whole group discussion and consideration. For example, in 2009 research protocols (e.g., protocols for classroom observation, video analysis and interviewing) were shared to foster discussions of possible cross-site collaboration. In 2012, the Working Group explicitly took up marginalization in the field of mathematics education with a discussion about the negotiation of equity language often necessary for getting published; this was done in the context of the 'Where's the mathematics in mathematics education' debate (see Heid, 2010; Martin, Gholson, & Leonard, 2010). Dr. Amy Parks was invited to join Working Group organizers to share reflections on their experiences. In 2013 the Working Group hosted its first panel in which scholars (Dr. Beatriz D'Ambrosio, Dr. Corey Drake, Dr. Danny Martin) shared their perspectives on the state of and new directions for mathematics education research with an equity focus. The success of prior panel discussions have encouraged us to use that format as a launching point for deepening conversations on lingering tensions in the field. We see this working group as questioning critical borders that persist within mathematics education.

Anticipated Follow-up Activities

As has happened following previous years of this working group, it is anticipated that scholars who make connections at the working group sessions will maintain contact and at least in some cases, this will lead to collaboration on research questions, conference presentations, manuscripts or research projects.

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DESIGNING AND RESEARCHING PEDAGOGIES OF REHEARSAL AND ENACTMENT FOR SECONDARY MATHEMATICS TEACHER DEVELOPMENT

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Pedagogies of teacher education centered on opportunities for rehearsal and enactment provide a promising way to support pre-service and practicing teachers in developing skill with a core set of instructional practices. These practices then support each student's engagement with a set of authentic disciplinary practices—what has been referred to as ambitious and equitable goals for instruction. The focus on these pedagogies of teacher education has grown, and efforts to build on and expand work at the elementary level have raised questions and challenges of using and researching these pedagogies with secondary mathematics teachers. A consortium of teacher educators and researchers who have worked and collaborated in this area leads this working group. This group will provide opportunities to engage the group leaders and other participants in discussions to frame the state of these efforts, identify challenges, develop common language, and establish plans to move forward in research on rehearsal and enactment with secondary mathematics teachers.

Keywords: Instructional Activities and Practices, Teacher Education-Preservice, Teacher Education-Inservice/Professional Development

Brief History of the Group

This is a new working group intended to advance the use and study of "pedagogies of enactment" (Grossman, Hammerness, & McDonald, 2009). Specifically, we study the use of rehearsals of instructional activities (Kazemi, Ghousseini, Cunard, & Turrou, 2016; Lampert et al., 2013) toward the development of core or high-leverage practices (Ball & Forzani, 2009; McDonald, Kazemi, & Kavanagh, 2013). We focus on this in the context of secondary mathematics teacher education and professional development (PD). For more than five years, a group of secondary mathematics teacher educators and researchers (represented, in part, by the leaders of this working group) have collaborated to consider the work of secondary teacher preparation in relation to recommendations from Grossman and her colleagues for teacher education and the work of Lampert, Kazemi, Franke and their colleagues in the area of elementary mathematics education. In addition to numerous calls

and in-person meetings at conferences, members of the group have come together to lead other conference sessions and, in the past year, subsets of the group have engaged in more focused data collection and inquiry. As the group has grown and evolved and the work has continued, the challenges and questions tied to doing this work in secondary mathematics teacher education and to collaborating with other teacher educators and across institutions have also grown and evolved. How do secondary mathematics teachers (and their students) and the setting of secondary mathematics classrooms make this work different or unique? What, exactly, gets rehearsed by teachers, and toward what end? How would this work, primarily discussed in the context of initial teacher preparation, extend to work in PD settings for practicing teachers? Given the increasing interest in the field (as is apparent in conference programs and journal articles) in the use of pedagogies of enactment with a focus on developing core practices, we see now as a crucial time to establish this work, answer foundational questions, and make plans for future collaborations and products.

Relevance to PME-NA

At PME-NA there has been a recent history of other working and discussion groups (and, likely, countless individual papers and posters) looking at the approaches to and content of preparing prospective mathematics teachers and supporting practicing teachers through PD. For example, Kastberg, Sanchez, Edenfield, Tyminski, and Stump (2012) led a group looking at the content and activities of mathematics methods courses. This group continued the next year (Kastberg, Sanchez, Tyminski, Lischka, & Lim, 2013) and also resulted in an NSF-funded conference on inquiry and practice of mathematics methods courses held in 2015. Winsor and colleagues (2013) focused specifically on research on the preparation and development of prospective secondary mathematics teachers, as we do in this work. We extend upon this developing history with a specific focus on a set of activities used in settings of teacher development toward the development of "skilled practice" (Grossman & McDonald, 2008) among prospective and practicing secondary mathematics teachers.

The focus of this working group also provides a contribution in the context of the theme of this year's conference: "Sin Fronteras: Questioning Borders with(in) Mathematics Education." We see the work we are engaged in with secondary mathematics teachers (and each other) as highlighting, questioning, and transforming various borders in mathematics education. Primarily, we see teacher development—whether the preparation of teacher candidates or the continued support of practicing mathematics teachers—as inherently cross-setting work. This may refer to multiple physical settings but, at the very least, the work of teacher development involves the intersection of settings with different cultures and histories that inform the practices in each. Researchers have put forth ideas for teacher development—from "practice based" (Ball & Cohen, 1999) approaches, such as the use of student work, classroom video, and written cases; to "practice focused" approaches as have recently been conceptualized by Ball & Forzani (2009) and McDonald and colleagues (2013), among others—that reframe (though not completely remove) that border. The work of our group continues on this trajectory, not only with approaches designed to better prepare teachers across settings, but with the consideration of how theory and research methods can help acknowledge, negotiate, and appreciate the cross-setting nature of teacher development. This and numerous other borders we discuss below serve as ways for us to focus this working group, as many of the questions that have emerged in this work are a product of coming up against areas of tension, divide, and differences.

Focal Issues

In order to increase student mathematics learning, secondary students need opportunities to productively engage with mathematical content and secondary teachers need to develop skill with instructional practices that provide students with those opportunities. In this section we first frame the focal issues of this working group by specifying what is meant by "ambitious and equitable mathematics instruction" and providing an overview of practice-based and practice-focused

approaches to teacher education and PD, specifically the use of "cycles of investigations and enactment" as a pedagogy for teacher development with a focus on rehearsal and enactment We then present emerging issues, challenges, and questions for conducting and researching this work in secondary mathematics teacher education and PD.

Ambitious and equitable mathematics instruction

Teachers must ensure that each student has access to rigorous academic work to develop mathematical proficiency and meet the demands of an increasing mathematically, statistically, and technologically complex society (Kilpatrick, Swafford, & Findell, 2001; National Council of Teachers of Mathematics [NCTM], 2014). These expectations are summed up, in part, by the way in which students must engage with and develop a set of mathematical practices (NGA & CCSSO, 2010). These practices represent the skills individuals in mathematics-related fields utilize in their work and the way in which all individuals make sense of, reason about, and make decisions regarding mathematical and quantitative situations. These opportunities must deliberately be made available to all students, drawing upon students' diverse cultural and linguistic resources in the mathematics classroom and positioning mathematics as a human practice and a tool for social change (Gutiérrez, 2011).

To be able to support these "ambitious and equitable" goals (Jackson & Cobb, 2010), mathematics teachers must not only hold particular beliefs, values, and knowledge around the teaching and learning of mathematics, but also "skilled practice" (Grossman & McDonald, 2008) to actually carry out the work out in the classroom. Specifically, teachers need skill with a set of core practices (e.g., Grossman et al., 2009; Forzani, 2014) that represent what is known in the fields of mathematics teaching and teacher education as integral aspects of ambitious and equitable mathematics teaching. These instructional practices include leading whole class discussions (Boerst, Sleep, Ball, & Bass, 2011; Chapin, O'Connor, & Anderson, 2009), eliciting and responding to students' reasoning through tasks and questioning (Lampert et al., 2013; Stein, Engle, Smith, & Hughes, 2008), and steering instruction toward a clear and worthwhile mathematical point (Baldinger, Selling, & Virmani, in press; Sleep, 2012). Teachers' capacities with these instructional practices have the potential to engage students in key practices of the discipline of mathematics. Efforts have been made to capture a set of core practices for teaching, such as the "high leverage practices" from TeachingWorks (2016) or the eight essential Mathematics Teaching Practices from NCTM's (2014) *Principles to Actions: Ensuring Mathematical Success for All.* This working group focuses on a set of tools and approaches of teacher education and PD experiences that support teachers' development of these skills.

Practice-based approaches to teacher learning and pedagogies of practice

There has been an increased focused on practice-based approaches to supporting teachers' professional learning. A practice-based approach uses "practice as a site of inquiry in order to center professional learning in practice" (Ball & Cohen, 1999, p. 19), and creates opportunities for teachers to examine the everyday aspects of teaching. Grossman and colleagues (2009) have called for the need to organize practice-based approaches to professional education around what they call representations, decompositions, and approximations of practice, with the latter referring, "to opportunities for novices to engage in practices that are more or less proximal to the practices of a profession" (p. 2058). One form of approximation of practice is what Lampert & Graziani (2009) call rehearsals of "instructional activities," or IAs, which serve as containers of core practices, pedagogical tools, and principles of high-quality teaching (Kazemi et al., 2009). Toward this end, IAs are designed to structure the relationship between the teacher, students, and content in order to put a teacher in position to engage in and develop skill with interactive practices around facilitating rich discussions about mathematics. IAs and the practices that they contain serve as a focus of a set of

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Tucson, AZ: The University of Arizona.

activities that provide teachers the opportunity to both investigate and enact the work of teaching (Lampert et al., 2013; McDonald et al., 2013), often referred to as a "cycle of investigation and enactment". This consists of observing, decomposing, and planning the IA; rehearsing the IA in the teacher education setting with in-the-moment coaching from a teacher educator; enacting the IA in a K-12 classroom setting; and using artifacts of practice such as video and student work to analyze instruction and make connections between teaching practices, student learning, and a broader vision of ambitious and equitable mathematics teaching. In this working group, we focus on the "enactment" elements of this cycle—both the coached rehearsals of IAs in a teacher education or PD setting and the enactments done in classrooms with students.

Emerging issues and challenges in this work

Much of the research around pedagogies of practice in mathematics education has tended to focus on elementary mathematics instruction (e.g., Lampert et al., 2013), but a number of scholars from different institutions and contexts (including members of the working group) have been working to extend this work to the secondary level, both in teacher education and PD (e.g. Aaron & Elliott, 2013; Baldinger, Selling, & Virmani, in press; Campbell & Elliott, 2015; Elliott, Aaron, & Maluangont, 2015; Ghousseini & Herbst, 2016). As we work to design, implement, and research pedagogies of rehearsal and enactment at the secondary level, a number of questions and challenges have emerged that could be explored and addressed through the types of collective work afforded by a working group. We see three main sets of challenges and questions to frame the proposed working group's foci—(1) issues of language around instruction and mathematics, (2) theoretical and methodological choices, and (3) considering multiple settings and boundaries.

Issues of common language around practice, practices, IAs, and mathematics. The first challenge emerges in the language used to communicate practice-focused work—among researchers and teacher educators, and in those stakeholders' interactions and collaborations with teacher candidates, teachers, and school administrators across settings. This includes different language for naming and decomposing instructional practices (Grossman et al., 2009) in ways that could be worked on by teachers and embedded within pedagogies of rehearsal and enactment. Issues of language also emerge when a language developed by teacher educators and researchers interacts with language about the work of teaching as defined in policy used by university and school-based stakeholders. One persistent issues is that of "grain size" (Boerst et al., 2011; Campbell & Elliott, 2015)—with some focusing on a practice of leading a whole class discussion (Baldinger, Selling, & Virmani, in press; Ghousseini & Herbst, 2016), while others focus on a smaller grain size, such as using particular questioning techniques (e.g., Arbaugh, Freeburn, Graysay, & Konuk, 2016). Challenges emerge among teacher education researchers and practitioners about what counts as an IA and how those decisions might be different in secondary contexts as compared to elementary. This is influenced, in part, by differences in the contexts of elementary and secondary classrooms and the mathematical demands of each of those settings as teacher educators work to be responsive to the work of the settings for which teachers are being prepared (Campbell & Elliott, 2015). What are the implications of the different names and decompositions of practice for helping teachers develop their skills—both instructionally as well as their mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008)? How could we build common language for practice or shared meanings from particular terms to describe practice, develop skilled practice, and communicate across settings? Answering these questions is important given the developing appeal of using pedagogies of rehearsal and enactment and other practice-based approaches.

Theoretical and methodological choices. A second set of challenges arises when thinking about how to conceptualize and track teachers' development through the enactment cycle, across cycles, and across settings. Past research on pedagogies of rehearsal and enactment has made use of theoretical lenses such as activity theory (Campbell, 2014; Campbell & Elliott, 2015) or theories of

knowing in practice (Cook & Brown, 1999; Lampert et al., 2013). How might we use different theories of teacher learning to help us define and explore a range of potential ways that teachers might learn in the context of the enactment cycles? For example, we might look to see changes in how teachers enact a particular instructional practice or we might examine the impact on teachers' noticing of student thinking or instruction (Sherin, Jacobs, & Philipp, 2010). Attention must also be paid to other outcomes of teacher development that are important to the various stakeholders invested in this work. Without explicit discussion of the theoretical constructs we employ we run the risk of leaving our goals unspecified, of designing instruments misaligned to our goals, and to make claims that are unwarranted given our research designs. A related challenge is identifying methodological tools that build on different theoretical framings that could allow us to systematically document evidence of change or growth in a range of data sources. As we and others look to make claims about the impact of pedagogies of rehearsal and enactment on teachers' practice, there needs to be more explicit discussion about the theoretical and methodological choices that contribute to research and development.

Crossing settings and boundaries. A final set of challenges and questions arise when considering that work on secondary enactments necessarily crosses a number of boundaries. The first boundary is between professional education settings, such as methods courses or PD workshops, and K-12 learning environments such as field placement classrooms, teachers' own classrooms, or on-site professional learning opportunities. How might the work in the enactment cycle be responsive to the contexts for which teachers are learning to teach or growing their practice? For example, Campbell and Elliott (2015) have highlighted the ways in which secondary mathematics classrooms pose unique structures and expectations to which teacher educators and researchers need to be responsive, resulting in designs that look to simultaneously build on these activity structures and provide students in those classroom settings with increased opportunities to reason mathematically. By designing and using IAs that are responsive to the "normal" workings of secondary mathematics classrooms they hope to be able to support teacher candidates in developing skills that they see as relevant to the work that they will be asked to do in the future and skills that align with what we, as a field, know about effective teaching. Striking a balance between responsive teacher education designs and promoting instructional change in mathematics classrooms is an important area of consideration in this work.

Another cross-setting challenge is to consider how work on pedagogies of rehearsal and enactment might look different for practicing teachers. Much of the work on practice-focused pedagogies has focused on teacher education, but recent efforts are extending this to PD with practicing teachers (e.g., Aaron & Elliott, 2013; Wilson, Webb, Martin, & Duggan, 2016). What aspects of the work might need to be different for in-service teachers and how might it influence the ways in which we study these efforts? Finally, a number of important questions have emerged around the relationship between pedagogies of rehearsal and enactment and efforts to help teachers develop more equitable instructional practices (Jackson & Cobb, 2010). How might situating teacher learning more directly into schools (e.g., methods course held in schools) provide opportunities to think about context and power dynamics in relation to instructional practices (Kazemi et al., 2016)? Kazemi and colleagues also highlight an area for further investigation around how pedagogies of rehearsal and enactment can provide a way to connect to "foundations" courses where issues of inequities in schooling are primarily addressed. How can we work to more explicitly link this body of research to other important efforts to focus on equity and justice in mathematics education?

Plan for Engagement of the Working Group

The three sessions of the proposed working group are structured around sets of focus questions that address these issues and challenges. We first focus on the core ideas of the work and the different language we are using to describe ideas about pedagogies of rehearsal and enactment. Next we consider theoretical and methodological issues involved in researching these ideas. Finally, we

will take up questions about the varied purposes and contexts for work by pursuing more focused interests in small groups. We describe each of the sessions in detail below.

Session 1. Enactments of what? Examining choices about instructional practice and content

The first session will open with opportunities for introductions and to briefly frame the purposes and the structure of the working group. The remainder of this first session will be comprised of two interactive panels, each exploring one of two questions: (1) How can practice be decomposed in different ways for the purpose of rehearsal and enactment and what are the implications for how practice is parsed in this work?, and (2) What mathematics do we work on when we work on instructional practice? The first interactive panel will take up some of the issues around language and terminology for describing instructional practice and work on enactments to help build shared meaning of different terms that are commonly used in our work, such as "core practice", "teaching move", "instructional activity", "rehearsal", and "enactment". Two leaders of the working group will provide short presentations highlighting their respective work on pedagogies of rehearsal and enactment, specifically about the distinct approaches to naming and parsing practice for the purpose of developing teachers' skill. Participants will then have opportunities to discuss different choices in decomposing practice and alignment to purposes and goals. This part of the session will provide key opportunities to develop shared understanding of the different language and terminology used by different groups in this work. The second interactive panel will take up some of the issues about mathematics content and practice in the context of pedagogies of rehearsal and enactment with secondary mathematics teachers. Two perspectives from leaders of the working group will be presented and discussed. Participants will then have opportunities to discuss the types of strategic choices that might be made around different ways to work on mathematics. In sum, the first session will establish shared understanding (or, at least, an understanding of the issues and questions) around decisions to be made in the work of secondary pedagogies of rehearsal and enactment.

Session 2. What constitutes evidence of "skilled practice" and its development?

The second session will build on the previous day's work to consider how we might conceptualize and trace teacher development as they participate in pedagogies of rehearsal and enactment. This session will focus on two main questions: (1) What theoretical and methodological tools are being used and what needs to be developed to track evidence of development within a cycle, across cycles, and across settings and institutions?; and (2) What are the conceptual/theoretical frameworks researchers are using to guide research on instructional practice? These questions are particularly complex to investigate as pedagogies of enactment and investigation cross borders between university coursework or PD and other settings for teacher learning. First, a panel of researchers who have previously published or otherwise shared findings that include claims of teacher learning will share how they are conceptualizing teacher learning and development and the methodological approaches they are using for identifying and characterizing this development in their research. We will then open discussion for other participants to ask further questions about challenges and strategies for researching teacher learning in this type of work. Next we will spend time in small groups that are organized around certain theory-methods perspectives to come up with takeaways about each perspective, including what kinds of research questions could be pursued, what types of data to collect, and what kinds of claims could be made. Participants will be invited to bring their own data about rehearsals and enactments to these small group conversations. We will leave time at the end of this session for the groups to report out about the nature of the discussions.

Session 3. Moving forward with a research and development agenda

The final working group session will provide opportunities for small groups to continue discussing collective takeaways of the first two days, to delve into particular areas of interest for

attendees, and to plan next steps for the working group. For example, some groups might choose to discuss how researching rehearsal and enactments might be different in teacher education or PD contexts. Another group might choose to discuss how pedagogies of enactment and investigation, and specifically choices about what practices to work on and why, allow for work on issues of access and equity. We will invite each group to record the different ideas discussed during this time to better inform the subsequent discussion of next steps. The final part of this session will be used to consider and discuss possible follow-up activities that could build on the work begun during the working group. We will offer a set of potential follow-up activities (see below) and invite participants to suggest additional activities that would build on the work begun at PME. Then we will discuss the group's interest in moving forward with on more more of these ideas and determine a possible work-plan for next steps. We will then ask for volunteers to take the lead in facilitating the next steps for follow-up work.

Anticipated Follow-Up Activities

There are a number of potential follow-up activities to build on the work that will begin during the proposed sessions. One possible activity is engaging in collective dissemination of the ideas that are shared and created through the working group. For example, submitting a set of papers to a journal like *Mathematics Teacher Educator* that address similar or contrasting ideas around secondary level enactments could provide an appropriate forum and audience for these ideas. Alternatively, collectively writing a research commentary would allow us to focus on theoretical or methodological challenges in researching teacher development in practice-based activities. This could be submitted to a venue such as the *Journal for Research in Mathematics Education* that publishes commentaries. Another set of follow-up activities is designing and studying new work around secondary rehearsals and enactments. Participants will have the opportunity to collaborate on projects that investigate one or more of the questions that emerge from the working group. For example, a group of researchers from different projects might choose to examine their data (or collect new data) using the same theoretical and methodological approach to compare ideas across contexts. The members of the working group have explored the conceptualization and preparation of a grant proposal to fund new collaborative research projects. The outcomes from the working group would further support this effort. We will also pursue options for funding future conferences for continued opportunities to bring together scholars for focused work around pedagogies of rehearsal and enactment. To support these collaborations, we will establish ways to communicate about our individual and collective work. In the past, we have used shared Dropbox folders, regular video calls, and in-person meetings at conferences. These strategies would likely be used to support future work.

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MATHEMATICS EDUCATION AND ENGLISH LEARNERS

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This Working Group will continue the work of last year's Working Group in considering multiple aspects of research and practice related to mathematics learning and teaching with English Learners. Our goals include: (1) developing further the collaborative endeavors that began during the 2015 meeting and (2) fostering new collaborations and supporting further connections among researchers and research projects in the future. In Session 1, the organizers will present brief reports on the work that resulted from the 2015 Working Group and preview the schedule for this year's Working Group. During Sessions 2 and 3 we will provide the smaller groups time to collaborate and continue their work. We will close Session 3 with time to review group progress and discuss next steps and products of our work.

Keywords: Equity and Diversity, Teacher Education-Pre-service, Teacher Education-Inservice/Professional Development, Research Methods

Brief History

The facilitators of this Working Group initially came to work together through the NSF-funded Center for the Mathematics Education of Latinas/os (CEMELA). CEMELA brought together researchers from across the country to collaborate on research focused specifically on critical issues related to Latinos/as in mathematics. Prior to CEMELA, researchers interested in such a focus worked mostly in isolation. The overarching framework for CEMELA included an examination of language, culture, and mathematics. In considering issues related to Latinos/as in US schools, the issues of language and English Learners (ELs) are closely related. While not all Latinos/as are English Learners, and not all ELs are Latinos/as, these two groups have significant overlap. For example, about 80% of ELs speak Spanish as a first language, and Spanish-speaking ELs appear to struggle on measures of academic achievement (Goldenberg, 2008).

CEMELA expanded the field's knowledge of ELs in mathematics by conducting studies in interdisciplinary teams. CEMELA focused on research in teacher education, research with parents, and research on student learning, resulting in well over 50 publications and presentations. Several of these studies involved the investigation of questions related to the interplay of language, culture, and mathematics education.

Following the conclusion of CEMELA's funding, Zandra de Araujo, Sarah Roberts, Craig Willey, and Bill Zahner continued to meet regularly. These meetings focused on examining intersections among these early career scholars' work related to the mathematics education of ELs. To date these meetings have resulted in a number of national presentations at the annual meetings of the National Council of Teachers of Mathematics, the American Educational Research Association, and PME-NA. Currently, this group is working on several manuscripts and follow-up studies related to the preparation of teachers to work with ELs.

The Mathematics Education and ELs Working Group met for the first time at PME-NA 2015 (de Araujo et al., 2015). At that meeting, we brought together a diverse group of about 20 researchers who started collaborating on several projects related to the mathematics education of ELs (see descriptions of these projects in the section titled *Prior Work*). Our aim for the 2016 Working Group is to provide a space for these scholars to continue their work and to bring new scholars into the fold.

Focal Issues

ELs are the fastest growing group of U.S. students (Verplaeste & Migliacci, 2008). In fact, U.S. schools have seen an increase of 152% in EL students over the past 20 years (National Clearinghouse for English Language Acquisition, 2008). The growing number of ELs across the country induces the need for teachers from all states to be prepared to attend to the needs of ELs in their mathematics classrooms. Considerations of how to support ELs in mathematics is no longer a concern for only states, like Arizona, Texas and California, with traditionally high numbers of EL students. Instead, with all but ten states across the country seeing increases in their EL populations between 2002-03 and 2011-12 (National Center for Educational Statistics (NCES), 2014), there is increasing pressure for support in addressing the needs of these students.

Despite the rise in the number and proportion of ELs, teacher preparation has not kept up with this trend. In 2002, the NCES reported that out of the 41% of teachers who worked with ELs in their classrooms, only 13% received EL-specific professional development. In 2008, Ballantyne, Sanderman, and Levy found that it was "likely that a majority of teachers have at least one English language learner in their classroom," although "only 29.5% of teachers with ELs in their classes have the training to do so effectively" (p. 9). This misalignment of the realities of today's classrooms and teacher preparation necessitates research into effective means for supporting current teachers and for preparing prospective teachers to meet the needs of linguistically diverse learners.

The implementation of the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices (NGA Center) and the Council of Chief State School Officers (CCSSO), 2010) is putting additional pressure on teachers of students who are still gaining proficiency in English. Teaching aligned with the CCSSM's content standards and the standards of mathematical practice will increase the language demands required to engage in mathematical discourse, which is typically in English (Moschkovich, 2012). Providing ELs with quality educational experiences is no longer relegated to only language specialists; it is a "mainstream" issue for which all mathematics teachers must be prepared (Bunch, 2013).

Many ELs or their families have literal experience crossing international borders. Issues related to educating ELs are also connected to metaphorical borders. This year, our Working Group aims to examine critically the borders with(in) mathematics education that hinder quality educational experience for bilingual and emergent bilingual students in mathematics. In particular, we will examine the artificial borders between research and practice, linguistic and mathematical resources, teacher preparation and practice, research with monolingual and bilingual learners, and research in mathematics education at large and research on the mathematics learning of ELs. To do this work, we will bring together researchers from diverse contexts to examine current and past research on ELs in mathematics education while also supporting burgeoning collaborations to establish future, imperative research directions.

Previous Work of the Group

The 2015 Working Group (de Araujo et al., 2015) began with whole group discussions aimed at examining a number of aspects related to the mathematics education of ELs including: a) Student Learning; b) Preservice & Inservice Teacher Education; c) Family and Community Resources; d) Curriculum; and e) Language Perspectives. In the following sections we briefly summarize our discussions of these areas.

Student Learning

Building upon situated and sociocultural perspectives (Moschkovich, 2002), we started from the premise that ELs, like all students, learn mathematics through participating in and appropriating discourse practices, tool use, and perspectives of mathematics. There exists a need to understand better how research in mathematics education at large is connected with research on the learning of ELs; much of the content-focused work in mathematics education is isolated from research on how ELs develop specific mathematical understandings. This has resulted in a gap in the literature and so we need to develop more linkages between mathematics education research on student learning of specific topics, and the mathematics education research on student learning of ELs. We also need to develop more connections among research with monolingual, bilingual, and multilingual mathematics learners or settings. Some results may apply across learners and settings, while other results may be specific to one population or setting. Researchers need to explore both these intersections and differences. Members of this group were involved in research that critically examines issues at the borders of learning mathematics and learning language.

Teacher Education

Despite findings that suggest ELs develop academic language in the content areas over a period of about 5-7 years, many ELs are mainstreamed within two years. Thus, it is essential that teachers support ELs in learning content in mainstream mathematics classes while also providing support to develop their academic language. Most inservice teachers have had few, if any, professional learning experiences around working with ELs in mathematics classrooms (Ballantyne et al., 2008). In fact, many mathematics teachers struggle to understand their role in supporting ELs' mathematics language development (Willey, 2013).

Preservice teachers typically have few opportunities to think specifically about how they will work with ELs in their mathematics classrooms. Preparation programs often include coursework on teaching ELs that is not specific to the content areas in which preservice teachers will work. Meanwhile, the content courses for preservice teacher often focus solely on content, devoid of exploring how to support ELs in mathematics classrooms. There is a need for much more content specific support for mathematics teachers of ELs. Recently researchers have shared strategies for engaging preservice teachers in working with ELs in mathematics classrooms. For example, Fernandes (2012) suggested a series of mathematics task-based interviews to engage preservice teachers in the process of noticing the linguistic challenges that ELs face and the resources these students bring to their mathematical communication. Additionally, the TEACH MATH (Drake et al., 2010) research team is using tasks in their content courses to support preservice teachers in drawing on students' funds of knowledge (Turner, Drake, Roth McDuffie, Aguirre, Bartell & Foote, 2012).

Much of the prior work on teacher education related to ELs has focused on more general strategies (e.g., sheltered instruction, as in Echevarria & Graves, 1998), such as using visuals, modifying texts or assignments, and using slower speech. At our previous meeting, we explored ways to support teachers, both preservice and inservice, in learning to better address the needs of ELs in the mathematics classroom.

Family and Community Resources

Families and communities can serve as resources for ELs in their mathematics learning in myriad ways. Families can advocate for their children and provide and support learning experiences both in and out of the classroom. Communities can also provide a wealth of support mechanisms and learning possibilities. Moll and colleagues (1992) described how students studied candy making and selling within their neighborhood to explore mathematics within this context, such as discussing and analyzing production and consumption. In doing so, the teachers and students acknowledged the value of these community experiences. Additionally, Civil and Bernier (2006) explored the

challenges and possibilities of involving parents in facilitating workshops for other parents around key mathematical topics. These studies and others like them illustrate the promising impact of family and community resources in fostering ELs' mathematics learning. At our prior meeting, we discussed the implications of different language policies on parental engagement and how teacher educators can draw on family and cultural resources in support of ELs.

Curriculum

Curriculum plays a key role in the teaching and learning of mathematics and teachers play a pivotal role in selecting and enacting curriculum materials for students. The choice of curriculum materials impacts students' opportunities for learning in the mathematics classroom (Kloosterman & Walcott, 2010). Early work on curriculum and English learners focused specifically on the challenges ELs encounter when completing or interpreting word problems (Téllez, Moschkovich, & Civil, 2011). More recent work that centers on both ELs and curriculum has focused on culturally relevant curricula. A subset of mathematics education, called ethnomathematics, focuses on this area (e.g., D'Ambrosio, 2006). Other studies have focused on the development of curriculum materials for ELs (e.g., Freeman & Crawford, 2008) or the evaluation of a curriculum's appropriateness for ELs (e.g., Khisty & Radosavljevic, 2010; Lipka et al., 2005). More recently, work related to ELs and curriculum has begun to look at teachers' use of curriculum (e.g., de Araujo, 2012). At our prior meeting, we investigated the intersection of teachers, curriculum, and ELs in discussing how curricula addresses (or not) the needs of ELs.

Language Perspectives

Teachers' and researchers' conceptions of language, second language acquisition, and bilingualism impact teaching and learning mathematics for ELs. At our prior meeting, we considered how perspectives of language, second language acquisition, and bilingualism appear in both theory and practice. We also contemplated, in particular, how work focused on ELs can draw on current work on language and communication in mathematics classrooms, classroom discourse, and linguistics. Looking for these intersections and connections was crucial because it ensures that work in mathematics education is both theoretically and empirically grounded in relevant research, and it will prevent researchers from reinventing wheels, and it can inform crucial research-based guidelines for curriculum, instruction, and assessment.

Aims for the 2016 Working Group

Following the Working Group's discussions of student learning, teacher education, family and community resources, curriculum, and language perspectives, three smaller groups formed to explore subsets of these ideas in depth. Since the conclusion of the 2015 Working Group, these smaller subgroups have continued the work they started in the Working Group. The aim of the 2016 Working Group is to provide updates on the work of these subgroups and continue to move the groups' research agendas forward. In the following section we provide an overview of each of these subgroups and the goals for the groups moving forward.

Group 1 (Curriculum)

In 2015, the curriculum subgroup focused on the role of textbooks, specifically teachers' guides and student work pages, in demonstrating how one might approach supporting ELs in building mathematical understanding and developing mathematics language. We inquired about the process publishers undergo to incorporate and offer support to teachers. What assumptions do they make? Who do they consult? What motivates them to invest in serving ELs better? What is/are their end goal(s)? The group decided to conduct an analysis of various middle grades curriculum to ascertain what supports and guidance are offered to teachers. It was suggested that we might build on the work

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of Pitvorec, Willey, and Khisty (2011), who explored the features of Finding Out/Descubrimiento (FO/D) that proved to be successful with bilingual children of migrant families in the 1980's (see Neves, 1997) and partially contributed to the development of complex instruction (Cohen, Lotan, Scarloss, & Arellano, 1999).

For the 2016 meeting, this group will continue to examine textbooks to understand better the supports they provide for ELs. The subgroup will will also consider language issues in mathematics texts for ELs, especially as related to word problems and assessment items. We will first share a short literature review of relevant research on linguistic complexity and vocabulary for mathematics word problems. Based on that research, we will summarize recommendations for addressing language complexity and vocabulary in designing word problems for instruction, curriculum, or assessment. We will then use examples of released sample Smarter Balanced Assessment Consortium items to illustrate how to apply those recommendations to designing word problems and to designing supports for ELs to work with word problems.

The subgroup will consider how to provide instructional support at three different levels for mathematical texts in terms of a) language issues: Light Support, Medium Support, and Strong Support, and b) the mathematical content. These three levels address the differing needs of students at differing English proficiency levels, but rather than working with a static student label, which is not consistent, dependable, or useful across settings, these labels refer to the level of support as the focus of instruction. For example, students who are newcomers may need strong support for language but differing support for mathematical content, and those need to be assessed separately but addressed concurrently. Students who are at an intermediate level of Academic English may need Medium Support for language and differing support for the math content. Students who are not immigrants, are native English speakers who belong to bilingual communities, or who speak an on-standard variety of English, or are labeled "long term ELs," may need strong support for reading comprehension in their first language (English) and differing support for mathematical content depending on their mathematics course taking experiences.

We will consider, in particular, how this subgroup can draw on current work on scaffolding in English Language Arts (ELA) to develop supports for comprehension of what ELA experts call "complex text." The following questions will guide the group's work related to this theme:

- What do we know from research regarding the linguistic complexity of mathematics texts, especially word problems and assessment items?
- How can instruction address these issues with mathematics texts?
- What are research based recommendations for designing word problems, revising text, and designing glossaries?

Group 2 (Student Learning)

During the 2015 working group meeting, the student learning sub group discussed multiple ongoing research projects that examine the intersection(s) of research on student learning of mathematics and research on learning language(s). Moschkovich shared examples from her past work that has illustrated the mediation of language and Discourses in students' mathematics learning (2007, 2008, 2015). For example, in one study Moschkovich traced the evolution of the phrase "went by" in reference to the intervals and scales on graphs. Zahner also shared results from an analysis of how assessment pressures shaped whole class discourse practices and opportunities for student learning in one ninth grade algebra classroom (Zahner, 2015). Zahner also shared a preliminary analysis of how teachers and students use formal and informal language about rates of change during classroom discussions (this work will be presented formally at ICME). The working group also discussed ideas for future analyses that build on and extend the prior work in this area.

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The collaborations started at the PME working group in 2015 have directly influenced work on two projects by Zahner that he will share at the working group. Zahner has also invited his collaborators on these projects to join the working group in 2016. In one study led by Phil Vahey at SRI International and Tracy Nobel at TERC, working group member Zahner is contributing to developing a coding scheme that captures EL's participation in the CCSSM Practice Standard "Reason Abstractly and Quantitatively." This coding scheme is being developed specifically to focus on interactions among trios of ELs solving problems about rates of change. The coding scheme builds on and extends prior work by members of the working group who have examined how ELs and their teachers use words, symbols, gestures, and discourse practices to engage in mathematical reasoning (Shein, 2012; Turner, Domínguez, Maldonado, & Empson, 2013; Zahner & Willey, 2014). The work with the team from SRI also includes developing dynamic representational technology for fostering abstract and quantitative reasoning among ELs in mathematics classrooms. Members of this project team will come to the working group in 2016 to share an update on progress and to present preliminary analyses of abstract and quantitative reasoning among ELs.

In a distinct study, Zahner is initiating a new project that has the goal of developing specialized learning trajectories that combine development students' mathematical reasoning and development of students' use of academic language related to specific mathematical topics. This development is occurring in collaboration with a working with a group of three high school teachers in in a Southwest high school, along with a research team including graduate and undergraduate students. The final goal is to design and develop curriculum materials related to linear and exponential growth that account for the special needs of ELs in secondary mathematics classrooms. At the working group Zahner and team members will share preliminary data and updates on this project. The working group will also function as a site where participants can offer feedback and build collaborations around these key ideas.

Group 3 (Teacher Education)

The teacher education subgroup focused on the primary issues that arise in the preparation of teachers to teach ELs at the various institutions. As a group we recognized that there were few attempts to include the teaching and learning of mathematics to ELs beyond the states where there was a high population of ELs. Given that some of the group members were meeting for the first time, a significant portion of the allotted time was spent sharing the details of our research we did and our interest in being part of this particular subgroup. One of the members shared a survey about examining preservice teachers conceptions about teaching mathematics to ELs and the other members agreed to administer the survey at their locations. Together, the responses will provide us with some insight about possible teacher conceptions that need to change and the steps we can take to make that happen. The group decided to stay in touch online and continue the discussions about potential collaborations. This subgroup aims to continue this line of work at the 2016 meeting. In particular, they seek to develop a multi-institution study that will investigate the issues raised at the 2015 meeting.

Plan for the Working Group

During the three sessions, participants will continue the work of the subgroups and bring new members up to date on the group's prior activities. In participating in the three sessions, participants will work to: a) clarify research questions, b) refine research tools, methods, and analyses, c) explore connections among different projects and studies, and d) discuss further collaborations and research on learning and teaching mathematics in classroom with ELs.

In Session 1, the organizers will present brief reports on the work that resulted from the 2015 Working Group and preview the schedule for this year's Working Group. During Sessions 2 and 3 we will provide the smaller groups time to collaborate and continue their work. We will close

Session 3 with time to review group progress and discuss next steps for our work as shown in Table 1. Meeting notes, work, and documents will continue to be shared and distributed via our Google Community (https://plus.google.com/communities/104376842129206334879) and corresponding Google Drive folder (https://goo.gl/EXhUVm). The use of Google Community allows members to create an institutional memory of activities during the Working Group that we have continued to use and add to following the 2015 Working Group. In addition to the Google Community and Drive resources, anticipated follow-up activities include planning for a continuation of the Working Group at PME-NA in 2017 and organizing collaborative writing projects on this topic.

Table 1: Overview of Proposed Working Group Sessions

	INTRODUCTIONS, UPDATES, & CLARIFYING AIMS	
	ACTIVITIES	GUIDING QUESTIONS
SESSION 1	1. Introduction and overview of the Working Group including introduction to the Google Community. 2. Brief presentations by panel members from each of the subgroups providing overviews of research projects with specific examples of how researchers have designed the studies. The purpose is to provide an overview and update of scholarly activities that were initiated at the 2015 Working Group. 3. Following the presentations, participants will discuss specific questions posed by the subgroups and aims for the subgroups' work at the 2016 meeting.	1. What research is being done in relation to each of the subgroups? 2. Which aspects of studies focusing on English learners do you find most puzzling? Most useful? Most misunderstood? 3. What goals do participants have for the 2016 conference?
	SUBGROUP WORK TIME	
	ACTIVITIES	GUIDING QUESTIONS
SESSION 2	1. Subgroups will work to establish goals and work toward those goals. 2. The second session will close with an allotment of time for subgroups to give brief updates on their work and pose questions for the whole group.	1. What theories and theoretical frameworks have informed the design of your research project(s)? 2. How might your work inform theory in mathematics learning and teaching? How can work on this student population expand our theoretical lenses? 3. What issues and challenges have you faced in designing studies?
	SUBGROUP WORK TIME & NEXT STEPS	
	ACTIVITIES	GUIDING QUESTIONS
SESSION 3	1. Work time for the subgroups to discuss directions for continued collaboration. Subgroups will also develop next steps as they plan for continued work. 2. Whole group discussion in which subgroups share goals and next steps developed by the subgroups. 3. Establish next steps for continued collaboration, including the Google Community.	1. How might other researchers pursue research projects on this topic and what can they learn from the work done so far? 2. What are the next steps in continuing this work? 3. How will your subgroups' work be disseminated?

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Follow-up Activities

We anticipate that this Working Group will attract other researchers interested in issues related to the mathematics education of ELs. Therefore, an important component of this second meeting of the Working Group will be to continue to establish connections with other interested researchers and to build opportunities for future collaborations. We will provide space for new researchers to contribute to our work, to suggest new directions, and to add to the growing body of research on mathematics and ELs. At the first session of our Working Group, we will share our ongoing online Google Community, which uses Google applications (Plus, Hangout, Groups, Drive, etc.). Google's applications are freely available and allow for a number of collaborative opportunities, including video conferencing, group messaging, collaborative document development, and shared web and social media space. Through this collaborative Google Community, we have organized follow up meetings both virtually and at conferences such as TODOS and the NCTM Research Conference. These meetings, both face-to-face and virtual, allow us to set concrete goals in preparation for the creation of a Working Group proposal for PME-NA 2017 to continue our work.

In addition to these short-term goals, we have several longer-term goals for this Working Group. First, we would like to seek funding for a conference where we can share the results of our work with other researchers and practitioners. A number of the facilitators will attend and present at the TODOS: Mathematics for All Conference in Summer 2016. The TODOS conference brings together practitioners and researchers and provides a venue for the dissemination and discussion of ideas and issues related to the mathematics education of diverse groups of learners. A related conference specifically focused on ELs, which we would organize, would be of great benefit to a multitude of stakeholders as we continue to examine how to best support ELs. In addition to the conference, we will also propose a special edition of a journal focused on issues related to mathematics education and ELs. This would allow the broader mathematics educational research audience access to current work being done in this area.

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EXAMINING SECONDARY MATHEMATICS TEACHERS' MATHEMATICAL MODELING KNOWLEDGE FOR TEACHING

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This working group is in its second year and was formed to address the following research question: What knowledge do secondary teachers need to possess to advance and foster mathematical modeling abilities in students? A task analysis framework with three distinct features – openness, authenticity, and complexity – was created to probe the differences between mathematical modeling tasks and other mathematics or modeling problems to clearly define mathematical modeling and modeling with mathematics. Using this task analysis framework, the group identified mathematical knowledge, modeling knowledge, and pedagogical knowledge as categories in a potential mathematical modeling knowledge for teaching (MMKT) framework. Recognizing that there are many types of mathematical models, e.g. empirical, deterministic, discrete, descriptive, analytical, etc., participants will consider the types of knowledge teachers' exhibit while implementing these different tasks in their classrooms. This session will focus on clearly defining features of these knowledge categories and refining research questions for study throughout the year. A well-defined MMKT framework built from theoretical, qualitative, and quantitative research can inform teacher training programs. Ongoing virtual discussions will continue, and the findings of the working group will be reported.

Keywords: Modeling, Mathematical Knowledge for Teaching, Teacher Knowledge, Teacher Education-Inservice/Professional Development

The Common Core State Standards for Mathematics (CCSSM) include modeling mathematics as a content standard and a mathematical practice in mathematics education and refers specifically to mathematical modeling in grades 9-12 (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). The topics of modeling with mathematics and mathematical modeling differ in the problem context and the degree of openness in the task. Modeling with mathematics activities rely on student generated representations of the either mathematical or real life context situations. These tasks are less open problem statements reducing the number of assumptions and decisions needed by the modeler to simplify the problem, choose variables, and identify relationships. The variety of mathematical approaches available to the solver is also limited resulting in students generating the same model and solution.

Mathematical models are products of an established mathematical modeling process that goes beyond detailing patterns and relationships to use equations, logical arguments, or computer code to explain the behavior of system that is authentic in real-world contexts. The success of the model is closely related to the selection and application of assumptions made by the modeler. These open decision-making problems require the modeler to interpret the ill-posed and less structured task situations, make assumptions, identify variables, develop a mathematical model, manipulate and solve the model, identify limitations, and interpret the solution in the problem's context to determine if the solutions is reasonable (Jensen, 2007). This problem solving process yields multiple solution paths dependent upon the modeler's experiences (Confrey & Maloney, 2007).

Secondary teachers' have limited experiences as learners and teachers of mathematical modeling, which makes implementation of standards and practices challenging (Goos, 2014). Preparing teachers to understand the complexity inherent in mathematical modeling situations and the pedagogical difficulties associated with facilitating and assessing mathematical modeling learning

poses a challenge for mathematics educators. There are few resources available specifically addressing secondary teachers' modeling cognition or pedagogical needs. The proposed working group will build upon its previous work addressing these research needs and increasing the body of research, focusing on the mathematical content knowledge, modeling content knowledge, and pedagogical knowledge that secondary mathematics' teachers need to facilitate rich learning opportunities for students.

Since the term *knowledge* is used repeatedly throughout this paper, it is important to establish a definition. According to the Oxford University Press (2016), knowledge can be defined as the "facts, information, and skills acquired by a person through experience or education; the theoretical or practical understanding of a subject; [and] awareness or familiarity gained by experience of a fact or situation". Research indicates that there are many types of mathematical knowledge called upon to solve mathematical problems, and mathematical knowledge can be increased through exposure to a variety of modeling types and activities (Doerr, 2007).

The fundamental research question for this working group asks "*what knowledge do secondary teachers need to possess in order to foster and advance mathematical modeling in their classrooms?*" In the previous working group sessions, participants, comprised of mathematics education graduate students, educators, and researchers, applied a mathematical modeling task analysis framework, discussed in detail later in this paper, to six distinct modeling tasks published in research journals and mathematics education textbook in order to explore facets of their own mathematical, pedagogical, and modeling knowledge. A groundwork was established defining mathematical modeling and examining distinctions in modeling mathematics, modeling with mathematics, and mathematical modeling tasks. Through the task analysis activity, discussions probed what information, facts, skills, and experiences were needed by teachers to engage students in mathematical modeling situations.

Mathematical Modeling Knowledge for Teaching Framework

The current framework, used by this working group to define mathematical modeling knowledge for teaching, is adapted from research in mathematical modeling competencies with competence defined as a "readiness to act in response to the challenges of a given situation" (Jensen, 2007, p. 142). Jensen's definition suggests that features of mathematical modeling knowledge in three dimensions can be demonstrated by the modeler and observed by educators. The *Degree of coverage* dimension describes the level of independence exhibited by the modeler while working with a mathematical modeling task. *Radius of action* is observed in the breadth of mathematical topics and modeling approaches accessible to the modeler. *Technical level* describes the extent of mathematical sophistication employed in the model.

In the initial research, Groshong, Gomez, and Manouchehri (2015) expanded and adapted Jensen's multi-dimensional approach and suggested that *modeling knowledge, mathematical knowledge,* and *pedagogical knowledge* are key elements of mathematical modeling knowledge for teaching framework (Figure 1). In this working group, the components of mathematical modeling knowledge for teaching are defined as follows: *Mathematical Knowledge* includes the breadth of mathematical content and skills as well as the application of necessary mathematics needed to solve mathematical modeling tasks; *Modeling Knowledge* includes the scope of extra-mathematical knowledge (e.g. contextual knowledge) required to solve mathematical modeling tasks, the understanding need to monitor progression through the various modeling sub-processes, and the awareness of various mathematical modeling approaches; and *Pedagogical Knowledge* includes the information and skills needed to plan and implement modeling lessons and to guide and assess student learning in mathematical modeling content.
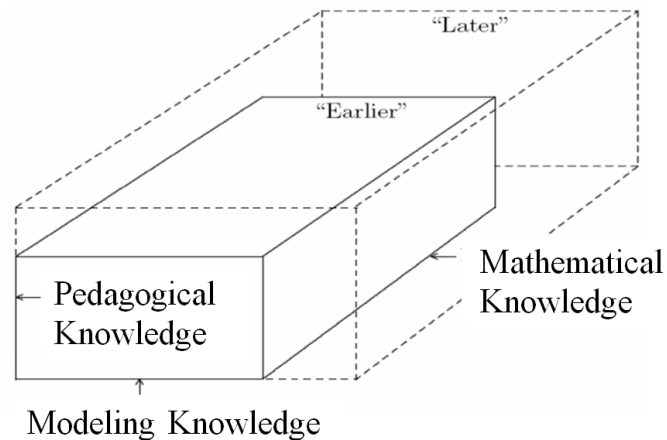
Figure 1. Mathematical modeling knowledge for teaching framework (Groshong, Gomez, & Manouchehri, 2015; Groshong & Park, 2015).

In the previous working group's task analysis, researchers noticed that when teachers' struggled in one dimension this impacted their performance in other areas. During the discussion of the "bending steel" modeling problem, participants sought to determine how high off the ground a railroad track would rise due to the expansion of the metal from winter to summer. Participants commented that from their perspectives, first as a modeler then as a teacher of modeling, there seems to be a continuum between providing too little supporting situational and modeling information and too much information (Groshong & Park, 2015). With too little information, participants lacked sufficient contextual background to understand the task and they lacked the necessary modeling skills to acquire the needed information, so they abandoned the task. Understanding of real life context, which is contextual knowledge, appears to be a necessary feature of modeling knowledge. The teachers' limited modeling knowledge impacted their pedagogical knowledge needed to prepare, implement, guide, and assess student learning with this task (Groshong & Park, 2015).

Mathematical knowledge for teaching

Mathematical knowledge for teaching (MKT) separates teachers' mathematical content knowledge from their pedagogical knowledge and further breaks each area into sub-categories (Ball, Thames, & Phelps, 2008). The working group is in the early stages of connecting an extensive literature review, task analysis discussions, Jensen's observable modeling competencies, and the MKT framework to gain insight into differences that may exist in a mathematical modeling knowledge for teaching framework and the knowledge needed for teaching other mathematics topics.

In the MKT framework, the *knowledge of content* (CK) has three subcategories of interest to aid in identifying a mathematical modeling knowledge for teaching framework. *Common content knowledge* (CCK) includes the mathematical knowledge that educated adults use in their daily lives, which provides modelers entry to mathematical modeling using less sophisticated but readily accessible daily mathematics. *Specialized content knowledge* (SCK) is specific to mathematics teaching and describes the mathematical knowledge teachers employ when predicting student errors, representing mathematical ideas, providing explanations, and recognizing different solution paths. This area identifies the knowledge that teachers use when presenting the mathematical modeling cycle and describing the problem-solving process.

Horizon content knowledge (HCK) refers to teachers' understanding of how mathematics is connected to other subjects, disciplines, and academic grade levels. Since mathematical modeling tasks arise from messy real-world situations, HCK includes the knowledge of how mathematics is applied in everyday life and in the world of work. Contextual understanding of the world is needed

for working with mathematical modeling tasks and consists of academic knowledge of various subject areas, encyclopedic and factual knowledge, and episodic knowledge arising from life experiences (Stillman, 2000). This knowledge area needs extensive research to identify features of HCK to determine if the broad knowledge of the world, extensive mathematical knowledge, organizational expertise, and communication skills used to make sense relevant information represent a type of knowledge that sets mathematical modeling learning apart from the learning other mathematics subjects (Jensen, 2007; Galbraith & Stillman, 2001). In secondary mathematics classrooms, many different types modeling tasks are used, and these tasks vary in their degree of authenticity as simulations of real life. The relationship between the task's authenticity and modeling teaching and learning is another area rich in research opportunities.

Ball, Thames, and Phelps' (2008) *pedagogical content knowledge* (PCK) also has three subcategories, which may be of interest to defining a MMKT framework. Distinguishing it from mathematical subject knowledge, *knowledge of content and students* (KCS) combines a teachers' knowledge about mathematics with knowing about students to anticipate what students think about topic and whether they will find the material difficult. Identifying barriers to students' progression in working with mathematical modeling activities, e.g. translating the real world context into mathematics and identifying relevant relationships, would demonstrate this type of knowledge. Teachers selecting examples and problems to explore, evaluating the advantages and disadvantages of representations, deciding when to wait or ask for student clarification, and posing new tasks to extend student thinking are all examples of *knowledge of content and teaching* (KCT). These knowledge examples are also relevant to the teaching of mathematical modeling. Due to the openness and complexity of mathematical modeling situations, sequencing mathematical modeling activities by content poses a challenge as each student may make different assumptions and identify different relationships resulting in different mathematical models being created. The *knowledge of curriculum* (KC) includes teachers' knowledge of their course curriculum, the curriculum of courses preceding and following their current course, and the curriculum of non-mathematics subjects the students study. This area also describes teachers' familiarity with and ability to critique available instructional materials. Given the real world contexts of mathematical modeling tasks, understanding the curricular topics studied in other subjects provides insight into the types of episodic and encyclopedic knowledge presented to students.

Secondary Teachers' Mathematical Modeling Knowledge Working Group

This working group will follow a similar path as previously established. The mathematical modeling task analysis framework (Figure 2) will be applied to very distinct mathematical modeling situations. The framework provides a means to encourage working group member participation, probe the differences in types of mathematical modeling approaches found in literature, and discuss the nature of the knowledge needed to solve various types of modeling situations.
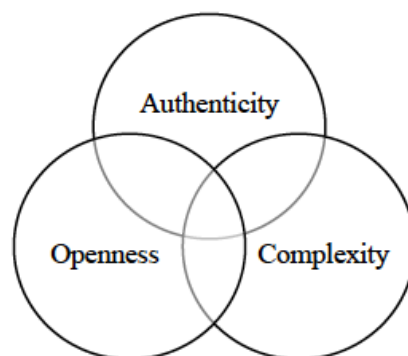
Figure 2. Mathematical modeling task analysis framework (Groshong & Park, 2015).

The operational definitions used in this framework are drawn from research. *Authentic* implies that the modeling problem serves more than just educational needs; the task is genuine in a subject area outside mathematics, and the task originates from reality outside classroom (Vos, 2011). Authenticity also refers to the degree of parallelism between the classroom experience and the research field of mathematical modeling including aspects of the problem solving approach, sources of data, modeling software, and modeling environment (Vos, 2011). *Complexity* has two components. Modeling complexity is contingent upon how much assistance is given to students, how many choices of techniques are available, and how many modeling techniques are required (Stillman, 2000). Mathematical complexity refers to the degree of mathematical sophistication incorporated in mathematical modeling process (Jensen, 2007). *Openness* refers to problem situations that permit multiple solutions, multiple interpretations and answers, and even multiple new questions (Abrams, 2001, p.18).

In distinguishing mathematical modeling from other types of modeling experiences, it is also important to identify the many different types of mathematical models found in mathematics education literature, educational standards and curriculum, and instructional materials. The differences identified in each of these taxonomies can provide more insight into the types of knowledge needed to develop, implement, and facilitate mathematical modeling as a standard and a practice in the classroom.

Different types of mathematical models

Since there are many different types of mathematical models which can be created, researching teachers' interactions with this broad field can reveal gaps in understanding teachers' knowledge and guide future teacher training programs (Groshong, 2016). For instance, some mathematical modeling types may be more accessible for students suggesting a sequencing in the types of tasks needed to increase mathematical modeling competencies as these modeling situations may vary in their degree of openness, authenticity, and complexity.

Mathematical models can be classified according to their general characteristics, e.g. the modeling situation and solution form, and mathematical features, e.g. the treatment of data and variability in the solution (Groshong, 2016). Many terms are used to describe different types of mathematical models in mathematics education literature. In addition, a single mathematical modeling activity can be described using several terms (Figure 3). *Empirical* models rely on data whereas *mechanistic* models are rooted in theoretical statements. *Deterministic* models are nonrandom and *stochastic* models include randomness in the model output. Combining these concepts suggests that empirical-deterministic models are built from regression relations; empirical-stochastic models may use analysis of variances techniques; mechanistic-deterministic models arise from theoretical equations; and mechanistic-stochastic models yield probabilistic equations (Edwards & Hamson, 2007).
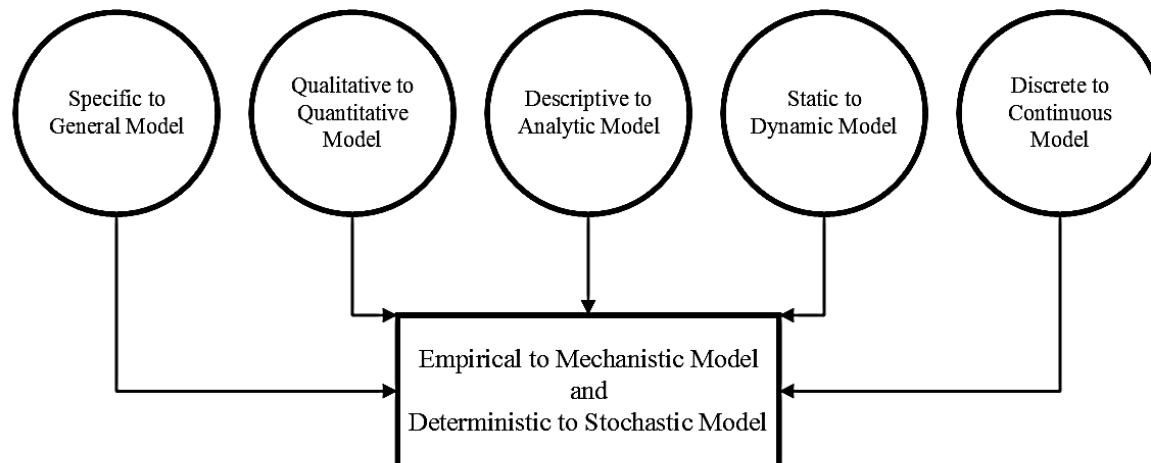
Figure 3. Different types of mathematical models (Groshong, 2016).

Solutions that address a single situation are *specific models*, and those addressing multiple versions of the situation are *general models*. When examining the overall model, *quantitative models* produce mathematical expressions that can predict or explain responses, and *qualitative models* are frequently use logical arguments to describe patterns and trends. In the initial modeling stages, *descriptive models* are often narratives or representations that include assumptions, define variables, and detail real-life issues (Tam, 2011). Mathematical methods are applied to yield exact answers from *analytical models*. *Static models* take a "snapshot" of time in an equilibrium condition. *Dynamic models* examine how the situation changes with the passage of time. Number values used in modeling my yield *continuous models* with all values between two endpoints acceptable, and *discrete models* treat variables as countable numbers.

Working group plan

The working group has begun and will continue the discussion of mapping the subcategories outlined by Ball, Thames, and Phelps (2008) to mathematical, modeling, and pedagogical content areas in hopes of more clearly defining mathematical modeling knowledge for teaching. Possible discussion topics in these MMKT dimensions include the following:

1. What mathematical knowledge is needed for working with different types of mathematical modeling tasks?
2. What modeling knowledge is needed for working with different types of mathematical modeling tasks?
    o How can modeling knowledge for working with different types of mathematical modeling tasks be identified?
    o How much contextual knowledge of the real world is needed to work with different types of mathematical modeling situations?
3. What pedagogical knowledge is needed for working with different types of mathematical modeling tasks?

Working Group Plan for Active Engagement of Participants

Participants are eagerly invited to join this working group to actively participate in defining this important research area. To continue examining this research question, we propose that members of the working group will divide into smaller groups providing an intellectual support system to critique the definitions, taxonomies, and framework as currently detailed, discuss the suggested focus

questions, and generate new research questions. The working group is committed to the goals of PME-NA while emphasizing collective reflection and collaborative inquiry, addressing challenges, and expanding the field of research. The working group will meet three times during the conference and virtually during the course of one year.

Session 1: Foundation

The first session will discuss common definitions in modeling research, summarize the previous year's working sessions, detail the progress made during the year, and outline the goals for the up-coming year. This will lay a common foundation for all participants to fully engage in the session discussions needed to define a mathematical modeling knowledge for teaching framework. Using the types of mathematical models' taxonomy and the mathematical modeling task analysis framework, sample mathematical modeling tasks will be provided to promote discussion, critique definitions and categorizations, and identify mathematical, modeling, and pedagogical knowledge needed by teachers for implementing these tasks in secondary mathematics lessons. Since it has shown to be successful for eliciting participation within this working group, members will again be invited to share mathematical modeling tasks with the group and generate research questions. The final part of the session will include sharing contact information and encouraging the formation of research groups based on the interests of the participants.

Session 2: Mathematical modeling knowledge for teaching framework

The second session will continue the work from the previous session to further define mathematical knowledge, modeling knowledge, and pedagogical knowledge. Employing the task analysis as a vehicle to elicit understanding of the different knowledge types, participants will discuss a mapping between Ball, Thames, and Phelp's (2008) MKT and a potential MMKT framework more clearly defining the content and pedagogical knowledge subcategories and providing specific exemplars. Participants will revisit last year's lively discussion about the influence of the real world contextual knowledge and experiences in modeling and, then, consider the placement of the knowledge of the real world in the MMKT framework.

Session 3: Modeling Knowledge

The last session will summarize the previous days' findings, articulate research goals, and outline a working plan for designing research projects to be conducted during the upcoming year. Participants will be encouraged to join the ongoing research discussions virtually.

Post-conference

To promote the working group's efforts, the results of all sessions and meetings will be documented and disseminated to all members. Following the conference, participants will be invited to continue discussing research interests in this area through a web-based discussion forum, which will continue to offer an environment for suggesting and reviewing research, posing and critiquing mathematical modeling tasks, posting and discussing teacher and student solutions, and developing an archive of useful and productive mathematical modeling tasks. Working group members will continue to participate in digital meetings throughout the year to review progress on scholarship. Findings of the working group and subsequent discussions will be summarized and reported.

Conclusion

While this working group is still in its infancy, considerable in-roads have been made in establishing a theoretical foundation built upon research principles. This groundwork has yielded operational definitions for distinguishing between the confusing yet often interchanged phrases of mathematical modeling, modeling mathematics, and modeling with mathematics. Drawing upon the theoretical research classifying different mathematical models by their mathematical features and

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general characteristics provides another tool for revealing facets of mathematical modeling knowledge for teaching. On-going research is needed to validate the working group's current thinking and refine the proposed MMKT framework. This research can expand our understanding of teaching knowledge and advice teacher training programs.

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EMBODIED MATHEMATICAL IMAGINATION AND COGNITION (EMIC) WORKING GROUP

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Embodied cognition is growing in theoretical importance and as a driving set of design principles for curriculum activities and technology innovations for mathematics education. The central aim of the EMIC (Embodied Mathematical Imagination and Cognition) Working Group is to attract engaged and inspired colleagues into a growing community of discourse around theoretical, technological, and methodological developments for advancing the study of embodied cognition for mathematics education. A thriving, informed, and interconnected community of scholars organized around embodied mathematical cognition will broaden the range of activities, practices, and emerging technologies that count as mathematical. EMIC builds upon our 2015 working group, and investigations in formal and informal education and workplace settings to bolster and refine the theoretical underpinnings of an embodied view of mathematical thinking and teaching, while reaching educational practitioners at all levels of administration and across the lifespan.

Keywords: Classroom Discourse, Cognition, Informal Education, Learning Theory

Motivations for This Working Group

Recent empirical, theoretical and methodological developments in embodied cognition and gesture studies provide a solid and generative foundation for the establishment of **an Embodied Mathematical Imagination and Cognition (EMIC) Working Group** for PME-NA. The central aim of EMIC is to attract engaged and inspired colleagues into a growing community of discourse around theoretical, technological, and methodological developments for advancing the study of embodied cognition for mathematics education, including, but not limited to, studies of mathematical reasoning, instruction, the design and use of technological innovations, learning in and outside of formal educational settings, and across the lifespan.

The interplay of multiple perspectives and intellectual trajectories is vital for the study of embodied mathematical cognition to flourish. Partial confluences and differences have to be maintained throughout the conversations; this is because instead of being oriented towards a single and unified theory of mathematical cognition, EMIC strives to establish a philosophical/educational "salon" in which entrenched dualisms, such as mind/body, language/materiality, or signifier/signified are subject to an ongoing and stirring criticism. A thriving, informed, and interconnected community of scholars organized around embodied mathematical cognition will broaden the range of activities and emerging technologies that count as mathematical, and envision alternative forms of engagement with mathematical ideas and practices (e.g., De Freitas & Sinclair, 2014). This broadening is particularly important at a time when schools and communities in North America face persistent achievement gaps between groups of students from many ethnic backgrounds, geographic regions, and socioeconomic circumstances (Ladson-Billings, 1995; Moses & Cobb, 2001; Rosebery, Warren, Ballenger & Ogonowski, 2005). There also is a need to articulate evidence-based findings and principles of embodied cognition to the research and development communities that are looking to generate and disseminate innovative programs for promoting mathematics learning through

movement (e.g., Petrick Smith, King, & Hoyte, 2014). Generating, evaluating, and curating empirically validated and reliable methods for promoting mathematical development and effective instruction through embodied activities that are engaging and curricularly relevant is an urgent societal goal.

The EMIC Working Group: A Brief History

The first meeting of the EMIC working group took place in East Lansing, MI during PME-NA 2015. It has a somewhat longer origin, however, growing out of several earlier collaborative efforts to review the existing literature, document embodied behaviors, and design theoretically motivated interventions. One early event was the organization of the 2007 AERA symposium, "Mathematics Learning and Embodied Cognition" (Nemirovsky, 2007). This and other gatherings led to a funded NSF "catalyst" grant to explore a Science of Learning Center, which was to involve scholars from multiple institutions and countries. Though unfunded, those SLC efforts shaped a subsequent 6-year NSF-REESE grant, "Tangibility for the Teaching, Learning, and Communicating of Mathematics," starting in 2008. Interest from the International PME community in this topic grew, and led to special issues of *Educational Studies in Mathematics* (2009), *The Journal of the Learning Sciences* (2012), and an NCTM 2013 research pre-session keynote panel, "Embodied cognition: What it means to know and do mathematics," along with a series of academic presentations, book chapters, and journal articles, as well as several masters' theses and doctoral dissertations. By now, several research programs have formed to investigate the embodied nature of mathematics (e.g., Abrahamson 2014; Alibali & Nathan, 2012; Arzarello et al., 2009; De Freitas & Sinclair, 2014; Edwards, Ferrara, & Moore-Russo, 2014; Lakoff & Núñez, 2000; Radford 2009), demonstrating a "critical mass" of projects, findings, senior and junior investigators, and conceptual frameworks to support an on-going community of likeminded scholars within the mathematics education research community.

It was within this historical context that approximately 22 members of PME-NA 2015 came together for three 90-min sessions of semi-structured activities. On **Day 1,** the organizers engaged attendees in some of the body-based math activities used in their research on proportional reasoning and geometry. We discussed how embodied theories are advancing our understanding of mathematical thinking, and how these ideas are shaping a new class of educational interventions. During **Day 2**, we used hands-on activities to expand our own understanding of topology. We then built on the emerging rapport among the group to hold a facilitated discussion of the potential intellectual benefits of forming a self-sustaining Working Group on embodied cognition, along with the necessary infrastructure it would need to maintain. Several concrete proposals led to the list of Future Steps on **Day 3**. However, before we tackled those matters, participants began the session doing math games and activities in small groups, including Spirograph, Set, Rush Hour, Tangrams, and Mastermind. We reflected on how some games and activities draw people into rich mathematical thinking and actions, and how we naturally engage in math through these activities. Day 3 culminated in an organized list of Future Steps, with some working group members assigned to specific tasks.

Since our first meeting at PME-NA 2015 **our accomplishments** include:

1. Creating a contact list with names and emails of attendees ($n = 22$) and other interested scholars who could not attend PME-NA 2015 ($n = 25$);
2. Developing a group website using the Google Sites platform to support ongoing interactions throughout the year
3. Joint submission of an NSF DRK-12 by members who first met during the 2015 EMIC sessions
4. Some senior members joining a junior member's NSF ITEST grant proposal
5. Submitting a proposal for the continuation of the EMIC WG to PME-NA 2016

6. Examining the potential for an NSF Research Coordination Network (RCN)

Focal Issues in the Psychology of Mathematics Education

Emerging, yet still influential, views of thinking and learning as embodied experiences have grown from several major intellectual developments in philosophy, psychology, anthropology, education, and the learning sciences that frame human communication as multi-modal interaction, and human thinking as multi-modal simulation of sensory-motor activity (Clark, 2008; Hostetter & Alibali, 2008; Lave, 1988; Nathan, 2014; Varela et al., 1992; Wilson, 2002). These views acknowledge the centrality of both unconscious and conscious motor and perceptual processes for influencing conscious awareness, and of embodied experience as following/producing pathways through social and cultural space. As Stevens (2012, p. 346) argues in his introduction to the *JLS* special issue on embodiment of mathematical reasoning,

> it will be hard to consign the body to the sidelines of mathematical cognition ever again if our goal is to make sense of how people make sense and take action with mathematical ideas, tools, and forms.

Four major ideas exemplify the plurality of ways that embodied cognition perspectives are relevant for the study of mathematical understanding: (1) Grounding of abstraction in perceptuo-motor activity as one alternative to representing concepts as purely amodal, abstract, arbitrary, and self-referential symbol systems. This conception shifts the locus of "thinking" from a central processor to a distributed web of perceptuo-motor activity situated within a physical and social setting. (2) Cognition is for action. This tenet proposes that things, including mathematical symbols and representations, are understood by the actions and practices we can perform with them, and by mentally simulating and imagining the actions and practices that underlie or constitute them. (3) Mathematics learning is always affective: there are no purely procedural or "neutral" forms of reasoning detached from the circulation of bodily-based feelings and interpretations surrounding our encounters with them. (4) Mathematical ideas are conveyed using rich, multimodal forms of communication, including gestures and tangible objects in the world.

Alongside these theoretical developments have been technical advances in multi-modal and spatial analysis, which allow scholars to collect new sources of evidence and subject them to powerful analytic procedures, from which they may propose new theories of embodied mathematical cognition and learning. Just as the "linguistic turn" in the social sciences was largely made possible by the innovation that enabled scholars to collect audio recordings of human speech and conversation *in situ*, growth of interest in multi-modal aspects of communication have been enabled by high quality video recording of human activity (e.g., Alibali et al., 2014; Levine & Scollon, 2004), motion capture technology (Hall, Ma, & Nemirovsky, 2015; Sinclair, 2014), and developments in brain imaging (e.g., Barsalou, 2008; Gallese & Lakoff, 2005).

Plan for Active Engagement of Participants

Our formula from PME-NA 2015 proved to be effective: By inviting participants into math activities at the beginning of each session, we were rapidly drawn into those very aspects of mathematics that we find most rewarding. Facilitated discussions (and we now have many effective members who can trade off in this role!) then help us all to "pull back" to the theoretical and methodological issues that are central to advancing math education research. Within this structure of beginning with mathematical activities and facilitated discussions, on **Day 1** we plan to introduce our new website, demonstrate the online resources for building sustained community, and revisit and further develop the items listed in our Future Steps, including assigning roles to EMIC members. On **Day 2**, we will discuss concrete goals and products. One example is the NSF Research Coordination Network (RCN), as a potential compliment to the PME-NA Working Group. The RCN is not

intended to promote any one particular research program, but rather to build the networked community of international scholars from which many fruitful lines of inquiry can emerge. Commensurate with the aims of the RCN, we will explore ways to

> share information and ideas, coordinate ongoing or planned research activities, foster synthesis and new collaborations, develop community standards, and in other ways advance science and education through communication and sharing of ideas.

This sharing and coordination will continue into **Day 3**. One proposed activity is to perform a live concept mapping activity that is displayed for all participants to explore the range of EMIC topics and identify common conceptual structure. Harkening back to the four major ideas that we developed earlier, sample seed topics for organizing this activity will be explored, such as:

1. Grounding Abstractions
   a. Conceptual blending (Tunner & Fauconnier, 1995) & metaphor (Lakoff & Núñez, 2000)
   b. Perceptuo-motor grounding of abstractions (Barsalou, 2008; Glenberg, 1997)
   c. Progressive formalization (Nathan, 2012; Romberg, 2001) & concreteness fading (Fyfe, McNeil, Son, & Goldstone, 2014)
   d. Use of manipulatives (Martin & Schwartz, 2005)
2. Cognition is for Action: Designing interactive learning environments for EMIC
   a. Development of spatial reasoning (Liu, Uttal, Marulis, & Newcombe, 2008)
   b. Math cognition through action (Abrahamson, 2014; Nathan et al., 2014)
   c. Perceptual boundedness (Bieda & Nathan, 2009)
   d. Perceptuomotor integration (Nemirovsky, Kelton, & Rhodehamel, 2013)
   e. Attentional anchors and the emergence of mathematical objects (Abrahamson & Sánchez–García, in press; Abrahamson, Shayan, Bakker, & Van der Schaaf, in press)
   f. Mathematical imagination (Nemirovsky, Kelton, & Rhodehamel, 2012)
   g. Students' integer arithmetic learning depends on their actions (Nurnberger-Haag, 2015).
3. Affective Mathematics
   a. Modal engagements (Hall & Nemirovsky, 2012; Nathan et al., 2013)
   b. Sensuous cognition (Radford, 2009)
4. Gesture and Multimodality
   a. Gesture & multimodal instruction (Alibali & Nathan 2012; Cook et al., 2008; Edwards, 2009)
   b. Bodily activity of professional mathematicians (Nemirovsky & Smith, 2013)
   c. Simulation of sensory-motor activity (Hostetter & Alibali, 2008; Nemirovsky & Ferrara, 2009)

Finally, we will introduce the EMIC website (see Figure 1) and invite members to join, and to encourage their interested colleagues to email Caro at cwilliamspierce@albany.edu for access. On this website, we have a list of members with their emails and bios, information about our PME-NA presence, and short personal introduction videos. We've also created a space for members to share information about their research activities – particularly for videos of the complex gesture and action-based interactions that are difficult to express in text format. In addition, we have a common publications repository to share files or links (including to ResearchGate or Academia.edu publication profiles, so members don't have to upload their files in multiple places). At our 2015 working group, some junior members expressed particular interest in this literature support for their pending theses, while more senior members were eager to share and organize the emerging body of

work on embodied math education. We've also linked the Google Sites platform directly to a Google Group, so members can participate in online forums (or the linked listserv), and discuss cutting edge topics, share in-progress working papers for review, or advertise for conferences, special issues, or other EMIC-relevant opportunities.

Figure 1. The EMIC website landing page serves as one of the ways EMIC members can share information about themselves and their work, support a common paper repository, post relevant announcements, and coordinate emerging collaborations.

Follow-up Activities

Even prior to our first anniversary, we have already seen a great deal of progress. This is perhaps best exemplified by coming together of the EMIC website and this proposal submission, which draws across multiple institutions. We envision an emergent process for the specific follow-up activities based on participant input and our multi-day discussions. At a minimum, we will continue to develop a list of interested participants and grant them all access to our common discussion forum and literature compilation. Those that are interested in the NSF RCN plan will work to form the international set of collaborations and articulate the intellectual topics that will knit the network together. One additional set of activities we hope to explore is to introduce educational practitioners at all levels of administration and across the lifespan to the power and utility of the EMIC

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perspective. We thus will strive to explore ways to reach farther outside of our young group to continually make our work relevant, while also seeking to bolster and refine the theoretical underpinnings of an embodied view of mathematical thinking and teaching.

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REFRAMING INTERVENTIONS IN MATHEMATICS EDUCATION: EMERGING CRITICAL PERSPECTIVES

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Mathematics interventions have proliferated within the field of mathematics education as schools are under increasing pressure to raise the mathematics test scores of low-achieving students, particularly students of color and students with disabilities. These interventions target individual students and are based on a deficit model; the problem is located within the individual student rather than in the social, discursive, political, or structural context. These interventions also tend to focus primarily on rote algorithms and calculation skills rather than the solving of rigorous, high cognitive demand problems. Our working group is composed of researchers and teachers who draw upon critical theories, such as Disability Studies in Education, Critical Race Theory, and DisCrit, in order to offer an alternative vision of mathematics education based around a different type of intervention. Rather than making an intervention into the deficits of an individual student, our working group suggests that we need to make an intervention into classroom practice, utilizing models like Complex Instruction (CI) and Universal Design for Learning (UDL) to create classrooms in which all students are able to access the curriculum in meaningful and rigorous ways.

Keywords: Classroom Discourse, Equity and Diversity, Instructional Activities and Practices

Rationale: Framing the Problems/Questions of the Working Group

The title of this working group, Reframing Interventions in Mathematics Education, Emerging Critical Perspectives, is intended to evoke a double meaning. It can be read as an intention to critique the practice of intervention in the field of mathematics education; alternatively, the title can be read as inviting critical interventions into mathematics and mathematics education. By pursuing both paths simultaneously, we intend to stimulate productive discussions and lay the foundation for future projects during our work in the span of three working group sessions.

Mathematics Intervention

Intervention programs have become ubiquitous within mathematics education as teachers and schools are under pressure to raise mathematics test scores. These programs utilize service delivery models such as Resource Specialist Programs (RSP) and Response to Intervention (RTI) and techniques such as Progress Monitoring in order to identify students in need of intervention and deliver interventions. These interventions rarely focus on increasing student participation in classroom discourse; they instead focus primarily on basic fact acquisition and algorithmic fluency. Most districts use older models that measure mathematics students against normative standards of development, with instruments such as the Woodcock-Johnson, the Wechsler Individual Achievement Test (WIAT), and the KeyMath being used to diagnose areas in which students' "achievement" fails to match what IQ tests show as their "ability." Some districts use newer models

such as RTI for diagnosis, which claim to identify and classify students based on their response to "scientific", "evidence-based" interventions. Progress monitoring techniques such as AIMSweb focus less on testing of "ability" and instead focus on repeated assessment of students' mathematical skills, but the data collected still tends to focus myopically on basic calculation skills. The problem, our working group believes, is not with the instruments being used to assess, but with the very concept of intervention itself.

The Problem with Intervention

To better understand the concerns we are raising about the concept of intervention, we invoke the history of the term, finding that the very etymology of the word points to some problematic assumptions imbedded within. The English word "intervention" comes from the Latin "inter," meaning "between," and "venire," meaning "to come." Etymologically then, intervention implies "coming between." Two core assumptions are embedded in the concept of intervention. The first assumption is that there is something wrong with the situation at hand that needs to be changed. The second assumption is that it is someone or something from outside the situation that must "come into" the situation and make the change. Often, in education generally and in mathematics education in particular, the need for intervention is taken-for-granted. However, acknowledging the assumptions upon which the concept of intervention rests raises numerous questions about the idea of intervention in education, and specifically in mathematics education. The critique that we are offering asks questions such as:

1. To what extent is "intervention" necessary or desired in the situations in which it is currently used? Necessary for what and to whom? Desired by whom and for what purpose?
2. What aspects of the situations are taken as "wrong" and requiring intervention?
3. Who benefits from interventions, and in what ways?
4. What are the implications of the fact that the concept of intervention is based on the assumption that someone from the outside must "come in" to make changes?
5. What counts as "intervention"; are interventions only narrow, technical methods used to correct problems with a particular student or can we expand the concept of intervention to create meaningful change within classrooms?

In our current historical moment in the United States we believe it is important to consider the way that "interventions" are used for low-achieving students (especially students of color and students with disabilities) and to offer critical alternatives to the practice and philosophy of intervention.

Drawing upon Theory

Gathering together a group of researchers, graduate students, undergraduate students, and classroom teachers interested in developing an alternate paradigm around intervention, we have begun to explore the different theoretical frameworks that we can use to analyze and change this situation within mathematics education. Heeding Lather's (1986) call for researchers to utilize multiple theoretical schemes in their work, our working group believes that this work requires the participation of multiple paradigms of critique. Our working group draws upon critical theories such as disability studies, critical race theory, and other poststructural theories (including queer studies and trans studies) in this analysis. A commonality among these perspectives is that each problematizes normativity. We welcome the participation of those with other perspectives as well. In the following sections, we describe a sampling of these perspectives in order to frame the work of the group.

Critical Theories

Although the notion of critique dates back to antiquity, the phrase critical theory originally referred to a body of work coming out of the Frankfurt School and Institute for Social Research in the late 1920s through the early 1940s. Critical theory seeks to reshape reality, not merely explaining things. As Karl Marx (1845) wrote almost a century earlier, "The philosophers have only interpreted the world, in various ways; the point is to change it." Critical theory has come to mean not just the work of the original critical theorists, but any theory that seeks to transform rather than merely explain society. The Stanford Encyclopedia of Philosophy gives three main criteria for a critical theory, that "it must explain what is wrong with current social reality, identify the actors to change it, and provide both clear norms for criticism and achievable practical goals for social transformation." All of the theoretical perspectives that our working group draws upon are in some ways "critical theories."

Disability Studies and Disability Studies in Education

Many of the members of our working group utilize disability studies (DS) and disability studies in education (DSE) as a mode of inquiry in order to questioning the taken-for-granted assumptions and practices in mathematics education and the intervention paradigm. DS and DSE provides a framework for exploring questions such as: who is labeling, who is being labeled, and how do we advance more equitable practices for all students. Disability studies calls into question the medical/individual model of disability in which disability is seen as a deficit within an individual that requires "curing."

In contrast to the medical model, many disability studies scholars and activists have adopted a social model of dis/ability, which locates dis/ability in an inaccessible environment. Those who adopt the social model of dis/ability make a distinction between impairment, as any physical or mental limitation, and disability, as the "social exclusions based on, and social meanings attributed to, that impairment" (Kafer, 2013, p. 7). Kafer (2013), however, argues that such a sharp distinction between impairment and dis/ability is unhelpful because it "fails to recognize that *both* impairment and disability are social" (p. 7). In the book *Feminist Queer Crip*, Kafer suggests the term "political/relational model" to refer to perspectives recognizing that both impairment and dis/ability are socially constructed.

In educational settings, this construction of dis/ability manifests in the double education system that splits general education and special education. Scholars have traced the ways in which special education "serves as a vehicle for preserving general education in the midst of ever increasing diversity" (Reid & Valle, 2004, p. 468, paraphrasing Dudley-Marling, 2001; also see Skrtic, 1991, 2005). Rather than using research-validated frameworks like Universal Design for Learning (UDL) and Complex Instruction (CI) to deliver rigorous, high-cognitive demand instruction to all mathematics students, the system of special education shunts certain students (especially students of color) into an inferior, segregated mathematics education, thus providing a band-aid to a broken general education system and preventing larger, more systematic changes. One line of research pursued by working group members involves developing understanding and theorizing the research divide between special education and mathematics.

Institutional schooling practices such as writing Individual Education Plans (IEPs) construct certain students as having disabilities; however, from a disability studies perspective, "the label of students with IEPs [can be viewed] not as an inherent and static determinant of individual ability, but as a school-based designation which reflects and recreates differential ability within the classroom" (Foote & Lambert, 2011, p. 250; also see Dudley-Marling, 2004; McDermott, Goldman & Varenne, 2006; Skrtic, 2005). Certain students are chosen for this assessment and intervention, and this selection process is not objective and often singles out those students who are not from a dominant cultural background.

Returning to the assumptions inherent in the concept of intervention, a disability studies perspective problematizes the taken-for-granted assumption that what is "wrong" with the situation requiring intervention is a pathology or deficit within students. Instead, the problem is located in the inaccessibility of the environment; in other words, what needs to be changed is not the student, but rather the environment to allow access for students who differ from one another. As Reid and Valle (2004) assert, "the responsibility for 'fitting in' has more to do with changing public attitudes and the development of welcoming classroom communities and with compensatory and differentiated instructional approaches than with individual learners (Shapiro, 1999). In other words, our focus is on redesigning the context, not on 'curing' or 'remediating' *individuals'* impairments" (p. 468). A related line of research of working group members involves conceptualizing interventions into *participation* rather than content. That is, what interventions might contribute to more equitable participation and deeper engagement across students in mathematics classrooms? For example, one of the working group members has conducted empirical research focused on equitable participation in a Cognitively Guided Instruction algebra routine. Moreover, a political/relational model suggests that inaccessibility is embedded in the context of power relations. Finding ways to "intervene" to make the environment accessible, then, also requires analyzing the power relations involved in maintaining inaccessibility.

Critical Race Theory and DisCrit

Critical race theory (CRT) is another theoretical framework that informs the work of the group. According to CRT, racism is 'normal' rather than an anomaly in U.S. society (Delgado, 1995). Critical race theorists assert that the U.S. was founded on property rights, and specifically the fact that enslaved African Americans were considered property, rather than civil rights (Ladson-Billings & Tate, 1995). CRT reveals the way race and racism continue to structure U.S. society. In relation to educational interventions, critical race theorists have addressed the issue of over-representation of students of color in special education.

Often, however, these analyses leave ableist assumptions in place; similarly, DS perspectives often fail to adequately consider race. DisCrit is a perspective that acknowledges that racism and ableism are both "normalizing processes that are interconnected and collusive. In other words, racism and ableism often work in ways that are unspoken, yet, racism validates and reinforces ableism, and ableism validates and reinforces racism" (Connor, Ferri, & Annamma, 2016, ch. 1). Studies of administrators' and teachers' perceptions related to overrepresentation of students of color in special education have revealed that their perceptions tend to be rooted in "deficit thinking and infused with racial and cultural factors" (Connor, Ferri, Annamma, 2016, ch. 1; also see Abram et al., 2001 and Skiba et al, 2006). DisCrit perspectives, therefore, identify the individual problematic attitudes of teachers and administrators as one "accessible entry point for intervention" (Connor, Ferri, Annamma, 2016, ch. 1).

Summary of the Problem

This working group will investigate the problem of interventions in mathematics education. Using multiple theoretical frameworks, the working group participants will analyze current practices in mathematics interventions, including the power relations involved, and develop and elaborate on alternatives. The working group participants will also plan ways to evaluate these alternatives in various educational settings and contexts.

Plan for Active Engagement of Working Group Participants

Prior to Session 1

The organizers will choose 2-3 articles or book chapters from different perspectives related to the theme of the working group. The organizers will send out the list of readings to participants who have expressed interest in the working group.

Session 1

In the first session, the organizers will introduce the rationale for the working group and the current state of related research. Participants will share related projects in which they are involved. The group will collaboratively refine the goals of the working group. The organizers will share the list of chosen readings with any participants who did not previous have access to them.

Session 2

In the second session, participants will discuss the chosen readings in relation to the goals refined in the first session. Participants will discuss the ways in which the readings relate to ongoing projects and possible future collaborative projects.

Session 3

In the second session, participants will discuss the chosen readings in relation to the goals refined in the first session. Participants will discuss the ways in which the readings relate to ongoing projects and possible future collaborative projects.

Plan for Sustainability: Anticipated Follow-up Activities

The working group sessions during the conference are designed to enable the participants to develop concrete plans for collaborative work beyond the end of the conference timeframe. Specifically, the third session is allotted for developing specific plans for future collaborative work.

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CONCEPTIONS AND CONSEQUENCES OF WHAT WE CALL ARGUMENTATION, JUSTIFICATION, AND PROOF

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Argumentation, justification, and proof are conceptualized in many ways in extant mathematics education literature. At times, the descriptions of these objects and processes are compatible or complementary; at other times, they are inconsistent and even contradictory. The inconsistencies in definitions and usages of the terms argumentation, justification, and proof highlight the need for scholarly conversations addressing these (and other related) constructs. Collaboration is needed to move toward, not one-size-fits-all definitions, but rather a framework that highlights connections among them and exploits ways in which they may be used in tandem to address overarching research questions. Working group leaders aim to facilitate discussions and collaborations among researchers and to advance our collective understanding of argumentation, justification and proof, particularly the relationships among these important mathematical constructs. The 2016 working group sessions will provide continued opportunities for participants to discuss existing definitions and descriptions as well as opportunities to consider how these definitions and descriptions are related to the positionality of researchers and the contexts of their research. Participants will examine data sources through a variety of lenses as a means to investigate the use of particular conceptualizations.

Keywords: Reasoning and Proof, Advanced Mathematical Thinking

Brief History of the Working Group

The *Conceptions and Consequences of What We Call Argumentation, Justification, and Proof* Working Group met for the first time in 2015 at Michigan State University in East Lansing, Michigan during the 37th Annual Meeting of the North American Chapter of the Psychology of Mathematics Education. The main goal of this working group is to advance the field's collective understanding of the interrelated objects and processes of argumentation, justification, and proof. Although past working groups had focused on proof or argumentation, this was the first working group at PME or PME-NA meetings focused specifically on the connections among these three constructs.

During the first year, we made progress on considering the interrelationships among argumentation, justification and proof, and we deepened our understandings of our own perspectives and the range of perspectives held by others in the group. Our goal was not consensus or deciding a best approach, but rather making sense of the diversity among and the priorities of individuals with respect to their research agendas. We were encouraged that our efforts were well received with 46 scholars, including at least 10 graduate students participating in the working group the first year.

A focal activity during these initial working group sessions was the development of Diagrams/Concept Maps in which each participant generated a representation of the relationships among argumentation, justification and proof from his or her perspective. Each participant also identified the concept that was most central to his or her work (i.e., argumentation, justification, or proof). This activity was followed by a presentation by Keith Weber, a co-author of the proof chapter

in the forthcoming NCTM compendium for research (Stylianides, Stylianides, & Weber, in press). He offered an historical view on the use of these constructs in mathematics and educational research, highlighting for us convergence and contradictions in definitions and research results in the proof literature.

Our second day featured a panel presentation, moderated by Samuel Otten, with Kristen Bieda, AnnaMarie Conner, and Pablo Mejía -Ramos serving as our expert panelists. Each panelist identified one of the three constructs as central to his or her work (i.e., Bieda - justification; Conner - argumentation; and Mejía-Ramos – proof). The panelists shared how they conceptualized the interrelationships among argumentation, justification and proof; they explicated how they came to use the central construct they use in their research and why they felt that choice is productive for their work; and they offered their thoughts on the current state of the field and what we might need to tackle next in relation to these constructs.

The final day of our work together offered the opportunity to revisit participants' Diagrams/Concept Maps, though now potentially informed by additional perspectives and questions gained from the prior two sessions. Participants viewed the full set of Diagrams/Concept Maps through a gallery walk and had the option to annotate and revise their diagrams as desired. The set of diagrams were artifacts for extensive and lively discussion on Day 3. All participants from the working group were invited to collaborate on a white paper that would include brief analyses of the diagrams.

As a result of our three days together, three products were generated. First, a pair of podcasts are available worldwide: The first podcast is of Weber's talk, and the second podcast is the moderated panel discussion. Thanks to Sam Otten, the podcasts are available at http://mathed.podomatic.com/entry/2015-11-16T07_01_19-08_00 and http://mathed.podomatic.com/entry/2015-11-19T07_19_37-08_00, respectively. The second product is a white paper which was developed by the working group organizers, the panelists, and several other participants from the working group who volunteered to participate in the online publication (Cirillo et al., 2016). The white paper summarizes the working group activities and discussions and also includes the set of 44 Diagrams/Concept Maps that were generated as well as annotations and analyses. The final product is a poster proposal for PME-NA 2016 that is based on the analyses of the Diagrams/Concept Maps. Last, we were able to begin creating a community of mathematics education professionals who we hope will continue these discussion over several years.

Summary of Focal Issues

There is a large and growing body of research in mathematics education focused on argumentation, justification, and proof. The research on proof, for example, includes studies on: the role of proof in the discipline; proof in school mathematics and at the undergraduate level; what counts as a proof; proof schemes and categories; teachers' conceptions of proof; students' abilities to write valid proofs; and what teaching proof looks like in classrooms at various levels (e.g., Boero, 2007; Harel & Sowder, 2007; Reid & Knipping, 2010; Stylianou, Blanton, & Knuth, 2009). At the same time, researchers and policy documents have issued calls to engage K-12 students with disciplinary practices such as constructing viable arguments, justifying conclusions, critiquing the reasoning of others, and constructing proofs for mathematical assertions (National Council of Teachers of Mathematics [NCTM], 2000; National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010).

Yet, as the field moves forward to maximize students' learning opportunities for engaging in these disciplinary practices, mathematics educators need to refine their notions of these terms in scholarly activities and in policy documents (Cai & Cirillo, 2014). How, when, and why decisions related to word choices are made (e.g., 'argument' versus 'proof') in curriculum materials, policy documents, and research is an open question. In fact, some researchers have hinted that these choices

are not always purposeful. For example, Lynn Steen, a member of the 1989 NCTM Standards Committee, claimed that uncertainty about the role of proof in school mathematics caused NCTM in its 1989 *Standards* document to resort to, what he called, "euphemisms" such as "'justify,' 'validate,' 'test conjectures,' [and] 'follow logical arguments'" (Steen, 1999, p. 274). Rarely, he stated, did the document use the term 'proof.' Although Steen's comments were published more than 15 years ago, we argue that his proposition, that the role of proof (as well as argumentation and justification) in school mathematics is uncertain, continues to be true today.

One additional challenge of reading extant research or developing a research agenda related to these disciplinary practices is that the classifications offered differ according to the perspective of the researcher, the focus of the research, and the particular data being analyzed (Reid & Knipping, 2010). Only recently have we begun to see mathematics educators offering explicit definitions of these constructs in their work; this is ironic given the importance of definitions in the field of mathematics itself.

Although proof has received more attention in extant research, argumentation as a concept, seems to be garnering new prominence, as it is a process that is playing an important role in policy and curricular documents across many disciplinary fields. The notion of constructing and analyzing arguments appears in the most recent standards in four core K-12 disciplines – English, mathematics, social studies and science. For example, the *Common Core State Standards for Mathematics (CCSSM)* includes as one of its standards for mathematical practice, "Construct viable arguments and critique the reasoning of others." Similarly, *CCSS for English Language Arts* devotes a portion of an appendix to "The Special Place of Argument in the Standards," emphasizing argumentation as critical for success in college and careers (Appendix A, pp. 24-25). The National Council of the Social Studies (NCSS) highlights argumentation as an aspect of historical thinking, with a focus on causation and argumentation (NCSS, 2013). A summary table of where and how references to arguments or argumentation appear in the policy documents is included in Table 1.

On a related note, although not an explicit Standard for Mathematical Practice, we find evidence in past and current mathematics standards documents that *justification* is considered an important mathematical practice. For example, in the 1989 *Standards*, it was noted by the authors that throughout the document, "verbs such as *explore, justify,…describe, develop,* and *predict* are used to convey this active physical and mental involvement of children in learning the content of the curriculum" (NCTM, 1989, p. 17). In particular, students are asked to "justify their answers and solution processes" (p. 29) as part of the Mathematics as Reasoning Standard. In *Principles and Standards for School Mathematics (PSSM)* (NCTM, 2000), geometry is positioned as a natural site for the development of students' "reasoning and justification skills" (p. 41). Justifying is also explicitly linked to the Reasoning and Proof, Communication, and Problem Solving Process Standards in *PSSM*. Finally, the authors of *CCSSM* (NGA & CCSSO, 2010) consider "the ability to justify, in a way appropriate to the student's mathematical maturity, *why* a particular mathematical statement is true or where a mathematical rule comes from" (p. 4) to be a hallmark of mathematical understanding. Looking across time, analyses of standards adopted prior to that of *CCSSM* suggest a relatively infrequent inclusion of the terms justify/justification in content standards language (Larnell & Smith, 2011), with a significant increase in such usage in *CCSSM* (Kosko & Gao, in press). Together, these examples demonstrate that justification has been considered to be important in school mathematics for more than 35 years, with increased focus in recent years.

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona.

Table 1: Summary of References to Argument(ation) in Policy Documents

Standards	Role in Standards	Specific Reference for Practice or Recommendation
Common Core State Standards for Mathematics	One of eight Standards for Mathematical Practice that should be developed in students	3. Construct viable arguments and critique the reasoning of others. (p. 6)
Common Core State Standards for English Language Arts & Literacy	Argumentation identified as the first of the Standards' "Three Text Types"	The Special Place of Argument in the Standards in Appendix A: Research Supporting Key Elements of the Standards (p. 24)
Next Generation Science Standards	One of eight Science and Engineering Practices in the NGSS identified as essential for all students to learn	7. Engaging in argument from evidence (Appendix F, p. 13)
National Curriculum Standards for Social Studies	One of three factors related to "powerful" social studies teaching and learning	Teachers show interest in and respect for students' thinking and demand well-reasoned arguments rather than opinions voiced without adequate thought or commitment. (p. 13)

Highlights from the Year 1 Working Group Discussions

As described above, the three main activities from the Year 1 Working Group included: the Concept Map/Diagram activity, Weber's overview of where the field is now with respect to proof, and the panel discussion that featured prominent researchers of argumentation, justification and proof. Here we summarize some of the key points of discussion from these activities.

Weber's presentation highlighted different traditions and points of disagreement, for example, citing Reid's (2001) observation that research simultaneously suggests that secondary students struggle to construct proofs, while at the same time suggesting that primary children are capable of engaging in proof. Weber prompted the group to consider how different traditions may inform each other in order to advance the field collectively. In particular, he outlined three broad traditions in proving: *proving as problem solving, proving as convincing,* and *proving as socially embedded activity.* Each corresponds to a different focus for research and/or instruction. Weber offered two thought-provoking suggestions, both of which may help us understand the lack of convergence in results and definitions. One suggestion was that proof may not be a singular, easily defined concept, but rather a *cluster concept,* as used by Lakoff (1987). In this sense, there is no list or decision procedure to identify a proof, but rather there is a set of features associated with the concept, and many - but not all - apply in any one instance.

The second suggestion was that the features or properties associated with proof may be closely interrelated for mathematicians, but not for students. In particular, for mathematicians, a convincing argument and socially sanctioned argument are often one-in-the-same. For students, however, those are not tightly connected and may describe very different types of arguments. He suggested, "Perhaps much of the disagreement amongst mathematics educators is that they are using proof as a shorthand to denote things that are different to students but similar for mathematicians" (Cirillo et al., 2016, p. 7).

The panel discussion on our second day further raised awareness of how crucial it is to not only defines one's terms, but also the context of one's work. For example, Bieda and Conner both work closely with students and teachers in secondary settings, and in that context, they have found proof

and proving to be terms that distance or invoke conceptions of end product and formality. Consequently in their work, they have chosen different focal constructs. Both Bieda and Conner articulated an explicit link of their work to students' proof-producing capacities at the tertiary levels, but do not centralize that term in their research in school mathematics.

Our discussions and subsequent analyses of the Concept Diagrams/Maps offered additional information about the variety of ways these three important constructs are understood in relation to one another. During the gallery walk and subsequent analyses across the Concept Diagrams/Maps, we could discern little to no agreement about the relationship between justification and argumentation. It seemed that participants held these two constructs (justification and argumentation) either fully distinct from the set of things we call proof, or that proof was a very specific version of each of these. Some considered justification a subset of arguments; others positioned arguments as a subset of justifications; and still others had them as overlapping, but not concurrent, sets.

Proof (and proving) seemed to be of a different nature than argumentation and justification for our participants and proof participants either positioned it at the far end of a continuum of the constructs, a plane above, or as a separate entity. Alternately, however, others considered proof to be a specific subset of arguments and justifications. Additional details can be found in the recent White Paper (Cirillo et al., 2016). A question raised in the discussion on Day 3 was whether proof was so valorized that we position it as the "desired end product" for all arguments, even when that might not be a productive or educative goal. This lack of not only convergence, but general clarity provides an important opportunity for further exploration and raises questions about the consequences of these different concepts. It implores us to continue to work to develop a framework to connect these constructs and clarify not only our commitments and definitions, but the interrelationships among these important ideas.

Conceptual Focus for Year 2

As we move into our second year as a Working Group, we propose to focus our efforts on getting clearer about definitions of argumentation, justification, and proof with a desired goal of developing a level of shared understandings about these constructs. One idea that emerged from the panel discussion last year was how the setting, or context, of one's research informs the definitions of argumentation, justification or proof (or their corresponding actions) that a researcher uses as they carry out the work. Balacheff (2002) wrote that proof "depends upon content and context" (p. 24). Even within applied and pure fields of mathematics, what scholars accept as proof may have different characteristics.

One way to examine our conceptions of justification, argumentation and proof could be to explicitly acknowledge our positionality and commitments as researchers, particularly related to the content and/or contexts in which we study these products or processes. Consider Table 2 that shows a variety of definitions for these constructs from existing literature (summarized in Cirillo, Kosko, Newton, Staples, & Weber, 2015).

Table 2: Varying Definitions for Argumentation, Justification, and Proof

Proof/Proving	Argumentation	Justification/Justify
"the process employed by an individual to remove or create doubts about the truth of an observation" (Harel & Sowder, 1998, p. 241)	"mathematical explanation intended to convince oneself or others about the truth of a mathematical idea" (Mueller, Yankelewitz, & Maher, 2012, p. 376)	"to provide sufficient reason for" (National Research Council, 2001, p. 130)
"arguments consisting of logically rigorous deductions of conclusions from hypotheses" (NCTM, 2000, p. 55)	"discursive exchange among participants for the purpose of convincing others through the use of certain modes of thought" (Wood, 1999, p. 172)	"an argument that demonstrates (or refutes) the truth of a claim that uses accepted statements and mathematical forms of reasoning" (Staples, Bartlo, & Thanheiser, 2012, p. 448)
"a mathematical argument, a connected sequence of assertions for or against a mathematical claim, with the following characteristics: (1) It uses statements accepted by the classroom community (*set of accepted statements*) that are true and available without further justification; (2) It employs forms of reasoning (*modes of argumentation*) that are valid and known to, or within the conceptual reach of, the classroom community; and (3) It is communicated with forms of expression (*modes of argument representation*) that are appropriate and known to, or within the conceptual reach of, the classroom community." (Stylianides, 2007, p. 291)	"the process of making an argument, that is, drawing conclusions based on a chain of reasoning" (Umland & Sriraman, 2014, p. 44)	

The definitions in Table 2 take on new meaning when considered in light of the contexts and/or content in which they arose in the researchers' work, or when considering the researchers' positionality and commitments. Might it be possible that researchers could define proof differently in studying a middle school mathematics classroom than a high school geometry classroom? Consider specifically the widely cited definition of proof by Harel and Sowder (1998). For research on proving in middle school classrooms, does it make sense to view proof being done by a middle school student as something that removes or confirms doubt? How might the findings change if we are able to capture how students' define proof for their work in class? We hope for the working group to address these, and other questions, by clarifying our assumptions and positionality as researchers to get clearer about the definitions for argumentation, justification, and proof we use in our research and to advance our understanding of how contexts and content informs our conceptions.

Plan for the Working Group

As we approach Year 2, our goal for the working group is to continue to facilitate communication and collaboration among members of the mathematics education community who are involved in research and scholarship using constructs that include argumentation, proof, and justification. One of

the goals for this year is to understand the multiple definitions of these terms that inform our work. We also acknowledge that the context in which we work influences our definitions and how we use them, and we intend to examine how context relates to definitions. A second goal for this year is to focus on connecting people with similar interests in hope of facilitating lasting collaborations to further knowledge related to argumentation, proof, and justification in the field.

Session 1: Existing definitions and influence of context

In Session 1, we will begin with introductions and collecting participants' informal definitions or descriptions of argumentation, proof, and justification. An interactive panel discussion will provide additional insights and a basis for the next two sessions. Panelists will discuss the definitions or descriptions of argumentation, proof, and justification central to their work. They will also discuss how they see the relationships between the context of their research and the definitions or descriptions of argumentation, proof, and justification they employ in their analysis and findings from their work.

We have invited three panelists to participate, chosen to provide a variety of perspectives to the working group. Mara Martinez is an Assistant Professor of Mathematics Education at University of Illinois at Chicago. Her research has investigated argumentation and proof in algebra classes, and she will focus her remarks on conceptions of argumentation. Eric Knuth is a Professor of Mathematics Education at University of Wisconsin at Madison. He has published on the development of middle school students' capacities to justify and prove, as well as, more recently, the role of examples in doing proof. He will focus his remarks on conceptions of justification. Finally, Orit Zaslavsky is a Professor of Mathematics Education at New York University. Her research also focuses on the interplay among mathematical examples, definitions and proof, and her remarks will focus on conceptions of proof. The discussion will be moderated, with the following potential questions to provide focus for the discussion: *(1) Which construct (argument/argumentation, proof/proving, justification/justify) do you use most in your work? How do you define this construct? What has informed that choice? (2) What are your views on the need for researchers to establish unique definitions for these constructs? (3) In what ways are your conceptions of the phenomena central to your research still evolving*? As was done when the working group convened in 2015, Sam Otten will record the panel discussion and post the recording to the MathEd Podcast.

Following the panel discussion, we will invite participants to meet with panelists in three breakout groups, so that attendees can interact with a panelist in a small group setting and continue the conversation about the panel discussion and their related work. These small group breakouts are intended to serve as a catalyst for networking groups that will develop throughout the sessions.

Session 2: Constructing definitions for different contexts

Session 2 will extend the discussion from Session 1 regarding the applicability and interplay of particular constructs and definitions to specific contexts for research and practice. We will begin the session with a shared, virtual representation of definitions and descriptions generated in Session 1 (i.e., using a Google Spreadsheet). We will engage participants in examining which aspects of the definitions are more useful for working in particular contexts and why those aspects seem to be important. To facilitate this discussion, participants will examine how various definitions apply when presented with artifacts of student argumentation, justification, and proof. Specifically, participants will be provided samples of data from prior studies to consider the flexibility, applicability, and limitations of posed definitions and descriptions. These data will include artifacts from published studies on argumentation, justification, and proof (used with permission) from elementary, middle, secondary, and tertiary levels such as students' writing and transcripts of verbal interactions.

Following exploration of the data artifacts, we will facilitate a group discussion of the affordances and constraints of different definitions given the nature of the data, context, and research

focus. The session will conclude with participants completing a survey identifying the definitions they wish to explore further. The survey results will be used to establish preliminary networking groups for the third session.

Session 3: Networking groups

Our goal for Session 3 is to establish networking groups among working group participants. The long-term goal for the establishment of the networking groups is for members to generate knowledge and products related to the overarching goal of the working group: to advance the field's collective understanding of argumentation, justification, and proof. We will use the surveys, definitions, and contexts from the first two sessions to create suggested networking groups at the start of Session 3. However, participants may re-assign themselves to other, or new, networking groups. Working group participants will be given one guiding question for Session 3, specifically, given the definition for argumentation/ justification/proof that they consider of interest, which potential research questions would move the field forward in understanding this topic (argumentation/justification/proof)?

We anticipate that allowing participants to spend time discussing such potential research questions will facilitate variations of networking groups to become reading groups, research discussion groups or research collaboration groups. Further, the discussion will allow for the introduction of a wiki page for members of the networking groups to share research related to specific foci and as a general resource for furthering the goals of the working group. We will conclude the session with participants identifying productive points of collaboration within their groups, as well as a survey of which network group participants wish to continue their work, and how, during the year.

Anticipated Follow-up Activities

We anticipate that the products and follow-up activities from Year 2 will build on the products from our previous year. We intend to expand the white paper to include new sections based on our working group activities. We anticipate new sections will include a collection of definitions of argumentation, justification, and proof along with rationale for their uses. We also plan to develop a wiki or website where we can share citations/papers related to these constructs, particularly articles that define one or more of them explicitly. We will conduct an analysis of the definitions/descriptions of the constructs and contexts collected during the working group to continue to develop a framework that illustrates the perceived relationships between and among them. Collaborations, should they materialize out of the networking groups, would also be an important product of the working group.

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IMPROVING PRESERVICE ELEMENTARY TEACHER EDUCATION THROUGH THE PREPARATION AND SUPPORT OF ELEMENTARY MATHEMATICS TEACHER EDUCATORS

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This working group will bring together researchers and practitioners to work collaboratively towards unifying goals centered around the work of elementary mathematics teacher educators (eMTEs, including mathematicians and graduate students) who teach elementary mathematics content courses. The working group will aim to conceptualize the work of eMTEs as a unique population of practitioners, their development, and the ways in which they may be supported in this work. Areas of focus will include, but are not limited to: (a) the development of prospective elementary teachers' learning in content courses; (b) the content and pedagogy enacted in elementary content courses; (c) the mathematical knowledge needed for teaching teachers; (d) the educational backgrounds and teaching experiences of eMTEs; (e) the preparation of eMTEs at the doctoral level; and (f) the institutional guidance and support from the greater community that eMTEs are currently being offered. Our vision is to create research-based practitioner resources for eMTEs, including: writing and publishing a handbook, based on research and policy recommendations, to serve as a practitioner guide for teaching elementary content courses; and building and maintaining an online repository of instructional resources that eMTEs may utilize in their work with prospective elementary teachers.

Keywords: Teacher Education-Preservice, Elementary School Education, Teacher Knowledge, Instructional Activities and Practices

Motivations for a New Working Group

History

This is a new working group that arose from interactions between the leaders of a PME-38 discussion group on the opportunities and challenges surrounding the preparation and support of mathematics teacher educators who work with elementary teachers (Welder, Jansen, & McCloskey, 2014), the presenters of a PME-39 research report on the purposes of mathematics teacher educators in their teaching of elementary content courses (Taylor & Appova, 2015), and the presenters of the Association of Mathematics Teacher Educators (AMTE) symposium on supporting mathematics teacher educators' work with elementary teachers through multiple perspectives (Taylor, Appova, Welder, & Feldman, 2016). Collective feedback received from participants (which included novice mathematics teacher educators and graduate students) highlighted a strong need for research and resources to support the development of elementary mathematics teacher educators (eMTEs), specifically in their teaching of mathematics content courses for prospective elementary teachers (ePTs).

Problem

Recent research and political initiatives have suggested that, while in teacher education programs, many ePTs receive inadequate preparation to become effective teachers of mathematics

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and do not develop the deep, conceptual knowledge of the mathematics they will teach (e.g., Conference Board of the Mathematical Sciences (CBMS), 2012; Greenberg & Walsh, 2008). These issues stem from, among other things, a lack of clarity and consensus within in the field about what ePTs should learn and experience while in mathematics content courses (e.g., Ball, Sleep, Boerst, & Bass, 2009; Zaslavsky, 2007) and a lack of preparation and support for the eMTEs who teach mathematics content courses for ePTs (e.g., Bergsten & Grevholm, 2008).

There has been a recent increase in the efforts of mathematics teacher educators to study and share the work they are doing with ePTs, primarily regarding the content (e.g., Ball, et al., 2009) and pedagogy (e.g., Lampert et al., 2013) enacted in mathematics methods courses; however, little of this work has specifically focused on elementary content courses (e.g., Bergsten & Grevholm, 2008). Research has indicated that these content courses are often taught by eMTEs, in mathematics departments, who have little to no experience working with elementary content or children and do not receive the training or support necessary to effectively address the needs of the ePTs with whom they work (Masingila, Olanoff, & Kwaka, 2012).

Goals

In response, the leaders have developed this working group to bring together researchers, mentors of mathematics teacher educators, and experts from the larger community to work collaboratively towards unifying goals regarding the teaching of elementary mathematics content courses and the preparation and support of those who teach these courses. More specifically, the aim of this new working group is to generate an understanding of how eMTEs develop as practitioners, particularly in light of recent recommendations, including the *Mathematics Education of Teachers I and II* (CBMS, 2001; 2012), the *National Council of Teachers of Mathematics for the Council of the Accreditation of Educator Preparation Standards* (NCTM CAEP, 2003; 2012), and the forthcoming *Standards for Mathematics Teacher Preparation* (AMTE, 2016).

Specifically, we aim to better understand, document, and address eMTEs' needs by:
(a) providing opportunities for eMTEs to make sense of the content and pedagogical recommendations provided in the aforementioned documents; (b) inviting eMTEs to reflect on their current practices in light of these recommendations; (c) understanding the challenges eMTEs face in addressing these recommendations in their teacher preparation courses/programs; (d) providing a platform for eMTEs to develop a shared vision for mathematics teaching and learning in elementary content courses; and (e) constructing resources, including research-based strategies and recommendations, for strengthening the preparation and support of eMTEs. From these discussions, we aim to open new scholarly and practitioner-based avenues for research and collaborations in the field of eMTE development.

Focal Issues

Jaworski (2008) defines mathematics teacher educators as "professionals who work with practicing teachers and/or prospective teachers to develop and improve the teaching of mathematics" (p. 1). Studying the work and practices of teacher educators is a fairly new area of research in the field of mathematics education. In the past two decades, the related research has mostly fallen into six overall categories: (1) knowledge needed by teacher educators to teach prospective teachers (e.g., Chick & Beswick, 2013; Rowland, Turner, & Thwaites, 2014; Superfine & Li, 2014); (2) general pedagogical practices of mathematics teacher educators (e.g., Dixon, Andreasen, & Stephan, 2009; Steele, 2008); (3) efforts of mathematics teacher educators to improve their own practices (e.g., Berk & Hiebert, 2009; Cady, Hopkins, & Hodges, 2008; Hiebert et al., 2003; 2007; Marin, 2014; Monroe, 2013; Nolan, 2015; Tzur, 2001); (4) actions, goals, purposes, and intentions enacted by experienced mathematics teacher educators in their content/methods courses (e.g., Appova & Taylor, 2014; Taylor, 2013; Taylor & Appova, 2015); (5) effects of particular activities, or series of activities, on

prospective teachers' learning (e.g., Castro 2006; Goodell, 2006); and (6) the design of tasks specifically for use in mathematics content courses for ePTs (e.g., Chval, Lannin, & Bowzer, 2008; Liljedahl, Chernoff, & Zazkis, 2007; Thanheiser, et al., 2016; Van Zoest & Stockero, 2008).

The working group leaders have been directly and indirectly involved in this research community, collecting, analyzing, and reporting data related to various aspects of the work of eMTEs. A major impetus for our work is our overlapping interests in strengthening the development of ePTs' mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008) through the preparation and support of the eMTEs who work with them. The leaders of the working group have critically considered several factors influencing the work and development of eMTEs, including: (a) educational backgrounds, research, and teaching experiences of eMTEs (especially those related to elementary mathematics); (b) specialized knowledge needed by eMTEs for teaching mathematics content to ePTs; (c) actions taken by eMTEs in elementary mathematics content courses and the purposes/intentions behind those actions; and (d) preparation and support of eMTEs for their work with ePTs. Below, we summarize the current research foci of the working group leaders and how we envision our work connecting to the larger goals of the working group, both during and after the conference. For the purpose of the working group, we will use the term "elementary" to broadly encompass PreK-8 grade-levels to be inclusive of varying state-certification requirements and institutional course offerings.

Educational Backgrounds and Teaching Experiences of eMTEs

Many, if not most, instructors of elementary mathematics content courses have not worked as elementary teachers themselves (Masingila, Olanoff, & Kwaka, 2012). Welder and McCloskey (under review) have been working to investigate this phenomenon through three lenses: (a) analyzing job opportunities for mathematics teacher educators, (b) surveying early-career mathematics teacher educators, and (c) conducting focus group interviews with educators who lack elementary experience yet work primarily with elementary teachers. Preliminary findings suggest that most mathematics teacher educators will work with elementary teachers in some capacity at some point in their careers and highlight that most will do this work without being able to claim the title or draw upon the expertise that only comes with having worked as a practicing elementary teacher (Welder, McCloskey, & Searle, 2013). By better understanding the nature of the challenges faced by eMTEs, especially those without elementary-specific educational backgrounds or teaching experience, the goal of this research is to find ways in which eMTEs may be better supported for their work with elementary teachers.

Connection to working group. In addition to better understanding the multitude of factors that may affect the quality of eMTEs' interactions with teachers, there is much work to be done to identify *where* and *how* eMTEs develop (or *could* or *should* develop) such knowledge, experience, and dispositions toward elementary mathematics teaching. Reys and Reys (2012) note, "[P]rofessionals continue to grow and adapt throughout their careers. It is unreasonable to expect that a Ph.D. program will adequately prepare mathematics educators for the wide range of challenges and expectations they will confront" (p. 290). How might responsibility for eMTE development be shared among doctoral programs, hiring institutions, and individual eMTEs themselves? The field of mathematics education could benefit from collectively identifying learning goals for the preparation of eMTEs and developing practices to support eMTEs in reaching these goals.

Specialized Knowledge of eMTEs Needed for Teaching Mathematics to ePTs

Much research has been done on the nature of mathematical knowledge for teaching (MKT) children (e.g., Ball, et al., 2008). However, much less research has looked at the knowledge required by mathematics teacher educators to facilitate the development of ePTs' MKT. It is generally assumed that teachers need to know *more* than their students, but in looking at the mathematical

knowledge for teaching teachers (MKTT), researchers are still investigating what this "more" entails. Deborah Zopf (2010) identifies three areas where the work of teaching mathematics to teachers differs from teaching the subject to students. First, the content is different; students are learning mathematics, whereas ePTs are learning MKT. Second, the audience is different; eMTEs are teaching adults who are already somewhat familiar with the mathematics content that they are learning, though mostly in procedural ways (Thanheiser et al., 2014). Third, the goals for instruction are different; students generally learn mathematics for academic purposes, whereas ePTs learn mathematics for the purpose of teaching it to children.

Connection to working group. Research is needed to look more deeply at how the differences between teaching students and ePTs influence the knowledge required by eMTEs. What challenges are involved in teaching mathematics to ePTs who already have some (albeit mostly procedural) knowledge of the content, but lack the deep, conceptual understandings needed to teach the subject to children? How should eMTEs determine the scope and sequence of mathematics content courses? Is less content, studied at a deeper level, more effective than covering more topics in less depth? How can eMTEs design formative and summative course assessments to capture the development of ePTs' conceptual understandings (Hill, Schilling, & Ball, 2004)? These are merely a few of the field's unanswered questions regarding MKTT.

Classroom Actions and Purposes of eMTEs in Mathematics Content Courses

Taylor and Appova (in preparation) closely examined the classroom actions, and purposes and intentions behind those actions, of experienced eMTEs. The authors define *classroom actions* as teaching practices eMTE employ during instruction (i.e., what eMTEs say and do while instructing ePTs, including the content that is visually presented) and *purposes/intentions* as what eMTEs want ePTs to learn from content and methods courses, which may or may not be listed in the course syllabi and/or curriculum. Preliminary results from this study indicate that experienced eMTEs situate their actions and purposes around specific learning opportunities that provide ePTs with essential foundations for developing two specific domains of MKT: *specialized content knowledge* (SCK), mathematical knowledge separate from pedagogy and knowledge of students, that is not needed in other professional settings (Ball et al., 2008), and *pedagogical content knowledge* (PCK), a special type of knowledge that blends "content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interest and abilities of learners" (Shulman, 1987, p. 8).

More specifically, in their content courses, experienced MTEs provided opportunities for ePTs to re-learn elementary mathematics in a way that helped them to make pedagogical connections to children's learning, teaching, and curriculum. They also focused on providing ePTs with opportunities to experience mathematics beyond the surface-level procedural learning, developing a deeper conceptual knowledge and problem-solving nature of mathematics through connections, representations, and modeling (echoed in the *Standards*-based documents, put forth by NCTM, 2000). These visions are described by Magnusson, Krajcik, and Borko (1999) as *orientations towards teaching*, encompassing "teachers' knowledge and beliefs about the purposes and goals for teaching [mathematics] at a particular grade level" (p. 97).

Connection to working group. The authors hope to open a window of collaborative research opportunities to extend their current work and provide the mathematics education research community with additional and new insights into the knowledge, practices, and purposes that eMTEs draw upon in their content and methods courses. Without a shared knowledge base, eMTEs will continue to teach and design courses with limited resources and support, contributing to even greater variability variability across teacher preparation programs (Floden & Philipp, 2003). Specifically, the authors are interested in exploring ways in which university faculty (especially non-mathematics educators, such as, graduate students, adjuncts, and mathematics faculty) can be supported in

developing ePTs' PCK and SCK through content and methods courses. Such work will help the mathematics education community develop a shared vision, curriculum, and knowledge base for the work of eMTEs, specifically in regards to their classroom actions, practices, and intentions/purposes when teaching courses for ePTs.

The Preparation and Support of eMTEs

With the recent policy documents and recommendations regarding elementary teacher preparation, there comes a greater need to prepare and support eMTEs in their work with ePTs. However, Masingila, Olanoff, and Kwaka (2012) found that, in their survey about who teaches mathematics content courses for ePTs, over half of the respondents indicated that there is no training or support for the eMTEs who teach content courses at their institutions. Of those who indicated that there is support, most described it as informal and infrequent (only occurring once or twice a year). Furthermore, findings from a study of novice mathematics teacher educators highlighted their collective need for gaining additional teaching experience during doctoral preparation programs and receiving mentorship during their initial years as university faculty (Yow, Eli, Beisiegel, McCloskey, & Welder, 2016). Kimani, Olanoff, and Masingila (2012) were able to successfully support the development of novice eMTEs by creating mentoring programs and relationships between novice and experienced eMTEs. In their research, the authors formed a community of practice to reflect on the processes of learning to teach mathematics content courses for ePTs. Through this community of practice, novice and experienced eMTEs were able to work together to establish learning goals for their ePTs and reflect on their abilities to help ePTs achieve these goals.

Connection to working group. This avenue of research is strongly aligned with all of the aforementioned areas of our working group; yet, little research has been conducted in this area suggesting that the field of mathematics education needs to identify best practices in preparing and supporting eMTEs and implement these practices throughout various stages of eMTEs' development. The leaders of this working group believe that this work is particularly important for novice mathematics teacher educators who are often assigned to teach content and/or methods courses for ePTs, within their first few years as faculty, regardless of their backgrounds or experiences (Welder & McCloskey, under review). Forming communities of practice to address the mathematical preparation of elementary teachers and conduct action research around the teaching of mathematics content courses for ePTs will help the field better recognize, understand, and address the needs of eMTEs. By helping the field better define the goals, purposes, and classroom practices that may best support the development of ePTs' learning, eMTEs will be able to confront their own knowledge, backgrounds, and experiences that may aid (or impede) their ability to facilitate ePTs' development. This knowledge may help us begin to develop a shared vision for effective mentorship practices and professional development structures specific to the needs of eMTEs (including mathematicians and graduate students).

Projected Working Group Outcome

Our primary goal is publishing a handbook containing research-based practitioner resources, based on theories of eMTE development, that will guide eMTEs (including mathematicians, graduate students, and those who prepare and mentor eMTEs), in their work with elementary teachers. Although the focus of our materials will be specifically on the teaching of content courses for ePTs, they will also be relevant to eMTEs who provide inservice teacher professional development. Our plan is to contact several publishers to find an appropriate outlet for our work, mainly focusing on opportunities for publishing research-based recommendations and guidelines (rather than individual research reports) such as those provided by special issues of Theory into Practice (published by Taylor & Francis).

We foresee the printed handbook being comprised of several interrelated chapters for eMTEs to draw upon when teaching elementary content courses. Potential chapters and/or topics may include, but are not limited to:

- An overview of elementary mathematics curriculum development in the United States (e.g., state-level standards and assessments; commonly used elementary curricula, state-specific teacher certification requirements);
- The mathematical content and practices recommended for supporting the development of ePTs' MKT through content coursework, as explicated in local and U.S. national standards and policies related to elementary teacher preparation (e.g., CBMS, NCTM CAEP, AMTE);
- Unique aspects of the ePT population (including an overview of the research on ePTs' knowledge, misconceptions, attitudes, beliefs, dispositions, etc.);
- The ways in which teaching mathematics content courses for ePTs differs from the teaching of other mathematics courses, including discussions of unique challenges faced by eMTEs when teaching teachers mathematical content they will eventually teach and asking ePTs to develop conceptual knowledge of content for which they are most likely procedurally fluent;
- Task design for content courses for ePTs (including samples of problem-based exploratory tasks, research-based recommendations for engaging teachers in cognitively challenging tasks [Stein, Grover, & Henningsen, 1996; Stein, & Smith, 1998], and resources for finding research-based tasks designed for elementary content courses);
- Benefits of having ePTs work directly with elementary children (through lab schools or fieldwork experiences) and/or exposing them to the mathematical thinking of children during content courses (e.g., children's math ingenuities, problem-solving strategies, and misconceptions, Cognitively Guided Instruction [CGI; Carpenter, Fennema, Franke, Levi, & Empson, 2015], van Hiele levels of thinking [Fuys, Geddes, & Tischler, 1988]);
- Sample course syllabi and suggestions for the scope and sequencing of content based on various university course offerings (e.g., how content goals and pacing may differ at institutions offering one, two, or three semester- or quarter-long courses);
- Valid and reliable assessments that eMTEs can use to measure various aspects of ePT development throughout content courses and teacher preparation programs, including MKT (e.g., Hill, Schilling, & Ball, 2004; Welder & Simonsen, 2011), attitudes, beliefs, and dispositions towards the teaching and learning of mathematics (e.g. Jong, Hodges, Royal, & Welder, 2015), self-efficacy (e.g., Bandura, 2006; Tschannen-Moran, Hoy, & Hoy, 1998), and math anxiety (e.g., Bursal & Paznokas, 2006; Vinson, 2001);
- A printed collection of resources relevant to the development of ePTs, including books, activity guides, articles, student artifacts, information regarding projects, conferences, and professional development opportunities for eMTEs, and online resources for tasks, lessons, and video collections (e.g., the IMAP Integrating Mathematics and Pedagogy: Searchable Collection of Children's-Mathematical-Thinking Video Clips, [Philipp, Cabral, & Schappelle, 2012]; Videos of How Children Learn [Children's Mathematical Learning, Feikes, 2016]; GDK Math Lessons [GDKMath, 2016]; CGI Online Video Clips [Carpenter, Fennema, Franke, Levi, & Empson, 2015]).

Plan for Working Group Sessions

The *first* session will begin with an overview of the literature on mathematics teacher educators to build rationale and purpose for this working group. The working group leaders will highlight gaps

in the extant literature surrounding eMTE development and use these as starting points for guiding the direction of the group's work. Key activities will include:

- Providing a platform for eMTEs to begin developing a shared vision for elementary content courses and the learning of ePTs;
- Identifying challenges faced by eMTEs and how they may differ from those of classroom mathematics teachers and other teacher educators;
- Sharing the leaders' goals for developing research-based resources for eMTEs teaching content courses;
- Providing the opportunity for participants to articulate their specific interests regarding the development of eMTEs and how they might fit into the overall goals of the working group.

The *second* session will provide opportunities for participants to break into smaller subgroups based on their areas of interest. Each subgroup will discuss the current research related to their area with the goal of making research-based recommendations and suggestions related to their topic. This may include identifying possible avenues for future research and resource development. Each group will have an opportunity to brainstorm and develop an outline for how their topic may contribute to the development of a practitioner handbook for eMTEs.

During the *third* session, each group will finalize their discussions, prepare a report to present to the larger group, and map-out a plan for "next steps" in moving forward with the work, goals, and outcomes of the working group. In particular, participants will identify specific opportunities for organizing future work and collaborations, including co-authoring the chapters of the upcoming publication and contributing to anticipated follow-up activities.

Anticipated Follow-up Activities

In addition to writing and publishing the practitioner handbook described above, the leaders of the working group intend to build, develop, and maintain an online repository of resources and instructional materials specifically designed for eMTEs. The goal of providing the repository is to support eMTEs in providing effective learning opportunities for ePTs and align their instruction with national standards and recommendations for mathematics teacher preparation (AMTE, 2016; CBMS, 2001; 2012). The online repository will offer explicit and concrete information about the instructional resources provided, such as mathematical tasks and activities, sample lesson plans, ideas for course scope and sequencing, videos of children's mathematical thinking and experienced eMTEs leading ePT instruction, and links to relevant online resources, such as virtual classrooms and manipulatives. The leaders plan on pursuing grant opportunities to secure funding sources to support the continued development and maintenance of this online repository. Together, the handbook and repository will provide an effective platform for building a community of practice of eMTEs and supporting the development of their work with ePTs.

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